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On barycentrically soft compacta

by

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Abstract. It is shown that a barycentrically soft compactum is necessarily an absolute retract of weight $\leq \omega_1$. Since softness of a map is the mapping version of the property of a space to be an absolute retract, the above mentioned result can be considered as mapping version of the Ditor–Haydon Theorem stating that if $P(X)$ is an absolute retract then the compactum X is of weight $\leq \omega_1$ [2].

All spaces considered are assumed to be compacta (compact Hausdorff spaces). For a compactum X let $C(X)$ be the space of all real-valued continuous functions on X metrized by sup-metric and let $P(X)$ be the space of all non-negative functionals $\mu : C(X) \to \mathbb{R}$ with norm 1, equipped with the weak* topology.

Recall that the base of the weak* topology in $P(X)$ consists of the sets of the form

$$O(\mu_0, f_1, \ldots, f_n, \varepsilon) = \{\mu \in P(X) \mid |\mu(f_i) - \mu_0(f_i)| < \varepsilon \text{ for every } 1 \leq i \leq n\}.$$

Let E be a locally convex vector space. Then for any convex compact subset $K \subset E$ there exists a map $b = b_K : P(X) \to K$ which is called the *barycentric map of probability measures*. It is defined by $b(\mu) = \int x\, d\mu(x)$, where $x = \mathrm{id}_E$. The map b_K is continuous [1]. It is not difficult to check that for $\mu = a_1\delta_{x_1} + \ldots + a_n\delta_{x_n}$, $a_i \in \mathbb{R}$, $x_i \in K$, we have $b_K(\mu) = a_1 x_1 + \ldots + a_n x_n$, where δ_{x_i} denotes the Dirac measure supported by x_i.

A map $f : X \to Y$ is said to be (0-)*soft* if for any (0-dimensional) paracompact space Z, any closed subspace A of Z and maps $\Phi : A \to X$ and $\Psi : Z \to Y$ with $\Psi|A = f \circ \Phi$ there exists a map $G : Z \to X$ such that $G|A = \Phi$ and $\Psi = f \circ G$. This notion was introduced by E. Shchepin [9].

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A convex compactum K is said to be *barycentrically soft* (*open*) if the barycentric map b_K is soft (open). V. Fedorchuk [3] has given a criterion of barycentric openness of compacta which, in particular, implies that for every compactum X the compactum $P(X)$ is barycentrically open. He has also shown in [4] that the product of a family of cardinality ω_1 of barycentrically soft compacta is barycentrically soft and in the survey article [5] he has formulated the following questions concerning barycentric softness of compacta:

1) [5, Question 7.13] Is there a barycentrically soft compactum of weight $\geq \omega_2$?

It is worth noticing that the space $P(X)$ is not an absolute retract for any compactum of weight $\geq \omega_2$ [2]. So we naturally obtain:

2) [5, Question 7.14] Is every barycentrically soft compactum an absolute retract?

3) [5, Question 7.15] Is every barycentrically open AR-compactum of weight ω_1 barycentrically soft?

The author has answered the last question negatively: it is shown in [8] that the barycentric softness of a compactum of the form $P(X)$ is equivalent to the metrizability of X.

In this paper we answer questions 1) and 2) showing that every barycentrically soft compactum must be an absolute retract of weight $\leq \omega_1$.

In the sequel we shall need some definitions and results. Let

$$\begin{array}{ccc} X_1 & \xrightarrow{p} & X_2 \\ \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} \\ Y_1 & \xrightarrow{q} & Y_2 \end{array}$$

be a commutative diagram. The map $\chi : X_1 \to X_2 \times_{Y_2} Y_1 = \{(x, y) \in X_2 \times Y_1 \mid f_2(x) = q(y)\}$ defined by $\chi(x) = (p(x), f_1(x))$ is called the *characteristic map* of this diagram. The diagram is called *bicommutative* (respectively *open*, 0-*soft*, *soft*) if the map χ is onto (respectively open, 0-soft, soft).

It is shown in [6] that softness of a map with compact convex fibers is equivalent to its 0-softness.

Let τ be an infinite cardinal number. A partially ordered set $\mathcal{A}$ is called τ-*complete* if every subset of cardinality $\leq \tau$ has a least upper bound in $\mathcal{A}$. An inverse system of compacta and surjective bonding maps over a τ-*complete* indexing set is called τ-*complete*. A continuous τ-complete system consisting of compacta of weight $\leq \tau$ is called a τ-*system*.

The following theorem from [10] gives a characterization of 0-soft maps:

THEOREM A. *A map $f : X \to Y$ is* 0*-soft if and only if there exist ω-systems S_X and S_Y with limits X and Y respectively and a morphism $\{f_\alpha\} : S_X \to S_Y$ with limit f such that*

1) *f_α is* 0*-soft for every α;*
2) *every limit square diagram is* 0*-soft.*

LEMMA 1. *Let X and Y be convex compacta and let $f : X \to Y$ be an affine non-open map. Then the diagram*

$$\begin{array}{ccc} P(X) & \xrightarrow{P(f)} & P(Y) \\ \downarrow{\scriptstyle b_X} & & \downarrow{\scriptstyle b_Y} \\ X & \xrightarrow{f} & Y \end{array}$$

is non-open.

Proof. Since f is non-open, the inverse map $f^{-1} : Y \to \exp X$ is not continuous. (By $\exp X$ we denote the *hyperspace of* X, i.e., the set of non-empty closed subsets of X endowed with the Vietoris topology.)

Let $y_0 \in Y$ be a discontinuity point of f^{-1}. Then there exist a net $\{y_\alpha\}_{\alpha\in\mathcal{A}}$ and a neighborhood U of $f^{-1}(y_0)$ in $\exp X$ such that $y_\alpha \to y_0$ and $f^{-1}(y_\alpha) \notin U$ for every $\alpha \in \mathcal{A}$. We can assume that $f^{-1}(y_\alpha) \to A \in \exp X$. Since f is a closed map, A is a proper convex subset of $f^{-1}(y_0)$.

Choose points $x_1 \in A$ and $x_2 \in f^{-1}(y_0) \setminus A$. Replacing x_1 by $x_1' = (1-\lambda)x_1 + \lambda x_2$, with $\lambda \in [0,1)$ chosen to be maximal subject to $x_1' \in A$, we may assume that $(x_1+x_2)/2 \notin A$. Since A is convex, there exists an affine map ψ that strictly separates the segment $[x_2;(x_1+x_2)/2]$ from A. We can assume that $\psi([x_2;(x_1+x_2)/2]) < 0$ and $\psi(A) > 0$. Since $f^{-1}(y_\alpha) \to A$, there exists $\alpha_0 \in \mathcal{A}$ such that $\psi|f^{-1}(y_\alpha) > 0$ for every $\alpha > \alpha_0$. Consider a net $\{x_\alpha\}_{\alpha>\alpha_0}$ such that $x_\alpha \to x_1$ and the corresponding net in $P(X)$ defined by $\mu_\alpha = (\delta_{x_2} + \delta_{x_\alpha})/2$. Then $\chi(\mu_\alpha) = ((x_2+x_\alpha)/2, (\delta_{y_0}+\delta_{y_\alpha})/2) \to ((x_2+x_1)/2, \delta_{y_0})$. Now take the measure $\delta_{(x_1+x_2)/2} \in \chi^{-1}((x_1+x_2)/2, \delta_{y_0})$ and its neighborhood $O(\delta_{(x_1+x_2)/2}, \phi, 1/4)$, where $\phi \in C(X)$ is a map such that $\phi((x_1+x_2)/2) = 1$ and $\phi(x) = 0$ for every $x \in X$ with $\psi(x) \geq 0$. Then $\mu(\phi) = 1/2$ for every measure $\mu \in P(f)^{-1}((\delta_{y_0} + \delta_{y_\alpha})/2)$, hence $\mu \notin O(\delta_{(x_1+x_2)/2}, \phi, 1/4)$ and the map χ is non-open. The lemma is proved.

A compactum X is called *openly generated* if X can be represented as the limit of an ω-system with open bonding maps.

THEOREM 1. *If a convex compactum K is barycentrically soft*, *then K is openly generated.*

Proof. Present K as the limit of an ω-system $S_K = \{K_\alpha, p_\alpha, \mathcal{A}\}$, where the K_α are convex compacta and the bonding maps p_α are affine for every $\alpha \in \mathcal{A}$.

If $b_K : P(K) \to K$ is soft, then, using the spectral theorem of E. V. Shchepin [7, Theorem 3.12] and Theorem A, we deduce that there exists a closed cofinal subset $\mathcal{B} \subset \mathcal{A}$ such that for each $\alpha \in \mathcal{B}$ the diagram

$$\begin{array}{ccc} P(K) & \xrightarrow{P(p_\alpha)} & P(K_\alpha) \\ \downarrow{\scriptstyle b_K} & & \downarrow{\scriptstyle b_{K_\alpha}} \\ K & \xrightarrow{p_\alpha} & K_\alpha \end{array}$$

is 0-soft and therefore open.

It follows from Lemma 1 that the map p_α is open for each $\alpha \in \mathcal{B}$. But since $K = \lim\{K_\alpha, p_\alpha, \mathcal{B}\}$, the compactum K is openly generated. The theorem is proved.

THEOREM 2. *Let K be a barycentrically soft compactum. Then the weight of K does not exceed ω_1.*

Proof. Let K be a convex barycentrically open compactum of weight $\tau > \omega_1$. Suppose that the barycentric map is soft. Then there exist an embedding $i : P(K) \to I^A \times K$ and a retraction $r : I^A \times K \to P(K)$ such that the diagram

$$\begin{array}{ccccc} P(K) & \xrightarrow{i} & I^A \times K & \xrightarrow{r} & P(K) \\ & {\scriptstyle b_K}\searrow & \downarrow{\scriptstyle \mathrm{pr}_2} & \swarrow{\scriptstyle b_K} & \\ & & K & & \end{array}$$

is commutative.

We can assume that the cardinality of A is τ. The compactum K is assumed to be embedded in I^A. Present K as the limit of an ω_1-system $S = \{K_\alpha, p_\alpha, \mathcal{A}\}$, where the K_α are convex compacta, the p_α are affine maps for every $\alpha \in \mathcal{A}$, and $\mathcal{A}$ is the set of all subsets of cardinality $\le \omega_1$ of A. Then $I^A \times K$ is the limit of the ω_1-system $S' = \{I^B \times K_B, q_B, B \in \mathcal{A}\}$ and pr_K is the limit of the morphisms determined by the family $\{\mathrm{pr}_B \mid B \in \mathcal{A}\}$. Since the map b_K is embedded in pr_K, we can assume that the restriction of the limit projections of the system S' onto $P(K)$ gives a morphism of inverse systems with the limit diagrams of the form

$$\begin{array}{ccc} P(K) & \xrightarrow{P(p_B)} & P(K_B) \\ \downarrow{\scriptstyle b_K} & & \downarrow{\scriptstyle b_{K_B}} \\ K & \xrightarrow{p_B} & K_B \end{array}$$

By Shchepin's Theorem (see [7]) there exists a cofinal closed subset $\mathcal{B} \subset \mathcal{A}$ such that for every $B \in \mathcal{B}$ there exist an embedding $i_B : P(K_B) \to I^B \times K_B$ and a retraction $r_B : I^B \times K_B \to P(K_B)$ for which the diagrams

$$\begin{array}{ccc} P(K) & \xrightarrow{\varrho_B} & P(K_B) \\ \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle i_B} \\ I^A \times K & \xrightarrow{q_B} & I^B \times K_B \\ \downarrow{\scriptstyle r} & & \downarrow{\scriptstyle r_B} \\ P(K) & \xrightarrow{\varrho_B} & P(K_B) \end{array}$$

and

$$\begin{array}{ccccc} P(K_B) & \xrightarrow{i_B} & I^B \times K_B & \xrightarrow{r_B} & P(K_B) \\ & {\scriptstyle b_{K_B}}\searrow & \downarrow{\scriptstyle \mathrm{pr}_2} & \swarrow{\scriptstyle b_{K_B}} & \\ & & K_B & & \end{array}$$

are commutative (here ϱ_B denotes $P(p_B)$ and q_B denotes the product of the corresponding projections from I^A to I^B and from K to K_B respectively).

Now choose sets $B, E \in \mathcal{B}$ such that $B \cap E = C \neq \emptyset$, $B \setminus C \neq \emptyset$ and $C \in \mathcal{B}$. We can do that by the method used in the proof of Theorem 3 of [2].

Let $T = \{(k_1, k_2) \in K_B \times K_E \mid p_B^C(k_1) = p_E^C(k_2)\}$, where $p_B^C : K_B \to K_C$ and $p_E^C : K_E \to K_C$ are the natural projections and $T_P = \{(\mu_1, \mu_2) \in P(K_B) \times P(K_E) \mid P(p_B^C)(\mu_1) = P(p_E^C)(\mu_2)\}$.

For every $(\mu_1, \mu_2) \in T_P$ we have $i_B(\mu_1) \in I^B \times K_B$, $i_E(\mu_2) \in I^E \times K_E$, and for each $l \in C = B \cap E$ the l-coordinates of the points $i_B(\mu_1)$ and $i_E(\mu_2)$ are equal. Using this fact define $i_T : T_P \to I^{B\cup E} \times T$ by the conditions $s_B \circ i_T(\mu_1, \mu_2) = i_B(\mu_1)$ and $s_E \circ i_T(\mu_1, \mu_2) = i_E(\mu_2)$, where $s_B : I^{B\cup E} \times T \to I^B \times K_B$ and $s_E : I^{B\cup E} \times T \to I^E \times K_E$ are the natural projections.

Define $r_T : I^{B\cup E} \times T \to T_P$ by the conditions $r_T \circ s_B = r_B$ and $r_T \circ s_E = r_E$. We can immediately check that $r_T \circ i_T = \mathrm{id}_{T_P}$.

The diagram

$$\begin{array}{ccc} I^A \times K & \xrightarrow{q_{B\cup E} \times h} & I^{B\cup E} \times T \\ \downarrow{\scriptstyle \mathrm{pr}_2} & & \downarrow{\scriptstyle \mathrm{pr}_2} \\ K & \xrightarrow{h} & T \end{array} \tag{1}$$

where $h = (p_B, p_E)$, is open.

But the diagram

$$\begin{array}{ccc} P(K) & \xrightarrow{h_P} & T_P \\ \downarrow{\scriptstyle b_K} & & \downarrow{\scriptstyle b_{K_B} \times b_{K_E}} \\ K & \xrightarrow{h} & T \end{array} \tag{2}$$

where $h_P = (P(p_B), P(p_E))$, is a retract of (1) and must be open as well. In order to finish the proof we have to show that (2) is non-open.

Consider the following three cases.

1) The map h is open. We can assume that p_B^C and p_E^C are not homeomorphisms. Then there is a point $c_1 \in K_C$ with non-singleton fiber with respect to p_B^C and a point $c_2 \in K_C$ with non-singleton fiber with respect to p_E^C. The fibers of $c = (c_1 + c_2)/2$ with respect to p_B^C and p_E^C are not single points. Now choose $b_1, b_2 \in (p_B^C)^{-1}(c)$ and $e_1, e_2 \in (p_E^C)^{-1}(c)$. We have $(b_i, e_j) \in T$ for every $i, j \in \{1, 2\}$. Since h is surjective, we can choose pairwise distinct points $k_{11}, k_{12}, k_{21}, k_{22}$ such that $p_B(k_{i1}) = p_B(k_{i2}) = b_i$ for $i \in \{1, 2\}$ and $p_E(k_{1j}) = p_E(k_{2j}) = e_j$ for $j \in \{1, 2\}$.

Let $\mu = \sum_{i,j=1}^2 \frac{1}{4}\delta_{k_{ij}}$. Then

$$\chi(\mu) = \left(\sum_{i,j=1}^{2} \frac{1}{4}k_{ij}, \frac{1}{2}\delta_{b_1} + \frac{1}{2}\delta_{b_2}, \frac{1}{2}\delta_{e_1} + \frac{1}{2}\delta_{e_2}\right) = (d, \nu, \eta).$$

Choose $k \in K$ such that $p_C(k) \neq c$. Let $b = p_B(k)$ and $e = p_E(k)$. Now put $b^i = \frac{1}{i+1}b + \frac{i}{i+1}b_2$ and $e^i = \frac{1}{i+1}e + \frac{i}{i+1}e_2$. Define the sequence of measures $\nu^i \in P(K_B)$ by $\nu^i = \frac{1}{2}\delta_{b^i} + \frac{1}{2}\delta_{b_1}$ and the sequence of measures $\eta^i \in P(K_E)$ by $\eta^i = \frac{1}{2}\delta_{e^i} + \frac{1}{2}\delta_{e_1}$.

It is obvious that ν^i converges to ν and η^i converges to η. Since h is open, we can choose a sequence d_{i_k} in K such that d_{i_k} converges to d and $p_B(d_{i_k}) = \frac{1}{2}b^{i_k} + \frac{1}{2}b_1$, $p_E(d_{i_k}) = \frac{1}{2}e^{i_k} + \frac{1}{2}e_1$. In $K \times_T T_P$ consider the sequence $(d_{i_k}, \nu^{i_k}, \eta^{i_k})$ converging to (d, ν, η). Choose neighborhoods V_1 and V_2 of e_1, e_2 such that $\operatorname{cl} V_1 \cap \operatorname{cl} V_2 = \emptyset$ and neighborhoods U_1, U_2 of b_1, b_2 with the same property. Let $O_{ij} = p_E^{-1}(V_j) \cap p_B^{-1}(U_i)$. There exist functions $\phi_{ij} \in C(K)$ such that $\phi_{ij}(k_{ij}) = 1$ and $\phi_{ij}|(K \setminus O_{ij}) = 0$ for every $i, j \in \{1, 2\}$. We can assume that $b^{i_k} \in U_2$ and $e^{i_k} \in V_2$ for every i_k. Let $\mu' \in \chi^{-1}(d_{i_l}, \nu^{i_l}, \eta^{i_l})$ for some i_l. Then $P(p_B)(\mu') = \frac{1}{2}\delta_{b^{i_l}} + \frac{1}{2}\delta_{b_1}$, i.e., the measure μ' takes on the value $1/2$ on the sets $p_B^{-1}(b^{i_l})$ and $p_B^{-1}(b_1)$. Since $p_B^C(b^{i_l}) \neq k$, we have $p_E^{-1}(e_1) \cap p_B^{-1}(b^{i_l}) = \emptyset$. But $P(p_E)(\mu') = \frac{1}{2}\delta_{e^{i_l}} + \frac{1}{2}\delta_{e_1}$ and hence μ' takes on the value $1/2$ on $p_E^{-1}(e_1) \cap p_B^{-1}(b_1) \subset O_{11}$. Reasoning similarly, we can prove that μ' takes on the value $1/2$ on O_{22}. But then $\mu'(O_{12}) = 0$ and hence $\mu' \notin O(\mu, \phi_{ij}, 1/8)$. We have shown that the diagram (2) is non-open whenever the map h is open.

2) Let h be non-open but surjective. In this case the proof is analogous to that of Lemma 1.

3) Let h be non-surjective. We show that so is χ. Let $(e_1, b_1) \in T$ and $(e_1, b_1) \notin h(K)$. Since p_B and p_E are surjective, we can choose $k_B \in p_B^{-1}(b_1)$ and $k_E \in p_E^{-1}(e_1)$. Let $p_E(k_B) = e_2$ and $p_B(k_E) = b_2$. Then $(e_1, b_2) \in h(K)$, $(e_2, b_1) \in h(K)$, and $(e_2, b_2) \in T$. But then $\frac{1}{2}(e_1, b_1) + \frac{1}{2}(e_2, b_2) =$

$\frac{1}{2}(e_2, b_1) + \frac{1}{2}(e_1, b_2) \in h(K)$. We can assume that $k = \frac{1}{4}(e_2, b_2) + \frac{3}{4}(e_1, b_1) \in h(K)$. Define $\nu = \frac{1}{4}\delta_{b_2} + \frac{3}{4}\delta_{b_1}$, $\eta = \frac{1}{4}\delta_{e_2} + \frac{3}{4}\delta_{e_1}$ and choose $z \in h^{-1}(k)$. Then $(z, \nu, \eta) \in K \times_T T_P$. But since $p_B^{-1}(b_1) \cap p_E^{-1}(e_1) = \emptyset$, we have $\chi^{-1}(z, \nu, \eta) = \emptyset$.

Thus the diagram (2) is non-open.

Since the properties of being an AR-compactum and of being an AE(0)-compactum coincide in the class of convex compacta and each openly generated compactum of weight $\le \omega_1$ is an AE(0)-compactum, the following theorem is an immediate consequence of Theorems 1 and 2.

THEOREM 3. *A barycentrically soft compactum is necessarily an AR-compactum of weight* $\le \omega_1$.

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