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On a perfect set theorem of A. H. Stone and N. N. Lusin's constituents

by

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Abstract. N. N. Lusin asked in 1935 if there exists a Borel sieve each constituent of which is a singleton. A negative answer based on metamathematical methods was given in 1981 by V. G. Kanoveĭ. We present a simple topological solution of Lusin's problem, and we also establish some new results on this topic. Our approach is based on a link between Lusin's constituents and certain results in the theory of non-separable Souslin sets developed by A. H. Stone.

1. Introduction. The aim of this note is to show a link between certain results in the theory of non-separable Souslin sets, developed by A. H. Stone [St1]–[St3], and some classical topics concerning N. N. Lusin's constituents of coanalytic sets.

Lusin asked in 1935 ([Lu1], Problème I in Sec. 8) if there exists a Borel sieve each constituent of which is a singleton (cf. Sec. 6.1). A negative answer was given in 1981 by V. G. Kanoveĭ [Ka1], [Ka2], by means of some advanced metamathematical methods. As was pointed out by V. A. Uspenskiĭ [Us], p. 111 (p. 128 of the English translation), the problem was almost unique, among other important questions set forth by Lusin, that could be solved without introducing new axioms for set theory, but was left open in the "classical period" of descriptive set theory (cf. [Kel], Sec. 1). It seems that no solution of Lusin's problem based only on standard topological arguments has been published (cf. Uspenskiĭ's comments ending [Us]).

We shall show in Section 3 that a simple version of a non-separable perfect set theorem of A. H. Stone (Theorem 4.1) easily provides such a solution.

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Refining Stone's theorem one can also get more information about Lusin's constituents. We illustrate this point formulating below twin statements, Theorem 1.2 and Corollary 1.3, connected by a resolution of well-orders described in Section 2. But first, let us clarify our terminology and notation.

1.1. *Terminology and notation.* Our terminology follows Kuratowski [Kur]. We denote by N the natural numbers. Let Q be the set of rational numbers and let 2^Q be the Cantor space of all subsets of Q, with the topology of pointwise convergence (cf. [Kech], 27.13). The order type of $A \subset Q$ is denoted by type(A). Let

$$WO = \{A \in 2^Q : A \text{ is well-ordered}\}, \tag{1}$$

$$WO_\xi = \{A \in WO : \text{type}(A) = \xi\}, \quad \xi < \omega_1. \tag{2}$$

Then WO is the complement of the analytic set sifted by the universal Lusin sieve and WO_ξ is the ξth constituent corresponding to the sieve (cf. [Kur], §3, XV, [Kech], 31.3).

The *Baire space* $B(\omega_1)$ of weight $\aleph_1$ is the countable product of the discrete space of cardinality $\aleph_1$. For our purpose, it is convenient to consider $B(\omega_1)$ as the space of functions $x : Q \to \omega_1$ from the rationals to the countable ordinals, with a "first difference" metric (i.e., for a fixed enumeration $e : N \to Q$, for $x \neq y$, $d(x, y) = 1/\min\{n : x(e(n)) \neq y(e(n))\}$ (cf. [St1], Sec. 2).

We denote by $[\alpha, \beta)$ the ordinal interval $\alpha \le \xi < \beta$, and for $x \in B(\omega_1)$, we set

$$\kappa(x) = \min\{\alpha : x(Q) \subset [0, \alpha)\}. \tag{3}$$

We shall call

$$B_\xi = \{x : \kappa(x) = \xi\} \tag{4}$$

the *layer at level* ξ of the Baire space $B(\omega_1)$.

A *Souslin set* S in a completely metrizable space X is the image under the projection parallel to the Cantor set 2^N of a G_δ-set in the product $X \times 2^N$ (equivalently, S is a result of the $\mathcal{A}$-operation applied to closed sets in X). Separable Souslin sets coincide with analytic sets (cf. [Kur], §3, XIV and §39, II, [Kech], 25, [Ro]).

A set of countable ordinals is a *c.u.b. set* if it is closed and unbounded in ω_1, and it is *stationary* if it intersects each c.u.b. set in ω_1 (cf. [Kun], II, §6).

The phrase "property $P(\xi)$ holds for all but non-stationary many ξ" means that $P(\xi)$ is satisfied for ξ in a set containing a c.u.b. set in ω_1.

We shall denote by Lim the set of countable limit ordinals.

1.2. Theorem. *Let S be a Souslin set in the Baire space $B(\omega_1)$ and let $C \subset S$ intersect stationary many layers B_ξ. Then for all but non-stationary many ξ, each F_σ-set containing $S \cap B_\xi$ intersects $C \cap \bigcup_{\alpha<\xi} B_\alpha$.*

1.3. Corollary. *Let A be an analytic set in the Cantor space 2^Q and let $E \subset A$ intersect stationary many constituents WO_ξ. Then for all but non-stationary many ξ, each F_σ-set containing $A \cap WO_\xi$ intersects $E \cap \bigcup_{\alpha<\xi} WO_\alpha$.*

Theorem 1.2 will be proved in Section 4 and Corollary 1.3 will be discussed in Section 5.

2. A complete non-separable resolution for WO.

We adopt the notation introduced in Section 1.1. For $x \in B(\omega_1)$, let $\operatorname{supp}(x) = \{q \in Q : x(q) > 0\}$. Evidently, the map $\sigma : B(\omega_1) \to 2^Q$ defined by $\sigma(x) = \operatorname{supp}(x)$ is continuous.

Our resolution is the restriction of σ to a subset M of $B(\omega_1)$ described in the following lemma, where B_ξ is the layer at level ξ in $B(\omega_1)$ defined in Section 1.1, (3) and (4).

2.1. Lemma. *Let M be the subspace of $B(\omega_1)$ consisting of functions x such that $x : \operatorname{supp}(x) \to [1, \kappa(x))$ is strictly increasing and onto. Then M is a G_δ-set in $B(\omega_1)$, $\sigma : M \to WO$ is one-to-one, onto, and $\sigma(M \cap B_\xi) = WO_\xi$ for $\xi \ge \omega_0$.*

Proof. We have to check that M is a G_δ-set, the remaining properties being transparent. To this end, consider

$$V(q, \alpha) = \{x \in B(\omega_1) : x(q) = \alpha\}.$$

The set of $x \in V(q, \alpha)$ such that $[0, \alpha) \not\subset x(Q)$,

$$F(q, \alpha) = V(q, \alpha) \setminus \bigcap_{\beta<\alpha} \bigcup_{r \in Q} V(r, \beta),$$

is an F_σ-set and since, for fixed q, $\{V(q, \alpha) : \alpha < \omega_1\}$ is a discrete open collection, the union $\bigcup_{\alpha<\omega_1} F(q, \alpha)$ is of type F_σ in $B(\omega_1)$.

We conclude that the set

$$G = B(\omega_1) \setminus \bigcup_{q \in Q} \bigcup_{\alpha<\omega_1} F(q, \alpha),$$

consisting of $x \in B(\omega_1)$ such that $[0, \kappa(x)) = x(Q)$, is a G_δ-set.

The set of functions x strictly increasing on $\operatorname{supp}(x)$,

$$H = B(\omega_1) \setminus \bigcup_{q<r} \{x : x(q) \ge x(r) > 0\},$$

is also of type G_δ.

Since $M = G \cap H$, this completes the proof.

3. A topological solution of Lusin's problem. As we have already mentioned in the introduction, V. G. Kanoveĭ [Ka1] solved Lusin's problem using metamathematical arguments (cf. Sec. 6.1).

We shall derive an answer to Lusin's question from a non-separable perfect set theorem. Let $A \subset 2^Q$ be an analytic set intersecting each constituent WO_ξ (cf. 6.1 for a reformulation in terms of Lusin sieves). We shall show that some intersection $A \cap WO_\xi$ is uncountable. By Lusin's covering theorem ([Kur], §39, VIII, Th. 5), it suffices to find a Cantor set L in $A \cap WO$.

To this end, consider the resolution $\sigma : M \to WO$ from Lemma 2.1 and let $S = \sigma^{-1}(A)$. Then S is a Souslin set in $B(\omega_1)$ intersecting every layer B_ξ. For each $\xi \in \mathrm{Lim}$ pick a point $x_\xi \in S \cap B_\xi$. Then x_ξ is a sequence of ordinals from $[0, \xi)$ with supremum ξ, and hence a result of Stone ([St2], Lemma 5.2) shows that $\{x_\xi : \xi \in \mathrm{Lim}\}$ is not σ-discrete. Therefore the non-σ-discrete Souslin set S contains a Cantor set L (cf. [St1], Sec. 4, [El]). Then $\sigma(L)$ is the Cantor set we were looking for.

4. A refinement of a perfect set theorem of A. H. Stone. The argument in the previous section was based on the fact that any non-σ-discrete Souslin set contains a perfect set, a simple instance of several more sophisticated non-separable perfect set theorems considered by A. H. Stone ([St1], Sec. 4, [St2], Sec. 4, or [St3], Sec. 3.4). We shall appeal to Stone's results in this section to get more information about the constituents.

Since we did not find a convenient reference for the perfect set theorem below, most suitable for our purpose, we include a brief proof, following closely Stone's ideas. This theorem will be further refined in Proposition 4.4.

4.1. THEOREM. *Let S be a Souslin set in a completely metrizable space X of weight $\aleph_1$. If $C \subset S$ is not a union of countably many locally separable sets, then S contains a copy K of $B(\omega_1)$ closed in X such that $K = \overline{K \cap C}$.*

Proof. We shall use Stone's terminology ([St3], Sec. 2.1), calling a set $\sigma Lw(<\aleph_1)$ if it is a union of countably many locally separable subsets.

Represent the Souslin set S as the image under the projection $p : X \times 2^N \to X$ of a G_δ-set G in $X \times 2^N$.

Let Y denote the product $X \times 2^N$ and fix sets G_i open in Y with $G = \bigcap_{i \ge 1} G_i$, and a subset D of G such that p maps D onto C in a one-to-one manner.

Note that p, being a projection parallel to a compact factor, maps closed subsets of Y onto closed subsets of X (cf. [Kur], §20, V, Theorem 7). Thus, the images of discrete in Y collections of subsets of D are discrete in X. In particular, D is not $\sigma Lw(<\aleph_1)$ (cf. [St3], Sec. 2.1), and removing from D a $\sigma Lw(<\aleph_1)$ set, we can assume that all relatively open non-empty sets in D have weight $\aleph_1$ (cf. [St3], Sec. 2.2).

Therefore, for each U open in Y and any $d \in D \cap U$, there is $F_U \subset D \cap U$ of cardinality $\aleph_1$, discrete in Y, with $d \in F_U$. Furthermore, we can find a collection $\mathcal{V} = \{V_y : y \in F_U\}$ of open subsets of Y such that, for $y \in F_U$, $y \in V_y \subset \overline{V}_y \subset U$ and $p(\mathcal{V}) = \{p(V_y) : y \in F_U\}$ is discrete in X.

Repeating this observation, we can define collections $\mathcal{V}_i$ discrete in Y of subsets of G_i open in Y with $\operatorname{mesh}(\mathcal{V}_i) \leq 1/i$, the closures of $\mathcal{V}_{i+1}$ in Y refining $\mathcal{V}_i$, and $p(\mathcal{V}_i)$ discrete in X. Moreover, we can fix, for each i, a subset F_i of D such that each $V \in \mathcal{V}_i$ contains exactly one point of F_i and $F_i \subset F_{i+1}$.

Then $L = \bigcap_{i\geq 1} \overline{\bigcup \mathcal{V}_i} \subset G$ is a copy of $B(\omega_1)$ closed in Y and $F = \bigcup_{i\geq 1} F_i \subset D$ is a dense subset of L.

Let $K = p(L) \subset S$. Our construction assures that K is closed in X and p maps L homeomorphically onto K. Thus K is a copy of $B(\omega_1)$ closed in X, $p(F) \subset C$ is dense in K, and the proof is complete.

Before stating Proposition 4.4, a basis for the next section, we shall consider some natural "approximations" of non-separable spaces by separable subspaces (more on this topic can be found in [Po2]).

Let X be a metrizable space of weight $\aleph_1$ and let a sequence $\{X_\xi\}_{\xi<\omega_1}$ satisfy

(1) $$X_1 \subset \ldots \subset X_\xi \subset \ldots,\ \xi < \omega_1,\ X_\xi \text{ is separable and closed in } X,$$

(2) $$X_\xi = \overline{\bigcup_{\alpha<\xi} X_\alpha} \quad \text{for } \xi \in \mathrm{Lim}, \quad X = \bigcup_{\xi<\omega_1} X_\xi.$$

We shall call such a sequence $\{X_\xi\}_{\xi<\omega_1}$ *admissible* in X. The set

(3) $$P_\xi = X_\xi \setminus \bigcup_{\alpha<\xi} X_\alpha$$

will be called the *layer at level* ξ determined by this sequence.

Clearly, the sequence $\{K_\xi\}_{\xi<\omega_1}$, where

(4) $$K_\xi = \{x \in B(\omega_1) : \kappa(x) \leq \xi\},$$

is admissible in $B(\omega_1)$, and the B_ξ defined in Section 1, (4), are the layers determined by this sequence.

Let us make two observations on admissible sequences.

4.2. LEMMA. *Let $Y \subset X$ and let $\{X_\xi\}_{\xi<\omega_1}$ be an admissible sequence in X. Then the sets X'_ξ defined, for $\xi \leq \omega_1$, by*

(5) $$X'_\xi = \overline{Y \cap \bigcup_{\alpha<\xi} X_\alpha}$$

form an admissible sequence in $\overline{Y}$. Moreover, if Y is closed in X and P'_ξ

denotes the layer at level ξ *determined by the sequence* $\{X'_\xi\}_{\xi<\omega_1}$, *then*

$$P'_\xi \subset P_\xi \quad \textit{for } \xi \in \mathrm{Lim}, \tag{6}$$

where P_ξ *denotes the layer at level* ξ *determined by the original sequence* $\{X_\xi\}_{\xi<\omega_1}$.

Clearly $\{X'_\xi\}_{\xi<\omega_1}$ satisfies condition (1). Thus in order to prove the first part of 4.2, it suffices to show that it satisfies (2). Observe that for a limit $\xi \le \omega_1$, we have $X'_\xi = \overline{Y \cap \bigcup_{\alpha<\xi} X_\alpha} = \overline{\bigcup_{\alpha<\xi}(Y \cap X_\alpha)} = \overline{\bigcup_{\alpha<\xi} \overline{Y \cap X_\alpha}} = \overline{\bigcup_{\alpha<\xi} X'_{\alpha+1}} = \overline{\bigcup_{\alpha<\xi} X'_\alpha}$.

In particular, $\overline{Y} = X'_{\omega_1} = \overline{\bigcup_{\alpha<\omega_1} X'_\alpha} = \bigcup_{\alpha<\omega_1} X'_\alpha$. The last equality follows from the fact that the closure in X is determined by sequences. This completes the first part of the proof.

If Y is closed in X, then for a limit $\xi < \omega_1$, we have $P'_\xi = X'_\xi \setminus \bigcup_{\alpha<\xi} X'_\alpha \subset X'_{\xi+1} \setminus \bigcup_{\alpha<\xi} X'_{\alpha+1} = Y \cap X_\xi \setminus \bigcup_{\alpha<\xi}(Y \cap X_\alpha) = Y \cap P_\xi \subset P_\xi$.

4.3. Lemma. *Let* $\{X_\xi\}_{\xi<\omega_1}$ *and* $\{X'_\xi\}_{\xi<\omega_1}$ *be two admissible sequences in* X *and let* P_ξ, P'_ξ *be the layers at level* ξ *corresponding to the sequences* $\{X_\xi\}_{\xi<\omega_1}$ *and* $\{X'_\xi\}_{\xi<\omega_1}$, *respectively. Then there exists a c.u.b. set* Γ *with* $X_\xi = X'_\xi$ *and* $P_\xi = P'_\xi$ *for* $\xi \in \Gamma$.

To check this, let $\psi(\alpha) = \min\{\beta : X_\alpha \subset X'_\beta \text{ and } X'_\alpha \subset X_\beta\}$, and let Γ be the set of limit ordinals $\xi < \omega_1$ with $\psi(\alpha) < \xi$ for $\alpha < \xi$. Then $\bigcup_{\alpha<\xi} X_\xi = \bigcup_{\alpha<\xi} X'_\xi$ for $\xi \in \Gamma$, hence, by (2) and (3), the set Γ has the required property.

4.4. Proposition. *Let* X *be a completely metrizable space of weight* $\aleph_1$ *with an admissible sequence* $\{X_\xi\}_{\xi<\omega_1}$ *and the corresponding layers* P_ξ. *Let* S *be a Souslin set in* X *and let* $C \subset S$ *intersect stationary many layers. Then* S *contains a copy* K *of* $B(\omega_1)$ *closed in* X *such that for all but non-stationary many* ξ, $C_\xi = C \cap K \cap \bigcup_{\alpha<\xi} P_\alpha$ *satisfies* $C_\xi \subset \overline{\overline{C}_\xi \cap P_\xi} \neq \emptyset$.

In Hausdorff's terminology (cf. [Kur], §12, VII), the relation $C_\xi \subset \overline{\overline{C}_\xi \setminus C_\xi}$ means that C_ξ is its own residue.

Theorem 1.2 follows from Proposition 4.4 instantly. Consider $X = B(\omega_1)$ with the admissible sequence given by (4). Then $P_\xi = B_\xi$ is the layer at level ξ for this sequence.

Let K and ξ be as in the assertion of Proposition 4.4, and let F be an F_σ-set containing $K \cap P_\xi$. Then F contains $\overline{C}_\xi \cap P_\xi$, a G_δ-set dense in $\overline{C}_\xi$, hence by the Baire Category Theorem, F must intersect C_ξ.

Proof of Proposition 4.4. By [Po1], Theorem 1, the set C is not a union of countably many locally separable sets. Thus Theorem 4.1 assures that S contains a copy K of $B(\omega_1)$ closed in X with $K \cap C$ dense in K.

We shall consider three admissible sequences $\{X'_\xi\}_{\xi<\omega_1}$, $\{X''_\xi\}_{\xi<\omega_1}$, $\{X'''_\xi\}_{\xi<\omega_1}$ in K with P'_ξ, P''_ξ, P'''_ξ being the corresponding layers at level ξ.

The first one is obtained by (5) of Lemma 4.2, where $Y = C \cap K$, so

$$X'_\xi = \overline{C}_\xi. \tag{7}$$

The second one is again obtained by Lemma 4.2(5), with $Y = K$ closed in X, so by (6),

$$P''_\xi \subset P_\xi \quad \text{for } \xi \in \mathrm{Lim}\,. \tag{8}$$

And finally, we set $X'''_\xi = h(K_\xi)$, where $h : B(\omega_1) \to K$ is a homeomorphism and K_ξ is defined by (4). Since B_ξ is dense in K_ξ and $P'''_\xi = h(B_\xi)$, we get

$$\overline{P'''_\xi} = X'''_\xi. \tag{9}$$

By Lemma 4.3, for all but non-stationary many ξ, the sets X'_ξ, X''_ξ, X'''_ξ and the layers P'_ξ, P''_ξ, P'''_ξ coincide. For any such ξ, by (7)–(9), we have $\overline{\overline{C}_\xi \cap P_\xi} \supset \overline{X'_\xi \cap P''_\xi} = \overline{P'''_\xi} = X'_\xi \supset C_\xi$.

5. Proof of Corollary 1.3. Let $\sigma : M \to WO$ be the resolution from Lemma 2.1, let $S = \sigma^{-1}(A)$ and $C = \sigma^{-1}(E)$. Then, by Lemma 2.1, C intersects stationary many layers B_ξ in the Baire space. Let ξ be any ordinal for which the assertion of Theorem 1.2 holds.

If F is an F_σ-set in 2^Q containing $A \cap WO_\xi$, then $\sigma^{-1}(F)$ is an F_σ-set in M containing $S \cap B_\xi$. Therefore $\sigma^{-1}(F)$ intersects $C \cap \bigcup_{\alpha<\xi} B_\alpha$ and hence F intersects $E \cap \bigcup_{\alpha<\xi} WO_\alpha$.

5.1. Remark. Let Z be a separable completely metrizable space, let A be an analytic set in $Z \times 2^Q$ and let $E \subset A$ intersect stationary many strips $Z \times WO_\xi$. Then, as in the assertion of Corollary 1.3, for all but non-stationary many ξ, each F_σ-set in $Z \times 2^Q$ containing $A \cap (Z \times WO_\xi)$ intersects $E \cap \bigcup_{\alpha<\xi}(Z \times WO_\alpha)$.

To see this, repeat the proof of Corollary 1.3 where the resolution $\sigma : M \to WO$ is replaced by $\mathrm{id} \times \sigma : Z \times M \to Z \times WO$ and the reference to Theorem 1.2 is replaced by Proposition 4.4 with $X = Z \times B(\omega_1)$ and $X_\xi = Z \times \{x : \kappa(x) \le \xi\}$ (then $P_\xi = Z \times B_\xi$).

6. Comments

6.1. *Lusin sieves.* A *Borel sieve* in a separable completely metrizable space Z is a collection $\mathcal{W} = \{W_q : q \in Q\}$ of Borel sets $W_q \subset Z$. The sieve $\mathcal{W}$ associates with each $z \in Z$ a set of rationals $\phi(z) = \{q \in Q : z \in W_q\}$; W_q being Borel, the map $\phi : Z \to 2^Q$ is Borel.

For $\xi < \omega_1$, the set $L_\xi = \{z : \text{type } \phi(z) = \xi\} = \phi^{-1}(WO_\xi)$ is the ξth constituent of $C = \phi^{-1}(WO)$ determined by the sieve $\mathcal{W}$ and $Z \setminus C$ is the analytic set sifted by $\mathcal{W}$ (cf. [Kur], §39, VIII).

Let $D \subset Z$ intersect stationary many constituents L_ξ. Then, using Remark 5.1 with $A = \{(z, \phi(z)) : z \in Z\}$ and $E = \{(z, \phi(z)) : z \in D\}$, we get the following conclusion: all but non-stationary many constituents L_ξ cannot be separated from $D \cap \bigcup_{\alpha<\xi} L_\alpha$ by any F_σ-set in Z.

Notice that if D intersects every constituent in at most one point, then each set $D_\xi = D \cap \bigcup_{\alpha<\xi} L_\alpha$ is countable, and therefore, for $\xi < \omega_1$, $Z \setminus D_\xi$ is a G_δ-set separating L_ξ from D_ξ.

V. G. Kanoveĭ ([Ka1], Theorem 4) proved that if "sufficiently many" constituents L_ξ are non-empty, then the Borel rank of sets which separate L_ξ from $\bigcup_{\alpha<\xi} L_\alpha$ must be unbounded.

6.2. *Borel additive families.* A family $\mathcal{A}$ of subsets of a metrizable space X is *Borel* (resp. *Souslin*)*-additive* if the union of each subfamily of $\mathcal{A}$ is Borel (or Souslin, respectively). Investigation of such families in non-separable spaces was originated by R. W. Hansell [Ha] and a discussion of the subject can be found in [Fr], Sec. 3. We shall indicate a connection between this topic and constituents.

In the proposition below, C is a coanalytic set in a separable completely metrizable space Z, and the constituents are determined by a Borel sieve through which $Z \setminus C$ is sifted (cf. 6.1).

PROPOSITION. *Each point-countable Borel-additive family $\mathcal{A}$ in a coanalytic set C has a disjoint refinement $\mathcal{E}$ with $\bigcup \mathcal{E} = \bigcup \mathcal{A}$ such that each selector for $\mathcal{E}$ intersects non-stationary many constituents of C.*

We sketch a proof of this fact. Consider $\phi : Z \to 2^Q$ such that ϕ is Borel and $\phi^{-1}(WO_\xi)$ is the ξth constituent of C (cf. 6.1).

For $Y \subset Z$ put $Y' = \{(z, \phi(z)) : z \in Y\}$. Then Z' is the graph of ϕ and $\mathcal{A}' = \{A' : A \in \mathcal{A}\}$ is a point-countable Borel-additive family in $Z' \cap (C \times 2^Q) = Z' \cap (Z \times WO)$ and, consequently, it is Borel-additive in $Z \times WO$.

As in Remark 5.1, use the one-to-one mapping $\widetilde{\sigma} = \mathrm{id} \times \sigma : Z \times M \to Z \times WO$ to transfer $\mathcal{A}'$ to the completely metrizable space $Z \times B(\omega_1)$ of weight $\aleph_1$. By [Po2], Theorem 1.3 (cf. [Fr], Theorem 3.J), there exists a disjoint refinement $\mathcal{E}'$ of $\mathcal{A}'$ such that $\bigcup \mathcal{E}' = \bigcup \mathcal{A}'$ and $\widetilde{\sigma}^{-1}(E)$ is σ-discrete for each selector E of $\mathcal{E}'$.

Since σ-discrete sets in $Z \times B(\omega_1)$ intersect at most non-stationary many layers $P_\xi = Z \times B_\xi$ (cf. [Po1], Theorem 1), it follows that each selector E of $\mathcal{E}'$ intersects at most non-stationary many strips $Z \times WO_\xi$.

Clearly the projections of the elements of $\mathcal{E}'$ onto Z form a refinement $\mathcal{E}$ of $\mathcal{A}$ with the required properties.

One can also interpret in a similar way Hansell's result, or its variations concerning Souslin-additive families (cf. [Fr], Sec. 3).

6.3. *The product* N^{ω_1}. Let N^{ω_1} be the $\aleph_1$-product of the natural numbers and let G be the subspace of N^{ω_1} consisting of functions x such that, for some $\alpha < \omega_1$, $x : [0,\alpha) \to N \setminus \{0\}$ is injective and $x(\beta) = 0$ for $\beta \ge \alpha$. The space G is closed in N^{ω_1} and locally homeomorphic to the irrationals.

The simple argument presented in Section 3 was a byproduct of our investigation [Ch-G-P] of the Borel structure of G. In particular, one can show that locally countable Borel sets in G are σ-discrete, and this in turn easily provides a negative answer to Lusin's question.

6.4. Remark. Lusin ([Lu2], Sec. 1) set forth the problem we considered in this note to test possibilities of describing a set of reals of cardinality $\aleph_1$ without transfinite induction.

Stone ([St2], Sec. 5) pointed out that the lack of "nice" way to choose, for each limit ordinal ξ, a sequence $x_\xi : N \to \omega_1$ with supremum ξ is reflected by the fact that the resulting set $\{x_\xi : \xi \in \mathrm{Lim}\}$ is not Borel (in fact, not Souslin) in the Baire space of weight $\aleph_1$.

The resolution defined in Section 2 confirms a connection between these two points of view, as indicated in Section 3.

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