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# On the category of modules of second kind for Galois coverings

by

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**Abstract.** Let $F : R \to R/G$ be a Galois covering and $\mathrm{mod}_1(R/G)$ (resp. $\mathrm{mod}_2(R/G)$) be a full subcategory of the module category $\mathrm{mod}(R/G)$, consisting of all $R/G$-modules of first (resp. second) kind with respect to $F$. The structure of the categories $(\mathrm{mod}(R/G))/[\mathrm{mod}_1(R/G)]$ and $\mathrm{mod}_2(R/G)$ is given in terms of representation categories of stabilizers of weakly-$G$-periodic modules for some class of coverings.

**0. Introduction.** The covering technique in representation theory was introduced and developed for the investigation of representation-finite algebras and computing their representations (see [G], [Gr], [BG], [R]). It has been generalized and applied to the study of representation-infinite algebras (see [DS1], [DLS], [DS2], [P]).

The covering methods in representation theory of algebras over a field $k$ are based on interpretation of modules over the algebra as representations of some quiver with relations, or more generally modules over a locally bounded category. Following [BG] a $k$-category $R$ is called *locally bounded* if all objects of $R$ have local endomorphism rings, the different objects are nonisomorphic, and both sums $\sum_{y\in R}\dim_k R(x,y)$ and $\sum_{y\in R}\dim_k R(y,x)$ are finite for each $x\in R$. $R$-modules are then contravariant $k$-linear functors from $R$ to the category of $k$-vector spaces. An $R$-module $M$ is *locally finite-dimensional* (resp. *finite-dimensional*) if $\dim_k M(x)$ is finite for each $x\in R$ (resp. $\sum_{x\in R}\dim_k M(x)$ is finite). We denote by $\mathrm{MOD}\,R$ the category of all $R$-modules, by $\mathrm{Mod}\,R$ (resp. $\mathrm{mod}\,R$) the full subcategory of all locally finite-dimensional (resp. finite-dimensional) $R$-modules and by $\mathrm{Ind}\,R$ (resp. $\mathrm{ind}\,R$) the full subcategory of indecomposable objects of $\mathrm{Mod}\,R$ (resp. $\mathrm{mod}\,R$). For any $M\in\mathrm{MOD}\,R$, $M^*$ is the $R^{\mathrm{op}}$-module dual to $M$, given by $M^*(x)=\mathrm{Hom}_k(M(x),k)$ for $x\in R$. The contravariant functor $(\ )^* : \mathrm{MOD}\,R \to$

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$\mathrm{MOD}\, R^{\mathrm{op}}$ mapping $M$ to $M^*$ induces an equivalence of categories $\mathrm{Mod}\, R \simeq (\mathrm{Mod}\, R^{\mathrm{op}})^{\mathrm{op}}$.

Let $\Lambda$ be a finite-dimensional algebra over an algebraically closed field $k$. Then the category of finite-dimensional right $\Lambda$-modules is equivalent to $\mathrm{mod}\, R_\Lambda$ for some uniquely determined (up to isomorphism), finite, locally bounded $k$-category $R_\Lambda$. Assume that $R_\Lambda$ is of the form $R/G$, where $R$ is some locally bounded $k$-category and $G$ some group of $k$-linear automorphims of $R$, acting freely on objects. This occurs for example if $R_\Lambda$ admits some nice group grading (see [Gr]). Then the Galois covering functor $F : R \to R/G$ induces a pair of functors

$$\mathrm{MOD}\, R \underset{F_\bullet}{\overset{F_\lambda}{\rightleftarrows}} \mathrm{MOD}(R/G),$$

where $F_\bullet$ is the pull-up functor given by $F_\bullet(M) = M \cdot F$ for $M \in \mathrm{MOD}\, R$, and $F_\lambda$ is the push-down functor, the left adjoint to $F_\bullet$ (see [BG]). If additionally $G$ acts freely on $(\mathrm{ind}\, R)/\simeq$, then $F_\lambda$ induces an injection from the set $((\mathrm{ind}\, R)/\simeq)/G$ of $G$-orbits of $(\mathrm{ind}\, R)/\simeq$ into $(\mathrm{ind}(R/G))/\simeq$ (see [G; 3.5]). In some cases the study of the module category for the algebra $\Lambda$ completely reduces to an analogous problem for the cover category $R$ of $R_\Lambda$ (see [G], [DS1], [DS2], [DLS]).

*Let $R$ be a locally bounded $k$-category and $G$ a group of $k$-linear automorphisms of $R$ acting freely on the isoclasses of indecomposable finite-dimensional $R$-modules. Assume that for any $x \in R$ the set $R_x$ consisting of all $y \in R$ such that there exists an indecomposable finite-dimensional $R$-module $M$ with nonzero $M(x)$ and $M(y)$, is finite. Then the push-down functor $F_\lambda : \mathrm{mod}\, R \to \mathrm{mod}(R/G)$ associated with the Galois covering $F : R \to R/G$ induces a bijection between the $G$-orbits of isoclasses of indecomposable finite-dimensional $R$-modules and the isoclasses of indecomposable finite-dimensional $R/G$-modules.*

In the general case the category $\mathrm{mod}(R/G)$ of finite-dimensional $R/G$-modules does not necessarily coincide with its full subcategory $\mathrm{mod}_1(R/G)$ formed by all modules of the form $F_\lambda M$, $M \in \mathrm{mod}\, R$. It was observed in [DS2] that the structure of the remaining indecomposable $R/G$-modules strongly depends on weakly-$G$-periodic $R$-modules, i.e. indecomposable locally finite-dimensional $R$-modules $B$ such that $\mathrm{supp}\, B$ is contained in finitely many $G_B$-orbits and $G_B$ is infinite, where $G_B = \{g \in G : {}^gB \simeq B\}$ is the *stabilizer* of the isoclass of $B$ and $\mathrm{supp}\, B = \{x \in R : B(x) \neq 0\}$ is the *support* of $B$.

The main aim of this paper is to give a description of the full subcategory $\mathrm{mod}_2(R/G)$ of $\mathrm{mod}(R/G)$ consisting of all modules having no direct summands from $\mathrm{mod}_1(R/G)$, for some class of Galois coverings. The elements from $\mathrm{mod}_2(R/G)$ (resp. $\mathrm{mod}_1(R/G)$) are usually called *modules of the second* (resp. *first*) *kind* with respect to the Galois covering $F$. The description is given in terms of the factor category $\mathrm{mod}(R/G)/[\mathrm{mod}_1(R/G)]$. This category carries essential information about the structure of the category $\mathrm{mod}_2(R/G)$, namely has the same indecomposable objects and irreducible maps (see [AR]).

Recall that for any subcategory $\mathcal{V}_0$ of an additive category $\mathcal{V}$, $\mathcal{V}/[\mathcal{V}_0]$ denotes the factor category of $\mathcal{V}$ modulo the ideal $[\mathcal{V}_0]$ of all morphisms in $\mathcal{V}$ factorizing through a direct sum of some objects of $\mathcal{V}_0$. If additionally $\mathcal{V}$ is a Krull–Schmidt category and $\mathcal{V}_0$ is closed under direct sums and summands, then for any $v, v' \in \mathcal{V}$ without direct summands in $\mathcal{V}_0$, $[\mathcal{V}_0](v, v')$ is contained in the square of the Jacobson radical of the category $\mathcal{V}$, and there exists a natural bijection between indecomposables from $\mathcal{V} \setminus \mathcal{V}_0$ and from $\mathcal{V}/[\mathcal{V}_0]$.

The first result describing the category of modules of the second kind was the reduction theorem proved in [DS2]:

*Let $R$ be a locally bounded $k$-category and $G$ a group of automorphisms of $R$ which acts freely on the isoclasses of finite-dimensional indecomposable modules. Assume that there exists a $G$-invariant family $\mathcal{S}$ of subcategories of $R$ with the following properties*:

(i) *for each $L \in \mathcal{S}$ and each $G$-orbit $\mathcal{O}$ of $R$, $\mathcal{O} \cap L$ is contained in finitely many $G_L$-orbits in $R$, where $G_L$ is the stabilizer of $L$ in $G$*,

(ii) *for any two different $L, L' \in \mathcal{S}$, $L \cap L'$ is locally support-finite*,

(iii) *for each weakly-$G$-periodic $R$-module $B$ there exists $L \in \mathcal{S}$ containing* $\mathrm{supp}\, B$.

*Then for any fixed set $\mathcal{S}_0$ of representatives of the $G$-orbits of $\mathcal{S}$ there exists an equivalence of factor categories*

$$\coprod_{L\in\mathcal{S}_0} (\mathrm{mod}(L/G))/[\mathrm{mod}_1(L/G_L)] \simeq (\mathrm{mod}(R/G))/[\mathrm{mod}_1(R/G)].$$

The above reduction theorem has very interesting consequences in situations similar to those when the supports of all weakly-$G$-periodic modules have linear ordinary quivers. In this case the family of all supports of weakly-$G$-periodic modules satisfies the assumptions of the theorem and the categories $L/G_L$ are simply the path categories of quivers of euclidean type $\widetilde{\mathbb{A}}_n$. Moreover, the supports of any two nonisomorphic weakly-$G$-periodic modules are different, and for each weakly-$G$-periodic $R$-module $B$ the group $G_B$ coincides with $G_{\mathrm{supp}\, B}$ and is an infinite cyclic group. Therefore $F_\lambda B$ has the structure of a $kG_B$-$R/G$-bimodule and induces a functor

$$\Phi^B = - \otimes_{k[\xi,\xi^{-1}]} F_\lambda B : \mathrm{mod}\, k[\xi, \xi^{-1}] \to \mathrm{mod}(R/G),$$

where $\operatorname{mod} k[\xi,\xi^{-1}]$ is the category of finite-dimensional modules over the algebra of Laurent polynomials in the variable $\xi$ over $k$.

Let $\mathcal{W}$ denote the set of all weakly-$G$-periodic modules and $\mathcal{W}_0$ a fixed set of representatives of the isoclasses representing $G$-orbits in $\mathcal{W}$. Then by the description of the module category for quivers of euclidean type $\widetilde{\mathbb{A}}_n$, the functors $(\Phi^B)_{B\in\mathcal{W}_0}$ induce equivalences

$$(*)_L \qquad \operatorname{mod} k[\xi,\xi^{-1}] \xrightarrow{\sim} (\operatorname{mod}(L/G_L))/[\operatorname{mod}_1(L/G_L)],$$

and the theorem yields the equivalence

$$\coprod_{\mathcal{W}_0} \operatorname{mod} k[\xi,\xi^{-1}] \xrightarrow{\sim} (\operatorname{mod}(R/G))/[\operatorname{mod}_1(R/G)].$$

The above equivalence allows us to understand better the structure of the module category for special biserial algebras. It has many applications (see [S1]–[S3]). Some generalization of this theorem has been given in [P].

In spite of many applications the reduction theorem is useless in the case when there exists a weakly-$G$-periodic $R$-module with support $R$, since then $\mathcal{S}$ has to be equal to $\{R\}$. This often happens when $G$ is the infinite cyclic group. The simplest example of this situation is the $\mathbb{Z}$ cover $R$ of the algebra $k[x,y]/(x^3,y^2,xy)$.

In the general case many weakly-$G$-periodic modules can have the same support $L$ and we cannot expect that the equivalence $(*)_L$ holds. The description of the category $\operatorname{mod}_2(R/G)$ in this situation cannot depend so strongly on the properties of supports of weakly-$G$-periodic modules and therefore some different approach is necessary. In this paper we propose a new strategy. It relies on a direct reduction to representation theory of stabilizers of weakly-$G$-periodic modules, without intermediate steps of the form $L/G$ and any knowledge of the module categories $\operatorname{mod}(L/G_L)$. The conditions imposed on weakly-$G$-periodic modules are expressed in terms of their tensor products and homomorphisms. We prove the following result (see Theorem 4.1):

*Let $R$ be a locally bounded $k$-category, where $k$ is algebraically closed, $G$ a group of automorphisms of $R$ acting freely on the isoclasses of indecomposable finite-dimensional $R$-modules, $\mathcal{W}$ the set of all weakly-$G$-periodic $R$-modules and $\mathcal{W}_0$ a fixed set of representatives of the $G$-orbits in $\mathcal{W}$ up to isomorphism. Assume that $R$ satisfies the following two conditions*:

(i) *for each $B\in\mathcal{W}$ the stabilizer $G_B$ is an infinite cyclic group and the endomorphism ring $\operatorname{End}_R(B)$ is isomorphic to $k$*,

(ii) *for any two different $B_1,B_2\in\mathcal{W}$ such that $G_{B_1}\cap G_{B_2}$ is nontrivial the tensor product $B_1\otimes_R B_2^*$ of $B_1$ and the $k$-dual of $B_2$ is a finitely generated free module over the group algebra $k(G_{B_1}\cap G_{B_2})$.*

*Then the functors*

$$\overline{\Phi}^B : \operatorname{mod} k[\xi,\xi^{-1}] \xrightarrow{\Phi^B} \operatorname{mod}(R/G) \to (\operatorname{mod}(R/G))/[\operatorname{mod}_1(R/G)],$$

$B \in \mathcal{W}_0$, *are full and faithful*, *the functor*

$$\overline{\Phi} : \coprod_{B\in\mathcal{W}_0} \operatorname{mod} k[\xi,\xi^{-1}] \to (\operatorname{mod}(R/G))/[\operatorname{mod}_1(R/G)]$$

*induced by* $(\overline{\Phi}^B)_{B\in\mathcal{W}_0}$ *is dense*, *and* $\overline{\Phi}$ *admits a left quasi-inverse*

$$\overline{\Psi} : (\operatorname{mod}(R/G))/[\operatorname{mod}_1(R/G)] \to \coprod_{B\in\mathcal{W}_0} \operatorname{mod} k[\xi,\xi^{-1}]$$

*whose kernel* $\operatorname{Ker}\overline{\Psi}$ *is an ideal contained in the Jacobson radical of the category* $(\operatorname{mod}(R/G))/[\operatorname{mod}_1(R/G)]$. *In particular*, $\overline{\Phi}$ *and* $\overline{\Psi}$ *induce a decomposition*

$$(\operatorname{mod}(R/G))/[\operatorname{mod}_1(R/G)] \simeq \coprod_{B\in\mathcal{W}_0} \operatorname{mod} k[\xi,\xi^{-1}] \oplus \operatorname{Ker}\overline{\Psi}$$

*and a bijection between the corresponding sets of isomorphism classes of indecomposable objects*, *and* $\operatorname{Ker}\overline{\Psi}$ *restricted to the image of* $\overline{\Phi}^B$ *is zero for each* $B \in \mathcal{W}_0$.

The class of examples covered by this theorem is not essentially larger than that covered by the previous one. The simplest example illustrating the theorem is the covering of the algebra $k[x,y]/(x^3,y^2,xy)$ with the group $\mathbb{Z}\times\mathbb{Z}$. In a subsequent publication a more general version of the above result without so strong restrictions on endomorphism rings of weakly-$G$-periodic $R$-modules will be proved.

The paper is organized as follows. Section 1 contains notations, terminology and the basic facts concerning Galois coverings of representation-infinite algebras. In Section 2 the operations on $R$-modules with $R$-actions of groups are studied and later applied to the description of the functors $\Phi^B$ and their adjoints in terms of $R$-modules. In Section 3 some technique for verifying whether certain representations of the infinite cyclic group are free is introduced. The whole Section 4 is devoted to the proof of the Theorem.

The methods we use here are very elementary. We assume the basic results on Galois coverings proved in [G] and [DS2], elementary properties of adjoint functors [M], the Krull–Warfield decomposition theorem [W], the description of indecomposable finitely generated modules over principal ideal domains and an elementary knowledge of representations of groups [L].

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## 1. Basic definitions and facts

**1.1.** Throughout this paper we denote by $k$ an algebraically closed field, by $R$ a locally bounded $k$-category (see [BG], [G]) and by $G$ a group of $k$-linear automorphisms of $R$. Then $G$ acts on $\mathrm{MOD}\, R$ by translations ${}^{g}(-)$ which assign to each $M \in \mathrm{MOD}\, R$ the $R$-module ${}^{g}M = M \circ g^{-1}$. For each $M \in \mathrm{MOD}\, R$ we denote by $G_M$ the stabilizer $\{g \in G : {}^{g}M \simeq M\}$. Throughout this paper we assume that $G$ acts freely on $(\mathrm{ind}\, R)/\simeq$.

By $\mathrm{MOD}^G R$ we denote the category of $R$-modules with an $R$-action of $G$. The objects of $\mathrm{MOD}^G R$ are pairs $(M, \mu)$, where $M \in \mathrm{MOD}\, R$ and $\mu$ is a family of $R$-homomorphisms $(\mu_g : M \to {}^{g^{-1}}M)_{g\in G}$ such that ${}^{g_1^{-1}}\mu_{g_2} \cdot \mu_{g_1} = \mu_{g_2 g_1}$ for all $g_1, g_2 \in G$. The set of morphisms from $(M, \mu)$ to $(M', \mu')$ in $\mathrm{MOD}^G R$, denoted by $\mathrm{Hom}_R^G(M, M')$, consists of all $f \in \mathrm{Hom}_R(M, M')$ such that $\mu'_g \cdot f = {}^{g^{-1}}f \cdot \mu_g$ for all $g \in G$.

$\mathrm{Mod}_{\mathrm{f}}^G R$ is the full subcategory of $\mathrm{MOD}^G R$ formed by all $(M, \mu) \in \mathrm{MOD}^G R$ such that $M \in \mathrm{Mod}\, R$ and $(\mathrm{supp}\, M)/G$ is finite. Then the pull-up functor $F_\bullet : \mathrm{MOD}(R/G) \to \mathrm{MOD}\, R$ associated with a Galois covering $F : R \to R/G$ induces an equivalence of categories [G; p. 94]

$$\mathrm{mod}(R/G) \xrightarrow{\sim} \mathrm{Mod}_{\mathrm{f}}^G R.$$

The group $G$ can also be interpreted as a group of $k$-linear automorphisms of $R^{\mathrm{op}}$. Then the functor $F^{\mathrm{op}} : R^{\mathrm{op}} \to (R/G)^{\mathrm{op}}$ is also a Galois covering since $(R/G)^{\mathrm{op}} = R^{\mathrm{op}}/G$. The corresponding pull-up and push-down functors are briefly denoted by $F_\bullet^{\mathrm{op}}$ and $F_\lambda^{\mathrm{op}}$.

The group $G^{\mathrm{op}}$ opposite to $G$ is isomorphic to $G$ via the map $(\ )^{-1} : G^{\mathrm{op}} \to G$. Therefore $G^{\mathrm{op}}$ can also be regarded as a group of $k$-linear automorphisms of $R$ and $G^{\mathrm{op}}$ acts on $\mathrm{MOD}\, R$ by translations ${}^{g^{-1}}(-)$, $g \in G^{\mathrm{op}}$.

**1.2.** Let $\mathrm{ind}_1(R/G)$ be the full subcategory of the category $\mathrm{ind}(R/G)$ of indecomposable finite-dimensional $R$-modules consisting of all objects isomorphic to $F_\lambda M$ for some $M \in \mathrm{ind}\, R$, and let $\mathrm{ind}_2(R/G)$ be the full subcategory of $\mathrm{ind}(R/G)$ formed by the remaining indecomposables. It is known [DS; 2.2] that a module $X \in \mathrm{ind}(R/G)$ belongs to $\mathrm{ind}_1(R/G)$ if $F_\bullet X$ has a finite-dimensional direct summand. Since each module $M \in \mathrm{Mod}\, R$ has a decomposition into a direct sum of indecomposables (with local endomorphism rings), therefore a module $X \in \mathrm{mod}(R/G)$ belongs to $\mathrm{mod}_2(R/G)$ if there exists a decomposition $F_\bullet X = \bigoplus_{i\in I} B_i$ in $\mathrm{Mod}\, R$ with all $B_i$ weakly-$G$-periodic (see [DS2; 2.3]).

**1.3.** For any $k$-algebra $A$ we denote by $\mathrm{MOD}\, A$ the category of all left $A$-modules and by $\mathrm{mod}\, A$ the full subcategory of $\mathrm{MOD}\, A$ formed by all finite-dimensional $A$-modules. By $A^{\mathrm{op}}$ we denote the algebra opposite to $A$ and by $(\ )^*$ the standard duality $\mathrm{Hom}_k(-, k) : \mathrm{MOD}\, A \to \mathrm{MOD}\, A^{\mathrm{op}}$.

**2. A description of the functors $\Phi_H^B$ and their adjoints.** Let $H$ be a subgroup of $G$ and $B=(B,\nu)\in \mathrm{MOD}^H R$. Then for each orbit $Gx\in R/G$, $F_\lambda B(Gx)=\bigoplus_{y\in Gx}B(y)$ carries via $\nu$ the structure of a free module over the group algebra $kH$ of $H$, which is finitely generated in case $(Gx\cap \operatorname{supp} B)/H$ is finite and $B\in \operatorname{Mod} R$. In fact, $F_\lambda B$ has the structure of a $kH$-$R/G$-bimodule and induces a functor

$$\Phi_H^B = -\otimes_{kH} F_\lambda B : \mathrm{MOD}\, kH \to \mathrm{MOD}(R/G)$$

(see [DS2; 3.6]). If additionally $B\in \mathrm{Mod}_{\mathrm{f}}^H(R/G)$ then the restriction of $\Phi_H^B$ to $\mathrm{mod}\, kH$ factors through $\mathrm{mod}(R/G)$. In case $H=G_B$ we write $\Phi^B=\Phi_{G_B}^B$. In this section we will study these functors and their adjoints in terms of the category $\mathrm{MOD}^G R$.

**2.1.** Let $A$ be a $k$ algebra, $\mathcal{C}$ a $k$-category and $Q:\mathcal{C}\to \mathrm{MOD}\, A^{\mathrm{op}}$ an $A$-$\mathcal{C}$-bimodule. Then we denote by $Q^A:\mathcal{C}^{\mathrm{op}}\to \mathrm{MOD}\, A$ the $A^{\mathrm{op}}$-$\mathcal{C}^{\mathrm{op}}$-bimodule defined by $Q^A(x)=Q(x)^A$, where $Q(x)^A=\mathrm{Hom}_A(Q(x),A)$. In particular, if $A=k$ then $Q^A=Q^*$.

For any subgroup $H$ of $G$ denote by $(\ )^{-1}:\mathrm{MOD}\, kH\to \mathrm{MOD}(kH)^{\mathrm{op}}$ the canonical isomorphism of categories given by $V^{-1}=V$ and $h\cdot v=vh^{-1}$ for $V\in \mathrm{MOD}\, kH$, $h\in H$, $v\in V$. The inverse functor is denoted in the same way. We set

$$(\ )^{\circledast}=(\ )^{-1}\circ(\ )^*:\mathrm{MOD}\, kH\to \mathrm{MOD}\, kH.$$

Analogously we denote by $(\ )^{-1}:\mathrm{MOD}^H R\to \mathrm{MOD}^{H^{\mathrm{op}}}R$ the isomorphism given by $(M,\mu)^{-1}=(M,\mu^{-1})$ for $(M,\mu)\in \mathrm{MOD}^H R$, where $(\mu^{-1})_h=\mu_{h^{-1}}$ for $h\in H$. The inverse functor will be denoted in the same way.

The usual duality $(\ )^*$ induces the contravariant functor $(\ )^*:\mathrm{MOD}^H R\to \mathrm{MOD}^{H^{\mathrm{op}}}R^{\mathrm{op}}$ mapping $M=(M,\mu)\in \mathrm{MOD}^H R$ to $M^*=(M^*,\mu^*)\in \mathrm{MOD}^{H^{\mathrm{op}}}R^{\mathrm{op}}$, where $(\mu^*)_h:M^*\to {}^h(M^*)$ for each $h\in H$ is given by the $R$-homomorphism

$$M^*={}^h({}^{h^{-1}}M^*)\xrightarrow{{}^h((\mu_h)^*)}{}^h(M^*).$$

We set

$$(\ )^{\circledast}=(\ )^{-1}\circ(\ )^*:\mathrm{MOD}^H R\to \mathrm{MOD}^H R^{\mathrm{op}}.$$

The composed functor $(\ )^{\circledast}$ maps $M=(M,\mu)\in \mathrm{MOD}^H R$ to $M^{\circledast}=(M^*,\mu^{\circledast})$, where $\mu_h^{\circledast}$ for each $h\in H$ is the $R$-homomorphism

$$M^*={}^{h^{-1}}({}^hM)^*\xrightarrow{{}^{h^{-1}}((\mu_{h^{-1}})^*)}{}^{h^{-1}}M^*.$$

LEMMA. *Let $B\in \mathrm{Mod}_{\mathrm{f}}^H R$. Then the $R/G$-$kH$-bimodules $F_\lambda^{\mathrm{op}}B^*$ and $F_\lambda B^{kH}$ are isomorphic.*

Proof. For any $a \in R/G$ fix a set $W_a$ of representatives of the $H$-orbits in $a \cap \operatorname{supp} B$. Then there exists a sequence of natural isomorphisms of right $kH$-modules

$$F_\lambda B^{kH}(a) = \operatorname{Hom}_{kH}(F_\lambda B(a), kH) \simeq \operatorname{Hom}_{kH}\Big({}_{kH}kH \otimes_k \Big(\bigoplus_{x\in W_a} B(x)\Big), kH\Big)$$
$$\simeq \operatorname{Hom}_k\Big(\bigoplus_{x\in W_a} B(x), \operatorname{Hom}_{kH}({}_{kH}kH, kH)\Big) \simeq \operatorname{Hom}_k\Big(\bigoplus_{x\in W_a} B(x), kH_{kH}\Big)$$
$$\simeq \Big(\bigoplus_{x\in W_a} B(x)\Big)^* \otimes_k kH_{kH} = F_\lambda^{\mathrm{op}} B^*(a). \ \blacksquare$$

Corollary. *The three functors*

$$- \otimes_{R/G} F_\lambda B, \operatorname{Hom}_{R/G}(F_\lambda B^{kH}, -), \operatorname{Hom}_{kH}(F_\lambda^{\mathrm{op}} B^*, -) : \operatorname{mod} kH \to \operatorname{mod}(R/G)$$

*are isomorphic.*

Proof. Since, for each $a \in R/G$, $F_\lambda B(a)$ is a finitely generated free $kH$-module, using Lemma 2.1 for any $V \in \operatorname{mod} kH$ we obtain a sequence of natural isomorphisms of $R/G$-modules

$$\begin{aligned}\operatorname{Hom}_{kH}(F_\lambda^{\mathrm{op}} B^*, V)(a) &\simeq \operatorname{Hom}_{kH}(F_\lambda B^{kH}(a), V)\\ &= \operatorname{Hom}_{kH}(F_\lambda B(a)^{kH}, V) \simeq V \otimes_{kH} (F_\lambda B(a)^{kH})^{kH}\\ &\simeq V \otimes_{kH} F_\lambda B(a) = (V \otimes_{kH} F_\lambda B)(a). \ \blacksquare\end{aligned}$$

**2.2.** Let $(M, \mu) \in \operatorname{MOD}^H R$ and $V \in \operatorname{MOD}(kH)^{\mathrm{op}}$. Then we denote by $V \otimes_k M$ the object $(V \otimes_k M, V \otimes_k \mu) \in \operatorname{MOD}^H R$ defined as follows: $(V \otimes_k M)(x) = V \otimes_k M(x)$ if $x \in R$, $(V \otimes_k M)(\alpha) = \mathrm{id}_V \otimes_k M(\alpha)$ if $\alpha$ is a morphism in $R$, and $(V \otimes_k \mu)_h : V \otimes_k M \to {}^{h^{-1}}(V \otimes_k M)$ for each $h \in H$ is the $R$-homomorphism given by $((V \otimes \mu)_h(x))(v \otimes m) = hv \otimes (\mu_h(x))(m)$ for $x \in R$, $m \in M(x)$ and $v \in V$.

Let $(N, \nu) \in \operatorname{MOD}^H R^{\mathrm{op}}$ and $V \in \operatorname{MOD}(kH)^{\mathrm{op}}$. Then by $\operatorname{Hom}_k(N, V)$ we mean the object $(\operatorname{Hom}_k(N, V), \operatorname{Hom}_k(\nu, V)) \in \operatorname{MOD}^H R$ defined as follows: $\operatorname{Hom}_k(N, V)(x) = \operatorname{Hom}_k(N(x), V)$ if $x \in R$, $\operatorname{Hom}_k(N, V)(\alpha) = \operatorname{Hom}_k(N(\alpha), V)$ if $\alpha$ is a morphism in $R$, and $\operatorname{Hom}_k(N, \nu)_h : \operatorname{Hom}_k(N, V) \to {}^{h^{-1}}(\operatorname{Hom}_k(N, V))$ for each $h \in H$ is the $R$-homomorphism given by $(\operatorname{Hom}_k(N, \nu)_h(x))(f_x) = f_x \cdot \nu_{h^{-1}}(hx)$ for $x \in R$ and $f \in \operatorname{Hom}_k(N(x), V)$.

Lemma. *Let $B \in \operatorname{Mod}^H R$. Then the two functors*

$$- \otimes_k B, \operatorname{Hom}_k(B^{\circledast}, -) : \operatorname{MOD}(kH)^{\mathrm{op}} \to \operatorname{MOD}^H R$$

*are isomorphic. If $B \in \operatorname{Mod}_{\mathrm{f}}^H R$ then the functor $- \otimes_k B$ restricted to $\operatorname{mod}(kH)^{\mathrm{op}}$ factors through $\operatorname{Mod}_{\mathrm{f}}^H R$.*

P r o o f. Since, for each $x \in R$, $B(x)$ is finite-dimensional, it follows that $V \otimes_k B(x) \simeq V \otimes_k B(x)^{**} \simeq \mathrm{Hom}_k(B(x)^*, V) = \mathrm{Hom}_k(B^{\circledast}, V)(x)$ for any $V \in \mathrm{MOD}\,(kH)^{\mathrm{op}}$. ■

**2.3.** Consider the restriction functor

$$\mathcal{R}_H : \mathrm{MOD}^G R \to \mathrm{MOD}^H R$$

mapping $N = (N, \nu) \in \mathrm{MOD}^G R$ to $(N, \nu|H) \in \mathrm{MOD}^H R$. Instead of $\mathcal{R}_H(N)$ we will simply write $N$. We give an explicit formula for its adjoint, the induction functor

$$\Theta_H : \mathrm{MOD}^H R \to \mathrm{MOD}^G R.$$

Denote by $S_H$ a fixed set of representatives of the left cosets $G \bmod H$. We define $\Theta_H(M, \mu) = (\bigoplus_{g \in S_H} {}^g M, \widetilde{\mu})$ for $M = (M, \mu) \in \mathrm{MOD}^H R$. Here the maps $\widetilde{\mu}_g : \bigoplus_{g_1 \in S_H} {}^{g_1}M \to \bigoplus_{g_2 \in S_H} {}^{g^{-1}g_2}M$, $g \in G$, are the $R$-homomorphisms defined by the family ${}^{g_1}\mu_h : {}^{g_1}M \to {}^{g^{-1}g_2}M$, $g_1 \in S_H$, where $g_2 \in S_H$ and $h \in H$ are determined by the equality $gg_1 = g_2 h$.

Lemma. *Let $M = (M, \mu) \in \mathrm{MOD}^H R$ and $N = (N, \nu) \in \mathrm{MOD}^G R$. Then there exists a natural isomorphism $\mathrm{Hom}_R^H(M, N) \simeq \mathrm{Hom}_R^G(\Theta_H(M), N)$. Moreover, if $(\mathrm{supp}\, M)/H$ is finite then also the isomorphism $\mathrm{Hom}_R^H(N, M) \simeq \mathrm{Hom}_R^G(N, \Theta_H(M))$ holds.*

P r o o f. Take $M, N$ as above. Then for any $f \in \mathrm{Hom}_R^H(N, M)$ denote by $\widetilde{f} : \bigoplus_{g_1} {}^{g_1}M \to N$ the $R$-homomorphism defined by the family

$${}^{g_1}M \xrightarrow{{}^{g_1}f} {}^{g_1}N \xrightarrow{{}^{g_1}\nu_{g_1}} N, \quad g_1 \in S_H.$$

It is easy to check that $\widetilde{f} \in \mathrm{Hom}_H^G(\Theta_H(M), N)$ and that the map $f \mapsto \widetilde{f}$ gives the required natural isomorphism. If now $(\mathrm{supp}\, M)/H$ is finite then $\bigoplus_{g_1 \in S_H} {}^{g_1}M = \prod_{g_1 \in S_H} {}^{g_1}M$ (see [DS2; 2.3]). For any $f \in \mathrm{Hom}_R^H(N, M)$ denote by $\overset{\approx}{f} : N \to \bigoplus_{g_1 \in S_H} {}^{g_1}M$ the $R$-homomorphism defined by the family

$$N \xrightarrow{\nu_{g_1^{-1}}} {}^{g_1}N \xrightarrow{{}^{g_1}f} {}^{g_1}M, \quad g_1 \in S_H.$$

It is easy to check that $\overset{\approx}{f} \in \mathrm{Hom}_H^G(N, \Theta_H(M))$ and that the map $f \mapsto \overset{\approx}{f}$ gives the second isomorphism. ■

Proposition. *Let $B$ be an object in $\mathrm{Mod}_{\mathrm{f}}^H R$.*

(i) *The two functors*

$$F_\bullet(- \otimes_{kH} F_\lambda B), \Theta_H((-)^{-1} \otimes_k B) : \mathrm{mod}\, kH \to \mathrm{Mod}_{\mathrm{f}}^G R$$

*are isomorphic.*

(ii) *The two functors*

$$F_\bullet \operatorname{Hom}_{kH}(F_\lambda B^{kH}, -), \Theta_H(\operatorname{Hom}_k(B^{\circledast}, (-)^{-1})) : \operatorname{mod} kH \to \operatorname{Mod}_{\mathrm{f}}^G R$$

*are isomorphic.*

Proof. (i) By Lemma 2.3 it is enough to construct a natural family of morphisms $f_V : V^{-1} \otimes_k B \to F_\bullet(V \otimes_{kH} F_\lambda B)$, $V \in \operatorname{mod} kH$ in $\operatorname{MOD}^H R$, and to show that all $R$-homomorphisms $\widetilde{f}_V : \Theta_H(V^{-1} \otimes_k B) \to F_\bullet(V \otimes_{kH} F_\lambda B)$ are isomorphisms.

Take any $V \in \operatorname{mod} kH$ and $x \in R$. Define the $k$-linear map $f_V(x) : V \otimes_k B(x) \to V \otimes_{kH} (\bigoplus_{y \in Gx} B(y))$ by setting $f_V(x)(v \otimes b) = v \otimes b$, where $v \in V$ and $b \in B(x)$. It is easy to verify that for each $V$ the family $(f_V(x))_{x \in R}$ defines a morphism $f_V$ in $\operatorname{MOD}^H R$, the family $(f_V)_{V \in \operatorname{mod} kH}$ induces a natural transformation of functors and all $R$-homomorphisms $\widetilde{f}_V : \Theta_H(V \otimes_k B) \to F_\bullet(V \otimes_{kH} F_\lambda B)$ induced by families $({}^{g_1}\mu_{g_1} \cdot {}^{g_1}f)_{g_1 \in S_H}$ are isomorphisms, where $\mu$ for each $V$ denotes the standard $R$-action of $G$ on $F_\bullet(V \otimes_{kH} F_\lambda B)$.

(ii) The proof is analogous. ∎

Remark. The above isomorphisms are compatible with those from Lemma 2.2 and Corollary 2.1.

**2.4.** In order to interpret the right and left adjoint functors

$$\operatorname{Hom}_{R/G}(F_\lambda B, -), - \otimes_{R/G} F_\lambda^{\mathrm{op}} B^* : \operatorname{mod}(R/G) \to \operatorname{MOD} kH$$

to $\Phi_B^H$ in terms of $\operatorname{MOD}^G R$, we first have to endow the homomorphism space and the tensor product of two modules from $\operatorname{MOD}^H R$ with the structure of a left $kH$-module. Given $(M, \mu), (N, \nu) \in \operatorname{MOD}^H R$ the map $H \times \operatorname{Hom}_R(M, N) \to \operatorname{Hom}_R(M, N)$, $(h, f) \mapsto {}^h\nu_h \cdot {}^h f \cdot \mu_{h^{-1}}$, defines the structure of a $kH$-module on $\operatorname{Hom}_R(M, N)$ with a corresponding $H$-action denoted by $\operatorname{Hom}_R(\mu, \nu)$.

Recall that for given $M \in \operatorname{MOD} R$ and $N \in \operatorname{MOD} R^{\mathrm{op}}$ the tensor product of $M$ and $N$ over $R$ is the factor space $M \otimes_R N = (M \otimes_k N)/I$, where $M \otimes_k N = \bigoplus_{x \in R} M(x) \otimes_k N(x)$ and $I = I(M, N)$ is the subspace of $M \otimes_k N$ generated by all vectors of the form $M(\alpha)(m_y) \otimes n_x - m_y \otimes N(\alpha)(n_x)$, for all $\alpha \in R(x, y)$, $n_x \in N(x)$, $m_y \in M(y)$.

Let now $M = (M, \mu) \in \operatorname{MOD}^H R$ and $N = (N, \nu) \in \operatorname{MOD}^H R^{\mathrm{op}}$. Then the maps $\mu_h(x) \otimes_k \nu_h(x) : M(x) \otimes_k N(x) \to M(hx) \otimes_k N(hx)$, $h \in H$, $x \in R$, furnish an action of $H$ on $M \otimes_k N$ denoted by $\mu \otimes_k \nu$. The subspace $I$ remains $H$-invariant under this action so $\mu \otimes_k \nu$ induces an $H$-action $\mu \otimes_R \nu$ on $M \otimes_R N$ and in consequence the structure of a left $kH$-module on $M \otimes_R N$.

R e m a r k. $M \otimes_k N$ is a free $kH$-module and can serve for a projective cover of $M \otimes_R N$, usually not minimal. Moreover, it is finitely generated if $M \in \mathrm{Mod}^H R$ and $N \in \mathrm{Mod}_{\mathrm{f}}^H R^{\mathrm{op}}$.

LEMMA. (i) *Let* $M \in \mathrm{MOD}^H R$, $N \in \mathrm{MOD}^H R^{\mathrm{op}}$ *and* $V \in \mathrm{MOD}\,(kH)^{\mathrm{op}}$. *Then there exist canonical natural isomorphisms of left* $kH$*-modules* $\mathrm{Hom}_k(M \otimes_R N, V) \simeq \mathrm{Hom}_R(M, \mathrm{Hom}_k(N, V))$ *and* $(V \otimes_k M) \otimes_R N \simeq V \otimes_k (M \otimes_R N)$. *In particular*, *there exists a natural isomorphism of left* $kH$*-modules* $(M \otimes_R N)^{\circledast} \simeq \mathrm{Hom}_R(M, N^{\circledast})$.

(ii) *Let* $M \in \mathrm{MOD}^H R$ *and* $V \in \mathrm{MOD}\,(kH)^{\mathrm{op}}$. *Then there exists a canonical natural isomorphism of left* $kH$*-modules* $\mathrm{Hom}_R(V \otimes_k M, N) = \mathrm{Hom}_k(V, \mathrm{Hom}_R(M, N))$.

P r o o f. (i) Use the isomorphisms

$$\mathrm{Hom}_k(M(x) \otimes_k N(x), V) \simeq \mathrm{Hom}_k(M(x), \mathrm{Hom}_k(N(x), V))$$

and

$$(V \otimes_k M(x)) \otimes_k N(x) \simeq V \otimes_k (M(x) \otimes_k N(x)), \qquad x \in R.$$

(ii) Use the isomorphism

$$\mathrm{Hom}_k(V \otimes_k M(x), N(x)) \simeq \mathrm{Hom}_k(V, \mathrm{Hom}_k(M(x), N(x))), \qquad x \in R. \ \blacksquare$$

COROLLARY. (i) *Let* $M \in \mathrm{MOD}^H R$ *and* $N \in \mathrm{MOD}^H R^{\mathrm{op}}$. *Then there exists a canonical natural embedding of* $kH$*-modules* $M \otimes_R N \hookrightarrow \mathrm{Hom}_R(M, N^{\circledast})^{\circledast}$.

(ii) *Let* $M \in \mathrm{MOD}^H R$ *and* $N \in \mathrm{MOD}^H R$. *Then there exists a natural isomorphism of* $kH$*-modules* $\mathrm{Hom}_R(M, N) \simeq (M \otimes_R N^{\circledast})^{\circledast}$ *and an embedding* $M \otimes_R N^{\circledast} \hookrightarrow \mathrm{Hom}_R(M, N)^{\circledast}$, *which is an isomorphism if* $\dim_k \mathrm{Hom}_R(M, N)$ *is finite.*

P r o o f. Use the standard embedding $V \hookrightarrow V^{\circledast\circledast}$ for $V \in \mathrm{MOD}\,(kH)^{\mathrm{op}}$. $\blacksquare$

**2.5.** PROPOSITION. *Let* $M \in \mathrm{MOD}^H R$, $N \in \mathrm{MOD}^{H^{\mathrm{op}}} R^{\mathrm{op}}$, $X \in \mathrm{mod}(R/G)$ *and* $Y \in \mathrm{MOD}\,(R/G)^{\mathrm{op}}$. *Then the following natural isomorphisms of left* $kH$*-modules hold.*

(i) $\mathrm{Hom}_{R/G}(F_\lambda M, X)^{-1} \simeq \mathrm{Hom}_R(M, F_\bullet X)$.

(ii) $F_\lambda M \otimes_{R/G} Y \simeq M \otimes_R F_\bullet^{\mathrm{op}} Y$.

(iii) $(X \otimes_{R/G} F_\lambda^{\mathrm{op}} N)^{-1} \simeq F_\bullet X \otimes_R N^{-1}$.

P r o o f. (i) This is a simple verification of $H$-invariance of the adjointness formula for the pair of functors $(F_\lambda, F_\bullet)$ (see [BG; 3.2]).

(ii) Take $M$ and $Y$ satisfying the assumptions. Then the canonical isomorphisms

$$F_\lambda M(Gx) \otimes_k Y(Gx) = \Big(\bigoplus_{y\in Gx} M(y)\Big) \otimes_k Y(Gx) \simeq \bigoplus_{y\in Gx} M(y) \otimes_k Y(Gx)$$
$$= \bigoplus_{y\in Gx} M(y) \otimes F_\bullet^{\mathrm{op}} Y(y)$$

are $H$-invariant and induce an isomorphism of $kH$-modules $f : M \otimes_k F_\bullet Y \xrightarrow{\sim} F_\lambda M \otimes_R Y$. Since $f(I(M, F_\bullet X)) \subset I(F_\lambda M, Y)$ the homomorphism $f$ induces an epimorphism $\bar{f} : M \otimes_R F_\bullet Y \to F_\lambda M \otimes_{R/G} Y$. By Corollary 2.4(i) and (i), $\bar{f}$ has to be an isomorphism.

(iii) Follows immediately from (ii). ■

Corollary. *Let $B = (B, \nu)$ be an object in* $\mathrm{Mod}_{\mathrm{f}}^H R$.

(i) *The two functors*

$$\mathrm{Hom}_{R/G}(F_\lambda B, -), (\mathrm{Hom}_R(B, F_\bullet(-)))^{-1} : \mathrm{mod}(R/G) \to \mathrm{MOD}\, kH$$

*are isomorphic.*

(ii) *The two functors*

$$(- \otimes_{R/G} F_\lambda B^{kH}), (F_\bullet(-) \otimes_R B^{\circledast})^{-1} : \mathrm{mod}(R/G) \to \mathrm{MOD}\, kH$$

*are isomorphic.*

Proof. (i) Obvious by Proposition 2.5(i).

(ii) Follows from Proposition 2.5(ii) and Lemma 2.1. ■

## 3. Free representations of an infinite cyclic group

**3.1.** In this section we will find some sufficient condition for a finitely generated module over the group algebra of an infinite cyclic group to be free.

Let $\phi : W \to W$ be a $k$-linear automorphism of a vector space $W$. Then to any decomposition $W = \bigoplus_{j\in J} W_j$ into a direct sum of subspaces we attach an oriented graph $\Gamma(\phi, J)$ of components of $\phi$ defined as follows. The set of points of $\Gamma(\phi, J)$ is $J$. The arrow $j_1 \to j_2$ in $\Gamma(\phi, J)$ exists if and only if $p_{j_2}\phi(W_{j_1}) \neq 0$, where $p_j : W \to W_j$ denotes the standard projection for each $j \in J$.

Proposition. *Let $H$ be an infinite cyclic group and $U$ be a finitely generated left $kH$-module. If for some $h \in H$ there exists a $k$-vector space decomposition $U = \bigoplus_{j\in J} U_j$ such that $\Gamma(h\cdot, J)$ has no oriented cycles, then $U$ is a finitely generated free $kH$-module.*

For the proof we need the following elementary facts.

Lemma. *Let $\phi : W \to W$ be a $k$-linear automorphism.*

(i) *If $\phi$ has a nonzero eigenvalue then so has each $\phi^n$, $n \in \mathbb{N}$.*

(ii) *If $W$ admits a decomposition $W = \bigoplus_{j \in J} W_j$ such that $\Gamma(\phi, J)$ has no oriented cycles then $\phi$ has no nonzero eigenvalue.*

Proof. (i) $\phi(w) = \lambda w$ implies $\phi^n(w) = \lambda^n w$.

(ii) The assumption of (ii) implies that $J$ is partially ordered with respect to the relation $\preceq$, where $j_1 \preceq j_2$ if and only if $j_1 = j_2$ or there exists an oriented path from $j_1$ to $j_2$ in $\Gamma(\phi, J)$. Assume now that $\phi(w) = \lambda w$, where $\lambda \in k$ and $0 \neq w = \sum_{j \in J} w_j \in W = \bigoplus_{j \in J} W_j$. The nonempty, finite set $J_0 = \{j \in J : w_j \neq 0\}$ has some minimal element $j_0$. Then $\lambda w = \phi(w) \in \bigoplus_{j \in J_1} W_j$, where $J_1$ is the set of direct successors of elements from $J_0$. Since $\Gamma(\phi, J)$ has no oriented loops the minimality of $j_0$ yields $\lambda = 0$. ∎

Proof of the Proposition. Since $kH$ is a principal ideal domain and $k$ is algebraically closed, a module $U$ in $\operatorname{mod}(kH)^{\mathrm{op}}$ is free if and only if it has no simple submodule isomorphic to $kH/(h_0 - \lambda)$ for some $\lambda \in k^*$, where $h_0$ is any fixed generator of $H$. In other words, $U$ is free if and only if the map $h_0 \cdot : U \to U$ has no nonzero eigenvalue. Now given $h \in H$ satisfying the assumptions, we choose a generator $h_0$ of $H$ such that $h = h_0^n$ for some $n \in \mathbb{N}$. Then by the Lemma, $h\cdot : U \to U$ and $h_0 \cdot : U \to U$ have no nonzero eigenvalues, and therefore the $kH$-module $U$ is free. ∎

**3.2.** Let $I$ be a set. We denote by $\mathcal{S}_0(I)$ the set of all finite subsets of $I$. Then to any subset $\mathcal{A} \subset \mathcal{S}_0(I)$ and any map $f : I \to I$ we attach the oriented graph $\Gamma(f, \mathcal{A})$ of intersections of $\mathcal{A}$ via $f$ defined as follows. The set of points of $\Gamma(f, \mathcal{A})$ is just $\mathcal{A}$. For any $A, B \in \mathcal{A}$ there exists an arrow $A \to B$ in $\Gamma(f, \mathcal{A})$ if and only if $f(A) \cap B$ is nonempty.

PROPOSITION. *Let $H$ be an infinite cyclic group and $U$ be a finitely generated left $kH$-module. Assume that the $k$-vector space $U$ has a decomposition $U = \bigoplus_{j \in J} U_j$, and there exists a function $s : J \to \mathcal{S}_0(I)$ and a free action $\bullet : H \times I \to I$ of $H$ on the set $I$ with the following properties*:

(i) *there exists a nontrivial subgroup $H' \subset H$ such that $s(J)$ is $H'$-stable and $s(J)/H'$ is finite*,

(ii) *for each $h \in H$, $s$ induces an oriented graph morphism*

$$s : \Gamma(h\cdot, J) \to \Gamma(h\bullet, s(J)).$$

*Then $U$ is a finitely generated free $kH$-module.*

Proof. The proof follows immediately from Proposition 3.1 and the lemma below. ∎

LEMMA. *Let $\bullet : H' \times I \to I$ be a free action of an infinite cyclic group $H'$ on some set $I$, and $\mathcal{A}$ be an $H'$-stable subset of $\mathcal{S}_0(I)$ such that $\mathcal{A}/H'$ is finite. Then there exists $h \in H'$ such that $\Gamma(h\bullet, \mathcal{A})$ has no oriented cycle.*

Proof. Without loss of generality we can assume $H' = \mathbb{Z}$. Since $\mathcal{A}/H'$ is finite there exists a finite subset $\mathcal{D}' \subset I$ such that for any $A \in \mathcal{A}$, $h' \bullet A$ is contained in $\mathcal{D}'$ for some $h' \in H'$. Denote by $\mathcal{D}$ the union of all sets $h' \bullet \mathcal{D}'$, where $h' \in H'$ is such that $h' \bullet \mathcal{D}' \cap \mathcal{D}' \neq \emptyset$. Since $\mathcal{D}$ is finite the set $H'_1$ consisting of all $h' \in H'$ such that $h' \bullet \mathcal{D} \cap \mathcal{D} \neq \emptyset$ is finite. Let $h$ be the smallest element of $H' = \mathbb{Z}$ such that $h > |h'|$ for all $h' \in H'_1$. In order to prove that $\Gamma(h\bullet, \mathcal{A})$ has no oriented cycles it is enough to show that for any $A_0, A_1, \ldots, A_n \in \mathcal{A}$, $n \in \mathbb{N}$, such that $A_0 \cap A_1 \neq \emptyset, A_1 \cap A_2 \neq \emptyset, \ldots, A_{n-1} \cap A_n \neq \emptyset$ we have $A_n \cap nh \bullet A_0 = \emptyset$.

Take $A_0, A_1, \ldots, A_n$ as above. Then there exist $h_0, h_1, \ldots, h_n \in H'$ such that $A_i \subset h_i \bullet \mathcal{D}'$ for each $i = 0, 1, \ldots, n$. Since $A_i \cap A_{i+1} \neq \emptyset$, both $A_i$, $A_{i+1}$ are contained in $h_i \bullet \mathcal{D}$, and $h_{i+1} - h_i \in H'_1$ for any $i = 0, 1, \ldots, n-1$. Without loss of generality we can assume $h_0 = 0$. Then $A_0 \cup A_1 \cup \ldots \cup A_n \subset \bigcup_{h' \in H'_2} h' \bullet \mathcal{D}$, where $H'_2 = \{h' \in H' : (1-n)h \leq h' \leq (n-1)h\}$. Therefore $A_n \cap nh \bullet D$ and consequently $A_n \cap nh \bullet A_0$ are empty, and the proof is finished. ∎

**3.3.** Let $M \in \operatorname{Mod} R$, $N \in \operatorname{Mod} R^{\mathrm{op}}$, $H$ be a subgroup of $G$ and $(\mu_h : M \to {}^{h^{-1}}M)_{h \in H}$ and $(\nu_h : N \to {}^{h^{-1}}N)_{h \in H}$ be families of $R$-homomorphisms. For any $h \in H$ we denote by $\mu_h \overline{\otimes}_R \nu_h$ the composed homomorphism

$$M \otimes_R N \xrightarrow{\mu_h \otimes_R \nu_h} {}^{h^{-1}}M \otimes_R {}^{h^{-1}}N \simeq M \otimes_R N.$$

Observe that in case $(\mu_h)_{h \in H}$ and $(\nu_h)_{h \in H}$ are both $R$-actions of $H$, for any $h \in H$ the map $\mu_h \overline{\otimes}_R \nu_h$ is equal to the value of the action $\mu \otimes_R \nu$ on $M \otimes_R N$ at $h$ (see 2.4).

Proposition. *Let $H$ be an infinite cyclic group, $(N, \nu) \in \operatorname{Mod}_{\mathrm{f}}^H R^{\mathrm{op}}$ and $M \in \operatorname{Mod} R$ a module such that $G_M$ contains $H$. Assume that $M$ has a decomposition $M = \bigoplus_{t \in T} M_t$ with the following properties*:

(i) *for each $t \in T$, all indecomposable direct summands of $M_t$ are isomorphic*,

(ii) *for any two different $t_1, t_2 \in T$ the modules $M_{t_1}$ and $M_{t_2}$ have no isomorphic direct summand*,

(iii) *for each $t \in T$ such that* $\operatorname{supp} M_t \cap \operatorname{supp} N$ *is nonempty and $G_{M_t} \cap H$ is nontrivial there exists an $R^{\mathrm{op}}$-action $\nu^t$ of $G_{M_t} \cap H$ on $N$ and an $R$-action $\mu^t$ of $G_{M_t} \cap H$ on $M_t$ such that $M_t \otimes_R N$ with the action $\mu^t \otimes_R \nu^t$ is a finitely generated free $k(G_{M_t} \cap H)$-module.*

*Then for any family of $R$-homomorphisms $(\mu_h : M \to {}^{h^{-1}}M)_{h \in H}$ such that $(\mu_h \overline{\otimes}_R \nu_h)_{h \in H}$ gives rise to an $H$-action on $M \otimes_R N$, the finitely generated $kH$-module $M \otimes_R N$ is free.*

This proposition is crucial for the main result of this paper. The rest of this section will be devoted to the preparation for the proof of Proposition 3.3 and the proof itself (given in 3.7).

**3.4.** Let $\pi : V \to U = \bigoplus_{j\in J} U_j$ be a $k$-linear map. For any subspace $V'$ of $V$ denote by $t_\pi(V')$ the set consisting of all $j \in J$ such that $p_j(\pi(V)) \neq 0$, where $p_j : U \to U_j$ denotes the canonical projection for each $j \in J$. Observe that $t_\pi(V')$ is a finite set if $\dim_k V'$ is finite. Assume that $V$ has a decomposition $V = \bigoplus_{i\in I} V_i$ into a direct sum of subspaces. Then for any $j \in J$ we denote by $o_\pi(U_j)$ the set of all $i \in I$ such that $p_j(\pi(V_i)) \neq 0$. Moreover, observe that if $\pi$ is surjective then for any finite-dimensional subspace $U' \subset U$ there exists a finite subset $I_0$ of $I$ such that $U' \subset \sum_{i\in I_0} \pi(V_i)$. The following simple fact explains the role of the above notation.

Lemma. *Let $\pi : \bigoplus_{i\in I} V_i \to \bigoplus_{j\in J} U_j$ and $\pi' : \bigoplus_{i'\in I'} V'_{i'} \to \bigoplus_{j'\in J'} U_{j'}$ be a surjective homomorphism of $k$-vector spaces, $\varphi : \bigoplus_{i\in I} V_i \to \bigoplus_{i'\in I'} V'_{i'}$ be the $k$-linear homomorphism induced by a family $\varphi_i : V_i \to V_{f(i)}$, $i \in I$, where $f : I \to I'$ is some function, and $\psi : \bigoplus_{j\in J} U_j \to \bigoplus_{j'\in J'} U'_{j'}$ be the homomorphism induced by a family of linear maps $\psi_{(j',j)} : U_j \to U'_{j'}$, $j \in J$, $j' \in J'$. Assume that $\psi\pi = \pi'\varphi$. Then for any $j \in J$, $j' \in J'$ and $I_0 \subset I$ such that $\psi_{(j',j)} \neq 0$ and $U_j \subset \sum_{i\in I_0} \pi(V_i)$, the intersection $f(I_0) \cap o_{\pi'}(U'_{j'})$ is nonempty.*

Proof. Obvious. ∎

**3.5.** Let $V = kH \otimes_k \overline{V}$, $U = kH \otimes_k \overline{U}$ and $\pi : V \to U$ be a $kH$-homomorphism, where $\overline{V} = \bigoplus_{\alpha=1}^{r} \overline{V}_\alpha$ and $\overline{U} = \bigoplus_{\beta=1}^{s} \overline{U}_\beta$ are $k$-vector spaces with some fixed decompositions into a finite direct sum of subspaces. Let us fix the notation $I = H \times \{1,\dots,r\}$, $J = H \times \{1,\dots,s\}$, $V_{(h,\alpha)} = kh \otimes_k \overline{V}_\alpha$ and $U_{(h,\beta)} = kh \otimes_k \overline{U}_\beta$ for $(h,\alpha) \in I$ and $(h,\beta) \in J$. The group $H$ acts on $I$ and $J$ in an obvious way compatible with multiplication by elements of $H$.

Lemma. *Let $\pi : \bigoplus_{i\in I} V_i \to \bigoplus_{j\in J} U_j$ be as above.*

(i) *If the free $kH$-module $V$ is finitely generated then all sets $o_\pi(U_j)$, $j \in J$, are finite and $o_\pi(U_{hj}) = h \cdot o_\pi(U_j)$ for any $j \in J$ and $h \in H$.*

(ii) *If the free $kH$-module $U$ is finitely generated and $\pi$ is surjective then there exists a finite subset $I_0 \subset I$ such that $U_{(h,\beta)} \subset \sum_{i\in hI_0} \pi(V_i)$ for any $h \in H$ and $\beta = 1,\dots,s$.*

Proof. (i) The assumption of (i) is equivalent to $\dim_k V_\alpha$ being finite for any $\alpha = 1,\dots,r$. Therefore all sets $t_\pi(V_i)$, $i \in I$, are finite. Take any $j \in J$ and suppose $o_\pi(U_j)$ is infinite. Then there exists $\alpha \in \{1,\dots,r\}$ and an infinite sequence of pairwise different elements $h_n \in H$, $n \in \mathbb{N}$, such that $p_j\pi(V_{(h_n,\alpha)}) \neq 0$. Since $h_n\pi(V_{(e,\alpha)}) = \pi(V_{(h_n,\alpha)})$, we have $p_{h_n^{-1}j}\pi(V_{(e,\alpha)}) \neq 0$

for each $n \in \mathbb{N}$ and thus $t_\pi(V_{(e,\alpha)})$ is infinite. This is a contradiction and therefore all sets $o_\pi(U_j)$, $j \in J$, are finite. The second part of (i) is obvious.

The proof of (ii) is easy and we leave it to the reader. ■

**3.6.** LEMMA. *Let $H_1$ and $H_2$ be subgroups of $G$, $L_1$ be an $H_1$-invariant subset of $R$, and $L_2$ an $H_2$-invariant subset of $R$, such that $H_1 \cap H_2 = \{e\}$ and $L_1/H$, $L_2/H$ are finite. Then $L_1 \cap L_2$ is finite.*

Proof. Suppose $L_1 \cap L_2$ is infinite. Since $L_1/H_1$ is finite there exists $x \in L_1 \cap L_2$ such that $H_1 x \cap L_1 \cap L_2$ is infinite. Analogously there exists $y \in H_2 x \cap L_1 \cap L_2$ such that $H_1 x \cap H_2 y \cap L_1 \cap L_2$ is infinite. Hence there exists $z \in L_1 \cap L_2$ and nontrivial elements $h_1 \in H_1$ and $h_2 \in H_2$ such that $h_1 z = h_2 z$ and we get a contradiction. Therefore $L_1 \cap L_2$ has to be finite. ■

Let now $(N, \nu) \in \operatorname{Mod}_{\mathrm{f}}^H R^{\mathrm{op}}$ and $M \in \operatorname{Mod} R$ be modules such that $(\operatorname{supp} M)/G_M$ is finite, where $H$ is an infinite cyclic group. Denote by $H_1$ the intersection $H \cap G_M$ and by $L$ the intersection $\operatorname{supp} M \cap \operatorname{supp} N$. The free $kH_1$-module $M \otimes_k N = \bigoplus_{i \in L} M(i) \otimes_k N(i)$ is finitely generated for any $R$-action of $H_1$ on $M$, if $H_1$ is nontrivial. Assume that $M$ admits some $R$-action $\mu_1$ of $H_1$ on itself such that $\mu_1 \otimes_R \nu$ induces the structure of a free $kH_1$-module on $M \otimes_R N$. Then for the subspace $U$ spanned by the set of free $kH_1$-generators of $M \otimes_R N$ we obtain a $k$-vector space decomposition $M \otimes_R N = \bigoplus_{h \in H_1} U_h$, where $U_h = h \cdot U$ for each $h \in H_1$. Therefore the canonical projection $\pi : M \otimes_k N \to M \otimes_R N$ can be viewed as a linear map $\pi : \bigoplus_{x \in L} M(x) \otimes_k N(x) \to \bigoplus_{h \in H_1} U_h$.

COROLLARY. *With the notation above, for any $U$ there exists a finite subset $s(U)$ of $L$ such that $U_h \subset \sum_{i \in s(U)} \pi(M(hi) \otimes_k N(hi))$ and $o_\pi(U_h) \subset hs(U)$ for each $h$. In particular, if $H_1$ is trivial then $U = M \otimes_R N$ and one can take the whole $L$ for $s(U)$.*

Proof. In case $H_1$ is nontrivial, the free $kH$-module $M \otimes_k N$ is finitely generated since $(\operatorname{supp} N)/H$ is finite and $H_1$ has a finite index in $H$. Therefore the assertion follows immediately from Lemma 3.5. In case $H_1$ is trivial the assertion follows from Lemma 3.6. ■

**3.7.** Proof of Proposition 3.3. In order to show that for given $M$ and $N$ satisfying the assumptions the finitely generated $kH$-module $M \otimes_R N$ is free we shall apply Proposition 3.2. Therefore we have to define a decomposition $M \otimes_R N = \bigoplus_{j \in J} U_j$, an action $\bullet : H \times I \to I$ and a map $s : \mathcal{J} \to \mathcal{S}_0(I)$ satisfying the required conditions.

Let $M = \bigoplus_{t \in T} M_t$ be a decomposition as in the assumption of the proposition. Denote by $T'$ the set of all $t \in T$ such that $\operatorname{supp} M_t \cap \operatorname{supp} N \neq \emptyset$. The group $H$ acts on $T'$ by the action $* : H \times T' \to T'$ given by the formula $h * t = t'$, where $t'$ is the unique element of $T'$ such that ${}^h M_t \simeq M_{t'}$. Since $(\operatorname{supp} N/H)$ is finite, $M \in \operatorname{Mod} R$ and $H$ is an infinite cyclic group, the

$H$-invariant subset $T_0 \subset T'$ consisting of all elements with trivial stabilizers $H_t = H \cap G_{M_t}$ in $H$ consists only of finitely many $H$-orbits and $T_1 = T' \setminus T_0$ is finite. By assumption (iii) and Lemma 3.6 all $M_t \otimes_R N$, $t \in T'$, are finitely generated $kH_t$-modules. Therefore for each $t \in T'$ there exists a finite-dimensional subspace $U_t \subset M_t \otimes_R N$ such that the $k$-vector space $M_t \otimes_R N$ has a decomposition $M_t \otimes_R N = \bigoplus_{h \in H_t} U_{(t,h)}$, where $U_{(t,h)} = h \cdot U_t$ for any $h \in H_t$. Then the $k$-vector space $M \otimes_R N = \bigoplus_{t \in T'} M_t \otimes_R N$ has a decomposition $M \otimes_R N = \bigoplus_{j \in J} U_j$, where $J$ is the disjoint union of $H_t$, $t \in T'$.

We now set $I = \operatorname{supp} N$. Then $H$ acts on $I$ in an obvious way. We define a function $s : J \to \mathcal{S}_0(I)$ by setting $s(t,h) = h \cdot s(U_t)$ if $t \in T_1$ and $s(t,h) = \operatorname{supp} M_t \cap I$ if $t \in T_0$ (see Corollary 3.6). Observe that the nontrivial intersection $H' = \bigcap_{t \in T_1} H_t$ satisfies the assumption (i) of Proposition 3.2, because $H'$ has a finite index in $H$ and in all $H_t$, $t \in T_1$. In order to verify the assumption (ii) of Proposition 3.2 one has to show that, for any $j_1, j_2 \in J$, $h \in H$, if the $(j_1, j_2)$th component $(\mu_h \mathbin{\overline{\otimes}_R} \nu_h)_{(j_2,j_1)} : U_{j_1} \to U_{j_2}$ of $\mu_h \mathbin{\overline{\otimes}_R} \nu_h$ is nonzero then $hs(j_1) \cap s(j_2)$ is nonempty. For any $t_1, t_2 \in T'$ denote by $\mu_h^{(t_2,t_1)} : M_{t_1} \to {}^{h^{-1}}M_{t_2}$ the $(t_2, t_1)$th component of $\mu_h : M \to {}^{h^{-1}}M$. Fix any elements $j_1 = (t_1, h_1)$, $j_2 = (t_2, h_2)$ of $J$ and assume $(\mu_h \mathbin{\overline{\otimes}_R} \nu_h)_{(j_2,j_1)} \neq 0$. Then $(\mu_h \mathbin{\overline{\otimes}_R} \nu_h)_{(j_2,j_1)}$ is the $(h_2, h_1)$th component of the $(t_2, t_1)$th component $\mu_h^{(t_2,t_1)} \mathbin{\overline{\otimes}_R} \nu_h : M_{t_1} \otimes_R N \to M_{t_2} \otimes_R N$ of $\mu_h \mathbin{\overline{\otimes}_R} \nu_h$. Now we set

$$\psi = \mu_h^{(t_2,t_1)} \mathbin{\overline{\otimes}_R} \nu_h : \bigoplus_{h_1' \in H_{t_1}} U_{(t_1,h_1')} \to \bigoplus_{h_2' \in H_{t_2}} U_{(t_2,h_2')}$$

and

$$\varphi = \bigoplus_{x \in R} \mu_h^{(t_1,t_2)}(x) \otimes_k \nu_h(x) : \bigoplus_{x \in R} M_{t_1}(x) \otimes_k N(x) \to \bigoplus_{x \in R} M_{t_2}(hx) \otimes_k N(hx).$$

Then by Lemma 3.4 and Corollary 3.6, $hs(j_1) \cap s(j_2)$ is nonempty. It follows that both assumptions of Proposition 3.2 are satisfied and the finitely generated $kH$-module $M \otimes_R N$ is free. ∎

**4. The Main Theorem.** In this section we first explain some technical details concerning the formulation of the main theorem and then we give a full proof.

**4.1.** We need the following fact.

Lemma. (i) *Let $M \in \operatorname{MOD} R$ be such that $G_M$ is an infinite cyclic group with a generator $g$. Then $(\mu_{g'})_{g' \in G_M} \mapsto \mu_g$ defines a bijective correspondence between the set of all $R$-actions of $G_M$ on $M$ and the set of all isomorphisms $f \in \operatorname{Hom}_R(M, {}^{g^{-1}}M)$.*

(ii) *Let $H$ be an infinite cyclic group and let $(M, \mu), (M, \mu') \in \operatorname{MOD}^H R$ and $(N, \nu), (N, \nu') \in \operatorname{MOD}^H R^{\mathrm{op}}$ be such that* $\operatorname{End}_R(M) \simeq k \simeq \operatorname{End}_{R^{\mathrm{op}}}(N)$.

*Then the left $kH$-module $(M \otimes_R N, \mu \otimes_R \nu)$ is free if and only if so is $(M \otimes_R N, \mu' \otimes_R \nu')$.*

Proof. Obvious. ■

Let $\mathcal{W}$ be the set of all weakly-$G$-periodic $R$-modules, $\mathcal{W}_0$ a fixed subset of nonisomorphic modules from $\mathcal{W}$ whose isoclasses form a set of representatives of the $G$-orbits of isoclasses of $\mathcal{W}$, and $\mathcal{W}^1$ the subset of $\mathcal{W}$ consisting of all $B \in \mathcal{W}$ such that $G_B$ is an infinite cyclic group; let $\mathcal{W}_0^1 = \mathcal{W}^1 \cap \mathcal{W}_0$. For each $B \in \mathcal{W}_0^1$ fix some $R$-action $\nu^B$ of $G_B$ on $B$, some generator $g_B$ of $G_B$ and some set $S_B$ of representatives of left cosets $G \bmod G_B$, containing the unit element $e$ of $G$. Thus $F_\lambda B$ is endowed with the structure of a left free $kG_B$-module, in fact a $kG_B$-$R/G$-bimodule. We can also identify $kG_B$ and $k[\xi, \xi^{-1}]$ by mapping $g_B$ onto $\xi$. Denote by $\underline{\underline{\mathrm{mod}}}(R/G)$ the residue category $(\mathrm{mod}(R/G))/[\mathrm{mod}_1(R/G)]$ and by

$$\overline{\Phi}^B : \mathrm{mod}\, kG_B \to \underline{\underline{\mathrm{mod}}}(R/G)$$

the functor induced by $\Phi^B = - \otimes_{kG_B} F_\lambda B : \mathrm{MOD}\, kG_B \to \mathrm{MOD}(R/G)$ (see Section 2). Let

$$\overline{\Phi} : \coprod_{B \in \mathcal{W}_0^1} \mathrm{mod}\, k[\xi, \xi^{-1}] \to \underline{\underline{\mathrm{mod}}}(R/G)$$

be the functor induced by the family $(\overline{\Phi}^B)_{B \in \mathcal{W}_0^1}$, which maps the object $(V_B)_{B \in \mathcal{W}_0^1}$ to the finite direct sum $\bigoplus_{B \in \Omega} \Phi^B(V_B)$, where $\Omega = \{B \in \mathcal{W}_0^1 : V_B \neq 0\}$.

THEOREM. *Let $R$ be a locally bounded $k$-category, where $k$ is algebraically closed, $G$ a group of $k$-linear automorphisms of $R$ acting freely on* $(\mathrm{ind}\, R)/\simeq$, *and $\mathcal{W}$, $\mathcal{W}_0$ as above. Assume that*

(i) *for each $B \in \mathcal{W}$, $G_B$ is an infinite cyclic group and* $\mathrm{End}_R(B) \simeq k$,

(ii) *for any two different $B_1, B_2 \in \mathcal{W}$ such that $G_{B_1} \cap G_{B_2}$ is nontrivial the tensor product $B_1 \otimes_R B_2^{\circledast}$ of $B_1$ and the $k$-dual $B_2^{\circledast}$ of $B_2$ with the structure defined above is a free left finitely generated module over $k(G_{B_1} \cap G_{B_2})$.*

*Then for any $B \in \mathcal{W}$ the functor*

$$\overline{\Phi}^B : \mathrm{mod}\, kG_B \to (\mathrm{mod}(R/G))/[\mathrm{mod}_1(R/G)]$$

*is full and faithful. The functor*

$$\overline{\Phi} : \coprod_{B \in \mathcal{W}_0} \mathrm{mod}\, k[\xi, \xi^{-1}] \to (\mathrm{mod}(R/G))/[\mathrm{mod}_1(R/G)]$$

*induced by* $(\overline{\Phi}^B)_{B\in\mathcal{W}_0}$ *is dense*, *and* $\overline{\Phi}$ *has a left quasi-inverse*

$$\overline{\Psi}:(\mathrm{mod}(R/G))/[\mathrm{mod}_1(R/G)]\to\coprod_{B\in\mathcal{W}_0}\mathrm{mod}\,k[\xi,\xi^{-1}]$$

*with kernel contained in the Jacobson radical of* $(\mathrm{mod}\,R/G)/[\mathrm{mod}_1(R/G)]$. *In particular*, $\overline{\Phi}$ *and* $\overline{\Psi}$ *induce a decomposition*

$$(\mathrm{mod}(R/G))/[\mathrm{mod}_1(R/G)]\simeq\coprod_{B\in\mathcal{W}_0}\mathrm{mod}\,k[\xi,\xi^{-1}]\oplus\mathrm{Ker}\,\overline{\Psi}$$

*and a bijection of the two corresponding sets of isoclasses of indecomposable objects*, *and* $\mathrm{Ker}\,\overline{\Psi}$ *restricted to the image of* $\overline{\Phi}^B$ *is zero for each* $B\in\mathcal{W}_0$.

Corollary. (i) *The functor* $\Phi:\coprod_{B\in\mathcal{W}_0}\mathrm{mod}\,k[\xi,\xi^{-1}]\to\mathrm{mod}_2(R/G)$ *induced by the family of functors* $(\Phi^B)_{B\in\mathcal{W}_0}$ *yields a bijection between the sets of indecomposable objects of both categories.*

(ii) *The Auslander–Reiten quiver* $\Gamma_{R/G}$ *of* $R/G$ (see [AR]) *is isomorphic to a disjoint union of translation quivers* $\Gamma_{R/G}\sqcup(\coprod_{B\in\mathcal{W}_0}\Gamma_{k[\xi,\xi^{-1}]})$, *where* $\Gamma_R$ (*resp.* $\Gamma_{k[\xi,\xi^{-1}]}$) *is the Auslander–Reiten quiver of* $R$ (*resp. of* $k[\xi,\xi^{-1}]$).

**4.2.** The proof of the Theorem will be done in several steps. For the rest of this section we will assume that

(i′) for each $B\in\mathcal{W}^1$, $\mathrm{End}_R(B)\simeq k$,

(ii′) for any two different $B_1,B_2\in\mathcal{W}^1$ the left finitely generated $k(G_{B_1}\cap G_{B_2})$-module $B_1\otimes B_2^{\circledast}$ is free.

Remark. (i) If $\mathcal{W}=\mathcal{W}^1$, then by Lemma 3.6 the conditions (ii) and (ii′) are equivalent.

(ii) The condition (i′) implies by Corollary 2.4(ii) that for any $(B,\nu_B)$ with $B\in\mathcal{W}^1$, the $kG_B$-module $B\otimes_R B^{\circledast}$ is isomorphic to the trivial $kG_B$-module $k$.

Given $V\in\mathrm{MOD}\,kG_B$ we denote by $\boldsymbol{t}(V)$ the maximal torsion submodule of $V$.

Let now $B\in\mathcal{W}_0^1$. Then we define the functor

$$\Psi^B:\mathrm{mod}(R/G)\to\mathrm{MOD}\,kG_B$$

by setting $\Psi^B(X)=\boldsymbol{t}(X\otimes_{R/G}F_\lambda B^{kG_B})$, where $X\in\mathrm{mod}(R/G)$.

Lemma. *For each* $B\in\mathcal{W}_0^1$ *the functor* $\Psi^B$ *induces a functor*

$$\overline{\Psi}^B:\underline{\underline{\mathrm{mod}}}(R/G)\to\mathrm{mod}\,kG_B.$$

Proof. The functor $\Psi^B$ factors through $\mathrm{mod}\,kG$, since $F_\lambda B$ is a finitely generated left $kG_B$-module. Let now $X\in\mathrm{mod}_1(R/G)$ be any module. Since by [DS2; 2.2], $F_\bullet X$ is a direct sum of finite-dimensional $R$-modules, Corollary 2.5 and Proposition 3.3 yield $\Psi^B(X)=\boldsymbol{t}(X\otimes_{R/G}F_\lambda B^{kG_B})\simeq\boldsymbol{t}((F_\bullet X\otimes_R B^{\circledast})^{-1})=0$ and therefore $\overline{\Psi}^B$ is well defined. ∎

**4.3.** LEMMA. *Let $B, B' \in \mathcal{W}_0^1$. Then*
$$\overline{\Psi}^{B'}\overline{\Phi}^{B} \simeq \begin{cases} \mathrm{id}_{\mathrm{mod}\,kG_B} & \textit{if } B = B', \\ 0 & \textit{if } B \neq B'. \end{cases}$$

Proof. Take any $V \in \mathrm{mod}\,kG_B$. If $B = B'$ then by the formulas of Section 2, Remark 4.2(ii) and Proposition 3.3 we obtain a sequence of natural isomorphisms of right $kG_B$-modules
$$\begin{aligned}\Psi^B\Phi^B(V) &= \boldsymbol{t}((V \otimes_{kG_B} F_\lambda B) \otimes_{R/G} F_\lambda B^{kG_B}) \\ &\simeq \boldsymbol{t}((F_\bullet(V \otimes_{kG_B} F_\lambda B) \otimes_R B^{\circledast})^{-1}) \\ &\simeq \boldsymbol{t}\Big(\Big(V^{-1} \otimes_k B \oplus \bigoplus_{g\in S_B,\, g\neq e} {}^g(V^{-1} \otimes_k B)\Big) \otimes_R B^{\circledast}\Big)^{-1} \\ &\simeq (V^{-1} \otimes_k (B \otimes_R B^{\circledast}))^{-1} \simeq (V^{-1} \otimes_k k)^{-1} \simeq V.\end{aligned}$$
Now if $B \neq B'$ then using analogous arguments we show that
$$\Psi^{B'}\Phi^B(V) = \boldsymbol{t}\Big(\bigoplus_{g\in S_B} (V^{-1} \otimes_k B) \otimes_R (B')^{\circledast}\Big) = 0. \ \blacksquare$$

Let
$$\overline{\Psi} : \underline{\underline{\mathrm{mod}}}(R/G) \to \prod_{B\in\mathcal{W}_0^1} \mathrm{mod}\,kG_B$$
be the functor defined by the family of functors $(\overline{\Psi}^B)_{B\in\mathcal{W}_0^1}$ and
$$I : \coprod_{B\in\mathcal{W}_0^1} \mathrm{mod}\,kG_B \to \prod_{B\in\mathcal{W}_0^1} \mathrm{mod}\,kG_B$$
be the canonical embedding.

COROLLARY. *The functors $I$ and $\overline{\Psi}\overline{\Phi}$ are isomorphic.*

**4.4.** PROPOSITION. (i) *$\overline{\Phi}^B$ is full for each $B \in \mathcal{W}_0^1$.*

(ii) *If $\mathcal{W} = \mathcal{W}^1$, then $\overline{\Psi}$ factors through $\coprod_{B\in\mathcal{W}_0^1} \mathrm{mod}\,kG_B$ and $\overline{\Phi}$ is dense.*

Proof. First we show that for any indecomposable $X \in \mathrm{mod}_2(R/G)$ such that all weakly-$G$-periodic direct summands of $F_\bullet X \in \mathrm{MOD}\,R$ belong to $\mathcal{W}^1$, $\Phi^B\Psi^B(X)$ is a nonzero direct summand of $X$ for some $B \in \mathcal{W}_0^1$. Take $X \in \mathrm{mod}_2(R/G)$ as above. Then there exists $B \in \mathcal{W}_0^1$ such that $F_\bullet X$ has a decomposition $F_\bullet X \simeq B^n \oplus M$ in $\mathrm{Mod}\,R$, where $M$ has no direct summand isomorphic to $B$. Denote by $\mu$ the induced $R$-action of $G$ on $B^n \oplus M$, by $H$ the group $G_B$ and by $\nu$ the action $\nu^B$. Then the results of Section 2 yield
$$\begin{aligned}\Psi^B(X) &= \boldsymbol{t}(X \otimes_{R/G} F_\lambda B^{kH}) \simeq \boldsymbol{t}(X \otimes_{R/G} F_\lambda^{\mathrm{op}} B^*) \\ &\simeq \boldsymbol{t}((F_\bullet X \otimes_R B^{\circledast})^{-1}) \simeq \boldsymbol{t}(((B^n \oplus M) \otimes_R B^{\circledast})^{-1}).\end{aligned}$$

Observe that $B^n \otimes B^{\circledast}$ is an $H$-invariant subspace of $(B^n \oplus M) \otimes_R B^{\circledast}$, since for any $h \in H$ the component

$$\mu_h^{M,B^n} \otimes_R \nu_h^{\circledast} : B^n \otimes_R B^{\circledast} \xrightarrow{h^{-1}} M \otimes_R {}^{h^{-1}}B^{\circledast}$$

of the map $\mu_h \otimes_R \nu_h^{\circledast}$ is zero by Corollary 2.4 and the assumption $\mathrm{End}_R(B) \simeq k$. Denote by $V_X$ the $kH$-module $B^n \otimes_R B^{\circledast}$, by $i' : V_X \to (B^n \oplus M) \otimes_R B^{\circledast}$ the canonical embedding and by $P_X$ the cokernel $\mathrm{Coker}\, i' = M \otimes_R B^{\circledast}$. By Proposition 3.3, $P_X$ is a free finitely generated left $kH$-module. Therefore $i'$ admits a retraction $p = (p_1, p_2) : (B \oplus M) \otimes_R B^{\circledast} \to V_X$ and $\Psi^B(X) = V_X^{-1}$. Denote by $p_X : X \to \mathrm{Hom}_{kH}(F_\lambda B^{kH}, V_X^{-1})$ the map adjoint to the composed map

$$\widetilde{p}_X : X \otimes_{R/G} F_\lambda B^{kH} \xrightarrow{\sim} ((B^n \oplus M) \otimes_R B^{\circledast})^{-1} \xrightarrow{p} V_X^{-1}.$$

We shall prove that $p_X$, or equivalently $F_\bullet(p_X)$, has a section. To this end, consider the canonical embedding $j : B \otimes_R (B^n)^{\circledast} \to B \otimes_R (B^n \oplus M)^{\circledast}$. As above, $B \otimes_R (B^n)^{\circledast}$ is an $H$-invariant subspace with respect to the action $\nu \otimes_R \mu^{\circledast}$ and $j$ has an $H$-equivariant retraction. Applying $\circledast$ we conclude by Corollary 2.4 that the kernel of the canonical projection $p' : \mathrm{Hom}_R(B, B^n \oplus M) \to \mathrm{Hom}_R(B, B^n)$ is $H$-invariant with respect to the action $\mathrm{Hom}_R(\nu, \mu)$ and $p'$ has an $H$-invariant section $i = \binom{i_1}{i_2}$. Denote by $i_X : \mathrm{Hom}_R(B, B^n)^{-1} \otimes_{kH} F_\lambda B \to X$ the $R/G$-homomorphism adjoint to the composed map

$$\widetilde{i}_X : \mathrm{Hom}_R(B, B^n)^{-1} \xrightarrow{i} \mathrm{Hom}_R(B, B^n \oplus M)^{-1} \xrightarrow{\sim} \mathrm{Hom}_{R/G}(F_\lambda B, X).$$

Now we prove that $F_\bullet(p_X) \circ F_\bullet(i_X)$ is an invertible $R$-homomorphism. Observe that via the isomorphisms $F_\bullet X \simeq B^n \oplus M$,

$$F_\bullet \mathrm{Hom}_{kH}(F_\lambda B^{kH}, (B^n \otimes_R B^{\circledast})^{-1}) \simeq \bigoplus_{g \in S_B} {}^g(\mathrm{Hom}(B^{\circledast}, B^n \otimes_R B^{\circledast}))$$

and

$$F_\bullet((\mathrm{Hom}_R(B, B^n))^{-1} \otimes_{kH} F_\lambda B) \simeq \bigoplus_{g \in S_B} {}^g(\mathrm{Hom}_R(B, B^n) \otimes_k B)$$

(see Proposition 2.3), the morphisms $F_\bullet(i_X)$ and $F_\bullet(p_X)$ are defined by the families of morphisms

$$\varphi_g : {}^g(\mathrm{Hom}_R(B, B^n) \otimes_k B) \xrightarrow{{}^g\tilde{i}} {}^g(B^n \oplus M) \xrightarrow{{}^g\mu_g} B^n \oplus M, \qquad g \in G,$$

and

$$\psi_g : B^n \oplus M \xrightarrow{\mu_{g^{-1}}} {}^g(B \oplus M) \xrightarrow{{}^g\tilde{p}} {}^g(\mathrm{Hom}_k(B^{\circledast}, B^n \otimes_R B^{\circledast})), \qquad g \in G,$$

where

$$\widetilde{i} = \begin{pmatrix} \widetilde{i}_1 \\ \widetilde{i}_2 \end{pmatrix} : \mathrm{Hom}_R(B, B^n) \otimes_k B \to B^n \oplus M$$

and

$$\widetilde{p} = (\widetilde{p}_1, \widetilde{p}_2) : B^n \oplus M \to \mathrm{Hom}_k(B^{\circledast}, B^n \otimes_R B^{\circledast})$$

are the maps adjoint to $i$ and $p$.

By [DS2; 2.1(iii)], it remains to show that all maps $\psi_g\varphi_g = {}^g(\widetilde{\widetilde{p}i})$, $g \in S_B$, are invertible. Since $\mathrm{End}_R(B) \simeq k$, by Lemma 2.2 the map $\widetilde{p}_2\widetilde{i}_2$ is zero. Both $i_1$ and $p_1$ are identity maps, therefore the map $\widetilde{p}_1\widetilde{i}_1 : \mathrm{Hom}_R(B, B^n) \otimes_k B \to \mathrm{Hom}_k(B^{\circledast}, B^n \otimes_R B^{\circledast})$ is given by $(\widetilde{p}_1\widetilde{i}_1(x)(f \otimes b_x))(\gamma_x) = f(x)(b_x) \otimes \gamma_x$, where $x \in R$, $b_x \in B(x)$, $\gamma_x \in B^{\circledast}(x)$, $f \in \mathrm{Hom}_R(B, B^n)$, and moreover it is invertible because $\mathrm{End}_R(B) \simeq k$. Therefore $\widetilde{\widetilde{p}i}$ is invertible, $F_\bullet(p_X)$ is a splittable epimorphism and consequently $\Phi^B\Psi^B X$ is a direct summand of $X$. Now (ii) follows immediately.

For the proof of (i), for each $B \in \mathcal{W}_0^1$ denote by $I_B$ the full subcategory of $\underline{\underline{\mathrm{mod}}}(R/G)$ consisting of all $X$ such that all weakly-$G$-periodic summands belong up to isomorphism to the $G$-orbit of $B$. Since $\overline{\Phi}^B$ factors through $I_B$ we shall prove that the restriction $\overline{\Psi}^B|I_B : I_B \to \mathrm{mod}\, kH$ of $\overline{\Psi}^B$ to $I_B$ is a quasi-inverse for $\overline{\Phi}^B$. For this purpose it is enough to check that the homomorphisms $p_X : X \to \Phi^B\Psi^B X$, $X \in I_B$, produce a natural transformation of functors $\mathrm{id}_{I_B} \to \overline{\Phi}^B\overline{\Psi}^B|I_B$, since $p_X$ is an isomorphism for each $X \in I_B$. Take any $X, Y \in I_B$ and $f \in \mathrm{Hom}_{R/G}(X, Y)$. One has to show that the map

$$u = \mathrm{Hom}_{kH}((F_\lambda B^{kH}, \Psi^B(f)) \circ p_X) - p_Y \circ f : X \to \mathrm{Hom}_{kH}(F_\lambda B^{kH}, \Psi^B(Y))$$

belongs to the ideal $[\mathrm{mod}_1(R/G)]$. But $u$ corresponds to the morphism

$$v = \Psi^B(f) \circ \widetilde{p}_X - \widetilde{p}_Y \circ (f \otimes_R F_\lambda B^{kH}) : X \otimes_{R/G} F_\lambda B^{kH} \to \Psi^B(Y).$$

Since $v$ restricted to $\Psi^B(X) = V_X^{-1}$ is zero, $v$ factors through $P_X^{-1}$ and in consequence $u$ factors through $\mathrm{Hom}_{R/G}(F_\lambda B^{kH}, P_X^{-1})$ which is isomorphic to $F_\lambda B^m$ for some $m \in \mathbb{N}$. Therefore by the Lemma below the residue class of $u$ in $\underline{\underline{\mathrm{mod}}}(R/G)$ is zero and the functors $\Phi^B$ and $\Psi^B|I_B$ are quasi-inverse. ∎

LEMMA. *For any module $X \in I_B$ each homomorphism $f \in \mathrm{Hom}_{R/G}(X, F_\lambda B)$ factors through a module from* $\mathrm{mod}_1(R/G)$.

Proof. Take any $X \in I_B$. Then there exists $V \in \mathrm{mod}(kH)^0$, where $H = G_B$, such that $X \simeq V^{-1} \otimes_{kH} F_\lambda B$. Using the results of Section 2 and [G; 3.2] we obtain

$$\begin{aligned}\mathrm{Hom}_{R/G}(X, F_\lambda B) &\simeq \mathrm{Hom}_R^G(F_\bullet(V^{-1} \otimes_R F_\lambda B), F_\bullet F_\lambda B)\\ &\simeq \mathrm{Hom}_R^G\Big(\bigoplus_{g\in S_B} {}^g(V \otimes_k B), \bigoplus_{g\in G} {}^gB\Big) \simeq \mathrm{Hom}_R^H\Big(V \otimes_k B, \bigoplus_{g\in G} {}^gB\Big)\\ &= \mathrm{Hom}_R^H\Big(V \otimes_k B, \bigoplus_{g_1\in U_B}\Big(\bigoplus_{h\in H} {}^{hg_1}B\Big)\Big),\end{aligned}$$

where $U_B$ is a fixed set of representatives of the right cosets $G \bmod H$. By Lemma 2.3 it is enough to prove that for each $f \in \mathrm{Hom}_R^H(V \otimes_k B, \bigoplus_{g_1 \in U_B}(\bigoplus_h {}^{hg_1}B))$, given by the family $f_g : V \otimes_k B \to {}^gB$, $g \in G$, the subobject $\bigoplus_{g\in G} \mathrm{Im}\, f_g = \bigoplus_{g_1 \in U_B}(\bigoplus_{h\in H} \mathrm{Im}\, f_{hg_1})$ of $\bigoplus_{g\in G} {}^gB$ in $\mathrm{MOD}^H R$, containing $\mathrm{Im}\, f$, is a direct sum of finite-dimensional $\tilde{R}$-modules. Fix such an $f$. Then the left $kH$-module $\bigoplus_{x\in \mathrm{supp}\, B} V \otimes B(x)$ is finitely generated and free. Therefore there exist $g_1, \ldots, g_r \in U_B$ such that $f_{hg} = 0$ for all $h \in H$ and $g \in U_B \setminus \{g_1, \ldots, g_r\}$, and hence $\bigoplus_{g\in G} \mathrm{Im}\, f_g = \bigoplus_{i=1}^r(\bigoplus_{h\in H} \mathrm{Im}\, f_{hg_i})$.

Now it remains to prove that $\dim \mathrm{Im}\, f_g$ or equivalently $\mathrm{supp}(\mathrm{Im}\, f_g)$ is finite for each $g \in G$. This is obvious in case $G_{{}^gB} \cap H = \{e\}$ since then $\mathrm{supp}\, B \cap \mathrm{supp}\, {}^gB$ is finite by Lemma 3.6. Assume that $G_{{}^gB} \cap H \neq \{e\}$ and suppose $\mathrm{supp}(\mathrm{Im}\, f_g)$ is infinite. Since $(\mathrm{supp}\, B)/(G_{{}^gB} \cap H)$ is finite there exists $x \in \mathrm{supp}\, B$ and an infinite sequence $h_n$, $n \in \mathbb{N}$, of pairwise different elements of $H \cap G_{{}^gB}$ such that $h_n x \in \mathrm{supp}(\mathrm{Im}\, f_g)$. Then $f_{h_n g} \neq 0$ for all $n \in \mathbb{N}$, and we get a contradiction with the fact that $\dim_k(V \otimes_k B(x))$ is finite. Therefore all spaces $\mathrm{Im}\, f_g$, $g \in G$, have finite dimension. This finishes the proof of the Lemma and of the main Theorem. ∎

R e m a r k. In fact, we proved that if $R$ satisfies only the conditions (i′) and (ii′) then all functors $\Phi^B$, $B \in \mathcal{W}_0^1$, are full and faithful and $\Phi$ induces an injection on the isomorphism classes of indecomposable objects.

**4.5.** P r o o f o f C o r o l l a r y 4.1. The first assertion is obvious. To prove the second recall that by [G; 3.6] the indecomposable modules of the first and the second kind are contained in different components of $\Gamma_{R/G}$ and that the union $(\Gamma_{R/G})_1$ of components containing indecomposable modules of the first kind has the form $\Gamma_R/G$.

It remains to show that the functor $\Phi^B$ preserves Auslander–Reiten sequences for any $B \in \mathcal{W}_0$. The functor $\Phi^B$ is exact and by Lemma 4.3 sends nonsplittable exact sequences into nonsplittable ones, therefore we only need to substantiate the preserving of the lifting property.

Denote by $\mathcal{J}_{R/G}$ (resp. $\mathcal{J}$) the Jacobson radical of the category $\mathrm{mod}(R/G)$ (resp. $\mathrm{mod}\, k[\xi, \xi^{-1}]$). Observe that if $X \in \mathrm{ind}(R/G)$ belongs to $\mathrm{mod}_1(R/G)$ or to the image of $\Phi^{B'}$, where $B' \in \mathcal{W}_0$ and $B' \neq B$, then the functor $\mathcal{J}_{R/G}(X, \Phi^B(-))$ coincides with $\mathrm{Hom}_{R/G}(X, \Phi^B(-))$, which by the results of Section 2 and Proposition 3.3 is exact. Moreover, if $X = \Phi^B(V)$ for some indecomposable $V \in \mathrm{mod}\, k[\xi, \xi^{-1}]$ then by Theorem 4.1 the functor $\Phi^B$ induces an isomorphism of functors

$$\mathcal{J}_{R/G}(X, \Phi^B(-)) \simeq [\mathrm{mod}_1(R/G)](X, \Phi^B(-)) \oplus \mathcal{J}(V, -).$$

The observation implies that $\Phi^B$ preserves the lifting property for exact sequences and in consequence Auslander–Reiten sequences, and the proof is finished. ∎

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