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On Radon measures on first-countable spaces

by

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Abstract. It is shown that every Radon measure on a first-countable Hausdorff space is separable provided ω_1 is a precaliber of every measurable algebra. As the latter is implied by MA(ω_1), the result answers a problem due to D. H. Fremlin.

Answering the problem posed by D. H. Fremlin ([4], 32R(c)), we show in this note that, assuming

(∗) ω_1 is a precaliber of every measurable Boolean algebra,

every Radon measure on a first-countable space is separable.

We treat here only finite measures. By the *Maharam type* of a measure μ we mean the density character of the Banach space $L^1(\mu)$ (see [4] or [5]). Thus the Maharam type of μ is the least cardinal κ for which there exists a family $\mathcal{D}$ of measurable sets such that $|\mathcal{D}| = \kappa$, and $\mathcal{D}$ approximates all measurable sets, that is, for every measurable B and $\varepsilon > 0$ there is $D \in \mathcal{D}$ with $\mu(B \triangle D) < \varepsilon$. In particular, a measure μ of Maharam type ω is called *separable*.

Basic facts concerning Radon measures can be found in [7] or [5]. Although one can use several definitions of a Radon measure, differences are not so important when the measure in question is finite. Let us agree that, given a topological space S, the statement "μ is a Radon measure on S" means that μ is defined on some σ-algebra containing all open subsets of S, and $\mu(B) = \sup\{\mu(K) : K \subseteq B,\ K \text{ compact}\}$ for every measurable set B.

Recall that ω_1 is said to be a *precaliber* of a Boolean algebra $\mathbf{A}$ if for every family $\{a_\xi : \xi < \omega_1\}$ of non-zero elements of $\mathbf{A}$ one can find an uncountable set $X \subseteq \omega_1$ such that the family $\{a_\xi : \xi \in X\}$ is centered, that is, $\prod_{\xi \in I} a_\xi \neq \mathbf{0}$ for every finite $I \subseteq X$ (see [6], A2T). Recall also that a

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measurable algebra is a complete Boolean algebra having a strictly positive and countably additive finite measure (see [5]).

It is known that $(*)$ is a consequence of Martin's axiom (more precisely of $\mathrm{MA}(\omega_1)$, see [4]), and of the existence of an atomlessly measurable cardinal (see [6], 6C). Using the Maharam theorem one can check that, to have $(*)$ granted, it suffices to assume that ω_1 is a precaliber of the measure algebra of the usual product measure on $\{0,1\}^{\omega_1}$.

Note that $(*)$ implies that ω_1 is a *caliber* of every Radon measure in the following sense: Given a Radon measure μ, for every family $\{B_\xi : \xi < \omega_1\}$ of μ-measurable sets of positive measure $\bigcap_{\xi\in X} B_\xi \neq \emptyset$ for some uncountable $X \subseteq \omega_1$. Indeed, we can find compact sets $F_\xi \subseteq B_\xi$ with $\mu(F_\xi) > 0$. Now $(*)$ applied to the measure algebra of μ implies that there is an uncountable $X \subseteq \omega_1$ such that $\{F_\xi : \xi \in X\}$ is centered. Hence, by compactness, $\bigcap_{\xi\in X} B_\xi \supseteq \bigcap_{\xi\in X} F_\xi \neq \emptyset$.

It is well known that CH implies that $(*)$ is false (see e.g. [2]). Moreover, CH implies the existence of first-countable compact spaces admitting non-separable Radon measures, see Haydon [9] and Kunen [10]. Thus the result we are aiming at is not provable in ZFC. The remark at the end of the paper explains that $(*)$ is in fact the weakest set-theoretic assumption we need.

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The main result is given below as Theorem 3. Its proof is based on two auxiliary facts we shall now present.

LEMMA 1. *Assume $(*)$ and let μ be a Radon measure on a space S. If $(X_\alpha)_{\alpha<\omega_1}$ is an increasing family of arbitrary subsets of S with $S = \bigcup_{\alpha<\omega_1} X_\alpha$ then there is a $\xi < \omega_1$ such that $\mu^*(X_\xi) = \mu(S)$.*

Proof. Suppose that $\mu^*(X_\alpha) < \mu(S)$ for every $\alpha < \omega_1$. This means that we can find, for every $\alpha < \omega_1$, a compact set $F_\alpha \subseteq S \setminus X_\alpha$ with $\mu(F) > 0$. By $(*)$, ω_1 is a caliber of μ, so there is an uncountable set $I \subseteq \omega_1$ such that $F = \bigcap_{\alpha\in I} F_\alpha \neq \emptyset$. But $F \subseteq S \setminus \bigcup_{\alpha<\omega_1} X_\alpha$, a contradiction.

The next lemma is, in essence, known; its proof closely follows the argument used in [8], Proposition 2.1.

LEMMA 2. *If a compact space K admits a non-separable Radon measure then there exists a Radon measure μ on K of Maharam type ω_1.*

Proof. Using the Maharam theorem (see part 3 of [5]), we can take a probability Radon measure λ on K whose measure algebra is isomorphic to the usual measure algebra on $\{0,1\}^\kappa$, where $\kappa \geq \omega_1$. We can thus find a sequence $(B_\alpha)_{\alpha<\omega_1}$ of λ-independent Borel subsets of K with $\lambda(B_\alpha) = 1/2$.

Choose for every $\alpha < \omega_1$ two compact sets F_α, H_α, each of measure at least 7/16, and such that $F_\alpha \subseteq B_\alpha$, $H_\alpha \subseteq K \setminus B_\alpha$. Note that

$$\lambda(F_\alpha \cap H_\beta) \geq \lambda(B_\alpha \cap (K \setminus B_\beta)) - 1/8 \geq 1/8,$$

whenever $\alpha \neq \beta$.

Let $g_\alpha : K \to [0,1]$ be a continuous function which is zero on F_α and equals 1 on H_α. Now consider the mapping

$$g = (g_\alpha)_{\alpha<\omega_1} : K \to [0,1]^{\omega_1},$$

and the induced measure $\nu = g(\lambda)$ on $[0,1]^{\omega_1}$.

Given $\alpha < \omega_1$ and $i \in \{0,1\}$, we put

$$Z^i_\alpha = \{x \in [0,1]^{\omega_1} : x(\alpha) = i\}.$$

Since for $\alpha \neq \beta$,

$$\nu(Z^0_\alpha \cap Z^1_\beta) = \lambda(g^{-1}(Z^0_\alpha \cap Z^1_\beta)) \geq \lambda(F_\alpha \cap H_\beta) \geq 1/8,$$

it follows that $\nu(Z^0_\alpha \bigtriangleup Z^0_\beta) \geq 1/4$; consequently, ν is not separable. On the other hand, the Maharam type of any Radon measure on $[0,1]^{\omega_1}$ is not greater than its topological weight. Thus ν is of type ω_1.

Now consider the set Λ of all Radon measures μ such that $g(\mu) = \nu$. Λ is non-empty convex and weak* compact so it has an extreme point, say μ_0. Now μ_0 is the required measure since the spaces $L^1(\mu_0)$ and $L^1(\nu)$ are isometric, see Douglas [3].

THEOREM 3. *If ω_1 is a precaliber of every measurable Boolean algebra then every Radon measure on a first-countable Hausdorff space is separable.*

Proof. It is clear that we can work in a compact space. By Lemma 2 it suffices to check that whenever K is a first-countable compact space and μ is a Radon measure on K of Maharam type less than or equal to ω_1 then μ is separable.

Suppose that $\mathcal{B} = \{B_\alpha : \alpha < \omega_1\}$ is a family of (Borel) subsets of K that approximates μ-measurable sets. For every $\xi < \omega_1$ we put $\mathcal{B}_\xi = \{B_\alpha : \alpha < \xi\}$. For every $x \in K$ we choose a countable base $(V_n(x))_{n\in\omega}$ at x.

Given $x \in K$, there is $\xi(x) < \omega_1$ such that $\mathcal{B}_{\xi(x)}$ approximates the family $(V_n(x))_{n\in\omega}$. Putting $X_\xi = \{x \in K : \xi(x) < \xi\}$ we thus have $K = \bigcup_{\xi<\omega_1} X_\xi$. It follows from Lemma 1 that $\mu^*(X_{\xi_0}) = \mu(X)$ for some $\xi_0 < \omega_1$; write $Y = X_{\xi_0}$ for simplicity. We let $\mathcal{D}$ be the closure of $\mathcal{B}_{\xi_0}$ under finite unions. As $\mathcal{D}$ is countable, it suffices to check that it approximates all open sets.

Take an open set $U \subseteq K$. For every $x \in U \cap Y$ we choose a natural number n_x such that $V_{n_x}(x) \subseteq U$. Putting

$$W = \bigcup_{x \in U \cap Y} V_{n_x}(x),$$

we have $U \cap Y \subseteq W \subseteq U$. As $\mu^*(U \cap Y) = \mu(U)$, we get $\mu(U \setminus W) = 0$. Since W can be approximated by finite sums of $V_{n_x}(x)$'s (which is due to τ-additivity of μ), it follows that W, as well as U, is approximated by $\mathcal{D}$. This completes the proof.

As D. H. Fremlin remarked, Theorem 3 can be generalized to higher cardinals, namely for every cardinal κ we have the following:

Assuming that κ^+ is a precaliber of every measurable algebra, if S is a space of character κ then every Radon measure on S is of Maharam type at most κ.

For this we can argue as before, adapting Lemma 1 and Lemma 2 in a straightforward manner.

We can slightly generalize the theorem above in another direction, replacing the assumption of first-countability by a certain covering property. A topological space S is called *metalindelöf* if every open cover of S has a point-countable refinement. This concept is very useful in topological measure theory; see [7], 4.9 (and [12] for further references).

COROLLARY 4. *Assume that ω_1 is a precaliber of every Boolean algebra. If S is a Hausdorff space such that $K \setminus \{x\}$ is metalindelöf for every $x \in S$ and every compact $K \subseteq S$ then every Radon measure on S is separable.*

Proof. Again it suffices to prove that whenever μ is a Radon measure on a compact space K, where $K \subseteq S$, then μ is separable. In turn, this reduces to the case when K is the support of μ, that is, $\mu(V) > 0$ for every non-empty V which is open in K.

It follows that K is first-countable. Indeed, take any $x \in K$, and let $\mathcal{U}$ be the family of open sets with $x \notin \overline{U}$. Then $\mathcal{U}$ is a cover of $K \setminus \{x\}$; since this space is assumed to be metalindelöf, there is an open refinement $\mathcal{V}$ of $\mathcal{U}$ which is point-countable. But a point-countable family of sets of positive measure has to be countable by our assumption on the caliber. Now the sets $K \setminus (\overline{V}_1 \cup \ldots \cup \overline{V}_n)$, where $V_i \in \mathcal{V}$, form a countable base at x. Thus μ is separable by Theorem 3 above and the proof is complete.

The class of topological spaces satisfying the assumption of Corollary 4 contains, of course, all first-countable spaces; besides, it contains all Σ-products of the real line. This is due to the fact that a Corson compact, i.e.

a compact space which is (homeomorphic to) a subset of some Σ-product of $\mathbb{R}$, is hereditarily metalindelöf (see e.g. [12]).

It is known that the negation of $(*)$ gives rise to the following construction of a Corson compact space (see [1]).

Let $\mathbf{A}$ be a probability measure algebra and let $\{a_\alpha : \alpha < \omega_1\}$ be a sequence in $\mathbf{A}$ witnessing the fact that ω_1 is not a precaliber of $\mathbf{A}$. Put

$$K = \Big\{C \subseteq \omega_1 : \prod_{\alpha\in I} a_\alpha \neq 0 \text{ for every finite } I \subseteq C\Big\}.$$

Such a K may be treated as a subspace of $\{0,1\}^{\omega_1}$; it is then compact. It is moreover Corson compact since every $C \in K$ is countable.

As shown in [13], K has a strictly positive non-separable Radon measure. More subtle results in this direction have been recently obtained by Kunen and van Mill [11]. Under the same assumption non-$(*)$ they constructed a first-countable Corson compact space carrying a non-separable measure. Thus the assertion *every Radon measure on a first-countable space is separable* is in fact equivalent to the axiom ω_1 *is a precaliber of every measurable algebra*.

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