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Algebraic properties of rings of continuous functions

by

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Abstract. This paper is devoted to the study of algebraic properties of rings of continuous functions. Our aim is to show that these rings, even if they are highly non-noetherian, have properties quite similar to the elementary properties of noetherian rings: we give going-up and going-down theorems, a characterization of z-ideals and of primary ideals having as radical a maximal ideal and a flatness criterion which is entirely analogous to the one for modules over principal ideal domains.

Introduction. Throughout this paper, $C(X)$ will denote the ring of real-valued continuous functions defined on a topological space X and $C^*(X)$ will be the subring of bounded functions.

The paper is divided into three sections. In the first one, we shall prove a theorem showing close relationships between topological properties of a continuous map $X \to S$ and algebraic properties of the induced morphism of rings $C(S) \to C(X)$. Explicitly, we shall prove the following

THEOREM. *If a continuous map $X \to S$ is open and closed* (*respectively*, *open and proper*) *then going-up and going-down theorems hold for the morphism $C^*(S) \to C^*(X)$* (*respectively*, *for $C(S) \to C(X)$*).

Using this theorem we shall prove that, under the same hypothesis, the continuous map between the prime spectra $\mathrm{Spec}(C^*(X)) \to \mathrm{Spec}(C^*(X))$ is open and closed. Since the Stone–Čech compactification βX is homeomorphic to the maximal spectrum of $C^*(X)$, this result generalizes that obtained by Isiwata [5] for the extension $\beta X \to \beta S$.

We think that this going-up and down theorem may also be used to establish other results concerning relationships between algebraic and topological properties. In fact, we have used it in [9] to characterize finite branched

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coverings between topological spaces by means of the algebraic properties of the corresponding morphisms of rings.

The second section deals with z-ideals. If $\mathfrak{m}$ is a maximal ideal in a noetherian ring, it is well known that any $\mathfrak{m}$-primary ideal contains some power $\mathfrak{m}^n$. If $\mathfrak{m}$ is a maximal ideal in $C(X)$, then any power $\mathfrak{m}^n$ coincides with $\mathfrak{m}$, and we prove that $\mathfrak{m}$ *is the unique* $\mathfrak{m}$-*primary ideal in* $C(X)$. In fact, we shall prove that if the radical of an ideal I of $C(X)$ is a z-ideal, then $I = \mathrm{rad}(I)$.

In the third section, these results about z-ideals are applied to obtain a pair of flatness criteria for $C(X)$-modules. We prove that the flatness criterion given by Neville [11] for F-spaces remains valid for $C(X)$-modules of finite presentation for any space X. Finally, we state the following criterion:

THEOREM. *If every closed set in* X *is a zero-set*, *then a* $C(X)$-*module* M *of finite presentation is flat if and only if it is torsion-free.*

Preliminaries. Concerning rings of continuous functions, we use the same notation and terminology as in [4] and, as usual in this framework, X and S will henceforth be completely regular spaces and every map $X \to S$ will be assumed to be continuous. For algebraic concepts, the reader may consult [1], [3] or [8]. Nevertheless, we review some notation that will be used in the paper.

The set of prime ideals in a ring A, i.e., the *prime spectrum*, will be denoted by Spec(A). We shall consider this space endowed with the Zariski topology: For any subset C of A, let $\mathrm{V}(C) = \{\mathfrak{p} \in \mathrm{Spec}(A) : C \subseteq \mathfrak{p}\}$ and take as closed sets in Spec(A) all subsets of the form $\mathrm{V}(C)$. If $f \in A$, we put $\mathrm{D}(f) = \mathrm{Spec}(A) - \mathrm{V}(f)$. The collection of those open sets forms a basis of open sets of Spec(A). Each morphism of rings $h : A \to B$ induces a continuous map $h^* : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ which sends $\mathfrak{p} \in \mathrm{Spec}(B)$ to $h^{-1}(\mathfrak{p}) \in \mathrm{Spec}(A)$.

The definition and basic properties of rings and modules of fractions may be found in [1] or in [8]. Let $\mathfrak{p}$ be a prime ideal in a ring A. If M is an A-module we denote by $M_{\mathfrak{p}}$ the localization, or module of fractions, of M with respect to the multiplicatively closed subset $A-\mathfrak{p}$ of A. If $h : A \to B$ is a morphism of rings, we denote by $B_{\mathfrak{p}}$ the ring of fractions of B with respect to the multiplicatively closed subset $h(A - \mathfrak{p})$.

LEMMA 0.1. *Let* $h : A \to B$ *be a morphism of rings and* $h^* : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ *the induced continuous map.*

(i) *If* h *is a surjective morphism*, *then* h^* *is a homeomorphism between* $\mathrm{Spec}(B)$ *and the closed subset* $\mathrm{V}(\ker(h))$ *of* $\mathrm{Spec}(A)$.

(ii) *If* B *is the ring of fractions of* A *with respect to a multiplicatively closed subset* S *of* A, *i.e.*, $B = \{a/s : a \in A,\ s \in S\}$ *and* $h : A \to B$

is the natural morphism which sends $a \in A$ *to* $a/1 \in B$, *then* h^* *is a homeomorphism between* $\mathrm{Spec}(B)$ *and the subset* $\{\mathfrak{p} \in \mathrm{Spec}(A) : \mathfrak{p} \cap S = \emptyset\}$ *of* $\mathrm{Spec}(A)$. *In particular*, $\mathrm{Spec}(A_{\mathfrak{p}}) = \{\mathfrak{q} \in \mathrm{Spec}(A) : \mathfrak{q} \subseteq \mathfrak{p}\}$.

(iii) *If* h *is an injective morphism*, *then every minimal prime ideal in* A *belongs to the image of* h^*. *Hence*, h^* *has dense image.*

Proof. (i) and (ii) are well known.

(iii) Let $\mathfrak{p}$ be a minimal prime ideal in A. If h is an injective morphism, then so is the morphism $A_{\mathfrak{p}} \to B_{\mathfrak{p}}$ induced by h (see Proposition 3.9 of [1]). Hence, $B_{\mathfrak{p}} \neq 0$ and therefore $\mathrm{Spec}(B_{\mathfrak{p}}) \neq \emptyset$. If $\mathfrak{q} \in \mathrm{Spec}(B_{\mathfrak{p}})$, then $h^*(\mathfrak{q}) \in \mathrm{Spec}(A_{\mathfrak{p}})$. Since $\mathfrak{p}$ is a minimal prime ideal in A, one has $\mathrm{Spec}(A_{\mathfrak{p}}) = \{\mathfrak{p}\}$. Hence $h^*(\mathfrak{q}) = \mathfrak{p}$. ∎

We shall use the homeomorphisms given in (i) and (ii) of the above lemma without further mention. Thus, if I is an ideal of a ring A, we shall not distinguish $\mathrm{Spec}(A/I)$ from $\mathrm{V}(I)$. By a *minimal prime over-ideal* of I we shall mean a minimal prime ideal in A/I.

The *radical* of an ideal I of A is the ideal $\mathrm{rad}(I) = \{f \in A : f^n \in I$ for some $n \in \mathbb{N}\}$. This ideal is just the intersection of all prime ideals containing I (see [8], (1.E)).

Recall that an A-module M is said to be *flat* if the tensor product $\otimes_A M$ is an exact functor. The *support* of M is the set $\mathrm{Supp}(M)$ of prime ideals $\mathfrak{p}$ of A such that $M_{\mathfrak{p}} \neq 0$.

Between the ring $C(X)$ and its subring of bounded functions $C^*(X)$ there exists an algebraic relation that we shall use here:

LEMMA 0.2. $C(X)$ *is the localization*, *or ring of fractions*, *of* $C^*(X)$ *with respect to the multiplicatively closed subset*

$$M_X = \{f \in C^*(X) : 0 \notin f(X)\}.$$

Proof. Every $f \in C(X)$ can be written as the fraction

$$f = \frac{f \cdot (1+f^2)^{-1}}{(1+f^2)^{-1}}. \ ∎$$

An immediate consequence of the above lemma is that $\mathrm{Spec}(C(X))$ is the following subspace of $\mathrm{Spec}(C^*(X))$:

$$\mathrm{Spec}(C(X)) = \{\mathfrak{p} \in \mathrm{Spec}(C^*(X)) : \mathfrak{p} \cap M_X = \emptyset\}.$$

The *maximal spectrum* of $C(X)$, i.e., the subspace of $\mathrm{Spec}(C(X))$ consisting of all maximal ideals in $C(X)$, will be denoted by $M(X)$, and the maximal spectrum of $C^*(X)$ will be denoted by $M^*(X)$. The topology induced on these subspaces by the Zariski topology is also known as the *Stone topology* or *hull-kernel topology*. It is well known that X can be identified with a dense subspace of $M(X)$: each point $x \in X$ defines the maximal ideal $\mathfrak{m}_x = \{f \in C(X) : f(x) = 0\}$; and the same holds for $M^*(X)$: the point

x defines the maximal ideal $\mathfrak{m}_x^* = \{f \in C^*(X) : f(x) = 0\}$. It is also well known that both $M(X)$ and $M^*(X)$ are compact Hausdorff spaces that can be identified with βX, the Stone–Čech compactification of X.

LEMMA 0.3. *Every prime ideal in $C(X)$ is contained in a unique maximal ideal and the map $r_X : \mathrm{Spec}(C(X)) \to M(X)$ that sends each prime ideal to the unique maximal ideal containing it, is a continuous retraction.*

Proof. It is proved in [4], Theorem 2.11, that every prime ideal in $C(X)$ is contained in a unique maximal ideal, and in [2] that r_X is continuous. ∎

Every map $\pi : X \to S$ defines, by composition, two morphisms of rings $C(S) \to C(X)$ and $C^*(S) \to C^*(X)$ that induce continuous maps $\pi_{\mathrm{s}} : \mathrm{Spec}(C(X)) \to \mathrm{Spec}(C(S))$ and $\pi^* : \mathrm{Spec}(C^*(X)) \to \mathrm{Spec}(C^*(S))$ which send $\mathfrak{p} \in \mathrm{Spec}(C(X))$ (respectively $\mathfrak{p} \in \mathrm{Spec}(C^*(X))$) to $\mathfrak{p} \cap C(S) = \{f \in C(S) : f \circ \pi \in \mathfrak{p}\}$ (respectively, to $\mathfrak{p} \cap C^*(S)$). The extension of π to the Stone–Čech compactification, $\pi_\beta : \beta X \to \beta S$, can be taken to be the following composition:

$$\beta X = M(X) \hookrightarrow \mathrm{Spec}(C(X)) \overset{\pi_{\mathrm{s}}}{\to} \mathrm{Spec}(C(S)) \overset{r_S}{\to} M(S) = \beta S.$$

1. Going-up and going-down theorems for open and closed maps

DEFINITION 1.1. Let $h : A \to B$ be a morphism of rings and $h^* : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ the induced continuous map. We say that *the going-up theorem holds for h* if for any $\mathfrak{p}, \mathfrak{p}' \in \mathrm{Spec}(A)$ such that $\mathfrak{p} \subseteq \mathfrak{p}'$, and for any $\mathfrak{q} \in (h^*)^{-1}(\mathfrak{p})$, there exists $\mathfrak{q}' \in \mathrm{Spec}(B)$ such that $h^*(\mathfrak{q}') = \mathfrak{p}'$ and $\mathfrak{q} \subseteq \mathfrak{q}'$. That is to say, $h^*(\mathrm{V}(\mathfrak{q})) = \mathrm{V}(\mathfrak{p})$.

Similarly, *the going-down theorem holds for h* if for any $\mathfrak{p}, \mathfrak{p}' \in \mathrm{Spec}(A)$ such that $\mathfrak{p} \subseteq \mathfrak{p}'$, and for any $\mathfrak{q}' \in (h^*)^{-1}(\mathfrak{p}')$, there exists $\mathfrak{q} \in \mathrm{Spec}(B)$ such that $h^*(\mathfrak{q}) = \mathfrak{p}$ and $\mathfrak{q} \subseteq \mathfrak{q}'$. That is to say, $h^*(\mathrm{Spec}(B_{\mathfrak{q}'})) = \mathrm{Spec}(A_{\mathfrak{p}'})$.

LEMMA 1.2. *Let $h : A \to B$ be a morphism of rings and $h^* : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ the induced continuous map. The going-up theorem holds for h if and only if h^* is closed.*

Proof. If h^* is closed, then for any $\mathfrak{q} \in \mathrm{Spec}(B)$, $h^*(\mathrm{V}(\mathfrak{q}))$ is a closed set, hence it contains the closure $\mathrm{V}(h^*(\mathfrak{q}))$ of $h^*(\mathfrak{q})$. Therefore, the going-up theorem holds for h.

Conversely, suppose that the going-up theorem holds for h. Any closed subset of $\mathrm{Spec}(B)$ is $\mathrm{V}(I)$ for some ideal I of B. It is easy to check that if I is an ideal of B, then the going-up theorem also holds for the morphism $A/h^{-1}(I) \to B/I$ induced by h. Since this morphism is injective, Lemma

0.1(iii) proves that any minimal prime over-ideal $\mathfrak{p}$ of $h^{-1}(I)$ comes from a prime ideal $\mathfrak{q}$ in B/I and then, by the going-up theorem, we have

$$h^*(\mathrm{V}(I)) = h^*(\mathrm{Spec}(B/I)) = \mathrm{V}(h^{-1}(I)) = \mathrm{Spec}(A/h^{-1}(I)). \ \blacksquare$$

Note that if $\pi : X \to S$ is an open and closed map then S is the union of the open and closed subsets $\pi(X)$ and $S - \pi(X)$. This implies that $C(S)$ is (isomorphic to) the direct product $C(S) = C(\pi(X)) \times C(S - \pi(X))$ and consequently $\mathrm{Spec}(C(S))$ is the union of the disjoint open and closed subsets $\mathrm{Spec}(C(\pi(X)))$ and $\mathrm{Spec}(C(S - \pi(X)))$ (and the same for $C^*(S)$). Thus, for our purposes there is no loss of generality in assuming from now on that every open and closed map is surjective.

The morphism of rings $C(S) \to C(X)$ induced by an open and closed map $\pi : X \to S$ is injective and its image is the subring of functions in $C(X)$ which are constant on every fibre of π. By abuse of notation, given a function $f \in C(S)$ we shall also write f for the function $f \circ \pi$.

Definition 1.3. Let $\pi : X \to S$ be open and closed. If a function $g \in C(X)$ is bounded on every fibre of π, we define the functions $g^*(s) = \sup\{g(x) : x \in \pi^{-1}(s)\}$ and $g_*(s) = \inf\{g(x) : x \in \pi^{-1}(s)\}$ on S. It is easy to prove that both functions are continuous.

Lemma 1.4. *Let $\pi : X \to S$ be open and closed. Let $\mathfrak{p}$ be a prime ideal in $C^*(S)$ and g a non-negative function in $C^*(X)$. If g becomes invertible in $C^*(X)_{\mathfrak{p}}$, then g_* becomes invertible in $C^*(S)_{\mathfrak{p}}$.*

Proof. Suppose that g becomes invertible in $C^*(X)_{\mathfrak{p}}$. Then, for some $g' \in C^*(X)$ and $s \in C^*(S) - \mathfrak{p}$, one has $(g/1) \cdot (g'/s) = 1$ in $C^*(X)_{\mathfrak{p}}$, i.e., $t \cdot g' \cdot g = t \cdot s$ for some $t \in C^*(S) - \mathfrak{p}$. Thus, $t \cdot g' \cdot g \in C^*(S) - \mathfrak{p}$. Since $\mathfrak{p}$ is an absolutely convex ideal (Theorem 5.5 of [4]), $|t \cdot g' \cdot g| = |t \cdot g'| \cdot g \in C^*(S) - \mathfrak{p}$. If $h = |t \cdot g'| \cdot (\sup_{x \in X} |t \cdot g'|(x))^{-1}$ then $h \cdot g \in C^*(S) - \mathfrak{p}$, $0 \le h \le 1$ and consequently $0 \le h \cdot g \le g$. Hence $h \cdot g \le g_*$ because $h \cdot g$, as a function on X, is constant on every fibre of π. Since $\mathfrak{p}$ is a convex ideal, one has $g_* \notin \mathfrak{p}$, i.e., g_* becomes invertible in $C^*(S)_{\mathfrak{p}}$. $\blacksquare$

Theorem 1.5. *If $\pi : X \to S$ is open and closed, then the going-up and going-down theorems hold for the morphism $C^*(S) \to C^*(X)$.*

Proof. Let $\mathfrak{q}$ be a prime ideal in $C^*(X)$ and $\mathfrak{p} = \pi^*(\mathfrak{q})$.

Going-up theorem: Let $\mathfrak{p}'$ be a prime ideal in $C^*(S)$ such that $\mathfrak{p} \subseteq \mathfrak{p}'$. The fibre of $\mathfrak{p}'$ by π^* is just (see [1] Chapter 3, Exercise 21)

$$(\pi^*)^{-1}(\mathfrak{p}') = \mathrm{Spec}(C^*(X)_{\mathfrak{p}'}/\mathfrak{p}' \cdot C^*(X)_{\mathfrak{p}'}) = \mathrm{V}(\mathfrak{p}' \cdot C^*(X)) \cap \mathrm{Spec}(C^*(X)_{\mathfrak{p}'}).$$

Suppose that in this fibre there are no prime ideals containing $\mathfrak{q}$, i.e.,

$$\mathrm{V}(\mathfrak{q}) \cap \mathrm{V}(\mathfrak{p}' \cdot C^*(X)) \cap \mathrm{Spec}(C^*(X)_{\mathfrak{p}'}) = \emptyset.$$

Since $\mathrm{V}(\mathfrak{q}) \cap \mathrm{V}(\mathfrak{p}' \cdot C^*(X)) = \mathrm{V}(\mathfrak{q} + \mathfrak{p}' \cdot C^*(X))$, the ideal $\mathfrak{q} + \mathfrak{p}' \cdot C^*(X)$ is not contained in any prime ideal in $C^*(X)_{\mathfrak{p}'}$ and therefore, there is $f + \sum g_i \cdot f_i \in \mathfrak{q} + \mathfrak{p}' \cdot C^*(X)$, where $f \in \mathfrak{q}$, $g_i \in \mathfrak{p}'$ and $f_i \in C^*(X)$, which becomes invertible in $C^*(X)_{\mathfrak{p}'}$, i.e., $f + \sum g_i \cdot f_i$ does not belong to any prime ideal in $C^*(X)_{\mathfrak{p}'}$. Since prime ideals in rings of continuous functions are convex (Theorem 5.5 of [4]), this implies that if $g = \sum g_i^2$, then $f^2 + g$ is invertible in $C^*(X)_{\mathfrak{p}'}$. But $g = \sum g_i^2 \in \mathfrak{p}'$ and since $0 \leq (f^2)_* \leq f^2$ and $f \in \mathfrak{q}$, by the convexity of $\mathfrak{q}$, one has $(f^2)_* \in \mathfrak{q} \cap C^*(S) = \mathfrak{p} \subseteq \mathfrak{p}'$ and therefore, $(f^2 + g)_* = (f^2)_* + g \in \mathfrak{p}'$. By 1.4, this is a contradiction. Hence, the going-up theorem holds.

Going-down theorem: We must prove that every prime ideal in $C^*(S)$ contained in $\mathfrak{p}$ is the image by π^* of some prime ideal in $C^*(X)$ contained in $\mathfrak{q}$, i.e., that the map $\mathrm{Spec}(C^*(X)_{\mathfrak{q}}) \to \mathrm{Spec}(C^*(S)_{\mathfrak{p}})$ is surjective. Since the prime ideals containing a given prime ideal in $C^*(X)$ form a chain (see 14.8 of [4]) and the going-up theorem holds for $C^*(S) \to C^*(X)$, it is not difficult to see that the going-up theorem also holds for $C^*(S)_{\mathfrak{p}} \to C^*(X)_{\mathfrak{q}}$. Hence, $\mathrm{Spec}(C^*(X)_{\mathfrak{q}}) \to \mathrm{Spec}(C^*(S)_{\mathfrak{p}})$ is closed (Lemma 1.2). Thus, by Lemma 0.1(iii), to prove that it is surjective, it is enough to check that $C^*(S)_{\mathfrak{p}} \to C^*(X)_{\mathfrak{q}}$ is injective. For this, assume that $f/s \in C^*(S)_{\mathfrak{p}}$ becomes zero in $C^*(X)_{\mathfrak{q}}$. Then there exists $g \in C^*(X) - \mathfrak{q}$, that clearly can be taken as $g \geq 0$, such that $f \cdot g = 0$. Since $g \notin \mathfrak{q}$ and $0 \leq g \leq g^*$, by the convexity of $\mathfrak{q}$ we have $g^* \notin \mathfrak{q}$ and therefore $g^* \notin \mathfrak{p} = \mathfrak{q} \cap C^*(S)$. It is clear that $g^* \cdot f = 0$. Hence, $f/s = 0$ in $C^*(S)_{\mathfrak{p}}$. ■

Corollary 1.6. *If* $\pi : X \to S$ *is open and closed*, *then so is* $\pi^* : \mathrm{Spec}(C^*(X)) \to \mathrm{Spec}(C^*(S))$.

Proof. Since the going-up theorem holds for $C^*(S) \to C^*(X)$, π^* is closed (Lemma 1.2).

To prove that it is open, we show that $\pi^*(\mathrm{D}(g)) = \mathrm{D}(g^*)$ for every non-negative $g \in C^*(X)$. The inclusion $\pi^*(\mathrm{D}(g)) \subseteq \mathrm{D}(g^*)$ is an immediate consequence of the convexity of prime ideals in $C^*(X)$. To prove the converse inclusion, consider $\mathfrak{p} \in \mathrm{Spec}(C^*(S))$ such that $\mathfrak{p} \notin \pi^*(\mathrm{D}(g))$, i.e., $(\pi^*)^{-1}(\mathfrak{p}) \subseteq \mathrm{V}(g)$. Since the going-down theorem holds for $C^*(S) \to C^*(X)$, it is not difficult to see that $\mathrm{V}(\mathfrak{p} \cdot C^*(X)) \subseteq \mathrm{V}(g)$ and therefore g belongs to every prime ideal containing $\mathfrak{p} \cdot C^*(X)$. Hence, $g^n \in \mathfrak{p} \cdot C^*(X)$ for some $n \in \mathbb{N}$, i.e. $g^n = \sum f_i \cdot g_i$ for some $f_i \in \mathfrak{p}$ and $g_i \in C^*(X)$. Let $(f_i)^+ = \sup(f_i, 0)$ and $(f_i)^- = \inf(f_i, 0)$. It is clear that $f_i = (f_i)^+ + (f_i)^-$ and $(f_i)^+ \cdot (f_i)^- = 0$. Since $f_i \in \mathfrak{p}$, both $(f_i)^+$ and $(f_i)^-$ belong to $\mathfrak{p}$. It is not difficult to check that $(f_i \cdot g_i)^* = (f_i)^+ \cdot (g_i)^* + (f_i)^- \cdot (g_i)_*$, since f_i is constant on the fibres of π. Hence, $(f_i \cdot g_i)^* \in \mathfrak{p}$. Since $\mathfrak{p}$ is a convex ideal, and $0 \leq (g^*)^n = (g^n)^* \leq \sum (f_i \cdot g_i)^*$, it follows that $(g^*)^n$, and consequently g^*, belongs to $\mathfrak{p}$, i.e., $\mathfrak{p} \notin \mathrm{D}(g^*)$. ■

Corollary 1.7 (Isiwata [5]). *If* $\pi : X \to S$ *is open and closed*, *then so is its extension* $\pi_\beta : \beta X \to \beta S$ *to the Stone–Čech compactifications.*

P r o o f. For any open subset U of βX, we have $\pi_\beta(U) = \pi^*(r_X^{-1}(U)) \cap \beta S$, where r_X is the continuous retraction of $\mathrm{Spec}(C^*(X))$ onto $\beta X = M^*(X)$. ■

Lemma 1.8. *If* $\pi : X \to S$ *is open and proper* (*closed and with all fibres compact*), *then* $C(X)$ *is the ring of fractions of* $C^*(X)$ *with respect to the multiplicatively closed subset* $M_S = \{f \in C^*(S) : 0 \notin f(S)\}$.

P r o o f. Since the fibres of π are compact, every $g \in C(X)$ is bounded on the fibres. Then we have on S the continuous functions g^* and g_* (see 1.3). It is clear that $g \cdot (1 + (g^2)^*)^{-1} \in C^*(X)$ and $(1 + (g^2)^*)^{-1} \in M_S$. Thus, g can be written as the fraction

$$g = \frac{g \cdot (1 + (g^2)^*)^{-1}}{(1 + (g^2)^*)^{-1}}. \ ■$$

Theorem 1.9. *If* $\pi : X \to S$ *is open and proper*, *then the going-up and going-down theorems hold for the morphism* $C(S) \to C(X)$ *and the map* $\pi_{\mathrm{s}} : \mathrm{Spec}(C(X)) \to \mathrm{Spec}(C(S))$ *is open and proper.*

P r o o f. This follows from 1.5 and 1.6, because by 1.8, the morphism $C(S) \to C(X)$ is a localization of $C^*(S) \to C^*(X)$, i.e.,

$$C(S) = C^*(S)_{M_S} \to C^*(X)_{M_S} = C(X),$$

and this implies that $\pi^* : \mathrm{Spec}(C^*(X)) \to \mathrm{Spec}(C^*(S))$ sends $\mathrm{Spec}(C^*(X)) - \mathrm{Spec}(C(X))$ to $\mathrm{Spec}(C^*(S)) - \mathrm{Spec}(C(S))$. ■

Example 1.10. Theorems 1.5 and 1.9 may be applied to some important classes of maps:

• It is a classical result that non-constant analytic maps between Riemann surfaces are open, hence any non-constant analytic map between compact Riemann surfaces is open and proper.

• It is well known and easy to prove that if a group G acts on a space X (i.e., there is a morphism of groups from G to the group of automorphisms of X) then the natural projection $\pi : X \to X/G$ is open. For G finite it is also closed.

• If $p : X \to S$ is a covering space in the classical sense (see definition in [7], p. 145), then it is open. It is easy to prove that if the cardinality of the fibres $p^{-1}(s)$ is finite then p is also closed.

2. On z-ideals and primary ideals. The *zero-set* of a function $f \in C(X)$ is the set

$$\mathrm{Z}(f) = \{x \in X : f(x) = 0\}.$$

Recall that an ideal I of $C(X)$ is said to be a *z-ideal* if $\mathrm{Z}(f) = \mathrm{Z}(g)$ and $f \in I$ imply $g \in I$. This condition is equivalent to the following one: $\mathrm{Z}(f) \subseteq \mathrm{Z}(g)$ and $f \in I$ imply $g \in I$, because $\mathrm{Z}(f) \cap \mathrm{Z}(g) = \mathrm{Z}(f^2 + g^2)$.

LEMMA 2.1. *For any $f_1, \ldots, f_n$ in $C(X)$, there exists $g \in C(X)$ such that any natural power of g divides every f_i and $\mathrm{Z}(g) = \mathrm{Z}(f_1) \cap \ldots \cap \mathrm{Z}(f_n)$.*

Proof. Let

$$f(x) = \sup\{|f_i(x)|(1 + f_i(x)^2)^{-1} : 1 \leq i \leq n\}$$

so that $0 \leq f(x) < 1$ for any $x \in X$. Define $g = (\log f(x))^{-1}$ in $X - \mathrm{Z}(f)$ and $g = 0$ in $\mathrm{Z}(f)$. Clearly, $\mathrm{Z}(g) = \mathrm{Z}(f_1) \cap \ldots \cap \mathrm{Z}(f_n)$. For $1 \leq i \leq n$ and $k \in \mathbb{N}$, define $h_{ik} = f_i \cdot (\log f)^k$ on $X - \mathrm{Z}(f)$ and $h_{ik} = 0$ on $\mathrm{Z}(f)$. It is clear that h_{ik} is continuous, so we have $f_i = h_{ik} \cdot g^k$ for every $k \in \mathbb{N}$. ∎

COROLLARY 2.2. (i) *Every finitely generated ideal in $C(X)$ is contained in a principal ideal.*

(ii) *Every z-ideal in $C(X)$ is an inductive limit* (*direct limit*) *of principal ideals.*

(iii) *Every z-ideal I is a flat $C(X)$-module.*

Proof. (i) This is an immediate consequence of 2.1.

(ii) Every module over a ring is the inductive limit of its finitely generated submodules (see [1], Chap. 2, Exercise 17). A finitely generated submodule of a z-ideal I is a finitely generated ideal $(f_1, \ldots, f_n) \subseteq I$. By Lemma 2.1, there exists a principal ideal (g) such that $(f_1, \ldots, f_n) \subseteq (g) \subseteq I$. Hence, principal ideals contained in I form a cofinal system of finitely generated submodules of I and therefore I is the inductive limit of these principal ideals.

(iii) It suffices to prove that for any ideal J of $C(X)$, the sequence $0 \to I \otimes_{C(X)} J \to I$ is exact (see [3] or [8]). For this, take any element $\sum f_i \otimes g_i \in I \otimes_{C(X)} J$ and set, as in 2.1, $f_i = h_{i2} \cdot g^2$. We can thus write $\sum f_i \otimes g_i = g \otimes (g \cdot \sum h_{i2} \cdot g_i)$. Obviously, if $\sum g_i \cdot f_i = 0$ then $g \cdot \sum h_{i2} \cdot g_i = 0$, and therefore $\sum f_i \otimes g_i = 0$. ∎

Note 2.3. A particular case of Corollary 2.2(iii), when the z-ideal I is the ideal of all functions vanishing on a given closed set of a locally compact metrizable space, was proved by Muñoz [10].

PROPOSITION 2.4. *Let I be a z-ideal and let J be an ideal of $C(X)$. If $\mathrm{V}(J) \subseteq \mathrm{V}(I)$, i.e., if $I \subseteq \mathrm{rad}(J)$, then $I \subseteq J$.*

Proof. By Lemma 2.1, given $f \in I$ we can take $g \in I$ such that f is a multiple of every power of g. Since $I \subseteq \mathrm{rad}(J)$, some power of g belongs to J, and then $f \in J$. ∎

COROLLARY 2.5. (i) *If the radical of an ideal I of $C(X)$ is a z-ideal then* $I = \mathrm{rad}(I)$.

(ii) *An ideal I of $C(X)$ is a z-ideal if and only if every minimal over-ideal of I is a z-ideal.*

Proof. (i) This follows immediately from Proposition 2.4.

(ii) The necessity of the condition was proved in [6] (see also [4], Theorem 14.7). The converse follows from (i), because if every minimal over-ideal of I is a z-ideal, then the intersection of these ideals, which is just $\mathrm{rad}(I)$, is also a z-ideal. ∎

Recall that an ideal $\mathfrak{q}$ of a ring A is called *primary* if the only zero-divisors of the quotient ring $A/\mathfrak{q}$ are nilpotent elements. Every ideal whose radical is a maximal ideal is a primary ideal. Since every maximal ideal in $C(X)$ is a z-ideal (see 2.7 of [4]), Corollary 2.5(ii) allows us to determine all primary ideals of $C(X)$ with maximal radical.

COROLLARY 2.6. *The only primary ideals in $C(X)$ having as radical a maximal ideal are the maximal ideals.*

We finish this section with another application of Proposition 2.4.

PROPOSITION 2.7. *Let I be a z-ideal in $C(X)$. A $C(X)$-module M is annihilated by I, i.e., $I \cdot M = 0$, if and only if* $\mathrm{Supp}(M) \subseteq \mathrm{V}(I)$.

Proof. The necessity is a well known general result. To prove the sufficiency it is enough to show that $I \cdot N = 0$ for every finitely generated submodule N of M. For any of these submodules we have

$$\mathrm{V}(\mathrm{Ann}(N)) = \mathrm{Supp}(N) \subseteq \mathrm{Supp}(M) \subseteq \mathrm{V}(I),$$

where $\mathrm{Ann}(N) = \{f \in C(X) : f \cdot m = 0, \forall m \in N\}$. From Proposition 2.4, it follows that $I \subseteq \mathrm{Ann}(N)$. ∎

3. Flat $C(X)$-modules. It is well known that a module over a principal ideal domain is flat if and only if it is torsion-free. Rings of continuous functions are not domains, so that to obtain similar flatness criteria for these rings, it is necessary to redefine the concept of torsion-free module.

DEFINITION 3.1 (Neville [11]). A $C(X)$-module M is *quasi-torsion-free* if, for every exact sequence of $C(X)$-modules $0 \to K \to F \to M \to 0$, where F is a flat module, and for every $f \in C(X)$, the equality $(f) \cdot F \cap K = (f) \cdot K$ is satisfied.

Since the $C(X)$-module F is assumed to be flat, it is an easy exercise to prove that this condition holds if and only if $\mathrm{Tor}_1(M, C(X)/I) = 0$ for every principal ideal I.

THEOREM 3.2 (Neville [11]). *If X is an F-space, then a $C(X)$-module M is flat if and only if it is quasi-torsion-free.*

P r o o f. If X is an F-space, then every finitely generated ideal in $C(X)$ is principal (see 14.25 of [4]). From a well known flatness criterion (see, for instance, Theorem 1 of Chapter 2 of [8]), it follows immediately that M is flat if and only if it is quasi-torsion-free. ∎

THEOREM 3.3. *A $C(X)$-module M of finite presentation is a flat* (*or equivalently, a projective*) *$C(X)$-module if and only if it is quasi-torsion-free.*

P r o o f. Since M is finitely presented, it is flat if and only if $\mathrm{Tor}_1(M, C(X)/\mathfrak{m}) = 0$ for every maximal ideal $\mathfrak{m}$ in $C(X)$ ([3], Corollary 2, Ch. II, 3). As every maximal ideal in $C(X)$ is a z-ideal, it is an inductive limit of principal ideals (Corollary 2.2(ii)), so that, if $\mathrm{Tor}_1(M, C(X)/I) = 0$ for every principal ideal I, then $\mathrm{Tor}_1(M, C(X)/\mathfrak{m}) = 0$ for every maximal ideal $\mathfrak{m}$ and therefore, M is flat. ∎

DEFINITION 3.4. We shall say that a module M over a ring A is *torsion-free* if no element of M, different from zero, is annihilated by a non-zero-divisor of A, in other words, if $\mathrm{Tor}_1(M, A/(a)) = 0$ whenever a does not divide zero.

THEOREM 3.5. *Assume that every closed set in X is a zero-set. Then a $C(X)$-module M of finite presentation is flat* (*or equivalently*, *projective*) *if and only if it is torsion-free.*

P r o o f. The module M is flat if and only if $\mathrm{Tor}_1(M, C(X)/\mathfrak{m}) = 0$ for every maximal ideal $\mathfrak{m}$ in $C(X)$ ([3], Corollary 2, Ch. II, 3). If a given maximal $\mathfrak{m}$ in $C(X)$ contains a non-zero-divisor function f then, by Lemma 2.1, every finite family $f_1, \ldots, f_n$ of elements of $\mathfrak{m}$ is contained in a principal ideal generated by a function g such that $\mathrm{Z}(g) = \mathrm{Z}(f_1) \cap \ldots \cap \mathrm{Z}(f_n) \cap \mathrm{Z}(f)$. This g is not a zero divisor and thus $\mathfrak{m}$ is an inductive limit of principal ideals generated by non-zero-divisor functions. Thus, if M is torsion-free then $\mathrm{Tor}_1(M, C(X)/\mathfrak{m}) = 0$.

For any other maximal ideal in $C(X)$ there is no problem, because if every element of a maximal ideal $\mathfrak{m}$ is a zero divisor, then every element of $\mathfrak{m}$ becomes zero in the local ring $C(X)_{\mathfrak{m}}$ and we conclude that this local ring is a field. Explicitly, since every closed set in X is a zero-set, given $f \in \mathfrak{m}$, $f \neq 0$, we may take a non-zero function g in $C(X)$ such that $\mathrm{Z}(g)$ is the closure of $X - \mathrm{Z}(f)$. This function g satisfies $g \cdot f = 0$ and $g \notin \mathfrak{m}$, the latter because $f^2 + g^2$ is not a zero divisor and so it does not belong to $\mathfrak{m}$. This implies that $f = 0$ in $C(X)_{\mathfrak{m}}$, and hence that $C(X)_{\mathfrak{m}}$ is a field. ∎

EXAMPLE 3.6. The following example shows that the condition of being torsion-free given in Definition 3.4 is weaker than the one given in Definition 3.1, and that the flatness criterion proved in Theorem 3.5 does not hold without the hypothesis of M being a $C(X)$-module of finite presentation.

Let $\mathfrak{p}$ be a minimal prime ideal of $C(X)$ contained in the maximal ideal $\mathfrak{m}$ of all functions vanishing at a given point x. Assume that the ideal η_x of all functions vanishing on some neighbourhood of x is not prime (for instance, take $X = \mathbb{R}$), so that η_x is strictly contained in $\mathfrak{p}$. The residue class ring $C(X)/\mathfrak{p}$ is torsion-free because every element of $\mathfrak{p}$ is a zero divisor. The condition $\mathrm{Tor}_1(C(X)/\mathfrak{p}, C(X)/(f)) = 0$ is equivalent to the exactness of the sequence

$$0 \to C(X)/\mathfrak{p} \otimes_{C(X)} (f) = (f)/(f) \cdot \mathfrak{p} \to C(X)/\mathfrak{p},$$

i.e., to the equality $\mathfrak{p} \cap (f) = (f) \cdot \mathfrak{p}$. If $f \in \mathfrak{p}$ this equality becomes $(f) = (f) \cdot \mathfrak{p}$, which implies the existence of a function g in $\mathfrak{p}$ such that $(g-1) \cdot f = 0$, and we conclude that f vanishes on a neighbourhood of x.

Therefore, $\mathrm{Tor}_1(C(X)/\mathfrak{p}, C(X)/(f)) \neq 0$ whenever f belongs to $\mathfrak{p} - \eta_x$. Hence, $C(X)/\mathfrak{p}$ is not quasi-torsion-free and, a fortiori, it is not flat.

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