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The relative coincidence Nielsen number

by

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Abstract. We define a relative coincidence Nielsen number $N_{\mathrm{rel}}(f,g)$ for pairs of maps between manifolds, prove a Wecken type theorem for this invariant and give some formulae expressing $N_{\mathrm{rel}}(f,g)$ by the ordinary Nielsen numbers.

Introduction. In [S2] pairs of spaces $A \subset X$ and maps $f : X \to X$ such that $f(A) \subset A$ were considered. A relative Nielsen number of such maps was defined, i.e. a lower bound of the cardinality of fixed points which is invariant with the respect to homotopies preserving A. In this paper we generalize this construction to coincidences. We consider pairs of maps $f, g : M \to N$ between n-manifolds sending a fixed k-submanifold $M^0 \subset M$ into a fixed k-submanifold $N^0 \subset N$. We define a relative coincidence Nielsen number $N_{\mathrm{rel}}(f,g)$ for such pairs of maps, i.e. a homotopy invariant which is a lower bound for the number of coincidence points. We prove that in dimension ≥ 3 it is the best such lower bound (a Wecken type theorem). Finally, we express $N_{\mathrm{rel}}(f,g)$ by similar invariants of lifts $\widetilde{f}, \widetilde{g}$ and we present some computations.

1. Preliminaries. We will base on the definitions of Nielsen and Reidemeister classes given in [Je1] (compare [Y]). In this section we recall them and show how the same definitions may be obtained by means of covering spaces. In fact, the identification of the sets $\nabla(f,g)$ and $\mathrm{lift}'(f,g)$ given below is the equivalence of the two approaches to coincidence theory: the first, "traditional", using the fundamental group (see [B] or [Y] for fixed points), and the approach via covering spaces [Ji].

Let X and Y be path connected spaces and $f, g : X \to Y$ a pair of maps. The *Nielsen relation* ($x \simeq y$ if there is a path ω from x to y such that $f\omega$ and $g\omega$ are fixed end point homotopic in Y) splits the *coincidence set* $\Phi(f,g) = \{x \in X : fx = gx\}$ into *Nielsen classes*, and the quotient

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set is denoted by $\Phi'(f,g)$. Fix a point $x_0 \in X$ and a path r joining fx_0 to gx_0. We will call such an (x_0, r) a *reference pair*. For fixed (x_0, r) we define a left action of the fundamental group $\pi_1(X, x_0)$ on $\pi_1(Y, fx_0)$ by $\beta \circ \alpha = f_\#\beta + \alpha + r - g_\#\beta - r$ (we denote compositions of paths and homotopy classes additively); we call the orbits of this action *Reidemeister classes* and denote their set by $\nabla(f, g : x_0, r)$. The sets $\{\nabla(f, g : x_0, r)\}_{(x_0,r)}$ can be canonically identified giving an abstract set $\nabla(f, g)$ [Je1, 1.3].

Fix a reference pair (x_0, r) and a coincidence point x. If u is a path from x_0 to x then $fu - gu - r$ is a loop based at fx_0, hence it defines an element in $\nabla(f, g : x_0, r)$. This yields a map $\Phi(f, g) \to \nabla(f, g : x_0, r)$ which determines an injection $\Phi'(f, g) \to \nabla(f, g)$, and hence any Nielsen class may be considered as a Reidemeister class.

The above Reidemeister classes can also be defined by the use of universal coverings as was done for fixed points in [Ji]. We will need this approach in the next section, hence we now give the necessary definitions and we show how to identify the classes from these two approaches.

Assume that X and Y are connected and admit universal coverings (i.e. they are locally connected and semi-locally simply connected). Fix universal coverings $p_X : \widetilde{X} \to X$ and $p_Y : \widetilde{Y} \to$ Y. Denote by $\pi_X = \pi_1 X$ and $\pi_Y = \pi_1 Y$ the groups of natural transformations of $\widetilde{X}$ and $\widetilde{Y}$ respectively. Let $\operatorname{lift}(f, g)$ denote the set of all pairs $(\widetilde{f}, \widetilde{g})$ of lifts for which the following diagram commutes:

$$\begin{array}{ccc} \widetilde{X} & \xrightarrow{\widetilde{f},\widetilde{g}} & \widetilde{Y} \\ {\scriptstyle p_X}\downarrow & & \downarrow{\scriptstyle p_Y} \\ X & \xrightarrow[f,g]{} & Y \end{array}$$

Then $\pi_X \times \pi_Y$ acts on $\operatorname{lift}(f, g)$ from the left by

$$(\alpha, \beta) \circ (\widetilde{f}, \widetilde{g}) = \beta(\widetilde{f}, \widetilde{g})\alpha^{-1}.$$

We denote by $\operatorname{lift}'(f, g)$ the orbit space; we will show that there is a natural bijection between $\operatorname{lift}'(f, g)$ and $\nabla(f, g)$.

We fix a reference pair (x_0, r_0) and we define a map $R : \operatorname{lift}(f, g) \to \nabla(f, g : x_0, r_0)$ as follows. Let $(\widetilde{f}, \widetilde{g}) \in \operatorname{lift}(f, g)$. Fix $\widetilde{x}_0 \in p_X^{-1}(x_0)$ and a path $\widetilde{\omega}$ joining $\widetilde{f}\widetilde{x}_0$ to $\widetilde{g}\widetilde{x}_0$ in $\widetilde{Y}$. Then we put

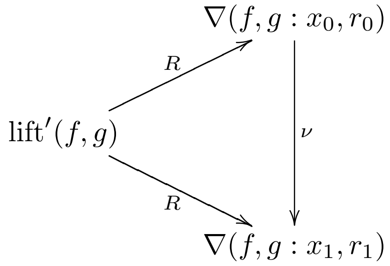
$$R(\widetilde{f}, \widetilde{g}) = [p_{Y\#}\widetilde{\omega} - r_0].$$

It is easy to check

(1.1) Lemma. *R is a correctly defined map inducing*

$$R : \operatorname{lift}'(f, g) \to \nabla(f, g : x_0, r_0).$$

Moreover, *for any other reference pair* (x_1, r_1) *the diagram*



commutes (*ν denotes the canonical identification of Reidemeister sets* [Je1, 1.3]). *Thus R induces a map* $R : \mathrm{lift}'(f, g) \to \nabla(f, g)$. ∎

Now we define the inverse map $S : \nabla(f, g) \to \mathrm{lift}'(f, g)$. Fix again a reference pair (x_0, r_0), a point $\widetilde{x}_0 \in p_X^{-1}(x_0)$ and $a \in \pi_1(Y, fx_0)$. Let $\widetilde{\omega}$ be a lift of the path $a + r_0$. Let $\widetilde{f}$ and $\widetilde{g}$ be the lifts satisfying $\widetilde{f}(\widetilde{x}_0) = \widetilde{\omega}(0)$ and $\widetilde{g}(\widetilde{x}_0) = \widetilde{\omega}(1)$. We define $S : \pi_1(Y, fx_0) \to \mathrm{lift}'(f, g)$ putting $S(a) = [\widetilde{f}, \widetilde{g}]$. Then it is easy to check

(1.2) Lemma. *S is a well defined map inducing* $S : \nabla(f, g : x_0, r_0) \to \mathrm{lift}'(f, g)$ *which is inverse to* $R : \mathrm{lift}'(f, g) \to \nabla(f, g : x_0, r_0)$. ∎

Thus we may identify the sets $\mathrm{lift}'(f, g)$ and $\nabla(f, g)$ by means of R and S. In Section 3 we will need the following relative version of (1.2). Let now $p_X : \widetilde{X} \to X$ and $p_Y : \widetilde{Y} \to Y$ be coverings corresponding to normal subgroups $H \subset \pi_1 X$ and $H' \subset \pi_1 Y$ such that $f_{\#} H \subset H'$ and $g_{\#} H \subset H'$. Then the corresponding set of lifts is nonempty and formulae similar to those above define mutually inverse maps $R : \mathrm{lift}'_{H'}(f, g) \to \nabla_{H'}(f, g)$ and $S : \nabla_{H'}(f, g) \to \mathrm{lift}'_{H'}(f, g)$, where $\mathrm{lift}'_{H'}(f, g)$ is obtained in a similar manner to $\mathrm{lift}'(f, g)$ above and $\nabla_{H'}(f, g)$ is defined in [Je1]. In fact, these two quotient sets do not depend on the subgroup H and hence the above symbols do not contain this letter.

2. The relative Nielsen number. A pair of maps $f, g : M \to N$ will be called *Φ-compact* if the coincidence set $\Phi(f, g) = \{x \in M : fx = gx\}$ is compact.

Now we recall the definition of the semi-index of a pair of maps $f, g : M \to N$ ([DJ], [Je4]). We assume that M and N are topological n-manifolds without boundary and that the coincidence set $\Phi(f, g)$ is compact. Replace f, g by a transverse pair and consider a subset $A \subset \Phi(f, g)$. Fix two points $x_0, x_1 \in A$ and a path ω establishing the Nielsen relation between them. Fix a local orientation α_0 of M at x_0 and denote by α_t its translation along ω. By transversality $(f, g)_{\#} \alpha_0$ determines an orientation β_0 of the normal bundle to the diagonal $\Delta N \subset N \times N$ at the point (fx_0, gx_0). We say that ω *establishes the R-relation* between x_0 and x_1 if the translation of β_0 along the path $(f\omega, f\omega)$ is opposite to the orientation of the normal bundle at the point (fx_1, gx_1) determined by $(f,g)_\#\alpha_1$. Then we say that x_0, x_1 are *R-related* and we write x_0Rx_1. Consider a presentation $A = \{a_1, b_1, \ldots, a_k, b_k : c_1, \ldots, c_s\}$, where a_iRb_i but never c_iRc_j $(i \neq j)$. Finally, we define the *semi-index*

$$|\mathrm{ind}|(f, g : A) = s$$

(number of free elements). For details see [DJ] in the smooth case, and [Je4] in the topological case. In the oriented case the above semi-index of a Nielsen class equals the absolute value of the ordinary coincidence index of this class.

Now we suppose that $\mathrm{bd}\, M$ and $\mathrm{bd}\, N$ are nonempty. A pair of maps $f, g : M \to N$ satisfying $g(\mathrm{bd}\, M) \subset \mathrm{bd}\, N$ will be called a *B-pair* [BS]. A homotopy $f_t, g_t : M \to N$ will be called a *B-homotopy* iff f_t, g_t is a B-pair for any t. Now we can follow [BS] to extend the above coincidence semi-index to Φ-compact B-pairs. Denote by $2M$ a double of M, i.e. a manifold without boundary obtained from two copies of M by identifying corresponding points on the boundaries: $2M = (M \cup (-M))/\simeq$. Let $r : 2M \to M$ be a retraction such that $r(-x) = x$ and let $i : N \to 2N$ be the inclusion. We define maps $\widehat{f}, 2g : 2M \to 2N$ by $\widehat{f}(x) = ifr(x)$ and $2g(x) = g(x)$, $2g(-x) = -g(x)$. Then the coincidence sets and Nielsen relations of the pairs f, g and $\widehat{f}, 2g$ coincide. For a Nielsen class $A \subset \Phi(f, g)$ we define

$$|\mathrm{ind}|(f, g : A) = |\mathrm{ind}|(\widehat{f}, 2g : A)$$

(the $|\mathrm{ind}|$ on the right side is already defined since $2M$ and $2N$ have no boundary). Now we define the Nielsen number $N(f, g)$ as the number of *essential* classes, i.e. classes with nonzero semi-index. It is easy to check that $|\mathrm{ind}|(f, g)$ and $N(f, g)$ are Φ-compact B-homotopy invariants (compare [BS]).

Notice that the above semi-index is not symmetric ($|\mathrm{ind}|(f, g)$ and $|\mathrm{ind}|(g, f)$ may be different if $\mathrm{bd}\, M \neq \emptyset$). In fact, even in the compact oriented but nonclosed case this index and the Lefschetz number from [BS] are not symmetric (if $f, g : (D^n, S^{n-1}) \to (D^n, S^{n-1})$, $f =$ identity, $g =$ constant, then $\mathrm{ind}(f, g) = 0$ while $\mathrm{ind}(g, f) = 1$). Here we are not going to discuss this question: the above semi-index will be used to introduce a relative Nielsen number which despite the lack of symmetry in its definition is, under some assumptions, the best lower bound for the number of coincidence points (Thm. (2.4)). In [Je5] we show that the above Lefschetz numbers differ by the Lefschetz number of restrictions to the boundaries. We prove there that this also holds for the coincidence Lefschetz numbers generalized to the nonorientable case.

Since the boundary of a connected manifold may be disconnected we have to generalize the definition of Reidemeister classes to the disconnected case. Consider a pair of maps $f,g : M \to N$ and let $M = \bigcup_{i\in I} M_i$ and $N = \bigcup_{j\in J} N_j$ be the decompositions into connected components. Let $I_0 = \{i \in I : f(M_i)$ and $g(M_i)$ are contained in the same component of $N\}$. For a fixed $i \in I_0$ let $j(i) \in J$ satisfy $f(M_i) \cup g(M_i) \subset N_{j(i)}$ and let $f_i, g_i : M_i \to N_{j(i)}$ denote the restrictions of f and g respectively. We define $\nabla(f,g)$ to be the disjoint sum $\bigcup_{i\in I_0} \nabla(f_i, g_i)$.

Let $\Phi'_{\mathrm{e}}(f,g)$, $\nabla_{\mathrm{e}}(f,g)$ denote the sets of essential Nielsen and Reidemester classes respectively. Notice that the natural inclusion $\Phi'(f,g) \to \nabla(f,g)$ identifies $\Phi'_{\mathrm{e}}(f,g)$ with $\nabla_{\mathrm{e}}(f,g)$.

Consider the following setting:

(2.0) M and N are connected topological n-manifolds (possibly with boundary), and $M^0 \subset M$ and $N^0 \subset N$ are fixed connected locally flat k-submanifolds (without boundary) such that either $M^0 \subset \operatorname{int} M$ and $N^0 \subset \operatorname{int} N$, or $M^0 \subset \operatorname{bd} M$ and $N^0 \subset \operatorname{bd} N$. Let $f,g : M \to N$ be a Φ-compact B-pair satisfying $f(M^0) \cup g(M^0) \subset N^0$.

It is also convenient to present the above setting as a commutative diagram

$$\begin{array}{ccc} M^0 & \xrightarrow{f^0,g^0} & N^0 \\ \downarrow & & \downarrow \\ M & \xrightarrow{f,g} & N \end{array}$$

where the vertical arrows denote inclusions.

(2.1) Definition ([S2]). An essential Nielsen class $A \subset \Phi(f,g)$ will be called *common essential* if it contains an essential class from $\Phi(f^0, g^0)$.

Denote by $N_\partial(f,g)$ the number of common essential classes of the setting (2.0). Notice that

(2.2)
$$\begin{aligned} N_\partial(f,g) &= \#\operatorname{im}\{\Phi'_{\mathrm{e}}(f^0,g^0) \to \Phi'_{\mathrm{e}}(f,g)\} \\ &= \#\{\nabla_{\mathrm{e}}(f,g) \cap \operatorname{im}(\nabla_{\mathrm{e}}(f^0,g^0) \to \nabla(f,g))\}, \end{aligned}$$

where the arrows denote the maps induced by the natural inclusions.

Now we define the *relative Nielsen number* $N_{\mathrm{rel}}(f,g)$ of the setting (2.0) to be

$$N_{\mathrm{rel}}(f,g) = N(f,g) + N(f^0,g^0) - N_\partial(f,g)$$

(we omit f^0, g^0 not to complicate the symbol $N_{\mathrm{rel}}(f,g)$). By (2.2) it is clear that $N_\partial(f,g)$ and hence also $N_{\mathrm{rel}}(f,g)$ are homotopy invariants with respect to Φ-compact B-homotopies, i.e. $F, G : M \times I \to N$ such that $F(M^0 \times I) \cup G(M^0 \times I) \subset N^0$, $G(\operatorname{bd} M \times I) \subset \operatorname{bd} N$ and $\Phi(F,G)$ is compact. We will call such homotopies *rel-admissible*. Now a standard modification of [S2, 3.1] gives

(2.3) Theorem. *Any pair f', g' rel-admissibly homotopic to f, g has at least $N_{\mathrm{rel}}(f,g)$ coincidence points.* ∎

To get a converse, i.e. to find some conditions for $N_{\mathrm{rel}}(f,g)$ to be also the best lower bound for $\#\Phi(f,g)$, we again recall a definition from [S2, 5.1]. We say that a subspace $X_0 \subset X$ can be *by-passed* iff $X - X_0$ is connected and $\pi_1(X - X_0) \to \pi_1 X$ is onto.

If we assume that X is a connected manifold and X_0 a locally flat submanifold of X, then any of conditions

(a) $X_0 \subset \mathrm{bd}\, X$,
(b) $\dim X - \dim X_0 \geq 2$

implies X_0 can be by-passed in X.

(2.4) Theorem. *Let $f, g : M \to N$ satisfy* (2.0), *where*

$$\dim M^0 = \dim N^0 \geq 3$$

and M^0 (resp. N^0) can be by-passed in M (resp. N). Then there is a pair rel-admissibly homotopic to f, g with exactly $N_{\mathrm{rel}}(f,g)$ coincidence points.

Proof. By applying the Whitney trick [Je4] to f^0, g^0 ($\dim M^0 \geq 3$) and to f, g we may assume that $\Phi(f^0, g^0)$ contains exactly $N(f^0, g^0)$ coincidence points, $\Phi(f,g)$ is finite, no two points in $\Phi(f,g) - M^0$ are Nielsen related and the semi-index of any $x \in \Phi(f,g)$ is nonzero. If $M^0 \subset \mathrm{int}\, M$ these homotopies may be chosen constant on the boundary and if $M^0 \subset \mathrm{bd}\, M$ we may require that during these homotopies $f(\mathrm{bd}\, M - M^0) \subset \mathrm{int}\, N$ and $g(\mathrm{bd}\, M) \subset \mathrm{bd}\, N$. In both cases we get a B-homotopy.

It remains to show that if two points $y \in \Phi(f,g) - M^0$ and $x^0 \in \Phi(f^0, g^0)$ are Nielsen related then there is a rel-admissible homotopy F, G constant on M^0 and in a neighbourhood of $\Phi(f,g) - \{x^0, y\}$ from f, g to a pair $\overline{f}, \overline{g}$ satisfying $\Phi(\overline{f}, \overline{g}) = \Phi(f,g) - \{y\}$.

First, consider the case $M^0 \subset \mathrm{bd}\, M$. The path establishing the Nielsen relation between x^0 and y may be chosen a locally flat (hence flat) arc satisfying $\omega(0,1] \subset \mathrm{int}\, M$ and $\omega(t) \notin \Phi(f,g)$ for $0 < t < 1$. Fix an open subset $U \subset M$ homeomorphic to $\mathbb{R}^{n-1} \times [0, \infty)$ such that under this homeomorphism $\omega(t) = (0,t) \in \mathbb{R}^{n-1} \times [0,1]$, $U \cap \mathrm{bd}\, M \subset \mathbb{R}^{n-1} \times 0$ and $U \cap \Phi(f,g) = \{x^0, y\}$. On the other hand, we find a flat arc τ joining $fx^0 = gx^0$ to $fy = gy$ in N and homotopic to $f\omega \simeq g\omega$. We fix a euclidean neighbourhood $V \simeq \mathbb{R}^{n-1} \times [0, \infty)$ of $\tau \subset N$.

We will show that there is a pair of B-homotopies F, G starting from f, g, constant outside U and satisfying $F_1(\omega) \cup G_1(\omega) \subset V$. Since $f\omega$ and $g\omega$ are fixed end point homotopic to τ, there exist homotopies $f_{|t}, g_{|t} : \omega[0,1] \to N$

from the restrictions of f and g to a pair of maps into V, constant on $\omega[0,\varepsilon]\cup\omega[1-\varepsilon,1]$ for an $\varepsilon>0$. Moreover, by the assumption $\dim N\geq 3$ we may assume that this pair of homotopies satisfies $\Phi(f_{|t},g_{|t})=\{x^0,y\}$ for each t. Fix two closed balls $K^0=D\times[0,\varepsilon]$, $K^1=D\times[1-\varepsilon,1+\varepsilon]$ in $\mathbb{R}^{n+1}\times[0,\infty)=U\subset M$ (Figure 1). Define $F':M\times 0\cup(K^0\cup\omega[0,1]\cup K^1)\times I\to N$ by

$$F'(x,t)=\begin{cases} f_{|t}(x) & \text{if } x\in\omega[\varepsilon,1-\varepsilon],\\ f(x) & \text{otherwise.}\end{cases}$$

Similarly we define G'.

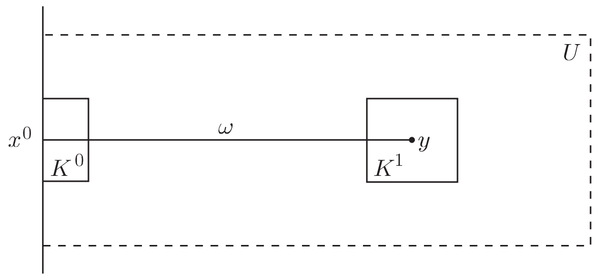


Fig. 1

Fix a retraction $r:M\times[0,1]\to(M\times 0)\cup(K^0\cup\omega[0,1]\cup K^1)\times[0,1]$ such that $r(x,t)=(x,0)$ for x lying outside U, $r(\operatorname{bd}M\times[0,1])\subset\operatorname{bd}M$ and $r^{-1}(\{x^0,y\}\times[0,1])=\{x^0,y\}\times[0,1]$. We put $F=F'r$, $G=G'r$ and we notice that $\Phi(F_1,G_1)=\Phi(f,g)$.

Now F_1 and G_1 send $\omega[0,1]$ into V, hence $F_1(U_1)\cup G_1(U_1)\subset V$ for some neighbourhood U_1 of $\omega[0,1]$ in U. We fix another euclidean neighbourhood $U_2\subset U_1$ so that $\omega(0,1]\subset U_2\subset U_1-M^0$ and any point $x\in\operatorname{cl}U_2-x^0$ is uniquely written as $x=tx^0+(1-t)x_1$, where $x_1\in(\operatorname{bd}U_2)-x^0$.

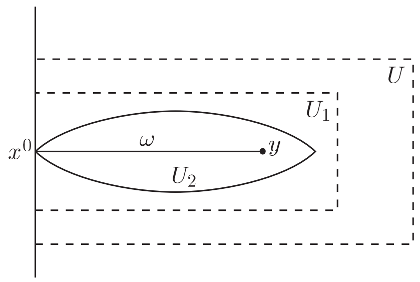


Fig. 2

Finally, we put

$$\bar{f}(x)=\begin{cases} tF_1(x^0)+(1-t)F_1(x_1) & \text{for } x=tx^0+(1-t)x_1\in\operatorname{cl}U_2,\\ F_1(x) & \text{otherwise,}\end{cases}$$

and

$$\overline{g}(x) = \begin{cases} tG_1(x^0) + (1-t)G_1(x_1) & \text{for } x = tx^0 + (1-t)x_1 \in \operatorname{cl} U_2, \\ G_1(x) & \text{otherwise.} \end{cases}$$

Now $\bar{f}, \overline{g}$ is homotopic to F_1, G_1 (by segments) $\operatorname{rel}(M - U_2)$, hence $\Phi(\bar{f}, \overline{g}) = \Phi(F_1, G_1) - \{y\}$.

Now assume that $x^0 \in \operatorname{int} M$. Then we proceed as above taking as $U \subset M$ a subset homeomorphic to $\mathbb{R}^n$ (x^0, y correspond to $(0,0), (0,1) \in \mathbb{R}^n \times \mathbb{R}$) and $K_0 = D \times [-\varepsilon, \varepsilon]$.

3. The relative Nielsen number and coverings. Let $f, g : M \to N$ be a pair satisfying (2.0). Let $p_M : \widetilde{M} \to M$ and $p_N : \widetilde{N} \to N$ be coverings corresponding to normal subgroups $H \subset \pi_1 M$ and $H' \subset \pi_1 N$. We assume that $f_\#$ and $g_\#$ send H into H'. Then f and g admit lifts

$$\begin{array}{ccc} \widetilde{M} & \xrightarrow{\widetilde{f}, \widetilde{g}} & \widetilde{N} \\ {\scriptstyle p_M}\big\downarrow & & \big\downarrow{\scriptstyle p_N} \\ M & \xrightarrow{f, g} & N \end{array} \tag{3.1}$$

Let

$$\begin{array}{ccc} p_M^{-1}(M^0) & \xrightarrow{\widetilde{f}^0, \widetilde{g}^0} & p^{-1}(N^0) \\ {\scriptstyle p_M}\big\downarrow & & \big\downarrow{\scriptstyle p_N} \\ M^0 & \xrightarrow{f^0, g^0} & N^0 \end{array}$$

denote the restriction of the above diagram over M^0 and N^0. Let $i_M : M^0 \to M$, $\widetilde{i}_M : p_M^{-1}(M^0) \to \widetilde{M}$, $i_N : N^0 \to N$ and $\widetilde{i}_N : p_N^{-1}(N^0) \to \widetilde{N}$ denote the inclusions, and $\nabla i = \nabla(i_M, i_N) : \nabla(f^0, g^0) \to \nabla(f, g)$ and $\nabla\widetilde{i} = \nabla(\widetilde{i}_M, \widetilde{i}_N) : \nabla(\widetilde{f}^0, \widetilde{g}^0) \to \nabla(\widetilde{f}, \widetilde{g})$ the induced maps of Reidemeister classes.

(3.2) Lemma. *Under the above notations*:

(i) $\Phi(f, g) = \bigcup p_M \Phi(\widetilde{f}, \widetilde{g})$, *where the summation runs over* $(\widetilde{f}, \widetilde{g}) \in \operatorname{lift}_{H'}(f, g)$.

(ii) *If for two pairs* $(\widetilde{f}, \widetilde{g}), (\widetilde{f}', \widetilde{g}') \in \operatorname{lift}(f, g)$ *we have* $p_M \Phi(\widetilde{f}, \widetilde{g}) \cap p_M \Phi(\widetilde{f}', \widetilde{g}') \neq \emptyset$ *then* $(\widetilde{f}', \widetilde{g}') = \beta(\widetilde{f}, \widetilde{g})\alpha$ *for some* $\alpha \in \pi_M$ *and* $\beta \in \pi_N$, *and hence the considered pairs belong to the same orbit in* $\operatorname{lift}'(f, g)$. *Then* $p_M \Phi(\widetilde{f}, \widetilde{g}) = p_M \Phi(\widetilde{f}', \widetilde{g}')$, *hence* $\Phi(f, g) = \bigcup p_M \Phi(\widetilde{f}, \widetilde{g})$ *is a disjoint sum if in the summation we take one representative* $(\widetilde{f}, \widetilde{g})$ *from each orbit in* $\operatorname{lift}'_{H'}(f, g)$.

(iii) *Let* $\widetilde{A} \subset \Phi(\widetilde{f}, \widetilde{g})$ *and* $A \subset \Phi(f, g)$ *be Nielsen classes such that neither* $A \subset \Phi(f, g)$ *nor* $A \cap M^0 \subset \Phi(f^0, g^0)$ *is defective and* $p_M \widetilde{A} = A$. *Then* A

is a common essential Nielsen class for f, g *iff* $\widetilde{A}$ *is common essential for* $\widetilde{f}, \widetilde{g}$. (For the definition of defective classes see [Je2].)

P r o o f. (i) and (ii) are easy to check. We prove (iii). First, we notice that

(1) if $\widetilde{A}$ is a Nielsen class of $\widetilde{f}, \widetilde{g}$ then $p_M\widetilde{A}$ is a Nielsen class of f, g,

(2) if $\widetilde{A}$ is essential then $p_M\widetilde{A}$ is also essential,

(3) if A is a nondefective class of f, g then either $p_M^{-1}A \cap \Phi(\widetilde{f}, \widetilde{g})$ is empty or it is a sum of m_A Nielsen classes of $\widetilde{f}, \widetilde{g}$ of semi-index $r_A|\mathrm{ind}|(f, g : A)$ each, where m_A and r_A are the natural numbers defined in [Je3, 2.2, 2.3]. In particular, A nondefective essential implies $\widetilde{A}$ essential.

Suppose $\widetilde{A} \subset \Phi(\widetilde{f}, \widetilde{g})$ is common essential. Then $\widetilde{A}$ contains an essential class $\widetilde{A}^0 \subset \Phi(\widetilde{f}^0, \widetilde{g}^0)$ and (2) implies $p_M\widetilde{A}^0 \subset p_M\widetilde{A} = A$ is essential, hence A is common essential.

Now let $A \subset \Phi(f, g)$ be common essential. Let $A^0 \subset \Phi(f^0, g^0)$ be an essential class contained in A. Now $p^{-1}A^0 \cap \widetilde{A} \neq \emptyset$; denote by $\widetilde{M}'$ a component of $p_M^{-1}M^0$ such that $\widetilde{M}' \cap p^{-1}A^0 \cap \widetilde{A} \neq \emptyset$ and let $\widetilde{N}'$ be the component of $p_N^{-1}N^0$ containing $\widetilde{f}\widetilde{M}' \cup \widetilde{g}\widetilde{M}'$. Consider the diagram

$$\begin{array}{ccc} \widetilde{M}' & \xrightarrow{\widetilde{f}', \widetilde{g}'} & \widetilde{N}' \\ {\scriptstyle p_M}\downarrow & & \downarrow{\scriptstyle p_N} \\ M^0 & \xrightarrow{f^0, g^0} & N^0 \end{array}$$

Since the covering spaces in this diagram are connected, (3) implies $p_M^{-1}A^0 \cap \widetilde{M}'$ is a disjoint sum of essential classes of $\widetilde{f}^0, \widetilde{g}^0$. But, as we have noticed, $\widetilde{M}' \cap p_M^{-1}A^0 \cap \widetilde{A} \neq \emptyset$, hence some of these classes are contained in $\widetilde{A}$. Thus $\widetilde{A}$ is common essential. ∎

By [Je3] any essential class $A \subset \Phi(f, g)$ is covered by m_A essential classes from $\Phi(\widetilde{f}, \widetilde{g})$ for some lifts $\widetilde{f}, \widetilde{g}$ and by the above proof if A is common essential then all the above classes from $\Phi(\widetilde{f}, \widetilde{g})$ are common essential, provided no class involved is defective. Thus we obtain

(3.3) Corollary. *If no Nielsen class of* f, g *or of* f^0, g^0 *is defective then*

$$\sum_{\widetilde{f}, \widetilde{g}} N(\widetilde{f}, \widetilde{g}) = \sum_{A} m_A,$$

where in the summation on the left we take one pair from each Reidemeister class in $\mathrm{lift}'_{H'}(f, g)$ *and on the right* A *runs over all the essential classes* *from $\Phi(f,g)$; moreover*,

$$\sum_{\widetilde{f},\widetilde{g}} N\partial(\widetilde{f},\widetilde{g}) = \sum_{A} m_A,$$

where the summation on the left is as above, while on the right A runs over all the common essential classes in $\Phi(f,g)$.

If $m_A = m$ does not depend on the class $A \in \Phi(f,g)$ then

$$\sum_{\widetilde{f},\widetilde{g}} N(\widetilde{f},\widetilde{g}) = mN(f,g), \qquad \sum_{\widetilde{f},\widetilde{g}} N\partial(\widetilde{f},\widetilde{g}) = mN\partial(f,g),$$

and hence

$$N(f,g) = (1/m)\sum_{\widetilde{f},\widetilde{g}} N(\widetilde{f},\widetilde{g}), \qquad N\partial(f,g) = (1/m)\sum_{\widetilde{f},\widetilde{g}} N\partial(\widetilde{f},\widetilde{g})$$

(*in all the above sums we take one representative from each class in* $\mathrm{lift}'(f,g)$).

If, moreover, $N(f^0,g^0) = (1/m)\sum_{\widetilde{f},\widetilde{g}} N(\widetilde{f}^0,\widetilde{g}^0)$ then we get a similar formula for the relative Nielsen numbers:

$$N_{\mathrm{rel}}(f,g) = (1/m)\sum_{\widetilde{f},\widetilde{g}} N_{\mathrm{rel}}(\widetilde{f},\widetilde{g}). \ ■$$

The last assumption is satisfied if for example the natural homomorphisms $\pi_1 M^0 \to (\pi_1 M)/H$ and $\pi_1 N^0 \to (\pi_1 N)/H'$ are epi and $f_\# h = g_\# h$ for any $h \in H$.

4. Computations. Suppose we are given two-fold coverings $p_M : \widetilde{M} \to M$ and $p_N : \widetilde{N} \to N$. Then their cones C_M and C_N are manifolds with boundaries $\widetilde{M}$ and $\widetilde{N}$ respectively. In this section we will give formulae for the relative Nielsen number of $f,g : C_M \to C_N$ preserving boundaries; more exactly, we will express $N_{\mathrm{rel}}(f,g)$ by the ordinary Nielsen numbers of suitable maps $\widetilde{M} \to \widetilde{N}$. In a special case $p_N : \widetilde{N} = S^n \to \mathbb{R}P^n = N$, $C_N = \mathbb{R}P^{n+1}-$ disk we may combine this result with Section 3 of [Je3], where a method of computing the (ordinary) Nielsen number of any pair of maps into $\mathbb{R}P^n$ and S^n is given, to get an algorithm for the relative Nielsen number of a pair of maps $C_M \to \mathbb{R}P^{n+1} - D^{n+1}$ sending boundary into boundary. By analogy with [Je3] one may regard maps into the pair $(\mathbb{R}P^{n+1} - D^{n+1}, S^n)$ as the simplest nontrivial case in the relative Nielsen theory since $\pi_1(\mathbb{R}P^{n+1} - D^{n+1}) = \mathbb{Z}_2$ and $\pi_1 S^n = 0$ $(n \ge 2)$.

Let $p_M : \widetilde{M} \to M$ be a two-fold covering of a connected n-manifold (without boundary) and let ϱ_M be the corresponding involution of $\widetilde{M}$. Then the cone of the above covering $C_M = (\widetilde{M} \times [0,1])/\simeq$ $((\widetilde{x},0) \simeq (\varrho_M(\widetilde{x}),0))$ is an $(n+1)$-manifold with the boundary $C_M^0 = \widetilde{M} \times 1$. Let $\widetilde{C}_M = \widetilde{M} \times [-1,1]$. Then the map $\overline{p}_M : \widetilde{C}_M \to C_M$ given by

$$\overline{p}_M(\widetilde{x}, t) = \begin{cases} [\widetilde{x}, t] & \text{for } t \geq 0, \\ [\varrho_M(\widetilde{x}), -t] & \text{for } t \leq 0, \end{cases}$$

is also a two-fold covering and $\overline{\varrho}_M(\widetilde{x}, t) = (\varrho_M \widetilde{x}, -t)$ is the corresponding involution.

Let $p_N : \widetilde{N} \to N$ be another two-fold covering. We will consider a pair of maps $f, g : C_M \to C_N$ sending boundary into boundary, i.e. $f(C_M^0) \cup g(C_M^0) \subset C_N^0$, and we will try to find the relative Nielsen number of such a pair (Thm. (4.5)). We start by classifying such maps.

(4.1) Lemma. *For any map $f : (C_M, C_M^0) \to (C_N, C_N^0)$ there exists a map $\widetilde{\phi} : \widetilde{M} \to \widetilde{N}$ such that $\widetilde{\phi}\varrho_M = \varrho_N \widetilde{\phi}$ and f is homotopic rel. boundary either to $f'[\widetilde{x}, t] = [\widetilde{\phi}(\widetilde{x}), t]$ or to $f''[\widetilde{x}, t] = [\widetilde{\phi}(\widetilde{x}), 1]$.*

Proof. First we show that f admits a lift $\widetilde{f}$:

$$\begin{array}{ccc} \widetilde{C}_M & \overset{\widetilde{f}}{\dashrightarrow} & \widetilde{C}_N \\ {\scriptstyle \overline{p}_M}\downarrow & & \downarrow{\scriptstyle \overline{p}_N} \\ C_M & \xrightarrow{f} & C_N \end{array}$$

Such a lift exists iff $(f\overline{p}_M)_\#(\pi_1 \widetilde{C}_M) \subset \overline{p}_{N\#}(\pi_1 \widetilde{C}_N)$. But we notice that $\overline{p}_{M\#}(\pi_1 \widetilde{C}_M) = i_{M\#}(\pi_1 C_M^0)$, where $i_M : C_M^0 \to C_M$ denotes the inclusion of the boundary, and similarly $\overline{p}_{N\#}(\pi_1 \widetilde{C}_N) = i_{N\#}(\pi_1 C_N^0)$. Now

$$\begin{aligned} (f\overline{p}_M)_\#(\pi_1 \widetilde{C}_M) &= (f i_M)_\#(\pi_1 C_M^0) \\ &= (i_N f)_\#(\pi_1 C_M^0) \subset i_{N\#}(\pi_1 C_N^0) = \overline{p}_{N\#}(\pi_1 \widetilde{C}_N). \end{aligned}$$

Since the covering p_N is two-fold, any lift $\widetilde{f}$ satisfies either $\widetilde{f}\overline{\varrho}_M = \overline{\varrho}_N \widetilde{f}$ or $\widetilde{f}\overline{\varrho}_M = \widetilde{f}$. In the first case we call $\widetilde{f}$ *odd* and in the second *even*. We notice that the following three conditions are equivalent:

(a) $\widetilde{f}$ is odd.

(b) The map $\pi_1 C_M / \operatorname{im} p_{M\#} \to \pi_1 C / \operatorname{im} p_{N\#}$ induced by f is nonzero.

(c) $\widetilde{f}$ sends the components of the boundary $\widetilde{C}_M^0 = (\widetilde{M} \times 1) \cup (\widetilde{M} \times (-1))$ into distinct components of $\widetilde{C}_N^0 = (\widetilde{N} \times 1) \cup (\widetilde{N} \times (-1))$.

Similarly

(a′) $\widetilde{f}$ is even.

(b′) The induced homotopy map is zero.

(c′) $\widetilde{f}$ carries both components of $\widetilde{C}_M^0$ into the same component of $\widetilde{C}_N^0$.

Without loss of generality we may assume that $\widetilde{f}:\widetilde{C}_M=\widetilde{M}\times[-1,1]\to\widetilde{C}_N=\widetilde{N}\times[-1,1]$, $\widetilde{f}(\widetilde{x},t)=(\widetilde{f}_1(\widetilde{x},t),\widetilde{f}_2(\widetilde{x},t))$, sends $\widetilde{M}\times 1$ into $\widetilde{N}\times 1$. Assume, moreover, that $\widetilde{f}$ is odd. Define a homotopy $\widetilde{H}_s:\widetilde{C}_M\to\widetilde{C}_N$ by

$$\widetilde{H}_s(\widetilde{x},t)=(\widetilde{f}_1(\widetilde{x},t(1-s)),(1-s)\widetilde{f}_2(\widetilde{x},t)+ts).$$

Since $\widetilde{H}_s\widetilde{\varrho}_M=\widetilde{\varrho}_N\widetilde{H}_s$, $\widetilde{H}_s$ defines a (boundary preserving) homotopy $H_s: C_M\to C_N$ from $H_0=f$ to the map $H_1[\widetilde{x},t]=[\widetilde{\phi}(\widetilde{x}),t]$, where $\widetilde{\phi}(\widetilde{x})=\widetilde{f}_1(\widetilde{x},0)$.

Now assume that $\widetilde{f}$ is even. Define

$$\widetilde{H}_s(\widetilde{x},t)=(\widetilde{f}_1(\widetilde{x},t(1-s)),(1-s)\widetilde{f}_2(\widetilde{x},t)+s).$$

Then $\widetilde{H}_s\widetilde{\varrho}_M=\widetilde{H}_s$ and hence $\widetilde{H}_s$ defines a (boundary preserving) homotopy from f to the map $H_1[\widetilde{x},t]=[\widetilde{\phi}(\widetilde{x}),1]$, where $\widetilde{\phi}(\widetilde{x})=\widetilde{f}_1(\widetilde{x},0)$. ■

Notice that if $\widetilde{\phi},\widetilde{\psi}:\widetilde{M}\to\widetilde{N}$ are maps from the above lemma corresponding to $f,g:C_M\to C_N$, then identifying C_M^0, C_N^0 with $\widetilde{M},\widetilde{N}$ we get $N(f^0,g^0)=N(\widetilde{\phi},\widetilde{\psi})$.

(4.2) Lemma. *There is a natural bijection of Reidemeister classes* $\nabla(f_0,g_0)\to\nabla(f_1,g_1)$ *preserving semi-index and defective classes provided one of the following assumptions is satisfied*:

(a) *The diagram*

$$\begin{array}{ccc} M_0 & \xrightarrow{f_0,g_0} & N_0 \\ \downarrow & & \downarrow \\ M_1 & \xrightarrow{f_1,g_1} & N_1 \end{array}$$

is commutative and the vertical lines are homeomorphisms.

(b) *The pairs* $(f_0,g_0),(f_1,g_1):M\to N$ *are B-homotopic.*

(c) *The above diagram is commutative*, *the vertical lines are inclusions*, $\dim M_i=\dim N_i$ $(i=0,1)$ *and* $f(M_1)\subset N_0$. *Moreover*, $M_0\subset M_1$ *and* $N_0\subset N_1$ *admit normal bundles* ν *and* ν' *such that the restriction* $g:\nu\to\nu'$ *is homeomorphic on fibres and any path establishing the Nielsen relation between two coincidence points of* f_1,g_1 *can be deformed into* M_0.

Proof. (a), (b) are evident. To prove (c) we notice that the last assumption implies that the map induced on the Nielsen classes is a bijection while the normal bundles give the equality of semi-indices. ■

Consider again a pair of maps $f,g:C_M\to C_N$ sending boundary into boundary and let $\widetilde{f},\widetilde{g}:\widetilde{C}_M\to\widetilde{C}_N$ be their lifts sending $\widetilde{M}\times 1$ into $\widetilde{N}\times 1$. We consider several cases.

(a) $\widetilde{g}$ *is even*. By (4.1) we may assume that the image of g is contained in $\widetilde{N}\times 1$. Composing f with a homotopy shifting $\widetilde{N}\times 1$ inside C_N we get $\Phi(f,g)=\emptyset$, hence $N(f,g)=0$. Thus $N_\partial(f,g)=0$, hence $N_{\mathrm{rel}}(f,g)=N(f^0,g^0)=N(\widetilde{\phi},\widetilde{\psi})$.

(b) $\widetilde{f}$ *is even*, $\widetilde{g}$ *is odd*. By (4.1) we may assume that $f[\widetilde{x},t]=[\widetilde{\phi}(\widetilde{x}),1]$ and $g[\widetilde{x},t]=[\psi(\widetilde{x}),t]$. We notice that the pairs (f^0,g^0) and (f,g) satisfy the assumptions of (4.2)(c). In fact, one needs only prove the last assumption. Fix a path ω in C_M joining two coincidence points of f^1,g^1 and such that $f\omega$ is homotopic to $g\omega$ in C_N. We prove that ω is homotopic to a path in C^0_M.

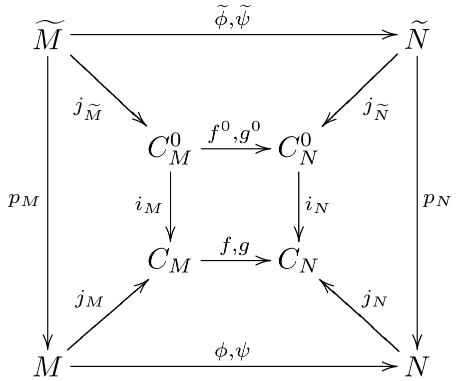
Suppose the contrary. Then for a lift $\widetilde{\omega}$ satisfying $\widetilde{\omega}(0)\in\widetilde{M}\times 1$, we have $\widetilde{\omega}(1)\in\widetilde{M}\times\{-1\}$. Let $\widetilde{f},\widetilde{g}$ be lifts of f,g sending $\widetilde{M}\times 1$ into $\widetilde{N}\times 1$. Then $\widetilde{f}\widetilde{\omega}(0)$, $\widetilde{g}\widetilde{\omega}(0)$ and $\widetilde{f}\widetilde{\omega}(1)$ belong to $\widetilde{N}\times 1$ ($\widetilde{f}$ is even) while $\widetilde{g}\widetilde{\omega}(1)\in\widetilde{N}\times\{-1\}$ (g is odd). Thus $f\omega$ is not homotopic to $g\omega$ (in C_N), which contradicts the assumption.

Now (4.2)(c) gives $N(f,g)=N(f^0,g^0)$, hence $N_{\mathrm{rel}}(f,g)=N(f^0,g^0)=N(\widetilde{\phi},\widetilde{\psi})$.

(c) $\widetilde{f}$ *and* $\widetilde{g}$ *are odd*. We show that $N_\partial(f,g)$ is the number of essential classes of $\Phi(\phi,\psi)$ which are images of essential classes of $\Phi(\widetilde{\phi},\widetilde{\psi})$ in the diagram

(4.3)
$$\begin{array}{ccc} \widetilde{M} & \xrightarrow{\widetilde{\phi},\widetilde{\psi}} & \widetilde{N} \\ {\scriptstyle p_M}\downarrow & & \downarrow{\scriptstyle p_N} \\ M & \xrightarrow{\phi,\psi} & N \end{array}$$

where $\widetilde{\phi}$ and $\widetilde{\psi}$ satisfy $\widetilde{f}[\widetilde{x},t]=[\widetilde{\phi}\widetilde{x},t]$ and $\widetilde{g}[\widetilde{x},t]=[\widetilde{\psi}\widetilde{x},t]$. Consider a homotopy commutative diagram



where the vertical arrows are natural coverings, $j_M : M\simeq M\times 0\to\widetilde{M}\times 0/{\simeq}\subset C_M$ and $j_{\widetilde{M}}:\widetilde{M}\simeq\widetilde{M}\times 1\to C^0_M$ are inclusions, and j_N and $j_{\widetilde{N}}$ are defined similarly. By [Je1, 2.1] it induces a commutative diagram of

Reidemeister sets

$$\begin{array}{ccc} \nabla(\widetilde{\phi},\widetilde{\psi}) & \xrightarrow{\varkappa} & \nabla(f^0,g^0) \\ {\scriptstyle\eta}\downarrow & & \downarrow{\scriptstyle\eta} \\ \nabla(\phi,\psi) & \xrightarrow{\varkappa} & \nabla(f,g) \end{array}$$

Now we notice that the upper horizontal arrow is induced by a pair of homeomorphisms $j_{\widetilde{M}}$, $j_{\widetilde{N}}$ while the lower arrow is induced by a pair of homotopy equivalences. Thus they are bijections of the Reidemeister sets [Je1]. It remains to show that they preserve semi-index. The upper arrow is evident by (4.2)(a). For the lower one we use the homotopy $\{f_s, g\}$ with $f_s[\widetilde{x}, t] = f[\widetilde{x}, st]$ and apply (4.2)(c).

Thus we may consider the diagram (4.3).

(4.4) Lemma. *The map* $\nabla(\widetilde{\phi},\widetilde{\psi}) \to \nabla(\phi,\psi)$ *induced by* (4.3) *sends essential classes to essential classes.*

Proof. Let $\widetilde{A} \subset \nabla(\widetilde{\phi},\widetilde{\psi})$ be a Nielsen class. Then $A = p_M\widetilde{A} \subseteq \nabla(\phi,\psi)$ is also a Nielsen class. Assume that A is not essential. Then $A = \{a_1, b_1, \ldots, a_k, b_k\}$ with a_iRb_i, and $\widetilde{A} \cap p_M^{-1}\{a_i, b_i\}$ also splits into pairs of R-related points. Now $\widetilde{A}$ also splits into such pairs, hence $\widetilde{A}$ is not essential. ■

It remains to find out how many Nielsen classes from $\nabla(\widetilde{\phi},\widetilde{\psi})$ are sent to a given $A \in \nabla(\phi,\psi)$. We fix a class $A \in \nabla(\phi,\psi)$ which is the image of an essential class from $\nabla(\widetilde{\phi},\widetilde{\psi})$. Since $\widetilde{\phi}$ and $\widetilde{\psi}$ are odd, $p_M^{-1}a \subset \Phi(\phi,\psi)$ for any $a \in A$. Write $p_M^{-1}a = \{\widetilde{a}_0, \widetilde{a}_1\}$. Notice that $\widetilde{a}_0, \widetilde{a}_1 \in \nabla(\widetilde{\phi},\widetilde{\psi})$ are Nielsen related iff $C_a(\phi_\#, \psi_\#) = \{\omega \in \pi_1(M, a) : \phi_\#\omega = \psi_\#\omega\}$ is not contained in $\operatorname{im} p_{M\#}$. Since $\operatorname{im} p_{M\#} \subset \pi_1 M$ is a subgroup of rank two, the above condition does not depend on the choice of $a \in \Phi(\phi,\psi)$. Thus if $C_a(\phi_\#,\psi_\#)$ is contained in $\operatorname{im} p_{M\#}$ for a point $a \in A$ then A is covered by two essential classes and otherwise by a single class.

The above and the equality $N(f^0, g^0) = N(\widetilde{\phi},\widetilde{\psi})$ imply

$$N_\partial(f,g) = \begin{cases} (1/2)N(\widetilde{\phi},\widetilde{\psi}) & \text{if } C_a(\phi_\#,\psi_\#) \subset \operatorname{im}[\pi_1\widetilde{M} \to \pi_1 M], \\ N(\widetilde{\phi},\widetilde{\psi}) & \text{otherwise.} \end{cases}$$

We sum up the results of this section:

(4.5) Theorem. *Let* $f, g : (C_M, C_M^0) \to (C_N, C_N^0)$ *be a* Φ*-compact pair of maps. Then*

$$N_{\text{rel}}(f,g) = \begin{cases} N(\widetilde{\phi},\widetilde{\psi}) & \textit{if at least one of } \widetilde{f},\widetilde{g} \textit{ is even}, \\ N(\phi,\psi) + (1/2)N(\widetilde{\phi},\widetilde{\psi}) & \textit{if } \widetilde{f},\widetilde{g} \textit{ are odd and} \\ & C_a(\phi_\#,\psi_\#) \subset \operatorname{im}[\pi_1\widetilde{M} \to \pi_1 M], \\ N(\phi,\psi) & \textit{if } \widetilde{f},\widetilde{g} \textit{ are odd and the above} \\ & \textit{inclusion does not hold.} \end{cases}$$

5. The relative Nielsen numbers of self-maps of projective spaces. In the last section we will compute $N_{\mathrm{rel}}(f,g)$ for a pair of self-maps $f,g:\mathbb{R}P^n\to\mathbb{R}P^n$ sending $\mathbb{R}P^l$ into itself ($l<n$), in other words, for a commutative diagram

$$(5.0)\qquad \begin{array}{ccc} \mathbb{R}P^l & \xrightarrow{f^0,g^0} & \mathbb{R}P^l \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle i} \\ \mathbb{R}P^n & \xrightarrow{f,g} & \mathbb{R}P^n \end{array}$$

($2\le l<n$). We will express $N_{\mathrm{rel}}(f,g)$ by the (ordinary) Nielsen numbers of the pair f,g and its restriction to $\mathbb{R}P^l$. To get explicit formulae one can combine these results with (5.1) and (6.1) of [Je3]. Notice that one of our results shows that the homotopy types of f,g and of their restrictions do not determine $N_{\mathrm{rel}}(f,g)$ (Remark (5.10)).

Let us start with some general remarks.

(5.1) Lemma. *Consider a diagram*

$$\begin{array}{ccc} M^0 & \xrightarrow{f^0,g^0} & N^0 \\ {\scriptstyle i_M}\downarrow & & \downarrow{\scriptstyle i_N} \\ M & \xrightarrow{f,g} & N \end{array}$$

satisfying (2.0). *Suppose that* $i_{M\#}:\pi_1M^0\to\pi_1M$ *and* $i_{N\#}:\pi_1N^0\to\pi_1N$ *are isomorphisms. Then the induced map of Reidemeister sets* $\nabla i:\nabla(f^0,g^0)\to\nabla(f,g)$ *is a bijection.* ∎

The assumptions of (5.1) are fulfilled for the diagram (5.0).

(5.2) Lemma. *If under the assumption of* (5.1), $\nabla(f,g)$ (*or resp.* $\nabla(f^0,g^0)$) *contains only essential classes then* $N_\partial(f,g)=N(f^0,g^0)$ (*resp.* $N_\partial(f,g)=N(f,g)$).

(5.3) Lemma. *If in the diagram* (5.0) *one of the dimensions* l *or* n *is odd then*

(a) $N(f^0,g^0)=0$ *or* $N(f,g)=0$ *implies* $N_\partial(f,g)=0$, *whereas if both* $N(f^0,g^0)$ *and* $N(f,g)$ *are nonzero then*
(b) l *odd implies* $N_\partial(f,g)=N(f,g)$,
(c) n *odd implies* $N_\partial(f,g)=N(f^0,g^0)$.

P r o o f. (a) is evident. To prove (b) notice that an odd-dimensional projective space is Jiang [Je3, Section 6], hence all the classes in $\nabla(f^0,g^0)$ are essential and (b) follows from (5.2). A similar argument proves (c). ∎

It remains to consider the case when both l and n are even. For reference we recall

(5.4) Lemma ([Je3, 5.1]). *For any pair of maps* $f,g:\mathbb{R}P^n\to\mathbb{R}P^n$ (*n even*),

$$N(f,g)=\begin{cases}0 & \text{if } f_\#=g_\#=0 \text{ and } f\simeq g,\\ 1 & \text{if } f_\#\neq g_\# \text{ or } (f_\#=g_\#=\mathrm{id} \text{ and } f\simeq g),\\ 2 & \text{if } f_\#=g_\# \text{ and } f,g \text{ are not homotopic.}\end{cases}\ \blacksquare$$

Assume first that $f_\#\neq g_\#$. Then $\nabla(f^0,g^0)$ and $\nabla(f,g)$ consist of one element, hence (5.4) implies $N(f,g)=1$ and $N(f^0,g^0)=1$. This yields

(5.5) Corollary. *If both* l,n *are even and* $f_\#\neq g_\#$ *then* $N_\partial(f,g)=N_{\mathrm{rel}}(f,g)=1$. ■

Now suppose $f_\#=g_\#$.

(5.6) Lemma. *Let* l,n *be even and* $f_\#=g_\#$. *Then*

- f,g *not homotopic implies* $N_\partial(f,g)=N(f^0,g^0)$,
- f^0,g^0 *not homotopic implies* $N_\partial(f,g)=N(f,g)$.

Proof. If f,g are not homotopic then by (5.4) all (i.e. two) classes in $\nabla(f,g)$ are essential and we may apply (5.2). Similarly if f^0,g^0 are not homotopic. ■

It remains to consider the case where f,g are homotopic and so are the restrictions f^0,g^0. First, we assume, moreover, that $f_\#=g_\#=0$. Then (5.4) implies

(5.7) Lemma. *If* l,n *are even*, f,g *are homotopic*, *so are the restrictions* f^0,g^0 *and* $f_\#=g_\#=0$ *then* $N_{\mathrm{rel}}(f,g)=N_\partial(f,g)=0$. ■

Now we assume that $f_\#=g_\#=\mathrm{id}$.

(5.8) Lemma. *Fix a map* $f:\mathbb{R}P^n\to\mathbb{R}P^n$ *such that* $f(\mathbb{R}P^l)\subset\mathbb{R}P^l$, $f_\#=\mathrm{id}$, $n>l$ *both even. Then the homotopy set* $[(\mathbb{R}P^n,\mathbb{R}P^l),(\mathbb{R}P^n,\mathbb{R}P^l)]$ *contains exactly two classes with a representative* g *satisfying* $f\simeq g$ *and* $f^0\simeq g^0$. *These two classes are represented by* f *and* Kf, *where* K *is the involution of* $\mathbb{R}P^n$ *given by the formula*

$$K\langle x_0,\dots,x_n\rangle=\langle x_0,\dots,x_{n-1},-x_n\rangle.$$

Proof. Define two forgetful functors

$$j_1:[(\mathbb{R}P^n,\mathbb{R}P^l),(\mathbb{R}P^n,\mathbb{R}P^l)]\to[\mathbb{R}P^n,\mathbb{R}P^n],$$
$$j_2:[(\mathbb{R}P^n,\mathbb{R}P^l),(\mathbb{R}P^n,\mathbb{R}P^l)]\to[\mathbb{R}P^l,\mathbb{R}P^l]$$

by $j_1[g]=[g]$ and $j_2[g]=[g^0]$. Now the lemma may be reformulated as asserting that

$$j_1^{-1}[f]\cap j_2^{-1}[f^0]=\{[f],[Kf]\}$$

and that $[f]$ and $[Kf]$ are distinct elements of $[(\mathbb{R}P^n,\mathbb{R}P^l),(\mathbb{R}P^n,\mathbb{R}P^l)]$.

We start by describing the set $j_2^{-1}[f^0]$. Let $[g] \in j_2^{-1}[f^0]$. By the homotopy extension property we may assume that $g^0 = f^0$. Since $\pi_l \mathbb{R}P^n = 0$ for $1 < l < n$, there is no obstruction to a homotopy from f to g on $\mathbb{R}P^{m-1}$ and therefore we assume that f and g are equal outside the unique n-cell of $\mathbb{R}P^n$. Thus g is homotopic to the connected sum $f\#s$, where $s : S^n \to \mathbb{R}P^n$ and $j_2^{-1}[f^0] = \{[h] : h = f\#s,\ s : S^n \to \mathbb{R}P^n\}$. Notice that any $s : S^n \to \mathbb{R}P^n$ is a composition $S^n \xrightarrow{s'} S^n \xrightarrow{p} \mathbb{R}P^n$. Let $k = \deg s'$. We then put $s_k = ps'$ and $h_k = f\#s_k$. Now $j_1^{-1}[f] \cap j_2^{-1}[f^0]$ consists of those $[h_k]$ which are freely homotopic to $f \simeq h_0$. But the degrees of the lifts $\widetilde{h}_k : S^n \to S^n$ are $\pm(\deg \widetilde{f} + 2k)$. If now h_k is freely homotopic to $f \simeq h_0$ then $\deg \widetilde{f} = \mp(\deg \widetilde{f} + 2k)$, hence either $k = 0$ or $k = -\deg \widetilde{f}$. Thus $j_1^{-1}[f] \cap j_2^{-1}[f^0]$ contains at most two elements.

It remains to show that f and Kf are not homotopic as self-maps of the pair $(\mathbb{R}P^n, \mathbb{R}P^l)$. Suppose otherwise: let H_t be a homotopy from f to Kf satisfying $H_t(\mathbb{R}P^l) \subset \mathbb{R}P^l$ and let $\widetilde{H}_t : S^n \to S^n$ be a lift of this homotopy. Consider the restriction $\widetilde{H}_t^0 : S^l \to S^l$ starting from $\widetilde{H}_0^0 = \widetilde{f}^0$. Since $\deg(-\widetilde{f}^0) = -\deg \widetilde{f}^0$ (n is even) and $\deg \widetilde{f}^0$ is odd ($f_\# = \mathrm{id}$), we have $\widetilde{H}_1^0 = \widetilde{f}^0$. This implies that the lift $\widetilde{H}_t$ of the homotopy H_t starting from $\widetilde{H}_0 = \widetilde{f}$ (the extension of $\widetilde{f}^0$) satisfies $\widetilde{H}_1 = \widetilde{K}\widetilde{f}$. But then $\deg \widetilde{H}_1 = \deg(\widetilde{K}\widetilde{f}) = -\deg \widetilde{f}$, contrary to $\deg \widetilde{H}_1 = \deg \widetilde{H}_0 = \deg \widetilde{f} \neq 0$. ∎

Thus we may consider two cases: $g = f$ and $g = Kf$. In the first case $\Phi(f^0, g^0) = \mathbb{R}P^l$ and $\Phi(f, g) = \mathbb{R}P^n$ are the unique (nonempty) Nielsen classes, and one of them is contained in the other. By (5.4) these two classes are essential, hence $N(f^0, g^0) = N(f, g) = 1$ and $N_\partial(f, g) = N_{\mathrm{rel}}(f, g) = 1$.

Now consider the pair $f, g = Kf$. The restriction of this pair to $\mathbb{R}P^l$ gives $f^0 = g^0$ and now $\Phi(f^0, g^0) = \mathbb{R}P^l$ is the unique essential Nielsen class of f^0, g^0. We will show that the inclusion $\mathbb{R}P^l \subset \mathbb{R}P^n$ sends this class to an inessential Nielsen class in $\Phi(f, Kf)$. We fix a lift $\widetilde{f}$ and consider the commutative diagram

$$\begin{array}{ccc} S^n & \xrightarrow{\widetilde{f},\, \widetilde{K}\widetilde{f}} & S^n \\ \downarrow & & \downarrow \\ \mathbb{R}P^n & \xrightarrow{f,\, Kf} & \mathbb{R}P^n \end{array}$$

Now the class $p(\Phi(\widetilde{f}, \widetilde{K}\widetilde{f})) \subset \Phi(f, Kf)$ contains $\mathbb{R}P^l$ and hence it remains to show that $p(\Phi(\widetilde{f}, \widetilde{K}\widetilde{f}))$ is inessential. Consider the homotopy $\widetilde{K}_s : S^n \to S^n$, $\widetilde{K}_s(z_1, \ldots, z_k, t) = (e^{is}z_1, \ldots, e^{is}z_k, -t)$ (we put $n = 2k$ and identify $\mathbb{R}^{n+1} = \mathbb{C}^k \times \mathbb{R}$). It induces a homotopy $K_s : \mathbb{R}P^n \to \mathbb{R}P^n$ starting from Kf and removing the class $p(\Phi(f, Kf))$ which turns out to be inessential. Thus $N_\partial(f, Kf) = 0$. ∎

(5.9) COROLLARY. *Assume that* l, n *are even*, f *is homotopic to* g, f^0 *is homotopic to* g^0, *and* $f_\# = g_\# = \mathrm{id}$. *If*, *moreover*, f *is homotopic to* g *as self-maps of the pair* $(\mathbb{R}P^n, \mathbb{R}P^l)$ *then* $N\partial(f,g) = 1$; *otherwise* $N\partial(f,g) = 0$. ■

(5.10) Remark. In any case ((5.3), (5.5), (5.6), (5.7), (5.9)), $N\partial(f,g)$ and $N_{\mathrm{rel}}(f,g)$ can be expressed by $N(f,g)$ and $N(f^0,g^0)$: we need only know the parity of dimensions, the induced homotopy homomorphisms $f_\#, g_\#$ and whether the maps f, g are homotopic (as maps of pairs of spaces). On the other hand, we may notice that the homotopy types of the maps f, g and of their restrictions f^0, g^0 do not determine $N\partial(f,g)$. Let $f : \mathbb{R}P^n \to \mathbb{R}P^n$ be a map inducing $f_\# = \mathrm{id}$ and mapping $\mathbb{R}P^l \subset \mathbb{R}P^n$ into itself (l, n even). Consider the pairs of maps f, f and f, Kf. These maps (as self-maps of $\mathbb{R}P^n$) are homotopic and their restrictions to $\mathbb{R}P^l$ are equal. Nevertheless, by the above $N\partial(f,f) = 1$ while $N\partial(f,Kf) = 0$. This shows that the homotopy types of f, g and f^0, g^0 do not determine $N\partial(f,g)$.

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