FUNDAMENTA
MATHEMATICAE
148 (1995)

Strolling through paradise

by

Jörg Brendle (Tübingen)

Abstract. With each of the classical tree-like forcings adjoining a new real, one can associate a σ-ideal on the reals in a natural way. For example, the ideal s^0 of Marczewski null sets corresponds to Sacks forcing $\mathbb{S}$, while the ideal r^0 of nowhere Ramsey sets corresponds to Mathias forcing $\mathbb{R}$. We show (in ZFC) that none of these ideals is included in any of the others. We also discuss Mycielski's ideal $\mathfrak{P}_2$, and start an investigation of the covering numbers of these ideals.

Introduction. In 1935, E. Marczewski [Mar] introduced on the reals the σ-ideal s^0, consisting of sets $X \subseteq 2^\omega$ so that for all perfect trees $T \subseteq 2^{<\omega}$ there is a perfect subtree $S \subseteq T$ with $[S] \cap X = \emptyset$, where $[S] := \{f \in \omega^\omega : \forall n\ (f{\restriction}n \in S)\}$ denotes the set of branches through S (see [JMS], [Mi2], [Ve] and others for recent results on s^0). Similarly a set $X \subseteq 2^\omega$ is called *s-measurable* iff for all perfect trees T there is a perfect subtree $S \subseteq T$ with either $[S] \cap X = \emptyset$ or $[S] \subseteq X$. Once forcing was born, the algebra of s-measurable sets modulo s^0-sets turned out to be of great interest; it was first studied by G. Sacks [Sa], and henceforth became known as *Sacks* (or *perfect set*) *forcing* $\mathbb{S}$. Since then, many Sacks-like partial orders have been investigated (e.g., Mathias forcing $\mathbb{R}$, Laver forcing $\mathbb{L}$, Miller forcing $\mathbb{M}$ etc.—see §1 for the definitions), and it is natural to ask how the corresponding σ-ideals (i.e. r^0, ℓ^0, m^0, respectively) look like. We note that the ideal r^0 of *Ramsey null* (or *nowhere Ramsey*) *sets* was first considered by Galvin and Prikry [GP], and has found much attention over the years (see [AFP], [Br], [Co], [Ma2], [Pl] and others), while ℓ^0 and m^0 were looked at only recently in work of Goldstern, Johnson, Repický, Shelah and Spinas (see [GJS] and [GRSS]).

1991 *Mathematics Subject Classification*: 03E05, 03E35, 03E40.
Key words and phrases: σ-ideal, Marczewski's ideal, nowhere Ramsey sets, Mycielski's ideal; Sacks forcing, Miller forcing, Laver forcing, Matet forcing, Silver forcing, Mathias forcing; covering number.
Research supported by the MINERVA foundation and by DFG grant Nr. Br 1420/1-1.

One of the fundamental questions one may ask about such ideals is whether an inclusion relation holds between any two of them. We shall show, in Sections 1 and 2 of the present work, that this is not the case by constructing in ZFC a set $X \in i^0 \setminus j^0$ for each pair (i^0, j^0) of such ideals. In case of (s^0, r^0), this was done previously under some additional set-theoretic assumptions by Aniszczyk, Frankiewicz, Plewik, Brown and Corazza (see [AFP], [Br] and [Co]), and our result answers questions of the latter [Co, Problems 6 and 10]. In case of (m^0, ℓ^0), this answers a question of O. Spinas (private communication).

These results bear some resemblance to the fact that the ideals corresponding to Cohen forcing $\mathbb{C}$ and random forcing $\mathbb{B}$, the meager sets $\mathcal{M}$ and the null sets $\mathcal{N}$, are not included in each other. There is even $A \subseteq 2^\omega$ with $A \in \mathcal{M}$ and $2^\omega \setminus A \in \mathcal{N}$. Two ideals with this property are called *orthogonal*. We also investigate the question which pairs of the ideals considered in our work are orthogonal and which are not.

Closely related to these ideals is one of the ideals introduced by J. Mycielski [My], the σ-ideal $\mathfrak{P}_2$, consisting of sets $X \subseteq 2^\omega$ so that for all infinite $A \subseteq \omega$ the restriction $X \restriction A := \{f \restriction A : f \in X\}$ is a proper subset of the restriction 2^A of the whole space. $\mathfrak{P}_2$ is easily seen to be included in the ideal v^0 of *Silver null sets* (corresponding to *Silver forcing* $\mathbb{V}$); we extend the work of Section 2 by showing that it is not included in any of the other previously considered ideals (Theorem 3.1).

Given an ideal $\mathcal{I}$ on the reals, let $\mathrm{cov}(\mathcal{I})$ be the size of the smallest $\mathcal{F} \subseteq \mathcal{I}$ covering the reals (i.e. satisfying $\forall f \in 2^\omega\ \exists F \in \mathcal{F}\ (f \in F)$). We shall prove (Theorem 3.3) that v^0 may be large in comparison with $\mathfrak{P}_2$ by showing the consistency of $\omega_1 = \mathrm{cov}(v^0) < \mathrm{cov}(\mathfrak{P}_2) = \omega_2 = \mathfrak{c}$. This answers a question addressed by Cichoń, Rosłanowski, Steprāns and Węglorz [CRSW, Question 1.3].

We will conclude our considerations with some remarks concerning the ideal $r^0_{\mathcal{U}}$ of *Ramsey null sets* with respect to a Ramsey ultrafilter $\mathcal{U}$ in Section 4. In particular, we shall relate the size of the smallest set not in $r^0_{\mathcal{U}}$ to the size of the smallest base of $\mathcal{U}$ and to a partition cardinal introduced by Blass [Bl3, Section 6].

Notation. Our set-theoretic notation is fairly standard (see [Je1] or [Ku]). $\mathfrak{c}$ denotes the cardinality of the continuum. Given two sets A, B, we say that A is almost included in B ($A \subseteq^* B$) iff $A \setminus B$ is finite. $\star$ is used for two-step iterations; we refer to [Bau], [Je 2] and [Sh] for iterated forcing constructions with countable support.

$\omega^{\uparrow\omega}$ is the space of strictly increasing functions from ω to ω, while $\omega^{\uparrow<\omega}$ is the set of strictly increasing finite sequences of natural numbers. For a finite sequence σ (i.e. $\sigma \in 2^{<\omega}, \omega^{<\omega}, \omega^{\uparrow<\omega}, \ldots$), we let $|\sigma| = \mathrm{dom}(\sigma)$, the size (or domain) of σ, and $\mathrm{rng}(\sigma)$, the range of σ. $\hat{}$ is used for concatenation of sequences; and $\langle\rangle$ stands for the empty sequence. The set of binary sequences of length n is lexicographically ordered as $\langle s_i : i < 2^n\rangle$ by $i < j \Leftrightarrow s_i(|s|) < s_j(|s|)$ where $s \subseteq s_i, s_j$.

A tree $T \subseteq \omega^{<\omega}$ is *perfect* iff given $s \in T$, there are $n \neq m$ and $t \supseteq s$ with $t\hat{}\langle n\rangle, t\hat{}\langle m\rangle \in T$; T is *superperfect* iff given $s \in T$, there are $t \supseteq s$ and infinitely many n with $t\hat{}\langle n\rangle \in T$. $\mathrm{split}(T) := \{s \in T : |\{n \in \omega : s\hat{}\langle n\rangle \in T\}| \geq 2\}$ is the set of split-nodes of T. Given $s \in T$, $\mathrm{succ}_T(s) := \{n \in \omega : s\hat{}\langle n\rangle \in T\}$ denotes the immediate successors of s in T, while $\mathrm{Succ}_T(s) := \{t \in T : s \subset t \in \mathrm{split}(T) \wedge t{\restriction}n \notin \mathrm{split}(T)$ for $|s| < n < |t|\}$ denotes the successor split-nodes of s in T. Finally, for $s \in T$, $T_s := \{t \in T : s \subseteq t \vee t \subseteq s\}$ is the restriction of T to s.

Acknowledgments. I thank Otmar Spinas for asking me the question which eventually led to this work. I am also grateful to the referee for many suggestions which improved the presentation of this paper.

1. Preliminaries

1.1. We will consider the following forcing notions.

• *Sacks forcing* $\mathbb{S}$ [Je2, Part 1, Section 3], also called perfect set forcing:

$$T \in \mathbb{S} \Leftrightarrow T \text{ is a perfect tree on } 2^{<\omega},$$
$$T \leq S \Leftrightarrow T \subseteq S.$$

• *Miller forcing* $\mathbb{M}$ [Mi1] (or superperfect tree forcing or rational perfect set forcing):

$$T \in \mathbb{M} \Leftrightarrow T \text{ is a superperfect tree on } \omega^{<\omega},$$
$$T \leq S \Leftrightarrow T \subseteq S.$$

We note that the conditions in $\mathbb{M}$ all of whose nodes have either infinitely many successor nodes or exactly one successor node are dense in $\mathbb{M}$, and we henceforth restrict our attention to such conditions.

• *Laver forcing* $\mathbb{L}$ [Je2, Part 1, Section 3]:

$$T \in \mathbb{L} \Leftrightarrow T \subseteq \omega^{<\omega} \text{ is a tree and } \forall \tau \in T\ (\mathrm{stem}(T) \subseteq \tau \rightarrow \exists^\infty n\ (\tau\hat{}\langle n\rangle \in T)),$$
$$T \leq S \Leftrightarrow T \subseteq S.$$

• *Willow tree forcing* $\mathbb{W}$ (see the end of 1.2 for the reason for introducing this forcing):

$$(f, A) \in \mathbb{W} \Leftrightarrow A \subseteq [\omega]^{<\omega} \text{ is infinite and consists of pairwise disjoint sets}$$
$$\wedge\ \mathrm{dom}(f) = \omega \setminus \bigcup A \wedge \mathrm{rng}(f) \subseteq 2,$$
$$(f, A) \leq (g, B) \Leftrightarrow f \supseteq g \wedge \forall a \in A\ \exists B' \subseteq B\ (a = \bigcup B')$$
$$\wedge\ \forall b \in B\ (b \subseteq \mathrm{dom}(f) \rightarrow f{\restriction}b \text{ is constant}).$$

In this p.o. conditions of the form (f, A), where $A = \{a_n : n \in \omega\}$ and $\max(a_n) < \min(a_{n+1})$, are dense, and we shall always work with such conditions.

• *Matet forcing* $\mathbb{T}$ [Ma1, Section 6] (see also [Bl1, Section 4] and [Bl2, Section 5]):

$$(s, A) \in \mathbb{T} \Leftrightarrow s \in \omega^{\uparrow<\omega} \wedge A \subseteq [\omega]^{<\omega} \text{ is infinite} \\ \wedge \forall a \in A\ (\max \operatorname{rng}(s) < \min(a)),$$

$$(s, A) \leq (t, B) \Leftrightarrow s \supseteq t \wedge \forall a \in A\ \exists B' \subseteq B\ (a = \bigcup B') \\ \wedge \exists B' \subseteq B(\operatorname{rng}(s) \setminus \operatorname{rng}(t) = \bigcup B').$$

Again, we may restrict our attention to conditions (s, A) with second coordinate $A = \{a_n : n \in \omega\}$ satisfying $\max(a_n) < \min(a_{n+1})$.

• *Silver forcing* $\mathbb{V}$ [Je 2, Part 1, Section 3]:

$$f \in \mathbb{V} \Leftrightarrow \operatorname{dom}(f) \subseteq \omega \text{ is coinfinite} \wedge \operatorname{rng}(f) \subseteq 2,$$

$$f \leq g \Leftrightarrow f \supseteq g.$$

• *Mathias forcing* $\mathbb{R}$ [Je2, Part 1, Section 3]:

$$(s, A) \in \mathbb{R} \Leftrightarrow s \in \omega^{\uparrow<\omega} \wedge A \subseteq \omega \text{ is infinite} \wedge \max \operatorname{rng}(s) < \min(A),$$

$$(s, A) \leq (t, B) \Leftrightarrow s \supseteq t \wedge A \subseteq B \wedge \operatorname{rng}(s) \setminus \operatorname{rng}(t) \subseteq B.$$

1.2. We note that these forcings are defined on different underlying sets, e.g. elements of Sacks forcing are subsets of $2^{<\omega}$, elements of Laver forcing are subsets of $\omega^{<\omega}$, and Mathias forcing consists of elements of $\omega^{\uparrow<\omega} \times [\omega]^\omega$. For our purposes we need, however, that all these forcings act on the same space, and we choose $\omega^{\uparrow\omega}$ to be this space; i.e. we shall think of each of the forcings as adding a new strictly increasing function from ω to ω.

To be more explicit, note first that Miller forcing is forcing-equivalent to

$$\mathbb{M}' = \{T \subseteq \omega^{\uparrow<\omega} : T \text{ is superperfect}\}.$$

Henceforth, when talking about Miller forcing, we shall mean the latter p.o. A similar remark applies to Laver forcing.

Next we remark that Mathias and Matet forcings are just uniform versions of Laver and Miller forcings, respectively, whereas both $\mathbb{W}$ and $\mathbb{V}$ are uniform versions of $\mathbb{S}$ ($\mathbb{V}$ even being a uniform version of $\mathbb{W}$). Namely, call a Laver tree T *uniform* iff there is $A_T \in [\omega]^\omega$ so that for all $\sigma \in T$ extending $\operatorname{stem}(T)$, we have $\operatorname{succ}_T(\sigma) = A_T \setminus (\sigma(|\sigma| - 1) + 1)$. Then we can identify $\mathbb{R}$ and $\mathbb{R}' := \{T \in \mathbb{L} : T \text{ is uniform}\}$: uniform trees $T \in \mathbb{R}'$ correspond to the pairs $(\operatorname{stem}(T), A_T) \in \mathbb{R}$. A similar argument works for the other forcings.

Finally, define $F : 2^{<\omega} \to \omega^{\uparrow<\omega}$ by

$$F(\sigma) := \text{the increasing enumeration of } \sigma^{-1}(\{1\}).$$

Then F extends to a map $\widehat{F} : 2^\omega \setminus \{f : |f^{-1}(\{1\})| < \omega\} \to \omega^{\uparrow\omega}$ defined by

$$\widehat{F}(f) := \bigcup\{F(f\restriction n) : n \in \omega\}.$$

$\widehat{F}$ is easily seen to be a homeomorphism. Now, given a Sacks tree $S \in \mathbb{S}$, we let

$$\widetilde{F}(S) := \{F(\sigma) : \sigma \in S\}.$$

Then $\widetilde{F}(S)$ is a perfect subtree of $\omega^{\uparrow<\omega}$; it is compact iff $\forall f \in [S]$ $(|f^{-1}(\{1\})| = \omega)$ [which we can assume, the set of such conditions being dense in $\mathbb{S}$]. By a further pruning argument, we may assume all $\widetilde{F}(S)$ are two-branching; i.e. $\forall \sigma \in \widetilde{F}(S)$ $(|\mathrm{succ}_{\widetilde{F}(S)}(\sigma)| \leq 2)$. Thus the copy of Sacks forcing on $\omega^{\uparrow\omega}$ looks exactly like the original Sacks forcing. Henceforth, when talking about Sacks forcing, we shall mean the p.o. $\{\widetilde{F}(S) : S \in \mathbb{S}\}$. Furthermore, we see that every Miller tree is a Sacks tree (more explicitly, it is of the form $\widetilde{F}(S)$ for some $S \in \mathbb{S}$). Using this (and similar remarks applied to $\mathbb{V}$ and $\mathbb{W}$) we see that the following inclusion relations between the p.o.s under consideration hold.

$$\begin{array}{ccccc} \mathbb{S} & \supset & \mathbb{M} & \supset & \mathbb{L} \\ \cup & & \cup & & \\ \mathbb{W} & \supset & \mathbb{T} & & \cup \\ \cup & & & \rotatebox{-45}{$\supset$} & \\ \mathbb{V} & & \supset & & \mathbb{R} \end{array}$$

We realize at this point that $\mathbb{W}$ arises in a natural way. The relation between $\mathbb{S}$ and $\mathbb{W}$ is like the one between $\mathbb{M}$ and $\mathbb{T}$, while the pair $(\mathbb{S}, \mathbb{V})$ corresponds to the pair $(\mathbb{L}, \mathbb{R})$.

We close this subsection with yet another remark concerning the uniform forcings. Let $\mathbb{P} := \mathcal{P}(\omega)/[\omega]^{<\omega}$, ordered by almost inclusion, i.e.

$$A \leq B \Leftrightarrow A \subseteq^* B;$$

and put $\mathbb{Q} := \mathcal{P}([\omega]^{<\omega})/[[\omega]^{<\omega}]^{<\omega}$, ordered by

$$A \leq B \Leftrightarrow \forall^\infty a \in A\ \exists B' \subseteq B\ (a = \bigcup B').$$

Both $\mathbb{P}$ and $\mathbb{Q}$ are σ-closed forcing notions; $\mathbb{P}$ adjoins a Ramsey ultrafilter $\mathcal{U}$ on ω [Ma], while $\mathbb{Q}$ adds a stable ordered-union ultrafilter $\mathcal{V}$ on $[\omega]^{<\omega}$ (see [Bl1] for this notion). It is well known that $\mathbb{R}$ is forcing-equivalent to the two-step iteration $\mathbb{P} \star \mathbb{R}_{\breve{\mathcal{U}}}$, where $\breve{\mathcal{U}}$ is the $\mathbb{P}$-name for the generic Ramsey ultrafilter, and $\mathbb{R}_{\mathcal{U}}$ is the σ-centered Mathias forcing with an ultrafilter $\mathcal{U}$ [Ma]; similarly $\mathbb{V}$ decomposes as $\mathbb{P} \star \mathbb{G}_{\breve{\mathcal{U}}}$, where $\mathbb{G}_{\mathcal{U}}$ is Grigorieff forcing [Gr]. To this corresponds that $\mathbb{T}$ is forcing-equivalent to $\mathbb{Q} \star \mathbb{T}_{\breve{\mathcal{V}}}$, where $\mathcal{V}$ is the $\mathbb{Q}$-name for the generic ultrafilter on $[\omega]^{<\omega}$, and $\mathbb{T}_{\mathcal{V}}$ is the σ-centered Matet forcing with an ordered-union ultrafilter $\mathcal{V}$ [Bl2]; similarly $\mathbb{W}$ decomposes as $\mathbb{Q} \star \mathbb{G}_{\breve{\mathcal{V}}}$, where $\mathbb{G}_{\mathcal{V}}$ is a Grigorieff-like forcing (we leave the details of this to the reader).

In case of the uniform forcings (i.e. $\mathbb{R}, \mathbb{T}, \mathbb{V}, \mathbb{W}$), we will sometimes have to go back to the original notation of the conditions; we shall always mark the places where we do so, and work in general with trees.

1.3. There is a natural way to associate the σ-ideal of $\mathbb{J}$-*null sets* with any of the tree forcings $\mathbb{J}$ defined above:

$$j^0 := \{A \subseteq \omega^{\uparrow\omega} : \forall T \in \mathbb{J}\ \exists S \le T\ ([S] \cap A = \emptyset)\}.$$

Of course, for the forcings with compact trees (i.e. $\mathbb{S}$, $\mathbb{W}$ and $\mathbb{V}$), one would rather define the corresponding ideals on 2^ω; e.g. Marczewski's ideal s^0 (see [Mar] and others) is usually defined as

$$s^0 := \{A \subseteq 2^\omega : \forall T \in \mathbb{S}\ \exists S \le T\ ([S] \cap A = \emptyset)\}.$$

However, putting

$$\overline{F}(s^0) := \{\widehat{F}[A] : A \in s^0 \wedge \forall f \in A\ (|f^{-1}(\{1\})| = \omega)\},$$

we get the corresponding ideal on $\omega^{\uparrow\omega}$, and shall henceforth work with the latter (and even call it s^0).

It is sometimes helpful to think of an i^0-set as the complement of the set of branches of all trees in some dense subset of $\mathbb{I}$ (or some maximal antichain of $\mathbb{I}$).

One of the main goals of this work will be to show that none of these ideals i^0 is included in any other, for the various forcing notions $\mathbb{I}$ introduced in 1.1. Note that we trivially have $i^0 \setminus j^0 \ne \emptyset$ whenever $\mathbb{J} \not\subseteq \mathbb{I}$: the set of branches of a tree $T \in \mathbb{J} \setminus \mathbb{I}$ must be a member of i^0, because given any $S \in \mathbb{I}$, there is $\sigma \in S \setminus T$, hence $[S_\sigma] \cap [T] = \emptyset$. Thus we are left with showing that i^0 is not included in j^0 in case $\mathbb{J} \subseteq \mathbb{I}$. This will be done in Section 2.

1.4. We make some general remarks concerning the constructions in Section 2. First note that given $\langle I_\alpha : \alpha < \mathfrak{c}\rangle \subseteq \mathbb{I}$ dense and letting $\langle J_\alpha : \alpha < \mathfrak{c}\rangle$ be an enumeration of $\mathbb{J}$, it suffices to construct $\langle x_\alpha : \alpha < \mathfrak{c}\rangle$ so that

(i) $x_\alpha \notin \bigcup_{\beta<\alpha}[I_\beta]$;
(ii) $x_\alpha \in [J_\alpha]$.

Then we will have $X = \{x_\alpha : \alpha < \mathfrak{c}\} \in i^0 \setminus j^0$. Indeed, $X \notin j^0$ is obvious. To see $X \in i^0$, fix $I \in \mathbb{I}$; find α so that $I_\alpha \le I$. Find $\{I'_\beta : \beta < \mathfrak{c}\} \subseteq \mathbb{I}$, an antichain of conditions below I_α with $[I'_\beta] \cap [I'_\gamma] = \emptyset$ for $\beta \ne \gamma$ (we leave it to the reader to verify that such I'_β can be found for each of our forcings $\mathbb{I}$, using an a.d. family of size $\mathfrak{c}$). As $|[I_\alpha] \cap X| < \mathfrak{c}$ we necessarily find some β with $[I'_\beta] \cap X = \emptyset$; thus we are done.

The main points of our proofs boil thus down to two steps:

- choose *carefully* a dense set $\mathcal{I} \subseteq \mathbb{I}$;

• find for each $J \in \mathbb{J}$ a subtree J' (which will usually be a homeomorphic copy of $2^{<\omega}$) so that $[J' \cap I]$ is small for all $I \in \mathcal{I}$ (in general the intersection will be at most countable).

In most cases, it is not difficult to do this. The hardest arguments are those concerning Marczewski's ideal s^0 (in 2.2 and 2.8).

Finally, note that if $\mathbb{I} \supseteq \mathbb{J}_1 \supseteq \mathbb{J}_2$, then a set $X \notin i^0 \setminus j_1^0$ constructed along the lines above will automatically not belong to j_2^0 either. Hence we are left with nine constructions; they are summarized in the following chart.

$X \in$ $\notin$	ℓ^0	m^0	s^0	t^0	w^0	r^0	v^0
ℓ^0		2.1	(2.2)	easy	easy	easy	easy
m^0	easy		2.2	easy	easy	easy	easy
s^0	easy	easy		easy	easy	easy	easy
t^0	easy	2.6	(2.2/8)		2.3	easy	easy
w^0	easy	easy	2.8	easy		easy	easy
r^0	2.5	(2.1/6)	(2.2/8)	2.9	(2.3/7)		2.4
v^0	easy	easy	(2.8)	easy	2.7	easy	

As all constructions are quite similar, we concentrate on the hard ones (2.2 and 2.8), and do only two of the others in detail (2.1 and 2.5). 2.3 and 2.4 follow from 3.1, and the remaining cases (2.6, 2.7 and 2.9) are left as exercises to the reader.

1.5. Given one of our tree forcings $\mathbb{J}$, call a set $A \subseteq \omega^{\uparrow\omega}$ *j-measurable* iff for all $T \in \mathbb{J}$ there is $S \leq T$ with either $[S] \subseteq A$ or $[S] \cap A = \emptyset$. If the first alternative always holds, then A is a *j^1-set* (this is equivalent to saying that $\omega^{\uparrow\omega} \setminus A \in j^0$). A is *j-positive* iff there is $T \in \mathbb{J}$ with $[T] \subseteq A$. Two ideals $\mathcal{I}$, $\mathcal{J}$ on the reals are said to be *orthogonal* iff there is $A \subseteq \omega^{\uparrow\omega}$ with $A \in \mathcal{I}$ and $\omega^{\uparrow\omega} \setminus A \in \mathcal{J}$.

We note that $\mathbb{I} \supseteq \mathbb{J}$ implies that j-positive sets are i-positive; in particular, i^0 and j^0 cannot be orthogonal. Furthermore, if all sets of reals are j-measurable, then $i^0 \subseteq j^0$. Using [GRSS, Section 2], it is easy to see that AD implies ℓ-measurability of all sets of reals; hence it implies $m^0 \subseteq \ell^0$ (this observation is due to O. Spinas; his original argument was somewhat different).

In Subsection 2.10 we shall return to the question whether i^0 and j^0 can be orthogonal in case $\mathbb{I} \not\subseteq \mathbb{J}$ and $\mathbb{J} \not\subseteq \mathbb{I}$.

2. The main results

2.1. THEOREM. $m^0 \setminus \ell^0 \neq \emptyset$.

Proof. Call a Miller tree $M \in \mathbb{M}$ an *apple tree* iff:

$$(*_1)\qquad \forall \sigma \in \mathrm{split}(M)\ (\text{if } n>m \wedge \sigma\hat{}\langle n\rangle, \sigma\hat{}\langle m\rangle \in M \wedge \sigma\hat{}\langle m\rangle \subseteq \tau \in \mathrm{Succ}_M(\sigma), \text{ then } \forall k \in |\tau|\ (\tau(k) < n)), \text{ and}$$

$$(*_2)\qquad \sigma \subset \tau,\ \sigma, \tau \in \mathrm{split}(M) \Rightarrow |\tau| \geq |\sigma| + 2.$$

A standard pruning argument shows that given $N \in \mathbb{M}$ there is an apple tree $M \leq N$.

We construct a *pear subtree* $P_L = \{\sigma_t : t \in 2^{<\omega}\} \subseteq L$ of a Laver tree L which is a copy of $2^{<\omega}$ such that:

(I) $\sigma_{\langle\rangle} = \mathrm{stem}(L)$;
(II) given $\sigma_t \in L$, $\sigma_{t\hat{}\langle 0\rangle} = \sigma\hat{}\langle k\rangle$ and $\sigma_{t\hat{}\langle 1\rangle} = \sigma\hat{}\langle l\rangle$ such that they are in L, $l > k > \max\{\max \mathrm{rng}(\sigma_{t'}) : |t'| = |t|\}$.

It is immediate from the definition of a Laver tree that this can be done.

Claim. $|[M \cap P_L]| \leq 1$ *whenever M is an apple tree and P_L is a pear tree.*

Proof. Assume to the contrary that $f_1 \neq f_2 \in [M \cap P_L]$; fix $\sigma \in \omega^{<\omega}$ so that $\sigma \subseteq f_1, f_2$ and $f_1(|\sigma|) < f_2(|\sigma|)$. As both f_1 and f_2 are branches of M, we must have $f_1(|\sigma|+1) < f_2(|\sigma|)$; on the other hand, both being branches of P_L, we get $f_1(|\sigma|+1) > f_2(|\sigma|)$, a contradiction. ∎

Now let $\langle M_\alpha : \alpha < \mathfrak{c}\rangle$ enumerate all apple trees, and let $\langle L_\alpha : \alpha < \mathfrak{c}\rangle$ enumerate all Laver trees. Using the above we easily construct $\langle x_\alpha : \alpha < \mathfrak{c}\rangle$ so that

(i) $x_\alpha \notin \bigcup_{\beta<\alpha}[M_\beta]$;
(ii) $x_\alpha \in [L_\alpha]$.

Then $X = \{x_\alpha : \alpha < \mathfrak{c}\} \in m^0 \setminus \ell^0$ by the remarks made in 1.4. ∎

2.2. Theorem. $s^0 \setminus m^0 \neq \emptyset$.

Proof. We proceed as before—but the argument is somewhat more involved. That is, we find $\langle S_\alpha : \alpha < \mathfrak{c}\rangle \subseteq \mathbb{S}$ dense, and construct $\langle x_\alpha : \alpha < \mathfrak{c}\rangle$ so that

(i) $x_\alpha \notin \bigcup_{\beta<\alpha}[S_\beta]$ and
(ii) $x_\alpha \in [M_\alpha]$,

where $\langle M_\alpha : \alpha < \mathfrak{c}\rangle$ is an enumeration of all Miller trees.

A partition result for Sacks trees. We start with thinning out the Sacks trees. Given $S \in \mathbb{S}$, $\sigma = \mathrm{stem}(S)$ and $i \neq j$ so that $\sigma\hat{}\langle i\rangle, \sigma\hat{}\langle j\rangle \in S$, we put $A_S^{\langle i,j\rangle} := \{\langle \varrho, \tau\rangle : \sigma\hat{}\langle i\rangle \subseteq \varrho \in \mathrm{split}(S) \wedge \sigma\hat{}\langle j\rangle \subseteq \tau \in S \wedge |\varrho| = |\tau|\}$.

LEMMA 1. *Let S be a Sacks tree, $\sigma = \operatorname{stem}(S)$, $i \neq j$ so that $\sigma\hat{}\langle i\rangle, \sigma\hat{}\langle j\rangle \in S$. Assume we have a two-place relation $R \subseteq A_S := A_S^{\langle i,j\rangle}$. Then there is $S' \leq S$ with the same stem so that either*

- $\forall\langle\varrho,\tau\rangle \in A_{S'}$ $(\langle\varrho,\tau\rangle \in R)$, *or*
- $\forall\langle\varrho,\tau\rangle \in A_{S'}$ $(\langle\varrho,\tau\rangle \notin R)$.

Proof. Assume first

(∗) there are $n \in \omega$ and $\varrho, \tau_k \in S$, $k < n$, with $\sigma\hat{}\langle i\rangle \subseteq \varrho$, $\sigma\hat{}\langle j\rangle \subseteq \tau_k$ and $|\tau_\ell| = |\tau_k|$ such that for all $\varrho' \supseteq \varrho$ in $\operatorname{split}(S)$ there is $k = k(\varrho') < n$ with

$$(\heartsuit) \qquad \forall\tau' \supseteq \tau_k \ ((|\varrho'| = |\tau'| \wedge \tau' \in S) \Rightarrow \langle\varrho',\tau'\rangle \notin R).$$

Then fix such $n, \varrho, \tau_k, k < n$. Notice that the function sending ϱ' to $k(\varrho')$ gives a coloring of $\operatorname{split}(S_\varrho)$ with finitely many colors. Thus there is a perfect $\widetilde{S} \subseteq S_\varrho$ homogeneous for one color, say k. Let $S' = \widetilde{S} \cup S_{\tau_k}$. Clearly the second alternative of the lemma holds for S'.

So suppose (∗) fails; we construct, by recursion on $|s|$, $\langle\varrho_s : s \in 2^{<\omega}\rangle$ and $\langle\tau_s : s \in 2^{<\omega}\rangle$ so that

(a) $\sigma\hat{}\langle i\rangle \subseteq \varrho_s \in \operatorname{split}(S) \wedge \sigma\hat{}\langle j\rangle \subseteq \tau_s \in S$;

(b) $(s \subseteq t \Rightarrow \varrho_s \subseteq \varrho_t, \tau_s \subseteq \tau_t) \wedge \varrho_{s\hat{}\langle 0\rangle}(|\varrho_s|) \neq \varrho_{s\hat{}\langle 1\rangle}(|\varrho_s|)$ and if s and t are incompatible, then so are τ_s and τ_t;

(c) $|s| = |t| \Rightarrow |\tau_s| = |\tau_t| \geq |\varrho_s|$ and $\langle\varrho_s, \tau_t\restriction|\varrho_s|\rangle \in R$.

Assume we are at step m in the construction; i.e. we have $\langle\varrho_s : s \in 2^{<m}\rangle$, $\langle\tau_s : s \in 2^{<m}\rangle$ as above. First choose $\langle\widetilde{\tau}_t : t \in 2^m\rangle$ and $\langle\widetilde{\varrho}_t : t \in 2^m\rangle \subseteq S$ pairwise incomparable so that $s \subset t$ implies $\tau_s \subseteq \widetilde{\tau}_t$ and $\varrho_s \subset \widetilde{\varrho}_t$, and also $\widetilde{\varrho}_{s\hat{}\langle 0\rangle}(|\varrho_s|) \neq \widetilde{\varrho}_{s\hat{}\langle 1\rangle}(|\varrho_s|)$ for $s \in 2^{m-1}$. Let $\{t_k : k < 2^m\}$ enumerate 2^m. By recursion on k find ϱ_{t_k} and $\widetilde{\tau}_t^k$ such that for all $t \in 2^m$,

(A) $\widetilde{\varrho}_{t_k} \subseteq \varrho_{t_k} \in \operatorname{split}(S)$;

(B) $\widetilde{\tau}_t \subseteq \widetilde{\tau}_t^{k-1} \subseteq \widetilde{\tau}_t^k$, $|\widetilde{\tau}_t^k| = |\varrho_{t_k}|$, and $\langle\varrho_{t_k}, \widetilde{\tau}_t^k\rangle \in R$.

This can be done, because (∗) fails for $2^m, \widetilde{\varrho}_{t_k}, \widetilde{\tau}_t^{k-1}$ $(t \in 2^m)$. Finally, put $\tau_t = \widetilde{\tau}_t^{2^m-1}$. This completes the construction.

Putting $S' = \{\varrho_s\restriction n, \tau_s\restriction n : s \in 2^{<\omega} \wedge n \in \omega\}$, we see that the first alternative of the lemma holds for S'. ∎

Now let us assume we have $S \in \mathbb{S}$ and finitely many pairwise disjoint relations $R_i \subseteq \{\langle\varrho,\tau\rangle : \varrho,\tau \in S \wedge |\varrho| = |\tau|\}$, $i < k$, with $\bigcup_{i<k} R_i = \{\langle\varrho,\tau\rangle : \varrho,\tau \in S \wedge |\varrho| = |\tau|\}$. We say a splitting node $\sigma \in S$ is *of type* $\langle i,j\rangle$ $(i,j \in k)$ in S iff letting $n_0 < n_1$ so that $\sigma\hat{}\langle n_0\rangle, \sigma\hat{}\langle n_1\rangle \in S$, we have

$$\forall\langle\varrho,\tau\rangle \in A_{S_\sigma}^{\langle n_0,n_1\rangle} \ (\langle\varrho,\tau\rangle \in R_i) \quad \text{and} \quad \forall\langle\varrho,\tau\rangle \in A_{S_\sigma}^{\langle n_1,n_0\rangle} \ (\langle\varrho,\tau\rangle \in R_j).$$

Using a standard fusion argument and Lemma 1 we see:

Lemma 2. *Given $S \in \mathbb{S}$, and R_i, $i < k$, as above, there are $S' \le S$ and $\langle i, j\rangle \in k^2$ so that each splitting node $\sigma \in S'$ is of type $\langle i, j\rangle$ (in which case we say S' is of type $\langle i, j\rangle$).* ■

Given $S \in \mathbb{S}$ so that $|\mathrm{split}(S) \cap \omega^n| \le 1$ for all $n \in \omega$ define relations R_i, $i < 3$, as follows: given $\sigma \in \mathrm{split}(S)$ and $\tau \in S \setminus \{\sigma\}$ with $|\sigma| = |\tau|$ arbitrarily, let $n_0 < n_1$ so that $\sigma\hat{}\langle n_0\rangle, \sigma\hat{}\langle n_1\rangle \in S$ and $\tau \subseteq \tau' \in S$ with $|\tau'| = |\tau| + 1$ (τ' being unique), and put

$$\langle \sigma, \tau\rangle \in R_0 \Leftrightarrow n_1 < \tau'(|\tau|),$$
$$\langle \sigma, \tau\rangle \in R_1 \Leftrightarrow n_0 \le \tau'(|\tau|) \le n_1,$$
$$\langle \sigma, \tau\rangle \in R_2 \Leftrightarrow n_0 > \tau'(|\tau|).$$

Applying Lemma 2, we get:

Corollary. *The set $\{S \in \mathbb{S} : \exists \langle i, j\rangle \in 3^2\ (S \text{ is of type } \langle i, j\rangle)\}$ is dense in $\mathbb{S}$.* ■

Subtrees of Miller trees. Assume we are given a family $\Sigma = \langle \sigma_s : s \in 2^{<\omega}\rangle \subseteq \omega^{\uparrow<\omega}$ satisfying

(I) $s \subset t \Rightarrow \sigma_s \subset \sigma_t$;

(II) $\sigma_{s\hat{}\langle 0\rangle}(|\sigma_s|) < \sigma_{s\hat{}\langle 1\rangle}(|\sigma_s|)$;

(III) given $s \in 2^{<\omega}$, $f_i \in 2^\omega$, $s\hat{}\langle i\rangle \subseteq f_i$ ($i \in 2$), and putting $\phi_i := \bigcup_n \sigma_{f_i\restriction n}$ we have for all $n \in \omega$,

$$\phi_0(|\sigma_{f_0\restriction(|s|+2n)}|) < \phi_1(|\sigma_{f_0\restriction(|s|+2n)}|),$$
$$\phi_0(|\sigma_{f_0\restriction(|s|+2n+1)}|) > \phi_1(|\sigma_{f_0\restriction(|s|+2n+1)}|),$$
$$\phi_1(|\sigma_{f_1\restriction(|s|+2n)}|) > \phi_0(|\sigma_{f_1\restriction(|s|+2n)}|),$$
$$\phi_1(|\sigma_{f_1\restriction(|s|+2n+1)}|) < \phi_0(|\sigma_{f_1\restriction(|s|+2n+1)}|);$$

then we call the closure $C(\Sigma)$ under initial segments a *cherry tree*.

Lemma 3. *A Miller tree M contains a cherry subtree $C(\Sigma_M)$.*

Proof. By recursion on the levels we construct the family $\Sigma_M = \langle \sigma_s : s \in 2^{<\omega}\rangle \subseteq \mathrm{split}(M)$ so that:

(α) (I)–(III) above are satisfied;

(β) given $s, t, t' \in 2^{<\omega}$ with $t(0) = 0$, $t'(0) = 1$ and $|t| = |t'|$ we have:

$$|\sigma_{s\hat{}t}| < |\sigma_{s\hat{}t'}| \quad \text{in case } |t| \text{ is odd},$$
$$|\sigma_{s\hat{}t}| > |\sigma_{s\hat{}t'}| \quad \text{in case } |t| \text{ is even}.$$

To start, let $\sigma_{\langle\rangle} := \mathrm{stem}(M)$, and choose splitting nodes $\sigma_{\langle 0\rangle}, \sigma_{\langle 1\rangle} \supseteq \sigma_{\langle\rangle}$ with $\sigma_{\langle 0\rangle}(|\sigma_{\langle\rangle}|) < \sigma_{\langle 1\rangle}(|\sigma_{\langle\rangle}|)$ and $|\sigma_{\langle 0\rangle}| < |\sigma_{\langle 1\rangle}|$.

Assume $\langle \sigma_t : t \in 2^{\le n}\rangle$ have been constructed satisfying (α) and (β) above. Enumerate $\langle t_k : k \in 2^n\rangle = 2^n$ in such a way that $k < l$ is equivalent to $|\sigma_{t_k}| > |\sigma_{t_l}|$ (this is possible by (β)); now recursively find $\sigma_{t_k} \subseteq \sigma_{t_k\hat{}\langle i\rangle} \in \mathrm{split}(M)$ $(i \in 2)$ so that:

- $|\sigma_{t_k\hat{}\langle i\rangle}| < |\sigma_{t_\ell\hat{}\langle j\rangle}|$ for $k < l$ or ($k = l$ and $i < j$);
- $\sigma_{t_k\hat{}\langle 0\rangle}(|\sigma_{t_k}|) < \sigma_{t_k\hat{}\langle 1\rangle}(|\sigma_{t_k}|)$;
- $\sigma_{t_l\hat{}\langle j\rangle}(|\sigma_{t_l}|) > \sigma_{t_k}(|\sigma_{t_l}|)$ and $\sigma_{t_\ell\hat{}\langle j\rangle}(|\sigma_{t_k}|) > \sigma_{t_k\hat{}\langle i\rangle}(|\sigma_{t_k}|)$ for $k < l$ and $i, j \in 2$.

This can be done easily. It is straightforward to verify that (α) and (β) are still satisfied. ∎

Using a similar—but much easier—construction, we see:

LEMMA 4. *A Miller tree has a subtree of type* $\langle 2, 2\rangle$. ∎

Unfortunately neither a cherry tree nor a type $\langle 2, 2\rangle$-tree will suffice for our purposes. We have to somehow "amalgamate" these two types of trees to prove the final lemmata (see below). So suppose we are given a system $\Sigma = \langle \sigma_{\langle s,t\rangle} : s, t \in 2^{<\omega} \wedge |s| = |t|\rangle \subseteq \omega^{\uparrow<\omega}$ such that, letting $M = M(\Sigma) := \{\sigma_{\langle s,t\rangle} \restriction n : n \in \omega \wedge \sigma_{\langle s,t\rangle} \in \Sigma\}$ and calling it a *mango tree*, we have:

(I) $(s, t) \subset (s', t') \Rightarrow \sigma_{\langle s,t\rangle} \subset \sigma_{\langle s',t'\rangle}$;

(II) $M^f := \{\sigma_{\langle f\restriction i,t\rangle} \restriction n : i, n \in \omega \wedge t \in 2^i\}$ is a cherry tree;

(III) whenever $f_i, g_i \in 2^\omega$ $(i \in 2)$, $f_0 \neq f_1$, $s \subseteq f_i$, $f_i(|s|) = i$, then, putting $\phi_i = \bigcup_n \sigma_{\langle f_i\restriction n, g_i\restriction n\rangle}$, we have

$$\phi_i(|\sigma_{\langle f_i\restriction n, g_i\restriction n\rangle}|) > \phi_j(|\sigma_{\langle f_i\restriction n, g_i\restriction n\rangle}|)$$

for $i \neq j$ and $n > |s|$, and

$$\phi_1(|\sigma_{\langle s, g_1\restriction |s|\rangle}|) > \phi_0(|\sigma_{\langle s, g_1\restriction |s|\rangle}|).$$

So a mango tree is a kind of "two-dimensional" tree, the vertical sections of which are cherry trees while the horizontal sections are of type $\langle 2, 2\rangle$ (this is a particular instance of (III), for $g_0 = g_1$).

To construct a $\Sigma = \langle \sigma_{\langle s,t\rangle} : s, t \in 2^{<\omega} \wedge |s| = |t|\rangle$ giving rise to a mango tree, proceed as in the proof of Lemma 3, guaranteeing along the way that:

$(\widetilde{\alpha})$ (I)–(III) are satisfied;

$(\widetilde{\beta})$ given $s, t, t' \in 2^{<\omega}$ with $t(0) = 0, t'(0) = 1$ and $|t| = |t'|$, and $f \in 2^\omega$, we have

$$|\sigma_{\langle f\restriction |s|+|t|, s\hat{}t\rangle}| < |\sigma_{\langle f\restriction |s|+|t|, s\hat{}t'\rangle}| \quad \text{in case } |t| \text{ is odd},$$
$$|\sigma_{\langle f\restriction |s|+|t|, s\hat{}t\rangle}| > |\sigma_{\langle f\restriction |s|+|t|, s\hat{}t'\rangle}| \quad \text{in case } |t| \text{ is even};$$

$(\widetilde{\gamma})$ in case $s, s', t, t' \in 2^n$ for some n, and s precedes s' in the lexicographic ordering of 2^n, we have

$$|\sigma_{\langle s,t\rangle}| < |\sigma_{\langle s',t'\rangle}|.$$

In step 0 of the construction, put $\sigma_{\langle\rangle} := \mathrm{stem}(M)$, and choose split-nodes $\sigma_{\langle i,j\rangle}$ $(i,j \in 2)$ extending $\sigma_{\langle\rangle}$ with $\sigma_{\langle 0,0\rangle}(|\sigma_{\langle\rangle}|) < \sigma_{\langle 0,1\rangle}(|\sigma_{\langle\rangle}|) < \sigma_{\langle 1,0\rangle}(|\sigma_{\langle\rangle}|) < \sigma_{\langle 1,1\rangle}(|\sigma_{\langle\rangle}|)$ and $|\sigma_{\langle 0,0\rangle}| < |\sigma_{\langle 0,1\rangle}| < |\sigma_{\langle 1,0\rangle}| < |\sigma_{\langle 1,1\rangle}|$.

In step n, let $\langle s_k : k \in 2^n\rangle$ enumerate 2^n lexicographically; and proceed by recursion on k. For fixed k, run the argument in the proof of Lemma 3 twice to get $\sigma_{\langle s_k\hat{\ }\langle i\rangle, t\rangle}$, where $i \in 2$, $t \in 2^{n+1}$.

Hence we proved:

Lemma 5. *A Miller tree contains a mango subtree.* ■

The final lemmata. We are now in a position to conclude our argument by looking at the intersections of a Sacks tree of one of the types $\langle i,j\rangle$ $(i,j \in 3)$ with a mango tree.

Let E denote the set of even numbers. Given a system $\Sigma = \langle\sigma_s : s \in 2^{<\omega}\rangle \subseteq \omega^{\uparrow<\omega}$ satisfying

$$s \subset t \Rightarrow \sigma_s \subset \sigma_t$$

(and thus defining a tree $T(\Sigma) := \{\sigma_s\restriction n : s \in 2^{<\omega} \wedge n \in \omega\}$) and a function $f \in 2^E$, we can form the tree $T(\Sigma_f) := \{\sigma_s\restriction n : s \in 2^{<\omega} \wedge n \in \omega \wedge \forall i \in E \cap \mathrm{dom}(s)\ (s(i) = f(i))\} \subseteq T(\Sigma)$.

Lemma 6. *Assume $M = M(\Sigma)$ is a mango tree constructed from the system $\Sigma = \langle\sigma_{\langle s,t\rangle} : s,t \in 2^{<\omega} \wedge |s| = |t|\rangle$, and S is a Sacks tree of one of the eight types $\langle i,j\rangle \in 3^2 \setminus \{\langle 2,2\rangle\}$. Then $|\{f \in 2^\omega : |[M^f \cap S]| \geq 1\}| \leq \omega$.*

Proof. We look at $\widehat{T} = \{\langle s,t\rangle : s,t \in 2^{<\omega} \wedge |s| = |t| \wedge \sigma_{\langle s,t\rangle} \in M \cap S\}$. This is a compact tree in the plane, hence its projection onto the first coordinate is also compact, and thus has either at most countably many branches or contains a perfect subtree T. In the first case, we are done, so assume the latter.

Put $s := \mathrm{stem}(T)$, and note that there must be $t_0, t_1 \in 2^{|s|+1}$ so that both $T_i := \widehat{T}_{\langle s\hat{\ }\langle i\rangle, t_i\rangle}$ $(i \in 2)$ contain perfect trees. Find incompatible extensions $\langle\langle s_i^j, t_i^j\rangle : i,j \in 2\rangle$, $\langle s_i^j, t_i^j\rangle \in T_i$, and let $\langle f_i^j, g_i^j\rangle$ be branches of T_i through $\langle s_i^j, t_i^j\rangle$. Put (as in (III)) $\phi_i^j := \bigcup_n \sigma_{\langle f_i^j\restriction n, g_i^j\restriction n\rangle}$, and let k_i be minimal with $\phi_i^0(k_i) \neq \phi_i^1(k_i)$. It is a consequence of (III) that we must have $\phi_i^j(k_i) > \phi_{1-i}^k(k_i)$ for $i,j,k \in 2$. This entails (by definition of the types) that S is of type $\langle 2,2\rangle$, a contradiction. ■

Lemma 7. *Assume S is a Sacks tree of type $\langle 2,2\rangle$, $C = C(\Sigma)$ is a cherry tree constructed from the system $\Sigma = \langle\sigma_s : s \in 2^{<\omega}\rangle$ and $f \in 2^E$. Then $|[C(\Sigma_f) \cap S]| \leq \omega$.*

Proof. Put $C_f = C(\Sigma_f)$ and assume the conclusion is false. Then $C_f \cap S$ must contain a perfect subtree; in particular, there are $s,t \in 2^{<\omega}$ so that $\sigma_s, \sigma_t \in \mathrm{split}(C_f \cap S)$ and $s\hat{\ }\langle 0\rangle \subseteq t$. Note that $|s|$ and $|t|$ must be odd. As S is of type $\langle 2,2\rangle$ we must have $\sigma_{t\hat{}\langle i\rangle}(|\sigma_t|) > g(|\sigma_t|)$ for any $g \in [C_f \cap S]$ extending $\sigma_{s\hat{}\langle 1\rangle}$. On the other hand, C being a cherry tree, we have $\sigma_{t\hat{}\langle i\rangle}(|\sigma_t|) < g(|\sigma_t|)$ for any such g, a contradiction. ■

COROLLARY. *If M is a mango tree, and $\mathcal{S}$ is a family of less than $\mathfrak{c}$ Sacks trees all of which are of type $\langle i,j\rangle$ for some $\langle i,j\rangle \in 3^2$, then $|[M] \setminus \bigcup_{S\in\mathcal{S}}[S]| = \mathfrak{c}$.*

Proof. First apply Lemma 6 to find $f \in 2^\omega$ so that $[M^f \cap S] = \emptyset$ for all trees in $\mathcal{S}$ which are not of type $\langle 2,2\rangle$. Choose $g \in 2^E$ arbitrarily and apply Lemma 7 to find $\mathfrak{c}$ many $\phi \in [C_g] \setminus \bigcup_{s\in\mathcal{S}}[S]$, where C is the cherry tree M^f. ■

We can now complete the proof of Theorem 2.2: let $\langle S_\alpha : \alpha < \mathfrak{c}\rangle$ enumerate the Sacks trees of type $\langle i,j\rangle$ for some $\langle i,j\rangle \in 3^2$, and construct $\langle x_\alpha : \alpha < \mathfrak{c}\rangle$ as required using Lemma 5 and the above Corollary. ■

2.3. THEOREM. $w^0 \setminus t^0 \neq \emptyset$.

Proof. This follows from Theorem 3.1. ■

2.4. THEOREM. $v^0 \setminus r^0 \neq \emptyset$.

Proof. This also follows from Theorem 3.1. ■

2.5. THEOREM. $\ell^0 \setminus r^0 \neq \emptyset$.

Proof. Call a Laver tree $T \in \mathbb{L}$ a *peach tree* iff

($\star$) for any $\sigma,\tau \in T$, if $\mathrm{stem}(T) \subseteq \sigma,\tau$ and $\sigma \neq \tau$, then $\mathrm{succ}_T(\sigma) \cap \mathrm{succ}_T(\tau) = \emptyset$.

Given $S \in \mathbb{L}$, there is $T \in \mathbb{L}$ so that $T \le S$ and T is a peach tree (this is a standard fusion argument).

Construct an *orange subtree* $O_M = \{\sigma_t : t \in 2^{<\omega}\} \subseteq M$ of a Mathias tree M as follows:

(I) $\sigma_{\langle\rangle} = \mathrm{stem}(M)$;

(II) suppose σ_t for $|t| \le n$ is defined; choose $l > k > \max\{\max \mathrm{rng}(\sigma_t) : |t| \le n\}$ such that $k,l \in A$ (where $(\sigma_{\langle\rangle}, A)$ is the Mathias condition in the usual notation corresponding to M); then $\sigma_{t\hat{}\langle 0\rangle} = \sigma_t\hat{}\langle k\rangle$ and $\sigma_{t\hat{}\langle 1\rangle} = \sigma_t\hat{}\langle l\rangle$ for any t with $|t| = n$.

CLAIM. $|[L \cap O_M]| \le 2$ *whenever L is a peach tree and O_M is an orange tree.*

Proof. Suppose $f,g,h \in [L \cap O_M]$ were three distinct elements. Find n such that $f{\restriction}n \neq g{\restriction}n \neq h{\restriction}n \neq f{\restriction}n$. Then (without loss) $f(n-1) = g(n-1)$ and $f{\restriction}(n-1) \neq g{\restriction}(n-1)$—by the properties of the orange tree O_M; this contradicts the fact that L is a peach tree. ■

Using peach and orange trees we complete the proof as in Theorem 2.1. ■

2.6. Theorem. $m^0 \setminus t^0 \neq \emptyset$.

Proof. This is very similar to the proof of Theorem 2.5. We therefore leave it to the reader. ∎

2.7. Theorem. $w^0 \setminus v^0 \neq \emptyset$.

Proof. Exercise! ∎

2.8. Theorem. $s^0 \setminus w^0 \neq \emptyset$.

Proof. We shall use the same dense subset of $\mathbb{S}$ as in 2.2, namely the set $\{S \in \mathbb{S} : \exists \langle i,j\rangle \in 3^2\ (S \text{ is of type } \langle i,j\rangle)\}$.

Subtrees of willow trees of type $\langle i,j\rangle$

Lemma 1. *A willow tree W has a type $\langle 0,0\rangle$ subtree T_W.*

Proof. Assume W corresponds to the pair (f_W, A_W) in the usual notation for willow conditions. Without loss $A_W = \{a_n : n \in \omega\}$, $\max(a_n) < \min(a_{n+1})$, $2|a_n| < |a_{n+1}|$, and also $f_W^{-1}(\{1\}) \cap (\min(a_n), \min(a_{n+1})) \neq \emptyset$ (otherwise go over to a stronger condition).

Let $\langle I_n : 1 \le n \in \omega\rangle$ be a partition of ω into intervals of size 2^{n+1}, $\max(I_n) + 1 = \min(I_{n+1})$. τ_j^n $(j < 2^{n+1})$ is the increasing enumeration of the set $a_i \cup [f_W^{-1}(\{1\}) \cap (\min(a_i), \min(a_{i+1}))]$ and ϱ_j^n $(j < 2^{n+1})$ is the increasing enumeration of $f_W^{-1}(\{1\}) \cap (\min(a_i), \min(a_{i+1}))$, where i is the jth element of I_n. Define recursively $\langle \sigma_s : s \in 2^{<\omega}\rangle \subseteq W$:

$$\sigma_{\langle\rangle} = \mathrm{stem}(W), \quad \sigma_{\langle 0\rangle} = \sigma_{\langle\rangle}\hat{\ }\tau_0^1, \quad \sigma_{\langle 1\rangle} = \sigma_{\langle\rangle}\hat{\ }\varrho_0^1\hat{\ }\varrho_1^1\hat{\ }\tau_2^1.$$

Assume $\langle \sigma_s : s \in 2^{\le n}\rangle$ have been defined, $n \ge 1$. Let $\langle s_i : i < 2^n\rangle$ be the lexicographic enumeration of 2^n. We put

$$\sigma_{s_i\hat{\ }\langle 0\rangle} = \sigma_{s_i}\hat{\ }\tau_{2i+1}^n\hat{\ }\varrho_{2i+2}^n\hat{\ }\ldots\hat{\ }\varrho_{2^{n+1}-1}^n\hat{\ }\varrho_0^{n+1}\hat{\ }\ldots\hat{\ }\varrho_{4i-1}^{n+1}\hat{\ }\tau_{4i}^{n+1},$$
$$\sigma_{s_i\hat{\ }\langle 1\rangle} = \sigma_{s_i}\hat{\ }\varrho_{2i+1}^n\hat{\ }\ldots\hat{\ }\varrho_{2^{n+1}-1}^n\hat{\ }\varrho_0^{n+1}\hat{\ }\ldots\hat{\ }\varrho_{4i+1}^{n+1}\hat{\ }\tau_{4i+2}^{n+1}.$$

Set $T_W := \{\sigma_s \restriction n : n \in \omega \wedge s \in 2^{<\omega}\}$. Note that the construction was set up in such a way that whenever $\sigma \in T_W$ is a split-node, then σ is of the form σ_s for some $s \in 2^{<\omega}$; thus the final part of this sequence is some τ_j^n. By our requirements on the $|a_m|$, this entails that σ_s is longer than any other sequence in the tree ending in the corresponding ϱ_j^n. Hence, if σ_1, σ_2 are two immediate successors of σ_s, then $\sigma_1(|\sigma_s|), \sigma_2(|\sigma_s|) < \tau(|\sigma_s|)$ for any sequence τ in the tree T_W which is incomparable with σ_s. Therefore T_W is of type $\langle 0,0\rangle$. ∎

We leave to the reader the proof of the following—easier—result:

Lemma 2. *A willow tree W has a type $\langle 0,2\rangle$ subtree S_W—in fact, we can construct a subtree S_W of W with split-nodes $\langle \sigma_s : s \in 2^{<\omega}\rangle$ so that $s \subset t$*

implies $\sigma_s \subset \sigma_t$ *which satisfies*: *whenever* $t_0 \supseteq s\hat{}\langle 0\rangle$ *and* $t_1 \supseteq s\hat{}\langle 1\rangle$, *then* $\forall n \geq |\sigma_s|\ (\sigma_{t_0}(n) < \sigma_{t_1}(n))$. ■

As in 2.2 we want a kind of "two-dimensional" subtree of a willow tree so that the sections in the two directions are of type $\langle 0,0\rangle$ and of type $\langle 0,2\rangle$, respectively. To this end we construct a system $\Sigma = \langle \sigma_{\langle s,t\rangle} : s,t \in 2^{<\omega} \wedge |s| = |t|\rangle \subseteq W$ so that, letting $P = P(\Sigma) := \{\sigma_{\langle s,t\rangle}\restriction n : n \in \omega \wedge \sigma_{\langle s,t\rangle} \in \Sigma\}$ and calling it a *poplar tree*, we have:

(I) $s' \subset s, t' \subset t \Rightarrow \sigma_{\langle s',t'\rangle} \subset \sigma_{\langle s,t\rangle}$;

(II) $P^f := \{\sigma_{\langle f\restriction i,t\rangle}\restriction n : i,n \in \omega \wedge t \in 2^i\}$ is of type $\langle 0,0\rangle$;

(III) $P_g := \{\sigma_{\langle s,g\restriction i\rangle}\restriction n : i,n \in \omega \wedge s \in 2^i\}$ is as in Lemma 2 (in particular, is of type $\langle 0,2\rangle$);

(IV) whenever $f_i, g_i \in 2^\omega$ $(i \in 2)$, $f_0 \neq f_1$, $s \subseteq f_i$, $f_0(|s|) = 0$, and $f_1(|s|) = 1$, then, putting $\phi_i = \bigcup_n \sigma_{\langle f_i\restriction n, g_i\restriction n\rangle}$, we have

$$\phi_0(m) < \phi_1(m) \quad \text{ for } m \geq |\sigma_{\langle f_1\restriction|s|+1, g_1\restriction|s|+1\rangle}|.$$

((IV) is a kind of strengthening of (III) which we shall need in Lemma 4 below; in fact, (III) itself will not be used.) This construction is done in a similar fashion to the construction in the proof of Lemma 1. Namely, we make the same initial assumptions about $W = (f_W, A_W)$. Next we take $\langle I_s : s \in 2^{<\omega} \wedge |s| \geq 1\rangle$, a partition of ω into intervals of size $2^{|s|+1}$ so that $\max(I_s) + 1 \leq \min(I_t)$ whenever $|s| < |t|$ and $\max(I_s) + 1 \leq \min(I_t)$ for s,t with $|s| = |t|$ iff s precedes t in the lexicographic ordering of $2^{|s|}$. We let $\langle \tau^s_j : j < 2^{|s|+1}\rangle$ and $\langle \varrho^s_j : j < 2^{|s|+1}\rangle$ be defined accordingly.

Construct recursively $\langle \sigma_{\langle s,t\rangle} : s,t \in 2^{<\omega} \wedge |s| = |t|\rangle \subseteq W$:

$$\sigma_{\langle\rangle} = \mathrm{stem}(W), \quad \sigma_{\langle 0,0\rangle} = \sigma_{\langle\rangle}\hat{}\tau_0^{\langle 0\rangle}, \quad \sigma_{\langle 0,1\rangle} = \sigma_{\langle\rangle}\hat{}\varrho_0^{\langle 0\rangle}\hat{}\varrho_1^{\langle 0\rangle}\hat{}\tau_2^{\langle 0\rangle},$$
$$\sigma_{\langle 1,0\rangle} = \sigma_{\langle\rangle}\hat{}\tau_0^{\langle 0\rangle}\hat{}\varrho_1^{\langle 0\rangle}\hat{}\ldots\hat{}\varrho_3^{\langle 0\rangle}\hat{}\tau_0^{\langle 1\rangle}, \quad \sigma_{\langle 1,1\rangle} = \sigma_{\langle\rangle}\hat{}\varrho_0^{\langle 0\rangle}\hat{}\ldots\hat{}\varrho_1^{\langle 1\rangle}\hat{}\tau_2^{\langle 1\rangle}.$$

Assume $\langle \sigma_{\langle s,t\rangle} : s,t \in 2^{\leq n} \wedge |s| = |t|\rangle$ have been defined, $n \geq 1$. Let $\langle s_i : i < 2^n\rangle$ be the lexicographic enumeration of 2^n. We put

$$\sigma_{\langle s_i,s_j\rangle\hat{}\langle 0,0\rangle}$$
$$= \sigma_{\langle s_i,s_j\rangle}\hat{}\tau^{s_i}_{2j+1}\hat{}\varrho^{s_i}_{2j+2}\hat{}\ldots\hat{}\varrho^{s_i}_{2^{n+1}-1}\hat{}\tau^{s_{i+1}}_0\hat{}\ldots\hat{}\tau^{s_{2^n-1}}_{2^{n+1}-1}\hat{}\varrho^{s_0\hat{}\langle 0\rangle}_0\hat{}\ldots\hat{}\varrho^{s_i\hat{}\langle 0\rangle}_{4j-1}\hat{}\tau^{s_i\hat{}\langle 0\rangle}_{4j},$$
$$\sigma_{\langle s_i,s_j\rangle\hat{}\langle 0,1\rangle}$$
$$= \sigma_{\langle s_i,s_j\rangle}\hat{}\varrho^{s_i}_{2j+1}\hat{}\ldots\hat{}\varrho^{s_i}_{2^{n+1}-1}\hat{}\tau^{s_{i+1}}_0\hat{}\ldots\hat{}\tau^{s_{2^n-1}}_{2^{n+1}-1}\hat{}\varrho^{s_0\hat{}\langle 0\rangle}_0\hat{}\ldots\hat{}\varrho^{s_i\hat{}\langle 0\rangle}_{4j+1}\hat{}\tau^{s_i\hat{}\langle 0\rangle}_{4j+2},$$
$$\sigma_{\langle s_i,s_j\rangle\hat{}\langle 1,0\rangle}$$
$$= \sigma_{\langle s_i,s_j\rangle}\hat{}\tau^{s_i}_{2j+1}\hat{}\varrho^{s_i}_{2j+2}\hat{}\ldots\hat{}\varrho^{s_i}_{2^{n+1}-1}\hat{}\tau^{s_{i+1}}_0\hat{}\ldots\hat{}\tau^{s_{2^n-1}}_{2^{n+1}-1}\hat{}\varrho^{s_0\hat{}\langle 0\rangle}_0\hat{}\ldots\hat{}\varrho^{s_i\hat{}\langle 1\rangle}_{4j-1}\hat{}\tau^{s_i\hat{}\langle 1\rangle}_{4j},$$
$$\sigma_{\langle s_i,s_j\rangle\hat{}\langle 1,1\rangle}$$
$$= \sigma_{\langle s_i,s_j\rangle}\hat{}\varrho^{s_i}_{2j+1}\hat{}\ldots\hat{}\varrho^{s_i}_{2^{n+1}-1}\hat{}\tau^{s_{i+1}}_0\hat{}\ldots\hat{}\tau^{s_{2^n-1}}_{2^{n+1}-1}\hat{}\varrho^{s_0\hat{}\langle 0\rangle}_0\hat{}\ldots\hat{}\varrho^{s_i\hat{}\langle 1\rangle}_{4j+1}\hat{}\tau^{s_i\hat{}\langle 1\rangle}_{4j+2}.$$

As in the proof of Lemma 1, we verify that this $\Sigma = \langle \sigma_{\langle s,t\rangle} : s,t \in 2^{<\omega} \wedge |s| = |t|\rangle$ satisfies (I)–(IV):

LEMMA 3. *A willow tree W has a poplar subtree $P(\Sigma_W)$.* ∎

We can now complete the proof of Theorem 2.8 along similar lines to the proof of 2.2 with the following lemmata:

LEMMA 4. *Assume $P = P(\Sigma_W)$ is a poplar subtree of a willow tree W, and S is a Sacks tree of one of the seven types $\langle i,j\rangle \in 3^2 \setminus \{\langle 0,2\rangle, \langle 2,0\rangle\}$. Then $|\{f \in 2^\omega : |[S \cap P^f]| \ge 1\}| \le \omega$.*

Proof. We look at $\widehat{T} := \{\langle s,t\rangle : s,t \in 2^{<\omega} \wedge |s| = |t| \wedge \sigma_{\langle s,t\rangle} \in S \cap P\}$. This is a compact tree in the plane, hence its projection onto the first coordinate is also compact, and thus has either at most countably many branches or contains a perfect subtree T. In the first case, we are done, so assume the latter.

Put $s := \mathrm{stem}(T)$, and note that there must be $t_0, t_1 \in 2^{|s|+1}$ so that both $T_i := \widehat{T}_{\langle s\hat{\ }\langle i\rangle, t_i\rangle}$ $(i \in 2)$ contain perfect trees. Find incompatible extensions $\langle\langle s_i^j, t_i^j\rangle : i,j < 2\rangle$, $\langle s_i^j, t_i^j\rangle \in T_i$, and let $\langle f_i^j, g_i^j\rangle$ be branches of T_i through $\langle s_i^j, t_i^j\rangle$. Put (as in (IV)) $\phi_i^j := \bigcup_n \sigma_{\langle f_i^j \restriction n, g_i^j \restriction n\rangle}$, and let k_i be minimal with $\phi_i^0(k_i) \ne \phi_i^1(k_i)$. It is a consequence of (IV) that we must have $\phi_0^j(k_i) < \phi_1^k(k_i)$ for $i,j,k \in 2$. This entails (by definition of the types) that S is either of type $\langle 0,2\rangle$ or of type $\langle 2,0\rangle$, a contradiction. ∎

The argument of the following result is similar, but much easier (just note that P^f is of type $\langle 0,0\rangle$ (II)):

LEMMA 5. *Assume $P = P(\Sigma_W)$ is a poplar subtree of a willow tree W, and S is a Sacks tree of type $\langle 0,2\rangle$ or $\langle 2,0\rangle$. Then for all $f \in 2^\omega$, $|[S \cap P^f]| \le \omega$.* ∎

Putting Lemmata 4 and 5 together we get:

COROLLARY. *Assume $P = P(\Sigma_W)$ is a poplar subtree of a willow tree W, and $\mathcal{S} \subseteq \{S \in \mathbb{S} : S$ is of type $\langle i,j\rangle$ for some $\langle i,j\rangle \in 3^2\}$ is a family of Sacks trees of size $< \mathfrak{c}$. Then $|[P] \setminus \bigcup_{S\in\mathcal{S}} [S]| = \mathfrak{c}$.*

Proof. Use Lemma 4 to find $f \in 2^\omega$ so that $[S \cap P^f] = \emptyset$ for all $S \in \mathcal{S}$ of one of the types $\langle i,j\rangle \in 3^2 \setminus \{\langle 0,2\rangle, \langle 2,0\rangle\}$. Then use Lemma 5 to find many g's so that $\phi = \bigcup_n \sigma_{\langle f\restriction n, g\restriction n\rangle}$ is as required. ∎

Now we can conclude the proof of Theorem 2.8 with the usual argument. ∎

2.9. THEOREM. $t^0 \setminus r^0 \ne \emptyset$.

Proof. Exercise! (This is again similar to 2.5.) ∎

2.10. We conclude this section with a result and a question about orthogonality of our ideals.

PROPOSITION. *The pairs of ideals* (m^0, w^0), (m^0, v^0), (ℓ^0, w^0), (ℓ^0, v^0) *and* (v^0, t^0) *are orthogonal.*

Proof. The first four follow from Proposition 3.2. To see the last, call $V \in \mathbb{V}$ an *oak tree* iff given $i_0 < i_1$, both in $\omega \setminus \mathrm{dom}(f_V)$, there is $i_2 \in f_V^{-1}(\{1\})$ with $i_0 < i_2 < i_1$ (here f_V denotes the Silver condition in the usual notation associated with V). $T \in \mathbb{T}$ is an *almond tree* iff $|a_n| \geq 2$ and $\max(a_n) < \min(a_{n+1})$, where $(s, \{a_n : n \in \omega\})$ is the Matet condition in the usual notation associated with T. The oak trees are dense in $\mathbb{V}$ and the almond trees are dense in $\mathbb{T}$. It is easily seen that $|[T \cap V]| \leq 1$ in case V is an oak tree and T is an almond tree. Hence, if $\langle V_\alpha : \alpha < \mathfrak{c}\rangle = \{\text{oak trees}\}$ and $\langle T_\alpha : \alpha < \mathfrak{c}\rangle = \{\text{almond trees}\}$, we can easily construct $\langle V'_\alpha : \alpha < \mathfrak{c}\rangle$ and $\langle T'_\alpha : \alpha < \mathfrak{c}\rangle$ so that for all α, β:

(i) $T'_\alpha \leq T_\alpha$, $V'_\alpha \leq V_\alpha$;
(ii) $[T'_\alpha \cap V'_\beta] = \emptyset$.

Then $\bigcup_{\alpha<\mathfrak{c}}[T'_\alpha]$ is a t^1-set in v^0. ∎

QUESTION. *Are* ℓ^0 *and* t^0 *orthogonal?*

As neither $\mathbb{L} \subseteq \mathbb{T}$ nor $\mathbb{T} \subseteq \mathbb{L}$, a positive answer seems plausible.

3. Some results concerning the Mycielski ideal $\mathfrak{P}_2$

3.1. Given $A \subseteq 2^\omega$ and $X \in [\omega]^\omega$ we let $A{\restriction}X := \{f{\restriction}X : f \in A\}$. Recall that the *Mycielski ideal* $\mathfrak{P}_2$ is defined as follows:

$$\mathfrak{P}_2 := \{A \subseteq 2^\omega : \forall X \in [\omega]^\omega\ (A{\restriction}X \neq 2^X)\}.$$

$\mathfrak{P}_2$ is easily seen to be a σ-ideal on the reals which is contained in both v^0 and w^0 (see [CRSW] for more on $\mathfrak{P}_2$).

As usual we shall be concerned with the isomorphic copy of $\mathfrak{P}_2$ in the space $\omega^{\uparrow\omega}$,

$$\overline{F}(\mathfrak{P}_2) := \{\widehat{F}[A] : A \in \mathfrak{P}_2 \wedge \forall f \in A\ (|f^{-1}(\{1\})| = \omega)\}$$

(compare 1.3), and actually mean the latter ideal when talking about $\mathfrak{P}_2$.

It follows from the results in Section 2 (and $\mathfrak{P}_2 \subseteq v^0, w^0$) that $i^0 \setminus \mathfrak{P}_2 \neq \emptyset$ for any of the ideals i^0 considered so far; similarly $\mathfrak{P}_2 \setminus s^0(m^0, \ell^0) \neq \emptyset$ is easy to see. We conclude this cycle of results by showing:

THEOREM. $\mathfrak{P}_2 \setminus t^0 \neq \emptyset$ (*and thus* $\mathfrak{P}_2 \setminus r^0 \neq \emptyset$).

Note. Theorems 2.3 and 2.4 immediately follow from this result.

Proof of Theorem. Given $X \in [\omega]^\omega$ and $f \in 2^X$ with $|f^{-1}(\{1\})| = \omega$, let $S_f := \{g \in 2^\omega : f \subseteq g\}$ and $R_f := \{\widehat{F}(g) : g \in S_f\} \subseteq \omega^{\uparrow\omega}$, the *closed* *set associated with* f (see 1.2 for the map $\widehat{F}$). Let $\langle X_\alpha : \alpha < \mathfrak{c}\rangle$ enumerate $[\omega]^\omega$ and $\langle T_\alpha : \alpha < \mathfrak{c}\rangle = \mathbb{T}$. We shall construct $\langle f_\alpha : \alpha < \mathfrak{c}\rangle$ and $\langle y_\alpha : \alpha < \mathfrak{c}\rangle$ so that for all α:

(i) $f_\alpha \in 2^{X_\alpha}$, $|f_\alpha^{-1}(\{1\})| = \omega$;
(ii) $y_\alpha \in [T_\alpha]$;
(iii) $y_\alpha \notin \bigcup_{\beta<\mathfrak{c}} R_{f_\beta}$.

Then we clearly have $Y := \{y_\alpha : \alpha < \mathfrak{c}\} \notin t^0, r^0$. But also $Y \cap R_{f_\alpha} = \emptyset$ for all $\alpha < \mathfrak{c}$, and thus $Y \in \mathfrak{P}_2$. Suppose we are at step α in the construction; then we can easily find $f_\alpha \in 2^{X_\alpha}$ so that $|f_\alpha^{-1}(\{1\})| = \omega$ and $\widehat{F}^{-1}(y_\beta)\restriction X_\alpha \neq f_\alpha$ for all $\beta < \alpha$ (this entails $y_\beta \notin R_{f_\alpha}$). Thus the main point is to find $y_\alpha \in [T_\alpha] \setminus (\bigcup_{\beta\le\alpha} R_{f_\beta})$.

Given a Matet tree $T \in \mathbb{T}$ construct an *elm subtree* $E_T = \{\sigma_s\restriction n : s \in 2^{<\omega} \wedge n \in \omega\}$ as follows:

(I) $\sigma_{\langle\rangle} = \mathrm{stem}(T)$;

(II) if σ_s are constructed for all s of length $\le n$, choose 2^{n+1} distinct finite subsets of ω, $\langle a_i : i \in 2^{n+1}\rangle$ with $\max\{\max \mathrm{rng}(\sigma_s) : s \in 2^n\} < \min(a_i)$ and $a_i \in A_T$ ($i \in 2^{n+1}$), where A_T is the second coordinate in the Matet condition in usual notation, and put

$$\sigma_{s_i\hat{}\langle 0\rangle} = \sigma_{s_i}\hat{}\tau_{2i}, \qquad \sigma_{s_i\hat{}\langle 1\rangle} = \sigma_{s_i}\hat{}\tau_{2i+1},$$

where $\{s_i : i \in 2^n\}$ enumerates 2^n and τ_j is the increasing enumeration of a_j ($j \in 2^{n+1}$).

Claim. *$|[E_T] \cap R_f| \le 1$ whenever E_T is an elm tree and $f \in 2^X$ ($X \in [\omega]^\omega$) is such that $|f^{-1}(\{1\})| = \omega$.*

Proof. Simply note that two distinct branches of E_T have almost disjoint ranges while $f^{-1}(\{1\})$ is contained in the range of any real in R_f. ∎

Using this claim we easily conclude the construction of y_α. ∎

3.2. Proposition. *$\mathfrak{P}_2$ is orthogonal to m^0 and ℓ^0; it is not orthogonal to any of the other ideals.*

Proof. Note that if i^0 and $\mathfrak{P}_2$ are orthogonal, then i^0 and w^0 (v^0) are orthogonal as well. This proves non-orthogonality for most ideals.

We proceed to show the orthogonality of $(\mathfrak{P}_2, m^0)$ (the orthogonality of $(\mathfrak{P}_2, \ell^0)$ is proved in a similar fashion). We use the notation of the proof of Theorem 3.1. Call a Miller tree M a *plum tree* iff given $\sigma, \tau \in \mathrm{split}(M)$ distinct, the sets $\bigcup\{\mathrm{rng}(\varrho) \setminus \mathrm{rng}(\sigma) : \varrho \in \mathrm{Succ}_M(\sigma)\}$ and $\bigcup\{\mathrm{rng}(\varrho) \setminus \mathrm{rng}(\tau) : \varrho \in \mathrm{Succ}_M(\tau)\}$ are disjoint. Let $\langle X_\alpha : \alpha < \mathfrak{c}\rangle$ enumerate $[\omega]^\omega$ and $\langle M_\alpha : \alpha < \mathfrak{c}\rangle = \{M \in \mathbb{M} : M \text{ is a plum tree}\}$. We construct recursively $\langle f_\alpha : \alpha < \mathfrak{c}\rangle$ and $\langle M'_\alpha : \alpha < \mathfrak{c}\rangle$ so that for all α:

(i) $f_\alpha \in 2^{X_\alpha}$, $|f_\alpha^{-1}(\{1\})| = \omega$;
(ii) $M'_\alpha \le M_\alpha$;
(iii) $[M'_\alpha] \cap \bigcup_{\beta<\mathfrak{c}} R_{f_\beta} = \emptyset$.

Then $Y := \bigcup_{\alpha<\mathfrak{c}} [M'_\alpha]$ is an m^1-set which lies in $\mathfrak{P}_2$. Assume we are at step α; find $f_\alpha \in 2^{X_\alpha}$ so that $|f_\alpha^{-1}(\{1\})| = \omega$ and $f_\alpha \notin \widehat{F}^{-1}([M'_\beta]) \restriction X_\alpha$ for $\beta < \alpha$ (this is easy to do because for $f \ne g \in [M'_\beta]$, $|(\widehat{F}^{-1}(f))^{-1}(\{1\}) \cap (\widehat{F}^{-1}(g))^{-1}(\{1\})| < \omega$, and $[\omega]^\omega$ is not the union of less than continuum many almost disjoint families of subsets of ω). Next note that, as in the proof of 3.1, we have $|[M_\alpha] \cap R_{f_\beta}| \le 1$ for $\beta \le \alpha$. Thus we find $M'_\alpha \le M_\alpha$ with $[M'_\alpha] \cap R_{f_\beta} = \emptyset$ for $\beta \le \alpha$. This concludes the construction. ∎

3.3. We next turn our attention to consistency results by showing that v^0 may be "much larger" than $\mathfrak{P}_2$:

THEOREM. *It is consistent with ZFC that* $\omega_1 = \mathrm{cov}(v^0) < \mathrm{cov}(\mathfrak{P}_2) = \omega_2 = \mathfrak{c}$.

This Theorem will be proved with an iterated forcing construction with countable support of length ω_2 over a model for CH. We start with the definition of the forcing we want to iterate, *Sacks forcing with uniform levels* $\mathbb{U}$:

$$S \in \mathbb{U} \Leftrightarrow S \in \mathbb{S} \wedge \exists A_S \in [\omega]^\omega\ \forall \sigma \in S\ (\sigma \in \mathrm{split}(S) \leftrightarrow |\sigma| \in A_S),$$
$$S \le T \Leftrightarrow S \subseteq T$$

(note that we work in $2^{<\omega}$ and 2^ω in this subsection). Standard arguments show that $\mathbb{U}$ is proper (satisfies even axiom A) and ω^ω-bounding (see [Bau], [Je2] or [Sh] for these notions). We shall be interested in showing that it shares with Sacks forcing $\mathbb{S}$ yet another nice property preserved in countable support iterations (which is not shared by Silver forcing). To do this recall that a non-principal ultrafilter $\mathcal{U}$ on ω is called a *P-point* iff for all $\langle A_n : n \in \omega\rangle$ with $A_n \in \mathcal{U}$ there is $B \in \mathcal{U}$ with $B \subseteq^* A_n$ for all n. A forcing notion $\mathbb{P} \in V$ is called *P-point-preserving* iff whenever $\mathcal{U} \in V$ is a P-point, then $\mathcal{U}$ generates a P-point in $V[G]$, where G is $\mathbb{P}$-generic over V.

LEMMA 1. $\mathbb{U}$ *is P-point-preserving.*

Remark. The proof is a mild variation of known arguments, due to Miller and Blass (see [Mi1, Proposition 4.2 and Remarks thereafter]), saying that $\mathbb{M}$ and $\mathbb{S}$ are P-point-preserving.

Proof. Let $\mathcal{U}$ be a P-point. It is well known that it follows from the properness of $\mathbb{U}$ that it suffices to show that $\mathcal{U}$ generates an ultrafilter in $V[G]$ (see, e.g., [BlSh, Lemma 3.2]).

Choose $S \in \mathbb{U}$, and let $\dot{B}$ be a $\mathbb{U}$-name for an infinite subset of ω. Let $\dot{\chi}$ be the $\mathbb{U}$-name for the characteristic function of $\dot{B}$. Let $\langle n_i : i \in \omega\rangle$ be the increasing enumeration of A_S. We may assume that if $\sigma \in S \cap 2^{n_i}$, then S_σ

decides $\dot\chi\restriction i$ (otherwise go over to a stronger $S' \le S$, using a standard fusion argument).

With each $f \in [S]$ we associate a set $B_f \subseteq \omega$ which reflects f's opinion about $\dot B$:

$$i \in B_f \Leftrightarrow S_{f\restriction n_{i+1}} \Vdash \dot\chi(i) = 1$$

(notice that $S_{f\restriction n_{i+1}}$ decides $\dot\chi(i)$). Assume that

$(*)$ for all $\sigma \in S$ there is $f \in [S_\sigma]$ so that $B_f \in \mathcal{U}$.

Then construct a decreasing sequence of sets $\langle B_i \in \mathcal{U} : i \in \omega\rangle$ as follows:

- choose $f \in [S]$ so that $B_f \in \mathcal{U}$, and put $B_0 := B_f$;
- assume B_{i-1} $(i \ge 1)$ is constructed; find for each of the 2^i split-nodes σ of S of length 2^{n_i} an $f_\sigma \in [S]$ extending σ with $B_{f_\sigma} \in \mathcal{U}$, and let B_i be the intersection of B_{i-1} and these 2^i B_f's.

In case $(*)$ fails for some $\sigma \in S$, we can make a similar construction of sets $\langle C_i \in \mathcal{U} : i \in \omega\rangle$ below σ, this time taking complements of sets of the form B_f, and intersecting them. The rest of the proof is the same, whether or not $(*)$ holds, so assume the former is the case.

$\mathcal{U}$ being a P-point, we find $B \in \mathcal{U}$ which is almost included in all B_i. Construct a sequence $\langle k_i : i \in \omega\rangle$ so that $k_0 = 0$ and $B \setminus B_{k_i+1} \subseteq k_{i+1}$. Note that either $B' := B \cap \bigcup_i [k_{2i}, k_{2i+1}) \in \mathcal{U}$ or $B \cap \bigcup_i [k_{2i+1}, k_{2i+2}) \in \mathcal{U}$. Without loss assume the former holds. We construct $S' \le S$ with $A_{S'} = \{n_{k_{2i+1}} : i \in \omega\}$ such that whenever $f \in [S']$, then $B' \setminus [k_0, k_1) \subseteq B_f$ (and thus $S' \Vdash B' \subseteq^* \dot B$):

- let $\sigma_{\langle\rangle} \in S$ of length n_{k_1} be arbitrary; choose two split-nodes $\widetilde\sigma_{\langle 0\rangle}, \widetilde\sigma_{\langle 1\rangle}$ of S extending $\sigma_{\langle\rangle}$ of length n_{k_1+1}; let $f_{\tilde\sigma_{\langle 0\rangle}}$ and $f_{\tilde\sigma_{\langle 1\rangle}}$ be as in the above construction (thus we will have $B_{f_{\tilde\sigma_{\langle j\rangle}}} \supseteq B_{k_1+1} \supseteq B' \setminus [k_0, k_1)$), and put $\sigma_{\langle j\rangle} := f_{\tilde\sigma_{\langle j\rangle}}\restriction n_{k_3}$;
- assume $\langle \sigma_s : s \in 2^i\rangle$, $i \ge 1$, are constructed, and of length $n_{k_{2i+1}}$; choose two split-nodes $\widetilde\sigma_{s\hat{}\langle j\rangle}$ $(j \in 2)$ of S extending σ_s of length $n_{k_{2i+1}+1}$; let $f_j \in [S]$ be the extension of $\widetilde\sigma_{s\hat{}\langle j\rangle}$ in the above construction (thus we will have $B_{f_j} \supseteq B_{k_{2i+1}+1} \supseteq B' \setminus [k_0, k_1)$), and put $\sigma_{s\hat{}\langle j\rangle} := f_j\restriction n_{k_{2i+3}}$.

This completes the construction, and thus the proof of the Lemma. ∎

LEMMA 2 (Shelah; see, e.g., [BlSh, Theorem 4.1]). *If* $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha : \alpha < \lambda\rangle$ *is a countable support iteration of proper forcing notions so that*

$$\Vdash_{\mathbb{P}_\alpha} \text{“}\dot{\mathbb{Q}}_\alpha \textit{ is } P\textit{-point-preserving”},$$

then $\mathbb{P}_\lambda$ *is* P*-point-preserving as well.* ∎

The *reaping number* $\mathfrak{r}$ is the size of the smallest family $\mathcal{F}$ of infinite subsets of ω so that given any $A \in [\omega]^\omega$, there is $B \in \mathcal{F}$ with either $B \subseteq A$ or $B \cap A = \emptyset$. A translation to cardinal invariants of the well-known fact that Silver forcing $\mathbb{V}$ adjoins a new subset of ω which neither contains nor is disjoint from an old infinite subset of ω leads to:

LEMMA 3 (Folklore). $\mathrm{cov}(v^0) \le \mathfrak{r}$.

Proof. Let $\widehat{G} : 2^\omega \to 2^\omega$ be defined by $\widehat{G}(f) = g$ iff $[g(0) = f(0)$ and $(g(n+1) = g(n)$ iff $f(n+1) = 0)]$. Then $\widehat{G}$ is a homeomorphism of 2^ω. Now let $\mathcal{F} \subseteq [\omega]^\omega$ be a witness for $\mathfrak{r}$ of size $\mathfrak{r}$. Given $A \in [\omega]^\omega$, let $\widetilde{A}_i = \{f \in 2^\omega : \forall n \in A\ (f(n) = i)\}$, and let $\widehat{A}_i = \widehat{G}^{-1}[\widetilde{A}_i]$ $(i = 0, 1)$.

We claim that $\widehat{A}_i \in v^0$. To see this let $f \in \mathbb{V}$; choose $n \in \omega$ minimal with $n \notin \mathrm{dom}(f)$. Let $m > n$ be minimal with $m \in A$. There are (at least) two ways to extend f to $\widehat{f}$ so that $(m+1) \subseteq \mathrm{dom}(\widehat{f})$; according to which way we choose we either have $\widehat{G}(g)(m) = 0$ for all $g \supseteq \widehat{f}$ or $\widehat{G}(g)(m) = 1$ for all $g \supseteq \widehat{f}$. Choosing the first we get $\widehat{G}(\{g : g \supseteq \widehat{f}\}) \cap \widetilde{A}_1 = \emptyset$, i.e. $\{g : g \supseteq \widehat{f}\} \cap \widehat{A}_1 = \emptyset$. Thus $\widehat{A}_1 \in v^0$. Similarly for $\widehat{A}_0$.

Next, $\{\widetilde{A}_i : A \in \mathcal{F} \wedge i \in 2\}$ is covering; hence so is the inverse image under $\widehat{G}$. ■

Proof of the Theorem. Our model is the generic extension by the countable support iteration of $\mathbb{U}$ of length ω_2 over a model for CH. By properness, no cardinals are collapsed and $\mathfrak{c} = \omega_2$ in the resulting model. By Lemmata 1 and 2, there is a P-point generated by ω_1 sets in the latter model; the base of this P-point is a reaping family, and thus witnesses $\mathfrak{r} = \omega_1$; hence—by Lemma 3—$\mathrm{cov}(v^0) = \omega_1$.

Next note that, letting u^0 be the σ-ideal corresponding to $\mathbb{U}$ (as in 1.3), we have $\mathfrak{P}_2 \subseteq u^0$ (this comes from the uniform levels); thus $\mathrm{cov}(\mathfrak{P}_2) \ge \mathrm{cov}(u^0)$. However, a standard Löwenheim–Skolem argument shows that $\mathrm{cov}(u^0) = \omega_2$ in our model (see [JMS, Theorem 1.2] for the corresponding result for Sacks forcing $\mathbb{S}$ and Marczewski's ideal s^0); hence $\mathrm{cov}(\mathfrak{P}_2) = \omega_2$, too. ■

3.4. We have seen in 3.3 that for two specific ideals $\mathcal{I}$, $\mathcal{J}$ with $\mathcal{I} \not\subseteq \mathcal{J}$, $\mathrm{cov}(\mathcal{I}) < \mathrm{cov}(\mathcal{J})$ is consistent. We conjecture that this is true for any such combination of our ideals. In many cases this is quite easy to see. For example, to show the consistency of $\mathrm{cov}(\ell^0) < \mathrm{cov}(m^0)$, simply iterate Miller forcing ω_2 times. Then a Löwenheim–Skolem argument shows $\mathrm{cov}(m^0) = \omega_2$ in the final model; on the other hand, it is well known [Mi1] that $\mathfrak{b} = \omega_1$ holds, where $\mathfrak{b}$, the *unbounding number*, is the smallest family $\mathcal{F}$ of functions from ω to ω so that for every $g \in \omega^\omega$ there is $f \in \mathcal{F}$ which is infinitely often above g. An easy translation (as in Lemma 3 in 3.3) of the folklore fact that Laver forcing adds a dominating real shows $\mathrm{cov}(\ell^0) \le \mathfrak{b}$; hence $\mathrm{cov}(\ell^0) = \omega_1$. However, the consistency of $\mathrm{cov}(m^0) < \mathrm{cov}(\ell^0)$ is open—and so are several similar problems.

As $\mathfrak{P}_2 \subseteq w^0$, the result in 3.3 leads to the following, more specific, problem.

QUESTION. *Is* $\mathrm{cov}(w^0) = \omega_1$ *in the model of Theorem* 3.3? *Is* $\mathrm{cov}(w^0) < \mathrm{cov}(\mathfrak{P}_2)$ *consistent*?

A positive answer seems plausible.

4. Odds and ends

4.1. Recall that a non-principal ultrafilter $\mathcal{U}$ on ω is *Ramsey* (or *selective*) iff given a partition $\pi : [\omega]^2 \to 2$ there is $A \in \mathcal{U}$ *homogeneous for* π (this means $|\pi[A]^2| = 1$) iff given a partition of ω into infinitely many (infinite) pieces $\langle A_n : n \in \omega\rangle$ not in $\mathcal{U}$ there is $A \in \mathcal{U}$ with $|A \cap A_n| \leq 1$ for all n (see [Je1, Lemma 38.1] for the latter equivalence). Ramsey ultrafilters may not exist [Je1, Theorem 91]; but for the remainder of this section we assume there is at least one Ramsey ultrafilter, called $\mathcal{U}$. $\mathbb{R}_{\mathcal{U}}$ is *Mathias forcing with respect to* $\mathcal{U}$; i.e. $(s, A) \in \mathbb{R}_{\mathcal{U}}$ iff $(s, A) \in \mathbb{R}$ and $A \in \mathcal{U}$, the ordering being inherited from $\mathbb{R}$ (as usual we can think of $\mathbb{R}_{\mathcal{U}}$ as forcing with certain subtrees of $\omega^{\uparrow<\omega}$). We denote by $r^0_{\mathcal{U}}$ the corresponding σ-ideal; i.e.

$$X \in r^0_{\mathcal{U}} \Leftrightarrow X \subseteq \omega^{\uparrow\omega} \wedge \forall T \in \mathbb{R}_{\mathcal{U}}\ \exists S \leq T\ (S \in \mathbb{R}_{\mathcal{U}} \wedge X \cap [S] = \emptyset).$$

$r^0_{\mathcal{U}}$ was studied by Louveau [Lo] and Corazza [Co]. It is easy to see that $r^0_{\mathcal{U}} \setminus i^0 \neq \emptyset$ for any of the ideals i^0 considered so far; on the other hand, it follows from our results in Section 2 that $i^0 \setminus r^0_{\mathcal{U}} \neq \emptyset$, as well (in case $i^0 = s^0$, this answers [Co, Problem 7]).

Given a σ-ideal $\mathcal{I}$ on the reals, let $\mathrm{non}(\mathcal{I})$ be the size of the smallest set not in $\mathcal{I}$ (note that $\mathrm{non}(\mathcal{I})$ is sometimes denoted by $\mathrm{unif}(\mathcal{I})$, the *uniformity* of $\mathcal{I}$). It is well known that $\mathrm{non}(\mathcal{I}) = \mathfrak{c}$ for any of the ideals $\mathcal{I}$ investigated in Sections 2 and 3; Corazza [Co, Theorem 4.3] proved that $\mathrm{non}(r^0_{\mathcal{U}}) = \omega_1 < \mathfrak{c}$ is consistent. We shall try to relate $\mathrm{non}(r^0_{\mathcal{U}})$ to more familiar cardinal invariants.

Given an ultrafilter $\mathcal{V}$ on ω, let $\mathfrak{g}(\mathcal{V})$, the *generating number* of $\mathcal{V}$, be the size of the smallest base for $\mathcal{V}$. The *homogeneity number* $\mathfrak{hom}$ [Bl3, Section 6] is the size of the smallest family $\mathcal{H}$ of subsets of ω so that for each partition $\pi : [\omega]^2 \to 2$ there is $H \in \mathcal{H}$ homogeneous for π.

PROPOSITION. $\mathfrak{hom} \leq \mathrm{non}(r^0_{\mathcal{U}}) \leq \mathfrak{g}(\mathcal{U})$.

Proof. Assume $\{A_\alpha : \alpha < \mathfrak{g}(\mathcal{U})\}$ generates $\mathcal{U}$. Let $\chi_\alpha \in 2^\omega$ be the characteristic function of A_α. Put $X := \{f \in 2^\omega : \exists \alpha < \mathfrak{g}(\mathcal{U})\ \forall^\infty n\ (f(n) = \chi_\alpha(n))\} \subseteq 2^\omega$. Clearly $|X| = \mathfrak{g}(\mathcal{U})$; on the other hand, $X \notin r^0_{\mathcal{U}}$ (here, we think of $r^0_{\mathcal{U}}$ as an ideal on 2^ω instead of ω^ω, i.e. we really work with $\overline{F}^{-1}(r^0_{\mathcal{U}})$—see 1.3). To see this take $(s, A) \in \mathbb{R}_{\mathcal{U}}$ arbitrarily; as $A \in \mathcal{U}$, choose $\alpha < \mathfrak{g}(\mathcal{U})$ with $A_\alpha \subseteq A$; let $f \in 2^\omega$ be such that $f(n) = 1$ iff

$n \in \mathrm{rng}(s) \cup A_\alpha$. Then $f \in X$, but f also lies in the subset of 2^ω defined by the condition (s, A) (by this we mean $\widehat{F}(f) \in T$, where T is the subtree of $\omega^{\uparrow<\omega}$ corresponding to the condition (s, A)). This proves $\mathrm{non}(r^0_{\mathcal{U}}) \le \mathfrak{g}(\mathcal{U})$.

To see $\mathfrak{hom} \le \mathrm{non}(r^0_{\mathcal{U}})$, let $\kappa < \mathfrak{hom}$ and $X = \{x_\alpha : \alpha < \kappa\} \subseteq 2^\omega$. Put $A_\alpha := x_\alpha^{-1}(\{1\})$. Choose a partition $\pi : [\omega]^2 \to 2$ so that no $A_\alpha \setminus n$ is homogeneous for π ($\alpha < \kappa$, $n \in \omega$). $\mathcal{U}$ being Ramsey, there is $A \in \mathcal{U}$ homogeneous for π. Now note that for every $(s, B) \in \mathbb{R}_{\mathcal{U}}$, $[T] \cap \widehat{F}[X] = \emptyset$ for the tree T associated with the condition $(s, A \cap B)$. Hence $X \in r^0_{\mathcal{U}}$. ∎

Corollary. (MA) $\mathrm{non}(r^0_{\mathcal{U}}) = \mathfrak{c}$. ∎

(This answers [Co, Problem 9].)

Question. *Does either* $\mathrm{non}(r^0_{\mathcal{U}}) = \mathfrak{g}(\mathcal{U})$ *or* $\mathrm{non}(r^0_{\mathcal{U}}) = \mathfrak{hom}$ *always hold? Or are there models* V_1, V_2 *of ZFC containing Ramsey ultrafilters* $\mathcal{U}_1 \in V_1$ *and* $\mathcal{U}_2 \in V_2$ *so that* $V_1 \models \mathrm{non}(r^0_{\mathcal{U}_1}) < \mathfrak{g}(\mathcal{U}_1)$ *and* $V_2 \models \mathfrak{hom} < \mathrm{non}(r^0_{\mathcal{U}_2})$?

4.2. We shall comment on the relation between $\mathfrak{hom}$ and other cardinal invariants of the continuum (though this is not directly related to our principal object of study, certain σ-ideals on the reals, we feel that what follows rounds off our work).

Let $\mathfrak{r}_\sigma$ be the size of the smallest family $\mathcal{F}$ of subsets of ω so that for each countable sequence $\langle Y_n : n \in \omega\rangle$ of infinite subsets of ω there is $F \in \mathcal{F}$ almost contained in either Y_n or $\omega \setminus Y_n$ for all n. Recall the definition of $\mathfrak{r}$ from 3.3. Finally, let $\mathfrak{d}$ be the *dominating number*, the size of the smallest family $\mathcal{F}$ of functions from ω to ω so that every function $f \in \omega^\omega$ is dominated everywhere by a function in $\mathcal{F}$. Blass proved [Bl3, Theorem 17]

$$\max(\mathfrak{r}, \mathfrak{d}) \le \mathfrak{hom} \le \max(\mathfrak{r}_\sigma, \mathfrak{d}).$$

We show (answering [Bl3, Question 5] positively):

Proposition. $\mathfrak{hom} = \max(\mathfrak{r}_\sigma, \mathfrak{d})$.

Proof. By Blass' result it suffices to prove that $\mathfrak{r}_\sigma \le \mathfrak{hom}$. To do this, let $\mathcal{H} \subseteq [\omega]^\omega$ be such that

$(\star)$ for all $\pi : [\omega]^2 \to 2$ there is $H \in \mathcal{H}$ homogeneous for π.

It suffices to show

$(\star\star)$ $\forall \langle Y_n : n \in \omega\rangle \subset [\omega]^\omega \ \exists X \in \mathcal{H} \ \forall n \in \omega \ (X \subseteq^* Y_n \vee X \subseteq^* \omega \setminus Y_n)$.

To do this we associate with $\langle Y_n : n \in \omega\rangle$ a partition $\pi : [\omega]^2 \to 2$ as follows.

Step 0. We first define π on all pairs $\{x, y\}$ so that $x \in Y_0$ and $y \in \omega \setminus Y_0$. For such pairs let

$$\pi(\{x, y\}) = \begin{cases} 0 & \text{iff } x < y, \\ 1 & \text{iff } x > y. \end{cases}$$

Step 1. Next we define π on all pairs $\{x, y\}$ so that

(i) π has not yet been defined on $\{x, y\}$;
(ii) $x \in Y_1$ and $y \in \omega \setminus Y_1$.

For such pairs let

$$\pi(\{x, y\}) = \begin{cases} 0 & \text{iff } x < y, \\ 1 & \text{iff } x > y, \end{cases}$$

etc.

If, in the end, π has not yet been defined on all pairs $\{x, y\}$ we define it arbitrarily on the remaining pairs. By $(\star)$ choose $H \in \mathcal{H}$ homogeneous for π. Let us check that H satisfies condition $(\star\star)$. To reach a contradiction, assume there is $n \in \omega$ so that

$$(+) \qquad |H \cap Y_n| = \omega = |H \cap (\omega \setminus Y_n)|.$$

Let n_0 be the least such n. Let $k \in \omega$ be so that for all $n < n_0$,

$$(++) \qquad \text{either } H \setminus k \subseteq Y_n \text{ or } H \setminus k \subseteq \omega \setminus Y_n.$$

By $(+)$ there are $x_0, x_1, x_2 \in H$ so that $k \le x_0 < x_1 < x_2$ and $x_0 \in Y_{n_0}$, $x_1 \in \omega \setminus Y_{n_0}$ and $x_2 \in Y_{n_0}$. By $(++)$, π was not defined for the pairs $\{x_0, x_1\}$ and $\{x_1, x_2\}$ in the steps up to step $n_0 - 1$ of the construction of π. Hence in step n_0 of the construction we set $\pi(\{x_0, x_1\}) = 0$ and $\pi(\{x_1, x_2\}) = 1$. This contradicts the homogeneity of H. ■

References

[AFP] B. Aniszczyk, R. Frankiewicz and S. Plewik, *Remarks on* (s) *and Ramsey-measurable functions*, Bull. Polish Acad. Sci. Math. 35 (1987), 479–485.

[Bau] J. Baumgartner, *Iterated forcing*, in: Surveys in Set Theory, A. R. D. Mathias (ed.), Cambridge University Press, Cambridge, 1983, 1–59.

[Bl1] A. Blass, *Ultrafilters related to Hindman's finite unions theorem and its extensions*, in: Logic and Combinatorics, S. Simpson (ed.), Contemp. Math. 65, Amer. Math. Soc., 1987, 89–124.

[Bl2] —, *Applications of superperfect forcing and its relatives*, in: Set Theory and its Applications, J. Steprāns and S. Watson (eds.), Lecture Notes in Math. 1401, Springer, 1989, 18–40.

[Bl3] —, *Simple cardinal characteristics of the continuum*, in: Set Theory of the Reals, H. Judah (ed.), Proceedings of the Bar-Ilan Conf. in Honour of Abraham Fraenkel, 1993, 63–90.

[BlSh] A. Blass and S. Shelah, *There may be simple* $P_{\aleph_1}$*- and* $P_{\aleph_2}$*-points and the Rudin–Keisler ordering may be downward directed*, Ann. Pure Appl. Logic 33 (1987), 213–243.

[Br] J. Brown, *The Ramsey sets and related sigma algebras and sets*, Fund. Math. 136 (1990), 179–185.

[CRSW] J. Cichoń, A. Rosłanowski, J. Steprāns and B. Węglorz, *Combinatorial properties of the ideal* $\mathfrak{P}_2$, J. Symbolic Logic 58 (1993), 42–54.

[Co] P. Corazza, *Ramsey sets, the Ramsey ideal, and other classes over* $\mathbb{R}$, ibid. 57 (1992), 1441–1468.

[GP] F. Galvin and K. Prikry, *Borel sets and Ramsey's theorem*, ibid. 38 (1973), 193–198.

[GJS] M. Goldstern, M. Johnson and O. Spinas, *Towers on trees*, Proc. Amer. Math. Soc. 122 (1994), 557–564.

[GRSS] M. Goldstern, M. Repický, S. Shelah and O. Spinas, *On tree ideals*, ibid. 123 (1995), 1573–1581.

[Gr] S. Grigorieff, *Combinatorics on ideals and forcing*, Ann. Math. Logic 3 (1971), 363–394.

[Je1] T. Jech, *Set Theory*, Academic Press, San Diego, 1978.

[Je2] —, *Multiple Forcing*, Cambridge University Press, Cambridge, 1986.

[JMS] H. Judah, A. Miller and S. Shelah, *Sacks forcing, Laver forcing, and Martin's Axiom*, Arch. Math. Logic 31 (1992), 145–162.

[Ku] K. Kunen, *Set Theory*, North-Holland, Amsterdam, 1980.

[Lo] A. Louveau, *Une méthode topologique pour l'étude de la propriété de Ramsey*, Israel J. Math. 23 (1976), 97–116.

[Mar] E. Marczewski, *Sur une classe de fonctions de W. Sierpiński et la classe correspondante d'ensembles*, Fund. Math. 24 (1935), 17–34.

[Ma1] P. Matet, *Some filters of partitions*, J. Symbolic Logic 53 (1988), 540–553.

[Ma2] —, *Happy families and completely Ramsey sets*, Arch. Math. Logic 32 (1993), 151–171.

[Ma] A. R. D. Mathias, *Happy families*, Ann. Math. Logic 12 (1977), 59–111.

[Mi1] A. Miller, *Rational perfect set forcing*, in: Axiomatic Set Theory, J. Baumgartner, D. Martin and S. Shelah (eds.), Contemp. Math. 31, Amer. Math. Soc., 1984, 143–159.

[Mi2] —, *Special subsets of the real line*, in: Handbook of Set-Theoretic Topology, K. Kunen and J. E. Vaughan (eds.), North-Holland, Amsterdam, 1984, 201–233.

[My] J. Mycielski, *Some new ideals of subsets on the real line*, Colloq. Math. 20 (1969), 71–76.

[Pl] S. Plewik, *On completely Ramsey sets*, Fund. Math. 127 (1986), 127–132.

[Sa] G. Sacks, *Forcing with perfect closed sets*, in: Axiomatic Set Theory, Proc. Sympos. Pure Math. 13, Amer. Math. Soc., 1971, 331–355.

[Sh] S. Shelah, *Proper and improper forcing*, in preparation.

[Ve] B. Veličković, *CCC posets of perfect trees*, Compositio Math. 79 (1991), 279–294.

MATHEMATISCHES INSTITUT DER UNIVERSITÄT
AUF DER MORGENSTELLE 10
72076 TÜBINGEN, GERMANY

Received 19 July 1994;
in revised form 10 March 1995