On Radon measures on first-countable spaces

by

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Abstract. It is shown that every Radon measure on a first-countable Hausdorff space is separable provided $\omega_1$ is a precaliber of every measurable algebra. As the latter is implied by MA($\omega_1$), the result answers a problem due to D. H. Fremlin.

Answering the problem posed by D. H. Fremlin ([4], 32R(c)), we show in this note that, assuming

(*) $\omega_1$ is a precaliber of every measurable Boolean algebra,

every Radon measure on a first-countable space is separable.

We treat here only finite measures. By the Maharam type of a measure $\mu$ we mean the density character of the Banach space $L^1(\mu)$ (see [4] or [5]). Thus the Maharam type of $\mu$ is the least cardinal $\kappa$ for which there exists a family $\mathcal{D}$ of measurable sets such that $|\mathcal{D}| = \kappa$, and $\mathcal{D}$ approximates all measurable sets, that is, for every measurable $B$ and $\varepsilon > 0$ there is $D \in \mathcal{D}$ with $\mu(B \Delta D) < \varepsilon$. In particular, a measure $\mu$ of Maharam type $\omega$ is called separable.

Basic facts concerning Radon measures can be found in [7] or [5]. Although one can use several definitions of a Radon measure, differences are not so important when the measure in question is finite. Let us agree that, given a topological space $S$, the statement “$\mu$ is a Radon measure on $S$” means that $\mu$ is defined on some $\sigma$-algebra containing all open subsets of $S$, and $\mu(\emptyset) = \sup\{\mu(K) : K \subseteq B, K \text{ compact}\}$ for every measurable set $B$.

Recall that $\omega_1$ is said to be a precaliber of a Boolean algebra $\mathbf{A}$ if for every family $\{a_\xi : \xi < \omega_1\}$ of non-zero elements of $\mathbf{A}$ one can find an uncountable set $X \subseteq \omega_1$ such that the family $\{a_\xi : \xi \in X\}$ is centered, that is, $\prod_{\xi \in I} a_\xi \neq 0$ for every finite $I \subseteq X$ (see [6], A2T). Recall also that a

1991 Mathematics Subject Classification: Primary 28C15; Secondary 54A25.

Partially supported by KBN grant 2 P 301 043 07.
measurable algebra is a complete Boolean algebra having a strictly positive and countably additive finite measure (see [5]).

It is known that (\*) is a consequence of Martin’s axiom (more precisely of \(\text{MA}(\omega_1)\), see [4]), and of the existence of an atomlessly measurable cardinal (see [6], 6C). Using the Maharam theorem one can check that, to have (\*) granted, it suffices to assume that \(\omega_1\) is a precaliber of the measure algebra of the usual product measure on \(\{0,1\}^{\omega_1}\).

Note that (\*) implies that \(\omega_1\) is a caliber of every Radon measure in the following sense: Given a Radon measure \(\mu\), for every family \(\{B_\xi : \xi < \omega_1\}\) of \(\mu\)-measurable sets of positive measure \(\bigcap_{\xi \in X} B_\xi \neq \emptyset\) for some uncountable \(X \subseteq \omega_1\). Indeed, we can find compact sets \(F_\xi \subseteq B_\xi\) with \(\mu(F_\xi) > 0\). Now (\*) applied to the measure algebra of \(\mu\) implies that there is an uncountable \(X \subseteq \omega_1\) such that \(\{F_\xi : \xi \in X\}\) is centered. Hence, by compactness, \(\bigcap_{\xi \in X} B_\xi \supseteq \bigcap_{\xi \in X} F_\xi \neq \emptyset\).

It is well known that \(\text{CH}\) implies that (\*) is false (see e.g. [2]). Moreover, \(\text{CH}\) implies the existence of first-countable compact spaces admitting non-separable Radon measures, see Haydon [9] and Kunen [10]. Thus the result we are aiming at is not provable in \(\text{ZFC}\). The remark at the end of the paper explains that (\*) is in fact the weakest set-theoretic assumption we need.

The author is very indebted to David Fremlin for several valuable suggestions.

The main result is given below as Theorem 3. Its proof is based on two auxiliary facts we shall now present.

\textbf{Lemma 1.} Assume (\*) and let \(\mu\) be a Radon measure on a space \(S\). If \((X_\alpha)_{\alpha < \omega_1}\) is an increasing family of arbitrary subsets of \(S\) with \(S = \bigcup_{\alpha < \omega_1} X_\alpha\) then there is a \(\xi < \omega_1\) such that \(\mu^*(X_\xi) = \mu(S)\).

\textbf{Proof.} Suppose that \(\mu^*(X_\alpha) < \mu(S)\) for every \(\alpha < \omega_1\). This means that we can find, for every \(\alpha < \omega_1\), a compact set \(F_\alpha \subseteq S \setminus X_\alpha\) with \(\mu(F) > 0\). By (\*), \(\omega_1\) is a caliber of \(\mu\), so there is an uncountable set \(I \subseteq \omega_1\) such that \(F = \bigcap_{\alpha \in I} F_\alpha \neq \emptyset\). But \(F \subseteq S \setminus \bigcup_{\alpha < \omega_1} X_\alpha\), a contradiction.

The next lemma is, in essence, known; its proof closely follows the argument used in [8], Proposition 2.1.

\textbf{Lemma 2.} If a compact space \(K\) admits a non-separable Radon measure then there exists a Radon measure \(\mu\) on \(K\) of Maharam type \(\omega_1\).

\textbf{Proof.} Using the Maharam theorem (see part 3 of [5]), we can take a probability Radon measure \(\lambda\) on \(K\) whose measure algebra is isomorphic to the usual measure algebra on \(\{0,1\}^\kappa\), where \(\kappa \geq \omega_1\). We can thus
find a sequence \((B_\alpha)_{\alpha<\omega_1}\) of \(\lambda\)-independent Borel subsets of \(K\) with \(\lambda(B_\alpha) = 1/2\).

Choose for every \(\alpha < \omega_1\) two compact sets \(F_\alpha, H_\alpha\), each of measure at least \(7/16\), and such that \(F_\alpha \subseteq B_\alpha, H_\alpha \subseteq K \setminus B_\alpha\). Note that
\[
\lambda(F_\alpha \cap H_\beta) \geq \lambda(B_\alpha \cap (K \setminus B_\beta)) - 1/8 \geq 1/8,
\]
whenever \(\alpha \neq \beta\).

Let \(g_\alpha : K \to [0, 1]\) be a continuous function which is zero on \(F_\alpha\) and equals 1 on \(H_\alpha\). Now consider the mapping
\[
g = (g_\alpha)_{\alpha<\omega_1} : K \to [0, 1]^{\omega_1},
\]
and the induced measure \(\nu = g(\lambda)\) on \([0, 1]^{\omega_1}\).

Given \(\alpha < \omega_1\) and \(i \in \{0, 1\}\), we put
\[
Z^i_\alpha = \{x \in [0, 1]^{\omega_1} : x(\alpha) = i\}.
\]

Since for \(\alpha \neq \beta\),
\[
\nu(Z^0_\alpha \cap Z^3_\beta) = \lambda(g^{-1}(Z^0_\alpha \cap Z^3_\beta)) \geq \lambda(F_\alpha \cap H_\beta) \geq 1/8,
\]
it follows that \(\nu(Z^0_\alpha \triangle Z^3_\beta) \geq 1/4\); consequently, \(\nu\) is not separable. On the other hand, the Maharam type of any Radon measure on \([0, 1]^{\omega_1}\) is not greater than its topological weight. Thus \(\nu\) is of type \(\omega_1\).

Now consider the set \(\Lambda\) of all Radon measures \(\mu\) such that \(g(\mu) = \nu\). \(\Lambda\) is non-empty convex and weak* compact so it has an extreme point, say \(\mu_0\). Now \(\mu_0\) is the required measure since the spaces \(L^1(\mu_0)\) and \(L^1(\nu)\) are isometric, see Douglas [3].

**Theorem 3.** If \(\omega_1\) is a precaliber of every measurable Boolean algebra then every Radon measure on a first-countable Hausdorff space is separable.

**Proof.** It is clear that we can work in a compact space. By Lemma 2 it suffices to check that whenever \(K\) is a first-countable compact space and \(\mu\) is a Radon measure on \(K\) of Maharam type less than or equal to \(\omega_1\) then \(\mu\) is separable.

Suppose that \(\mathcal{B} = \{B_\alpha : \alpha < \omega_1\}\) is a family of (Borel) subsets of \(K\) that approximates \(\mu\)-measurable sets. For every \(\xi < \omega_1\) we put \(\mathcal{B}_\xi = \{B_\alpha : \alpha < \xi\}\). For every \(x \in K\) we choose a countable base \((V_n(x))_{n \in \omega}\) at \(x\).

Given \(x \in K\), there is \(\xi(x) < \omega_1\) such that \(\mathcal{B}_{\xi(x)}\) approximates the family \((V_n(x))_{n \in \omega}\). Putting \(X_\xi = \{x \in K : \xi(x) < \xi\}\) we thus have \(K = \bigcup_{\xi < \omega_1} X_\xi\). It follows from Lemma 1 that \(\mu^*(X_{\xi_0}) = \mu(X)\) for some \(\xi_0 < \omega_1\); write \(Y = X_{\xi_0}\) for simplicity. We let \(\mathcal{D}\) be the closure of \(\mathcal{B}_{\xi_0}\) under finite unions. As \(\mathcal{D}\) is countable, it suffices to check that it approximates all open sets.
Take an open set $U \subseteq K$. For every $x \in U \cap Y$ we choose a natural number $n_x$ such that $V_{n_x}(x) \subseteq U$. Putting
\[ W = \bigcup_{x \in U \cap Y} V_{n_x}(x), \]
we have $U \cap Y \subseteq W \subseteq U$. As $\mu^*(U \cap Y) = \mu(U)$, we get $\mu(U \setminus W) = 0$.
Since $W$ can be approximated by finite sums of $V_{n_x}(x)$'s (which is due to $\tau$-additivity of $\mu$), it follows that $W$, as well as $U$, is approximated by $D$.
This completes the proof.

As D. H. Fremlin remarked, Theorem 3 can be generalized to higher cardinals, namely for every cardinal $\kappa$ we have the following:

**Assuming that $\kappa^+$ is a precaliber of every measurable algebra, if $S$ is a space of character $\kappa$ then every Radon measure on $S$ is of Maharam type at most $\kappa$.**

For this we can argue as before, adapting Lemma 1 and Lemma 2 in a straightforward manner.

We can slightly generalize the theorem above in another direction, replacing the assumption of first-countability by a certain covering property. A topological space $S$ is called *metalindelöf* if every open cover of $S$ has a point-countable refinement. This concept is very useful in topological measure theory; see [7], 4.9 (and [12] for further references).

**Corollary 4.** Assume that $\omega_1$ is a precaliber of every Boolean algebra. If $S$ is a Hausdorff space such that $K \setminus \{x\}$ is metalindelöf for every $x \in S$ and every compact $K \subseteq S$ then every Radon measure on $S$ is separable.

**Proof.** Again it suffices to prove that whenever $\mu$ is a Radon measure on a compact space $K$, where $K \subseteq S$, then $\mu$ is separable. In turn, this reduces to the case when $K$ is the support of $\mu$, that is, $\mu(V) > 0$ for every non-empty $V$ which is open in $K$.
It follows that $K$ is first-countable. Indeed, take any $x \in K$, and let $\mathcal{U}$ be the family of open sets with $x \notin \overline{U}$. Then $\mathcal{U}$ is a cover of $K \setminus \{x\}$; since this space is assumed to be metalindelöf, there is an open refinement $\mathcal{V}$ of $\mathcal{U}$ which is point-countable. But a point-countable family of sets of positive measure has to be countable by our assumption on the caliber. Now the sets $K \setminus (V_1 \cup \ldots \cup V_n)$, where $V_i \in \mathcal{V}$, form a countable base at $x$. Thus $\mu$ is separable by Theorem 3 above and the proof is complete.

The class of topological spaces satisfying the assumption of Corollary 4 contains, of course, all first-countable spaces; besides, it contains all $\Sigma$-products of the real line. This is due to the fact that a Corson compact, i.e.
a compact space which is (homeomorphic to) a subset of some Σ-product of \( \mathbb{R} \), is hereditarily metalindelöf (see e.g. [12]).

It is known that the negation of \((\ast)\) gives rise to the following construction of a Corson compact space (see [1]).

Let \( A \) be a probability measure algebra and let \( \{a_\alpha : \alpha < \omega_1\} \) be a sequence in \( A \) witnessing the fact that \( \omega_1 \) is not a precaliber of \( A \). Put

\[
K = \left\{ C \subseteq \omega_1 : \prod_{\alpha \in I} a_\alpha \neq 0 \text{ for every finite } I \subseteq C \right\}.
\]

Such a \( K \) may be treated as a subspace of \( \{0, 1\}^{\omega_1} \); it is then compact. It is moreover Corson compact since every \( C \in K \) is countable.

As shown in [13], \( K \) has a strictly positive non-separable Radon measure. More subtle results in this direction have been recently obtained by Kunen and van Mill [11]. Under the same assumption non-(\( \ast \)) they constructed a first-countable Corson compact space carrying a non-separable measure. Thus the assertion every Radon measure on a first-countable space is separable is in fact equivalent to the axiom \( \omega_1 \) is a precaliber of every measurable algebra.

References


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Received 23 January 1995