On barycentrically soft compacta

by

T. Radul (L’viv)

Abstract. It is shown that a barycentrically soft compactum is necessarily an absolute retract of weight \( \leq \omega_1 \). Since softness of a map is the mapping version of the property of a space to be an absolute retract, the above mentioned result can be considered as mapping version of the Ditor–Haydon Theorem stating that if \( P(X) \) is an absolute retract then the compactum \( X \) is of weight \( \leq \omega_1 \) [2].

All spaces considered are assumed to be compacta (compact Hausdorff spaces). For a compactum \( X \) let \( C(X) \) be the space of all real-valued continuous functions on \( X \) metrized by sup-metric and let \( P(X) \) be the space of all non-negative functionals \( \mu : C(X) \to \mathbb{R} \) with norm 1, equipped with the weak* topology.

Recall that the base of the weak* topology in \( P(X) \) consists of the sets of the form

\[ O(\mu_0, f_1, \ldots, f_n, \varepsilon) = \{ \mu \in P(X) \mid |\mu(f_i) - \mu_0(f_i)| < \varepsilon \text{ for every } 1 \leq i \leq n \}. \]

Let \( E \) be a locally convex vector space. Then for any convex compact subset \( K \subset E \) there exists a map \( b = b_K : P(X) \to K \) which is called the barycentric map of probability measures. It is defined by \( b(\mu) = \int x \, d\mu(x) \), where \( x = \text{id}_E \). The map \( b_K \) is continuous [1]. It is not difficult to check that for \( \mu = a_1 \delta_{x_1} + \ldots + a_n \delta_{x_n} \), \( a_i \in \mathbb{R} \), \( x_i \in K \), we have \( b_K(\mu) = a_1 x_1 + \ldots + a_n x_n \), where \( \delta_{x_i} \) denotes the Dirac measure supported by \( x_i \).

A map \( f : X \to Y \) is said to be (0-)soft if for any (0-dimensional) paracompact space \( Z \), any closed subspace \( A \) of \( Z \) and maps \( \Phi : A \to X \) and \( \Psi : Z \to Y \) with \( \Psi|A = f \circ \Phi \) there exists a map \( G : Z \to X \) such that \( G|A = \Phi \) and \( \Psi = f \circ G \). This notion was introduced by E. Shchepin [9].

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A convex compactum $K$ is said to be *barycentrically soft (open)* if the barycentric map $b_K$ is soft (open). V. Fedorchuk [3] has given a criterion of barycentric openness of compacta which, in particular, implies that for every compactum $X$ the compactum $P(X)$ is barycentrically open. He has also shown in [4] that the product of a family of cardinality $\omega_1$ of barycentrically soft compacta is barycentrically soft and in the survey article [5] he has formulated the following questions concerning barycentric softness of compacta:

1) [5, Question 7.13] Is there a barycentrically soft compactum of weight $\geq \omega_2$?

It is worth noticing that the space $P(X)$ is not an absolute retract for any compactum of weight $\geq \omega_2$ [2]. So we naturally obtain:

2) [5, Question 7.14] Is every barycentrically soft compactum an absolute retract?

3) [5, Question 7.15] Is every barycentrically open AR-compactum of weight $\omega_1$ barycentrically soft?

The author has answered the last question negatively: it is shown in [8] that the barycentric softness of a compactum of the form $P(X)$ is equivalent to the metrizability of $X$.

In this paper we answer questions 1) and 2) showing that every barycentrically soft compactum must be an absolute retract of weight $\leq \omega_1$.

In the sequel we shall need some definitions and results. Let

\[
\begin{array}{c}
X_1 \rightarrow^p X_2 \\
\downarrow^{f_1} \quad \downarrow^{f_2} \\
Y_1 \rightarrow^q Y_2
\end{array}
\]

be a commutative diagram. The map $\chi : X_1 \to X_2 \times Y_2$ defined by $\chi(x) = (p(x), f_1(x))$ is called the characteristic map of this diagram. The diagram is called bicommutative (respectively open, 0-soft, soft) if the map $\chi$ is onto (respectively open, 0-soft, soft).

It is shown in [6] that softness of a map with compact convex fibers is equivalent to its 0-softness.

Let $\tau$ be an infinite cardinal number. A partially ordered set $\mathcal{A}$ is called $\tau$-complete if every subset of cardinality $\leq \tau$ has a least upper bound in $\mathcal{A}$. An inverse system of compacta and surjective bonding maps over a $\tau$-complete indexing set is called $\tau$-complete. A continuous $\tau$-complete system consisting of compacta of weight $\leq \tau$ is called a $\tau$-system.

The following theorem from [10] gives a characterization of 0-soft maps:
THEOREM A. A map \( f : X \to Y \) is 0-soft if and only if there exist \( \omega \)-systems \( S_X \) and \( S_Y \) with limits \( X \) and \( Y \) respectively and a morphism \( \{f_\alpha\} : S_X \to S_Y \) with limit \( f \) such that
1) \( f_\alpha \) is 0-soft for every \( \alpha \); 
2) every limit square diagram is 0-soft.

LEMMA 1. Let \( X \) and \( Y \) be convex compacta and let \( f : X \to Y \) be an affine non-open map. Then the diagram

\[
P(X) \xrightarrow{P(f)} P(Y) \\
\downarrow b_X \quad \downarrow b_Y \\
X \xrightarrow{f} Y
\]

is non-open.

Proof. Since \( f \) is non-open, the inverse map \( f^{-1} : Y \to \exp X \) is not continuous. (By \( \exp X \) we denote the hyperspace of \( X \), i.e., the set of non-empty closed subsets of \( X \) endowed with the Vietoris topology.)

Let \( y_0 \in Y \) be a discontinuity point of \( f^{-1} \). Then there exist a net \( \{y_\alpha\}_{\alpha \in A} \) and a neighborhood \( U \) of \( f^{-1}(y_0) \) in \( \exp X \) such that \( y_\alpha \to y_0 \) and \( f^{-1}(y_\alpha) \notin U \) for every \( \alpha \in A \). We can assume that \( f^{-1}(y_\alpha) \to A \in \exp X \).

Since \( f \) is a closed map, \( A \) is a proper convex subset of \( f^{-1}(y_0) \).

Choose points \( x_1 \in A \) and \( x_2 \in f^{-1}(y_0) \setminus A \). Replacing \( x_1 \) by \( x'_1 = (1 - \lambda)x_1 + \lambda x_2 \), with \( \lambda \in [0, 1] \) chosen to be maximal subject to \( x'_1 \in A \), we may assume that \( (x_1 + x_2)/2 \notin A \). Since \( A \) is convex, there exists an affine map \( \psi \) that strictly separates the segment \( [x_2; (x_1 + x_2)/2] \) from \( A \). We can assume that \( \psi([x_2; (x_1 + x_2)/2]) < 0 \) and \( \psi(A) > 0 \). Since \( f^{-1}(y_\alpha) \to A \), there exists \( \alpha_0 \in A \) such that \( \psi|f^{-1}(y_\alpha) > 0 \) for every \( \alpha > \alpha_0 \).

Consider a net \( \{x_\alpha\}_{\alpha > \alpha_0} \) such that \( x_\alpha \to x_1 \) and the corresponding net in \( P(X) \) defined by \( \mu_\alpha = (\delta x_\alpha + \delta x_\alpha)/2 \). Then \( \chi(\mu_\alpha) = ((x_2 + x_\alpha)/2, (\delta y_\alpha + \delta y_\alpha)/2) \to ((x_2 + x_1)/2, \delta y_0) \). Now take the measure \( \delta(x_1 + x_2)/2 \in \chi^{-1}((x_1 + x_2)/2, \delta y_0) \) and its neighborhood \( O(\delta(x_1 + x_2)/2, \phi, 1/4) \), where \( \phi \in \exp C(X) \) is a map such that \( \phi((x_1 + x_2)/2) = 1 \) and \( \phi(x) = 0 \) for every \( x \in X \) with \( \psi(x) > 0 \). Then \( \mu(\phi) = 1/2 \) for every measure \( \mu \in P(f^{-1}(\delta y_\alpha + \delta y_\alpha)/2) \), hence \( \mu \notin O(\delta(x_1 + x_2)/2, \phi, 1/4) \) and the map \( \chi \) is non-open. The lemma is proved.

A compactum \( X \) is called openly generated if \( X \) can be represented as the limit of an \( \omega \)-system with open bonding maps.

THEOREM 1. If a convex compactum \( K \) is barycentrically soft, then \( K \) is openly generated.

Proof. Present \( K \) as the limit of an \( \omega \)-system \( S_K = \{K_\alpha, p_\alpha, \mathcal{A}\} \), where the \( K_\alpha \) are convex compacta and the bonding maps \( p_\alpha \) are affine for every \( \alpha \in A \).
If $b_K : P(K) \to K$ is soft, then, using the spectral theorem of E. V. Shchepin [7, Theorem 3.12] and Theorem A, we deduce that there exists a closed cofinal subset $B \subset A$ such that for each $\alpha \in B$ the diagram

$$
\begin{array}{ccc}
P(K) & \xrightarrow{P(p_\alpha)} & P(K_\alpha) \\
\downarrow{b_K} & & \downarrow{b_{K_\alpha}} \\
K & \xrightarrow{p_\alpha} & K_\alpha
\end{array}
$$

is 0-soft and therefore open.

It follows from Lemma 1 that the map $p_\alpha$ is open for each $\alpha \in B$. But since $K = \lim \{K_\alpha, p_\alpha, B\}$, the compactum $K$ is openly generated. The theorem is proved.

**Theorem 2.** Let $K$ be a barycentrically soft compactum. Then the weight of $K$ does not exceed $\omega_1$.

**Proof.** Let $K$ be a convex barycentrically open compactum of weight $\tau > \omega_1$. Suppose that the barycentric map is soft. Then there exist an embedding $i : P(K) \to I^A \times K$ and a retraction $r : I^A \times K \to P(K)$ such that the diagram

$$
\begin{array}{ccc}
P(K) & \xrightarrow{i} & I^A \times K \\
\downarrow{b_K} & & \downarrow{b_{K_\alpha}} \\
K & \xrightarrow{pr_2} & P(K)
\end{array}
$$

is commutative.

We can assume that the cardinality of $A$ is $\tau$. The compactum $K$ is assumed to be embedded in $I^A$. Present $K$ as the limit of an $\omega_1$-system $S = \{K_\alpha, p_\alpha, A\}$, where the $K_\alpha$ are convex compacta, the $p_\alpha$ are affine maps for every $\alpha \in A$, and $A$ is the set of all subsets of cardinality $\leq \omega_1$ of $A$. Then $I^A \times K$ is the limit of the $\omega_1$-system $S' = \{I^B \times K_B, q_B, B \in A\}$ and $pr_K$ is the limit of the morphisms determined by the family $\{pr_B | B \in A\}$. Since the map $b_K$ is embedded in $pr_K$, we can assume that the restriction of the limit projections of the system $S'$ onto $P(K)$ gives a morphism of inverse systems with the limit diagrams of the form

$$
\begin{array}{ccc}
P(K) & \xrightarrow{P(p_B)} & P(K_B) \\
\downarrow{b_K} & & \downarrow{b_{K_B}} \\
K & \xrightarrow{p_B} & K_B
\end{array}
$$

By Shchepin’s Theorem (see [7]) there exists a cofinal closed subset $B \subset A$ such that for every $B \in B$ there exist an embedding $i_B : P(K_B) \to I^B \times K_B$ and a retraction $r_B : I^B \times K_B \to P(K_B)$ for which the diagrams
Barycentrically soft compacta

\[ P(K) \xrightarrow{\varrho_B} P(K_B) \]
\[ I^A \times K \xrightarrow{q_B} I^B \times K_B \]
\[ P(K) \xrightarrow{\varrho_B} P(K_B) \]

and

\[ P(K_B) \xrightarrow{i_B} I^B \times K_B \xrightarrow{r_B} P(K_B) \]
\[ b_{K_B} \]
\[ \text{pr}_2 \]

are commutative (here \( \varrho_B \) denotes \( P(p_B) \) and \( q_B \) denotes the product of the corresponding projections from \( I^A \) to \( I^B \) and from \( K \) to \( K_B \) respectively).

Now choose sets \( B, E \in \mathcal{B} \) such that \( B \cap E = C \neq \emptyset, B \setminus C \neq \emptyset \) and \( C \in \mathcal{B} \). We can do that by the method used in the proof of Theorem 3 of [2].

Let \( T = \{(k_1, k_2) \in K_B \times K_E \mid p_C^B(k_1) = p_C^E(k_2)\} \), where \( p_C^B : K_B \to K_C \) and \( p_C^E : K_E \to K_C \) are the natural projections and \( T_P = \{(\mu_1, \mu_2) \in P(K_B) \times P(K_E) \mid P(p_C^B)(\mu_1) = P(p_C^E)(\mu_2)\} \).

For every \((\mu_1, \mu_2) \in T_P \) we have \( i_B(\mu_1) \in I^B \times K_B, i_E(\mu_2) \in I^E \times K_E \), and for each \( l \in C = B \cap E \) the \( l \)-coordinates of the points \( i_B(\mu_1) \) and \( i_E(\mu_2) \) are equal. Using this fact define \( i_T : T_P \to I^{B \cup E} \times T \) by the conditions \( s_B \circ i_T(\mu_1, \mu_2) = i_B(\mu_1) \) and \( s_E \circ i_T(\mu_1, \mu_2) = i_E(\mu_2) \), where \( s_B : I^{B \cup E} \times T \to I^B \times K_B \) and \( s_E : I^{B \cup E} \times T \to I^E \times K_E \) are the natural projections.

Define \( r_T : I^{B \cup E} \times T \to T_P \) by the conditions \( r_T \circ s_B = r_B \) and \( r_T \circ s_E = r_E \). We can immediately check that \( r_T \circ i_T = \text{id}_{T_P} \).

The diagram

\[ I^A \times K \xrightarrow{q_{B \cup E} \times h} I^{B \cup E} \times T \]
\[ \text{pr}_2 \]
\[ h \]
\[ K \]
\[ T \]

where \( h = (p_B, p_E) \), is open.

But the diagram

\[ P(K) \xrightarrow{h} T_P \]
\[ b_K \]
\[ h \]
\[ K \]
\[ T \]
where \( h_P = (P(p_B), P(p_E)) \), is a retract of (1) and must be open as well. In order to finish the proof we have to show that (2) is non-open.

Consider the following three cases.

1) The map \( h \) is open. We can assume that \( p_B^C \) and \( p_E^C \) are not homeomorphisms. Then there is a point \( c_1 \in K_C \) with non-singleton fiber with respect to \( p_B^C \) and a point \( c_2 \in K_C \) with non-singleton fiber with respect to \( p_E^C \). The fibers of \( c = (c_1 + c_2)/2 \) with respect to \( p_B^C \) and \( p_E^C \) are not single points. Now choose \( b_1, b_2 \in (p_B^C)^{-1}(c) \) and \( e_1, e_2 \in (p_E^C)^{-1}(c) \). We have \( (b_i, e_j) \in T \) for every \( i, j \in \{1, 2\} \). Since \( h \) is surjective, we can choose pairwise distinct points \( k_{ij} \) such that \( p_B(k_{ij}) = p_B(k_{ij}) = b_i \) for \( i \in \{1, 2\} \) and \( p_E(k_{ij}) = p_E(k_{ij}) = e_j \) for \( j \in \{1, 2\} \).

Let \( \chi = \sum_{i,j=1}^2 \delta_{k_{ij}} + 1 \). Then

\[
\chi(\mu) = \left( \sum_{i,j=1}^2 \frac{1}{4} \delta_{k_{ij}} + \frac{1}{2} \delta_{b_i} + \frac{1}{2} \delta_{e_j} \right) = (d, \nu, \eta).
\]

Choose \( k \in K \) such that \( p_C(k) \neq c \). Let \( b = p_B(k) \) and \( e = p_E(k) \). Now put \( b^i = \frac{1}{i+1} b + \frac{i}{i+1} b_2 \) and \( e^i = \frac{1}{i+1} e + \frac{i}{i+1} e_2 \). Define the sequence of measures \( \nu^i = P(K_B) \) by \( \nu^i = \frac{1}{2} \delta_{b^i} + \frac{1}{2} \delta_{e^i} \) and the sequence of measures \( \eta^i = P(K_E) \) by \( \eta^i = \frac{1}{2} \delta_{b_i} + \frac{1}{2} \delta_{e_i} \).

It is obvious that \( \nu^i \) converges to \( \nu \) and \( \eta^i \) converges to \( \eta \). Since \( h \) is open, we can choose a sequence \( d_{ik} \) in \( K \) such that \( d_{ik} \) converges to \( d \) and \( p_B(d_{ik}) = \frac{1}{2} b^i + \frac{1}{2} b_1 \) and \( p_E(d_{ik}) = \frac{1}{2} e^i + \frac{1}{2} e_1 \). In \( K \times T \) we consider the sequence \( (d_{ik}, \nu^i, \eta^i) \) converging to \( (d, \nu, \eta) \). Choose neighborhoods \( V_1 \) and \( V_2 \) of \( e_1, e_2 \) such that \( V_1 \cap V_2 = \emptyset \) and neighborhoods \( U_1, U_2 \) of \( b_1, b_2 \) with the same property. Let \( O_{ij} = p_E^{-1}(V_j) \cap p_B^{-1}(U_i) \). There exist functions \( \phi_{ij} \in C(K) \) such that \( \phi_{ij}(k_{ij}) = 1 \) and \( \phi_{ij}(K \setminus O_{ij}) = 0 \) for every \( i, j \in \{1, 2\} \). We can assume that \( b^i \in U_2 \) and \( e^i \in V_2 \) for every \( i_k \). Let \( \mu^i \in \chi^{-1}(d_{ik}, \nu^i, \eta^i) \) for some \( i_k \). Then \( P(p_B)(\mu^i) = \frac{1}{2} \delta_{b_i} + \frac{1}{2} \delta_{b_1} \), i.e., the measure \( \mu^i \) takes on the value \( 1/2 \) on the sets \( p_B^{-1}(b^i) \) and \( p_B^{-1}(b_1) \). Since \( p_C(b^i) \neq k \), we have \( p_E^{-1}(e_1) \cap p_B^{-1}(b^i) = \emptyset \). But \( P(p_E)(\mu^i) = \frac{1}{2} \delta_{e_i} + \frac{1}{2} \delta_{e_1} \) and hence \( \mu^i \) takes on the value \( 1/2 \) on \( p_E^{-1}(e_1) \cap p_B^{-1}(b_1) \subset O_{11} \). Reasoning similarly, we can prove that \( \mu^i \) takes on the value \( 1/2 \) on \( O_{22} \). But then \( \mu^i(O_{12}) = 0 \) and hence \( \mu^i \notin O(\mu, \phi_{ij}, 1/8) \). We have shown that the diagram (2) is non-open whenever the map \( h \) is open.

2) Let \( h \) be non-open but surjective. In this case the proof is analogous to that of Lemma 1.

3) Let \( h \) be non-surjective. We show that so is \( \chi \). Let \( (e_1, b_1) \in T \) and \( (e_1, b_1) \notin h(K) \). Since \( p_B \) and \( p_E \) are surjective, we can choose \( k_B \in p_B^{-1}(b_1) \) and \( k_E \in p_E^{-1}(e_1) \). Let \( p_E(k_B) = e_2 \) and \( p_B(k_E) = b_2 \). Then \( (e_1, b_1) \in h(K) \), \( (e_2, b_1) \in h(K) \), and \( (e_2, b_2) \in T \). But then \( \frac{1}{2}(e_1, b_1) + \frac{1}{2}(e_2, b_2) = \frac{1}{2}(e_1, b_1) + \frac{1}{2}(e_2, b_2) \).
\( \frac{1}{2}(e_2, b_1) + \frac{1}{2}(e_1, b_2) \in h(K) \). We can assume that \( k = \frac{1}{4}(e_2, b_1) + \frac{3}{4}(e_1, b_1) \in h(K) \). Define \( \nu = \frac{1}{4}\delta_{b_2} + \frac{3}{4}\delta_{b_1}, \eta = \frac{1}{4}\delta_{e_2} + \frac{3}{4}\delta_{e_1} \) and choose \( z \in h^{-1}(k) \).

Then \( (z, \nu, \eta) \in K \times T P \). But since \( p_B^{-1}(b_1) \cap p_E^{-1}(e_1) = \emptyset \), we have \( \chi^{-1}(z, \nu, \eta) = \emptyset \).

Thus the diagram (2) is non-open.

Since the properties of being an AR-compactum and of being an AE(0)-compactum coincide in the class of convex compacta and each openly generated compactum of weight \( \leq \omega_1 \) is an AE(0)-compactum, the following theorem is an immediate consequence of Theorems 1 and 2.

**Theorem 3.** A barycentrically soft compactum is necessarily an AR-compactum of weight \( \leq \omega_1 \).

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**References**