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A free group acting without fixed points on the rational unit sphere

by

Kenzi Satô (Tokyo)

Abstract. We prove the existence of a free group of rotations of rank 2 which acts on the rational unit sphere without non-trivial fixed points.

Introduction. The purpose of this paper is to prove that the group $SO_3(\mathbb{Q})$ of all proper orthogonal 3×3 matrices with rational entries has a free subgroup F_2 of rank 2 such that for all $w \in F_2$ different from the identity and for all $\vec{r} \in \mathbb{S}^2 \cap \mathbb{Q}^3$ we have $w(\vec{r}) \neq \vec{r}$ (Theorem 2). The question if such a group exists was raised by Professor J. Mycielski. Theorem 2 has the following corollary. The rational unit sphere $\mathbb{S}^2 \cap \mathbb{Q}^3$ $(= \{\vec{r} \in \mathbb{Q}^3 : |\vec{r}| = 1\})$ has all possible kinds of Banach–Tarski paradoxical decompositions, e.g. a partition into three sets A, B, and C such that

$$A \approx B \approx C \approx A \cup B \approx B \cup C \approx C \cup A,$$

where $\approx$ denotes congruence by a transformation of F_2 (such a partition is called a *Hausdorff decomposition*). The proof of this corollary of Theorem 2 is well known (see e.g. [W, Cor. 4.12]). Moreover, since in this case the space $\mathbb{S}^2 \cap \mathbb{Q}^3$ is countable, the proof does not require the axiom of choice. A Hausdorff decomposition is not possible for the real sphere $\mathbb{S}^2$ $(= \{\vec{r} \in \mathbb{R}^3 : |\vec{r}| = 1\})$ relative to $SO_3(\mathbb{R})$ (= the group of all proper orthogonal matrices) since every rotation of $\mathbb{S}^2$ has fixed points (thus $C \approx A \cup B$ cannot hold). However, it is possible if reflections are allowed (see [A] or [W, Theorem 4.16]).

Other constructions of free subgroups of $SO_3(\mathbb{Q})$ are known. S. Świercz-

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kowski ([Sw0], [Sw1]) has shown that the transformations

$$\begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{pmatrix}$$

are free generators if $\cos\phi \in \mathbb{Q} \setminus \{-1, -1/2, 0, 1/2, 1\}$. But of course these generators have fixed points in $\mathbb{S}^2 \cap \mathbb{Q}^3$.

Theorem 2 gives a concrete example of a pair of free generators of a free group acting without non-trivial fixed points, namely

$$\mu = \frac{1}{7}\begin{pmatrix} 6 & 2 & 3 \\ 2 & 3 & -6 \\ -3 & 6 & 2 \end{pmatrix} \quad \text{and} \quad \nu = \frac{1}{7}\begin{pmatrix} 2 & -6 & 3 \\ 6 & 3 & 2 \\ -3 & 2 & 6 \end{pmatrix}.$$

Preliminaries. Thus our aim is to prove that:

For every non-empty reduced word w in $\{\mu^{-1}, \nu^{-1}, \mu, \nu\}$, the rotation $w \in SO_3(\mathbb{Q})$ is not the identity and its axis intersects the sphere $\mathbb{S}^2$ at irrational points.

We will use Hamilton's quaternion field $\mathbb{R} \times \mathbb{R}^3$ with $*$, where

$$(c', \vec{s'}) * (c, \vec{s}) = (c'c - \vec{s'} \cdot \vec{s}, c\vec{s'} + c'\vec{s} + \vec{s'} \times \vec{s}).$$

If $c \in \mathbb{R}$, $\vec{s} \in \mathbb{R}^3$ and $c^2 + |\vec{s}|^2 = 1$, then the pair of quaternions $\pm(c, \vec{s})$ represents a single rotation γ on $\mathbb{S}^2$ (see also [Sa]). The rotation $\gamma \in SO_3(\mathbb{R})$ is the identity rotation iff $\vec{s} = \vec{0}$. Otherwise γ is determined as an anti-clockwise rotation on $\mathbb{S}^2$ around the vector $\vec{s}$, whose angle θ is such that $c = |\sin(\theta/2)|/\tan(\theta/2)$, i.e.,

$$\gamma(\vec{r}) = 2(\vec{s} \cdot \vec{r})\vec{s} + (c^2 - |\vec{s}|^2)\vec{r} + 2c\vec{s} \times \vec{r} \quad \text{for } \vec{r} \in \mathbb{S}^2.$$

We denote the pair which represents the rotation γ by $\pm(c_\gamma, \vec{s}_\gamma)$. The pair of quaternions $\pm(c_\beta, \vec{s}_\beta)*(c_\alpha, \vec{s}_\alpha)$ represents the rotation $\beta\circ\alpha$, since $(0, \gamma(\vec{r})) = (c_\gamma, \vec{s}_\gamma)*(0, \vec{r})*(c_\gamma, -\vec{s}_\gamma)$ for all $\vec{r} \in \mathbb{S}^2$. And γ^{-1} is represented by $\pm(c_\gamma, -\vec{s}_\gamma)$.

The two rotations $\mu, \nu \in SO_3(\mathbb{Q})$ defined above are represented by the quaternion pairs

$$\pm(c_{\mu^\varepsilon}, \vec{s}_{\mu^\varepsilon}) = \pm\frac{1}{\sqrt{14}}\left(3, \begin{pmatrix} 2\varepsilon \\ \varepsilon \\ 0 \end{pmatrix}\right) \quad \text{and}$$

$$\pm(c_{\nu^\delta}, \vec{s}_{\nu^\delta}) = \pm\frac{1}{\sqrt{14}}\left(3, \begin{pmatrix} 0 \\ \delta \\ 2\delta \end{pmatrix}\right),$$

where $\varepsilon, \delta \in \{-1, 1\}$. Let $|w|$ be the length of the word w, i.e., the number of occurrences of μ^{-1}, ν^{-1}, μ, and ν in w. Then it suffices to show that if

$$\left(C_w, \begin{pmatrix} X_w \\ Y_w \\ Z_w \end{pmatrix}\right) = \sqrt{14}^{|w|}(c_w, \vec{s}_w) \in \mathbb{Z} \times \mathbb{Z}^3$$

then the integer $X_w^2 + Y_w^2 + Z_w^2$ is not a square. We will show more: $X_w^2 + Y_w^2 + Z_w^2$ is not a square mod 7. To prove this, we define an equivalence relation $\equiv$ on $\mathbb{Z} \times \mathbb{Z}^3$. We write

$$\left(C, \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}\right) \equiv \left(C', \begin{pmatrix} X' \\ Y' \\ Z' \end{pmatrix}\right)$$

if $C \equiv C'$, $X \equiv X'$, $Y \equiv Y'$, and $Z \equiv Z'$, where $p \equiv q$ means that $p - q$ is divisible by 7. Notice that $(\mathbb{Z}\times\mathbb{Z}^3)/\equiv$ is not a field but a (non-commutative) ring. Thus we have to prove that $X_w^2 + Y_w^2 + Z_w^2$ is not a square mod 7. We shall use an additional simplification. We write

$$\left(C, \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}\right) \asymp \left(C', \begin{pmatrix} X' \\ Y' \\ Z' \end{pmatrix}\right)$$

if there exists $t \in \{-3, -2, -1, 1, 2, 3\}$ such that

$$\left(C, \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}\right) \equiv t \left(C', \begin{pmatrix} X' \\ Y' \\ Z' \end{pmatrix}\right).$$

An easy calculation shows that if

$$\left(C, \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}\right) \asymp \left(C', \begin{pmatrix} X' \\ Y' \\ Z' \end{pmatrix}\right)$$

then $X^2 + Y^2 + Z^2$ is not a square mod 7 iff $X'^2 + Y'^2 + Z'^2$ is not a square mod 7. Thus in our computation of w we do not have to worry about $\equiv$ but only about $\asymp$.

Main result. First, we get the following lemma.

LEMMA 0. *Let w be a non-empty reduced word in $\{\mu^{-1}, \nu^{-1}, \mu, \nu\}$.*

- *If $w = \mu^{\varepsilon k}$ then*

$$\left(C_w, \begin{pmatrix} X_w \\ Y_w \\ Z_w \end{pmatrix}\right) \asymp \left(3, \begin{pmatrix} 2\varepsilon \\ \varepsilon \\ 0 \end{pmatrix}\right).$$

• *If* $w=\nu^{\delta l}$ *then*

$$\left(C_w, \begin{pmatrix} X_w \\ Y_w \\ Z_w \end{pmatrix}\right) \asymp \left(3, \begin{pmatrix} 0 \\ \delta \\ 2\delta \end{pmatrix}\right),$$

where $\varepsilon,\delta\in\{-1,1\}$, $k,l\in\mathbb{N}\setminus\{0\}$.

• *If* $w=\mu^{\varepsilon_m k_m}\nu^{\delta_m l_m}\cdots\mu^{\varepsilon_0 k_0}\nu^{\delta_0 l_0}$ *then*

$$\left(C_w, \begin{pmatrix} X_w \\ Y_w \\ Z_w \end{pmatrix}\right) \asymp \left(2-\varepsilon_m\delta_0, \begin{pmatrix} -\varepsilon_m+2\varepsilon_m\delta_0 \\ 3\varepsilon_m+3\delta_0+3\varepsilon_m\delta_0 \\ -\delta_0+2\varepsilon_m\delta_0 \end{pmatrix}\right),$$

where $m\in\mathbb{N}$, $\varepsilon_m,\delta_m,\ldots,\varepsilon_0,\delta_0\in\{-1,1\}$, $k_m,l_m,\ldots,k_0,l_0\in\mathbb{N}\setminus\{0\}$.

Proof. To get the first two equivalence relations, we use the following two equations respectively:

$$\left(3, \begin{pmatrix} 2\varepsilon \\ \varepsilon \\ 0 \end{pmatrix}\right) * \left(3, \begin{pmatrix} 2\varepsilon \\ \varepsilon \\ 0 \end{pmatrix}\right) = -\left(3, \begin{pmatrix} 2\varepsilon \\ \varepsilon \\ 0 \end{pmatrix}\right) + 7\left(1, \begin{pmatrix} 2\varepsilon \\ \varepsilon \\ 0 \end{pmatrix}\right),$$

$$\left(3, \begin{pmatrix} 0 \\ \delta \\ 2\delta \end{pmatrix}\right) * \left(3, \begin{pmatrix} 0 \\ \delta \\ 2\delta \end{pmatrix}\right) = -\left(3, \begin{pmatrix} 0 \\ \delta \\ 2\delta \end{pmatrix}\right) + 7\left(1, \begin{pmatrix} 0 \\ \delta \\ 2\delta \end{pmatrix}\right).$$

We have the last equivalence relation from the following two:

$$\left(3, \begin{pmatrix} 2\varepsilon \\ \varepsilon \\ 0 \end{pmatrix}\right) * \left(3, \begin{pmatrix} 0 \\ \delta \\ 2\delta \end{pmatrix}\right) = \left(2-\varepsilon\delta, \begin{pmatrix} -\varepsilon+2\varepsilon\delta \\ 3\varepsilon+3\delta+3\varepsilon\delta \\ -\delta+2\varepsilon\delta \end{pmatrix}\right) + 7\left(1, \begin{pmatrix} \varepsilon \\ -\varepsilon\delta \\ \delta \end{pmatrix}\right),$$

$$\left(2-\varepsilon'\delta', \begin{pmatrix} -\varepsilon'+2\varepsilon'\delta' \\ 3\varepsilon'+3\delta'+3\varepsilon'\delta' \\ -\delta'+2\varepsilon'\delta' \end{pmatrix}\right) * \left(2-\varepsilon\delta, \begin{pmatrix} -\varepsilon+2\varepsilon\delta \\ 3\varepsilon+3\delta+3\varepsilon\delta \\ -\delta+2\varepsilon\delta \end{pmatrix}\right)$$
$$\equiv 2(1+\varepsilon'\varepsilon-\delta'\varepsilon+\delta'\delta-\varepsilon'\delta'\varepsilon\delta)\left(2-\varepsilon'\delta, \begin{pmatrix} -\varepsilon'+2\varepsilon'\delta \\ 3\varepsilon'+3\delta+3\varepsilon'\delta \\ -\delta+2\varepsilon'\delta \end{pmatrix}\right),$$

where $\varepsilon,\delta,\varepsilon',\delta'\in\{-1,1\}$. We show the latter. Let

$$\vec{\imath}=\begin{pmatrix} -1 \\ 3 \\ 0 \end{pmatrix}, \quad \vec{\jmath}=\begin{pmatrix} 0 \\ 3 \\ -1 \end{pmatrix}, \quad \text{and} \quad \vec{k}=\begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix}.$$

Then we have

$$\vec{i}\cdot\vec{i}=10\equiv -4,\quad \vec{j}\cdot\vec{j}=10\equiv -4,\quad \vec{k}\cdot\vec{k}=17\equiv 3,$$
$$\vec{i}\cdot\vec{j}=\vec{j}\cdot\vec{i}=9\equiv 2,\quad \vec{i}\cdot\vec{k}=\vec{k}\cdot\vec{i}=7\equiv 0,\quad \vec{j}\cdot\vec{k}=\vec{k}\cdot\vec{j}=7\equiv 0,$$
$$\vec{i}\times\vec{j}=-\vec{j}\times\vec{i}=\begin{pmatrix}-3\\-1\\-3\end{pmatrix}=2\vec{k}+\begin{pmatrix}-7\\-7\\-7\end{pmatrix},$$
$$\vec{i}\times\vec{k}=-\vec{k}\times\vec{i}=\begin{pmatrix}6\\2\\-9\end{pmatrix}=\vec{i}+2\vec{j}+\begin{pmatrix}7\\-7\\-7\end{pmatrix},$$
$$\vec{j}\times\vec{k}=-\vec{k}\times\vec{j}=\begin{pmatrix}9\\-2\\-6\end{pmatrix}=-2\vec{i}-\vec{j}+\begin{pmatrix}7\\7\\-7\end{pmatrix}.$$

Hence we obtain

$$\begin{aligned}
&\left(2-\varepsilon'\delta',\begin{pmatrix}-\varepsilon'+2\varepsilon'\delta'\\3\varepsilon'+3\delta'+3\varepsilon'\delta'\\-\delta'+2\varepsilon'\delta'\end{pmatrix}\right)*\left(2-\varepsilon\delta,\begin{pmatrix}-\varepsilon+2\varepsilon\delta\\3\varepsilon+3\delta+3\varepsilon\delta\\-\delta+2\varepsilon\delta\end{pmatrix}\right)\\
&=(2-\varepsilon'\delta',\varepsilon'\vec{i}+\delta'\vec{j}+\varepsilon'\delta'\vec{k})*(2-\varepsilon\delta,\varepsilon\vec{i}+\delta\vec{j}+\varepsilon\delta\vec{k})\\
&\equiv((4-2\varepsilon'\delta'-2\varepsilon\delta+\varepsilon'\delta'\varepsilon\delta)\\
&\quad-((-4\varepsilon'\varepsilon+2\delta'\varepsilon+2\varepsilon'\delta-4\delta'\delta)+3\varepsilon'\delta'\varepsilon\delta),\\
&\quad(\varepsilon'(2-\varepsilon\delta)\vec{i}+\delta'(2-\varepsilon\delta)\vec{j}+\varepsilon'\delta'(2-\varepsilon\delta)\vec{k})\\
&\quad+((2-\varepsilon'\delta')\varepsilon\vec{i}+(2-\varepsilon'\delta')\delta\vec{j}+(2-\varepsilon'\delta')\varepsilon\delta\vec{k})\\
&\quad+(\delta'\varepsilon(-2\vec{k})+\varepsilon'\delta'\varepsilon(-\vec{i}-2\vec{j})+\varepsilon'\delta(2\vec{k})\\
&\quad+\varepsilon'\delta'\delta(2\vec{i}+\vec{j})+\varepsilon'\varepsilon\delta(\vec{i}+2\vec{j})+\delta'\varepsilon\delta(-2\vec{i}-\vec{j})))\\
&=((4+4\varepsilon'\varepsilon-4\delta'\varepsilon+4\delta'\delta-4\varepsilon'\delta'\varepsilon\delta)\\
&\quad-(2\varepsilon'\delta+2\varepsilon\delta-2\varepsilon'\delta'\varepsilon\delta+2\varepsilon'\delta'-2\delta'\varepsilon),\\
&\quad(2\varepsilon'+2\varepsilon-2\varepsilon'\delta'\varepsilon+2\varepsilon'\delta'\delta-2\delta'\varepsilon\delta)\vec{i}\\
&\quad+(2\delta+2\varepsilon'\varepsilon\delta-2\delta'\varepsilon\delta+2\delta'-2\varepsilon'\delta'\varepsilon)\vec{j}\\
&\quad+(2\varepsilon'\delta+2\varepsilon\delta-2\varepsilon'\delta'\varepsilon\delta+2\varepsilon'\delta'-2\delta'\varepsilon)\vec{k})\\
&=2(1+\varepsilon'\varepsilon-\delta'\varepsilon+\delta'\delta-\varepsilon'\delta'\varepsilon\delta)(2-\varepsilon'\delta,\varepsilon'\vec{i}+\delta\vec{j}+\varepsilon'\delta\vec{k})\\
&=2(1+\varepsilon'\varepsilon-\delta'\varepsilon+\delta'\delta-\varepsilon'\delta'\varepsilon\delta)\left(2-\varepsilon'\delta,\begin{pmatrix}-\varepsilon'+2\varepsilon'\delta\\3\varepsilon'+3\delta+3\varepsilon'\delta\\-\delta+2\varepsilon'\delta\end{pmatrix}\right).\ \blacksquare
\end{aligned}$$

Secondly, Lemma 0 implies

LEMMA 1. *Let the word w be of the form $\mu^{\varepsilon k}$, $\nu^{\delta l}$, or $\mu^{\varepsilon_m k_m}\nu^{\delta_m l_m}\cdots\mu^{\varepsilon_0 k_0}\nu^{\delta_0 l_0}$. Then $X_w^2+Y_w^2+Z_w^2\equiv -2,-1,$ or 3.*

P r o o f. If $w=\mu^{\varepsilon k}$ then there exists $t\in\{-3,-2,-1,1,2,3\}$ (actually $t\in\{-1,1\}$) such that

$$\left(C_w,\begin{pmatrix}X_w\\Y_w\\Z_w\end{pmatrix}\right)\equiv t\left(3,\begin{pmatrix}2\varepsilon\\\varepsilon\\0\end{pmatrix}\right)$$

from Lemma 0, so

$$X_w^2+Y_w^2+Z_w^2\equiv t^2((2\varepsilon)^2+\varepsilon^2+0^2)=5t^2\equiv 5,20,\ \text{or } 45\equiv -2,-1,\ \text{or } 3.$$

If $w=\nu^{\delta l}$ then there exists $t\in\{-3,-2,-1,1,2,3\}$ (actually $t\in\{-1,1\}$) such that

$$\left(C_w,\begin{pmatrix}X_w\\Y_w\\Z_w\end{pmatrix}\right)\equiv t\left(3,\begin{pmatrix}0\\\delta\\2\delta\end{pmatrix}\right)$$

from Lemma 0, so

$$X_w^2+Y_w^2+Z_w^2\equiv t^2(0^2+\delta^2+(2\delta)^2)=5t^2\equiv 5,20,\ \text{or } 45\equiv -2,-1,\ \text{or } 3.$$

If $w=\mu^{\varepsilon_m k_m}\nu^{\delta_m l_m}\cdots\mu^{\varepsilon_0 k_0}\nu^{\delta_0 l_0}$ then there exists $t\in\{-3,-2,-1,1,2,3\}$ such that

$$\left(C_w,\begin{pmatrix}X_w\\Y_w\\Z_w\end{pmatrix}\right)\equiv t\left(2-\varepsilon_m\delta_0,\begin{pmatrix}-\varepsilon_m+2\varepsilon_m\delta_0\\3\varepsilon_m+3\delta_0+3\varepsilon_m\delta_0\\-\delta_0+2\varepsilon_m\delta_0\end{pmatrix}\right)$$

from Lemma 0, so

$$\begin{aligned}X_w^2&+Y_w^2+Z_w^2\\&\equiv t^2((-\varepsilon_m+2\varepsilon_m\delta_0)^2+(3\varepsilon_m+3\delta_0+3\varepsilon_m\delta_0)^2+(-\delta_0+2\varepsilon_m\delta_0)^2)\\&=t^2((5-4\delta_0)+(27+18\varepsilon_m+18\delta_0+18\varepsilon_m\delta_0)+(5-4\varepsilon_m))\\&=t^2(37+14\varepsilon_m+14\delta_0+18\varepsilon_m\delta_0)\equiv t^2(2-3\varepsilon_m\delta_0)\\&=-t^2\ \text{or } 5t^2=-1,-4,-9,5,20,\ \text{or } 45\equiv -2,-1,\ \text{or } 3. \ \blacksquare\end{aligned}$$

Lemma 1 implies the main result of this paper.

THEOREM 2. *μ and ν are free generators of a free group acting on $\mathbb{S}^2\cap\mathbb{Q}^3$ without non-trivial fixed points.*

P r o o f. If a word w has no fixed point on $\mathbb{S}^2\cap\mathbb{Q}^3$ then $\mu^{-1}w\mu$, $\mu w\mu^{-1}$, $\nu^{-1}w\nu$, $\nu w\nu^{-1}$, and w^{-1} have no fixed point on $\mathbb{S}^2\cap\mathbb{Q}^3$. So it is sufficient to show that $\vec{s}_w$ is non-zero and $\vec{s}_w/|\vec{s}_w|$ does not belong to $\mathbb{S}^2\cap\mathbb{Q}^3$ for a non-empty reduced word, w, of the form $\mu^{\pm1}\cdots\nu^{\pm1}$ (i.e., w starts with $\mu^{\pm1}$ and ends with $\nu^{\pm1}$) or simply a power of μ or of ν. For such a non-empty reduced word w, we get $X_w^2+Y_w^2+Z_w^2\equiv -2,-1,$ or 3 from Lemma 1. But

$a^2 \equiv -3, 0, 1,$ or 2 for $a \in \mathbb{Z}$. Hence $\sqrt{X_w^2 + Y_w^2 + Z_w^2} \notin \mathbb{N}$. Therefore we obtain

$$\frac{\vec{s}_w}{|\vec{s}_w|} = \frac{1}{\sqrt{X_w^2 + Y_w^2 + Z_w^2}} \begin{pmatrix} X_w \\ Y_w \\ Z_w \end{pmatrix} \notin \mathbb{Q}^3. \ \blacksquare$$

The following problems are raised by Professor J. Mycielski.

PROBLEM A. *For $n \in \mathbb{N}$, n even, $n \geq 4$, does $SO_n(\mathbb{Q})$ have a free non-abelian subgroup F_2 such that all the elements of F_2 different from the identity have no eigenvectors in $\mathbb{Q}^n$?*

PROBLEM B. *For $n \in \mathbb{N}$, n odd, $n \geq 5$, does $SO_n(\mathbb{Q})$ have a free non-abelian subgroup F_2 which acts without fixed points on $\mathbb{S}^{n-1} \cap \mathbb{Q}^n$ and is such that if two elements $f, g \in F_2$ have a common eigenvector in $\mathbb{Q}^n$ then $fg = gf$?*

Both problems can be solved for all n except $n = 5$ provided one solves Problem A for $n = 4$ and $n = 6$. Problem A can be easily solved for $n = 4$ using Dekker's method ([Dek]), but it does not seem possible to solve Problem A for $n = 6$ using Deligne & Sullivan's method ([DelSu]).

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DEPARTMENT OF INFORMATION SCIENCE
TOKYO INSTITUTE OF TECHNOLOGY
OH-OKAYAMA, MEGURO-KU
TOKYO 152, JAPAN
E-mail: KENJI@IS.TITECH.AC.JP

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