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Product splittings for p-compact groups

by

W. G. Dwyer (Notre Dame, Ind.) and
C. W. Wilkerson (West Lafayette, Ind.)

Abstract. We show that a connected p-compact group with a trivial center is equivalent to a product of simple p-compact groups. More generally, we show that product splittings of any connected p-compact group correspond bijectively to algebraic splittings of the fundamental group of the maximal torus as a module over the Weyl group. These are analogues for p-compact groups of well-known theorems about compact Lie groups.

1. Introduction. One of the fundamental results from the theory of compact Lie groups is the product splitting theorem:

1.1. THEOREM. *Any connected compact Lie group with a trivial center is isomorphic to a product of simple compact Lie groups.*

The purpose of this paper is to prove a theorem similar to 1.1 for p-compact groups, which are homotopy-theoretic analogues of compact Lie groups. Suppose that $\mathcal{X}$ is a p-compact group (see §2) with maximal torus $T_{\mathcal{X}}$ and Weyl group $W_{\mathcal{X}}$ (see 2.4). The finitely generated free $\mathbb{Z}_p$-module $L_{\mathcal{X}}=\pi_1 T_{\mathcal{X}}$ has a natural action of $W_{\mathcal{X}}$ and is called the *dual weight lattice* of $\mathcal{X}$.

1.2. DEFINITION. A p-compact group $\mathcal{X}$ is said to be *simple* if the center of $\mathcal{X}$ is trivial (see 2.3) and the action of $W_{\mathcal{X}}$ on $\mathbb{Q}\otimes L_{\mathcal{X}}$ gives an irreducible representation of $W_{\mathcal{X}}$ over $\mathbb{Q}_p$.

The main result of this paper is the following one.

1.3. THEOREM. *Any connected p-compact group $\mathcal{X}$ with a trivial center is equivalent to a product of simple p-compact groups.*

Remark. For p odd, this has been proved independently by Notbohm (see 1.7). The word "equivalent" here means that the classifying space of $\mathcal{X}$

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is weakly equivalent to a product of classifying spaces of simple p-compact groups.

Remark. If G is a connected compact Lie group with maximal torus T_G and Weyl group W_G, the finitely generated free $\mathbb{Z}$-module $L_G=\pi_1 T_G$ has a natural action of W_G and is called the *dual weight lattice* of G. The group G is simple in the usual sense if and only if the center of G is trivial and the action of W_G on $\mathbb{Q}\otimes L_G$ gives an irreducible representation of W_G over $\mathbb{Q}$. From this point of view 1.3 is a direct analogue of 1.1.

In order to prove Theorem 1.3 we will first give a general splitting criterion for a p-compact group and then show that the criterion applies in a strong way to a p-compact group with a trivial center. Suppose that $\mathcal{X}$ is the product $\prod_i \mathcal{X}_i$, where each $\mathcal{X}_i$ $(i=1,\ldots,n)$ is p-compact group with Weyl group W_i and dual weight lattice L_i. It is easy to see from the definitions that $W_{\mathcal{X}}$ is isomorphic to the product group $\prod_i W_i$ and that $L_{\mathcal{X}}$ is isomorphic to the product module $\prod_i L_i$. In particular, the product decomposition for $\mathcal{X}$ is reflected by a product decomposition for $L_{\mathcal{X}}$ as a module over $W_{\mathcal{X}}$. A product decomposition of $L_{\mathcal{X}}$ which arises in this way is said to be *realized by a splitting of* $\mathcal{X}$. The general splitting criterion is this.

1.4. Theorem. *If $\mathcal{X}$ is a connected p-compact group, then any product decomposition of $L_{\mathcal{X}}$ as a module over $W_{\mathcal{X}}$ can be realized up to equivalence by a splitting of $\mathcal{X}$.*

If $\mathcal{X}$ is p-compact group then the action of $W_{\mathcal{X}}$ on $\mathbb{Q}\otimes L_{\mathcal{X}}$ is generated by reflections (see 2.4). This implies that no irreducible representation of $W_{\mathcal{X}}$ over $\mathbb{Q}_p$, except possibly the trivial one, can appear with multiplicity greater than one in $\mathbb{Q}\otimes L_{\mathcal{X}}$. If $\mathcal{X}$ has trivial center then the trivial representation of $W_{\mathcal{X}}$ does not appear in $\mathbb{Q}\otimes L_{\mathcal{X}}$ (see 7.2), and so there is a unique expression of $\mathbb{Q}\otimes L_{\mathcal{X}}$ as a direct sum of simple $\mathbb{Q}_p[W_{\mathcal{X}}]$-submodules.

1.5. Theorem. *Suppose that $\mathcal{X}$ is a connected p-compact group with trivial center, and that*

$$\mathbb{Q}\otimes L_{\mathcal{X}}=M_1+\ldots+M_n$$

is the unique expression of $\mathbb{Q}\otimes L_{\mathcal{X}}$ as a direct sum of simple $\mathbb{Q}_p[W_{\mathcal{X}}]$-submodules. Let $L_i=L_{\mathcal{X}}\cap M_i$. Then the addition map

$$\alpha: L_1\times\ldots\times L_n\to L_{\mathcal{X}}$$

is an isomorphism of $W_{\mathcal{X}}$-modules.

If $\mathcal{X}$ is p-compact group with trivial center, Theorem 1.5 gives a product splitting of $L_{\mathcal{X}}$ as a module over $W_{\mathcal{X}}$, and Theorem 1.4 provides a corresponding product splitting of $\mathcal{X}$. It is clear that the factors in this product splitting of $\mathcal{X}$ are simple p-compact groups. This proves 1.3.

1.6. Remark. We call a connected p-compact group $\mathcal{X}$ *almost simple* if the center of $\mathcal{X}$ is finite and the quotient of $\mathcal{X}$ by its center is simple. Theorem 1.3 implies that if $\mathcal{X}$ is p-compact group which is connected and simply connected, then $\mathcal{X}$ splits as a product of almost simple factors. One way to prove this is to deduce from 6.11 and 7.2 that the center C of $\mathcal{X}$ is finite, use 1.3 to split $\mathcal{X}/C$ as a product $\prod_i \mathcal{Y}_i$ of simple factors (cf. [6, 6.3]), and then lift the product decomposition $\mathrm{B}(\mathcal{X}/C) \simeq \prod_i \mathrm{B}\mathcal{Y}_i$ to a product decomposition for the 2-connected cover $\mathrm{B}\mathcal{X}$ of $\mathrm{B}(\mathcal{X}/C)$. See also [10].

1.7. *Relationship to* [11]. Our results were originally obtained independently of [11]. While we were preparing this manuscript, however, we learned of Notbohm's work and discovered that our arguments could be drastically simplified by using a beautiful idea which he described to us (5.1, 5.7). We are very grateful to him for his communication.

Guide to the proof. The proof of 1.4 given in this paper is completely parallel to a (slightly unconventional) proof of the corresponding fact for compact Lie groups. We will sketch the group-theoretic argument and indicate how it fits in with the argument below. Recall the following classical result:

1.8. Proposition. *Suppose that G is a connected compact Lie group. Then*

(1) *any element of G is conjugate to some element in the maximal torus T_G* [6, proof of 2.14], *and*

(2) *if x and y are two elements of T_G which are conjugate in G, then there is an element of the Weyl group W_G which carries x to y* (see 3.4).

Suppose now that G is a connected compact Lie group with dual weight lattice $L_G = \pi_1 T_G$, and that $L_G \cong L_1 \times L_2$ is a product decomposition of modules over W_G. The problem is to produce a group splitting $G \cong G_1 \times G_2$. The isomorphism $T_G \cong (\mathbb{R} \otimes L_G)/L_G$ (2.5) gives a splitting $T_G \cong T_1 \times T_2$ of modules over W_G, where $T_i \cong (\mathbb{R} \otimes L_i)/L_i$. Let H' be the centralizer of T_2 in G and K' the centralizer of T_1. Denote the quotients H'/T_2 and K'/T_1 by H and K respectively. All of these groups are connected, and they can be arranged in a large commutative diagram similar to (6.2). The composite $T_1 \to T_G \to H' \to H$ is a maximal torus inclusion, as is the corresponding composite $T_2 \to K$; for simplicity, then, denote T_1 by T_H and T_2 by T_K.

One checks by elementary methods (see 6.5) that the central quotient map $H' \to H$ has a unique section which on T_H agrees with the existing homomorphism $T_H \to H'$; there is a similar section for the quotient map $K' \to K$. These sections give direct product decompositions $H' \cong H \times T_K$ and $K' \cong T_H \times K$. Composing the sections with the inclusions $H' \to G$ and $K' \to G$ gives homomorphisms $H \to G$ and $K \to G$, which allow H and K to be considered as subgroups of G. The Weyl group W_H is isomorphic to the subgroup of W_G which pointwise fixes T_K; similarly, W_K is isomorphic to the subgroup of W_G which pointwise fixes T_H. These two subgroups of W_G commute, and the product map $W_H \times W_K \to W_G$ is an isomorphism (see 6.3).

By 1.8(2), no element of T_H is conjugate in G to any element of T_K, and so by 1.8(1) (see 3.3) no element of H is conjugate in G to any element of K. Another way of expressing this last fact is to say that the left action of H on G/K is free, so that the double coset space $H\backslash G/K$ is a manifold and there is a principal fibre bundle sequence

$$H \times K \to G \to H\backslash G/K.$$

It follows from dimension counting (see 3.8, 3.9) that $H\backslash G/K$ is a point (see 6.7), and thus that the multiplication map $H \times K \to G$ is a diffeomorphism. From this it is easy to see that the natural map

$$H \cong (H \times T_K)/T_K = H'/T_K \to G/K \cong H$$

is also a diffeomorphism (see 6.9). It is clear that this diffeomorphism respects the left actions of T_K on the spaces involved. Since the action of T_K on $(H \times T_K)/T_K$ is trivial, it follows that so is the action of T_K on G/K, and thus by 1.8(1) also the action of K on G/K (see 5.1). This is one way of saying that K is a normal subgroup of G (see 4.8). Elementary considerations show that the composite map

$$H \to G \to G/K$$

is an isomorphism. By symmetry, H is normal in G, and it follows easily that the product map

$$G \to (G/K) \times (G/H) \cong H \times K$$

is an isomorphism.

Organization of the paper. Section 2 sketches some background material about p-compact groups. Sections 3 and 4 study respectively the notions of *double coset space* and *normalizer*. In §5 there is a method (due to Notbohm) for recognizing the triviality of certain bundles over classifying spaces. Finally, §6 has the proof of 1.4 and §7 the proof of 1.3.

Notation and terminology. The prime p is fixed for the paper; $\mathbb{F}_p$ is the field with p elements, $\mathbb{Z}_p$ the ring of p-adic integers, and $\mathbb{Q}_p = \mathbb{Q} \otimes \mathbb{Z}_p$ the field of p-adic numbers. All unspecified homology and cohomology is with coefficients in $\mathbb{F}_p$. A map f of spaces is an $\mathbb{F}_p$-*equivalence* if $\mathrm{H}^*(f)$ is an isomorphism. A space $\mathcal{X}$ is said to be $\mathbb{F}_p$-*finite* if $\mathrm{H}^*\mathcal{X}$ is finite, and $\mathbb{F}_p$-*complete* if it is $\mathbb{F}_p$-complete in the sense of [2]. The symbol $\mathrm{H}^i_{\mathbb{Q}_p}(\mathcal{X})$ stands for the $\mathbb{Q}_p$-module $\mathbb{Q} \otimes \mathrm{H}^i(\mathcal{X}, \mathbb{Z}_p)$.

We assume that all spaces which arise have been replaced, if necessary, by weakly equivalent CW-complexes. The word *equivalence* applied to a map between spaces means homotopy equivalence.

2. p-compact goups. The purpose of this section is to recall some material connected with p-compact groups and establish notation. The basic references are [5], [6] and [10].

2.1. *Loop spaces and p-compact groups.* A *loop space* is a triple $(\mathcal{X}, \mathrm{B}\mathcal{X}, e)$ consisting of a space $\mathcal{X}$, a pointed space $\mathrm{B}\mathcal{X}$, and an equivalence $e : \mathcal{X} \simeq \Omega\mathrm{B}\mathcal{X}$. Usually $\mathrm{B}\mathcal{X}$ and e are omitted from the notation. A loop space $\mathcal{X}$ is said to be a *p-compact group* if $\mathcal{X}$ is $\mathbb{F}_p$-finite, $\mathcal{X}$ is $\mathbb{F}_p$-complete, and $\pi_0\mathcal{X}$ is a finite p-group, or equivalently if $\mathcal{X}$ is $\mathbb{F}_p$-finite and $\mathrm{B}\mathcal{X}$ is $\mathbb{F}_p$-complete. The p-compact group $\mathcal{X}$ is a *p-compact torus* if it is the $\mathbb{F}_p$-completion of an ordinary torus, and a *p-compact toral group* if its identity component is a p-compact torus. We will use script letters (e.g., $\mathcal{X}, \mathcal{Y}$) for spaces and p-compact groups, and roman letters (G, T) for p-compact toral groups.

2.2. *Homomorphisms and centralizers.* A *homomorphism* $f : \mathcal{X} \to \mathcal{Y}$ of loop spaces is by definition a pointed map $\mathrm{B}f : \mathrm{B}\mathcal{X} \to \mathrm{B}\mathcal{Y}$; two homomorphisms f and g are *homotopic* (resp. *conjugate*) if $\mathrm{B}f$ and $\mathrm{B}g$ are homotopic as pointed (resp. unpointed) maps. Note that if $\mathcal{Y}$ is connected there is no distinction between homotopy and conjugacy. A homomorphism f is *trivial* if $\mathrm{B}f$ is homotopic to a constant map. Suppose that $f : \mathcal{X} \to \mathcal{Y}$ is a homomorphism of loop spaces. The *centralizer* of $f(\mathcal{X})$ in $\mathcal{Y}$, denoted by $\mathbf{C}_{\mathcal{Y}}(f(\mathcal{X}))$ or $\mathbf{C}_{\mathcal{Y}}(\mathcal{X})$, is the loop space $\Omega\,\mathrm{Map}(\mathrm{B}\mathcal{X}, \mathrm{B}\mathcal{Y})_{\mathrm{B}f}$ and the *homogeneous space* $\mathcal{Y}/f(\mathcal{X})$ (or $\mathcal{Y}/\mathcal{X}$) is the homotopy fibre of $\mathrm{B}f$ over the basepoint of $\mathrm{B}\mathcal{Y}$. There is a natural loop space homomorphism $\mathbf{C}_{\mathcal{Y}}(f(\mathcal{X})) \to \mathcal{Y}$ given by a construction which involves evaluating maps at the basepoint of $\mathrm{B}\mathcal{X}$. The homomorphism f is said to be *central* if $\mathbf{C}_{\mathcal{Y}}(\mathcal{X}) \to \mathcal{Y}$ is an equivalence. If f is a homomorphism of p-compact groups, the f is said to be a *monomorphism* if $\mathcal{Y}/f(\mathcal{X})$ is $\mathbb{F}_p$-finite. If $\mathcal{X}$ is a p-compact toral group and $\mathcal{Y}$ is p-compact group then $\mathbf{C}_{\mathcal{Y}}(f(\mathcal{X}))$ is p-compact group and the map $\mathbf{C}_{\mathcal{Y}}(f(\mathcal{X})) \to \mathcal{Y}$ is a monomorphism.

2.3. *Abelian p-compact groups and central quotients.* A loop space $\mathcal{X}$ is said to be *abelian* if the identity homomorphism of $\mathcal{X}$ is central. A p-compact group is abelian if and only if it is equivalent to the product of a p-compact torus and a finite abelian p-group. If A is abelian and $f : A \to \mathcal{X}$ is a homomorphism of p-compact groups, then f lifts to a central homomorphism $A \to \mathbf{C}_{\mathcal{X}}(A)$ which is a monomorphism if f is. In general, if $f : A \to \mathcal{X}$ is a central monomorphism of p-compact groups (sometimes expressed by saying that A *is a central subgroup of* $\mathcal{X}$) then it is possible to form a "quotient"

p-compact group $\mathcal{X}/A$. For any p-compact group $\mathcal{X}$ there is a universal central monomorphism $C \to \mathcal{X}$ called the *center* of $\mathcal{X}$.

2.4. *Maximal tori and Weyl groups*. A *maximal torus* for p-compact group $\mathcal{X}$ is a p-compact torus T together with a monomorphism $T \to \mathcal{X}$ with the property that the mod p cohomology Euler characteristic $\chi(\mathcal{X}/T)$ is nonzero. Any p-compact group $\mathcal{X}$ has a maximal torus, unique up to conjugacy; a chosen representative is denoted by $i_{\mathcal{X}} : T_{\mathcal{X}} \to \mathcal{X}$ or just $T_{\mathcal{X}}$. The *Weyl space* $\mathbf{W}_{\mathcal{X}}(T_{\mathcal{X}})$ of self-maps of $\mathrm{B}T_{\mathcal{X}}$ over $\mathrm{B}\mathcal{X}$ (computed after replacing the map $\mathrm{B}T_{\mathcal{X}} \to \mathrm{B}\mathcal{X}$ by an equivalent fibration) is homotopically discrete, and $\pi_0\mathbf{W}_{\mathcal{X}}(T_{\mathcal{X}})$ is a finite group called the *Weyl group* $W_{\mathcal{X}}$ of $\mathcal{X}$. The *dual weight lattice* $L_{\mathcal{X}}$ of $\mathcal{X}$ is defined to be the finitely generated free $\mathbb{Z}_p$-module $\pi_1 T_{\mathcal{X}} = \pi_2 \mathrm{B}T_{\mathcal{X}}$; this is a module over $W_{\mathcal{X}}$ in an evident way. If $\mathcal{X}$ is connected then $W_{\mathcal{X}}$ acts faithfully on $L_{\mathcal{X}}$ and the image of $W_{\mathcal{X}}$ in $\mathrm{Aut}_{\mathbb{Q}_p}(\mathbb{Q}\otimes L_{\mathcal{X}})$ is generated by *reflections*, that is, by automorphisms which are diagonalizable and pointwise fix a codimension 1 subspace. (Sometimes such automorphisms are called *pseudoreflections*.) An element $s \in W_{\mathcal{X}}$ is itself called a *reflection* if its image in $\mathrm{Aut}_{\mathbb{Q}_p}(\mathbb{Q}\otimes L_{\mathcal{X}})$ is a reflection.

The Borel construction of the action of $\mathbf{W}_{\mathcal{X}}$ on $\mathrm{B}T_{\mathcal{X}}$ is denoted by $\mathrm{B}\mathbf{N}(T_{\mathcal{X}})$ and maps naturally to $\mathrm{B}\mathcal{X}$; the loop space $\mathbf{N}(T_{\mathcal{X}}) = \Omega\mathrm{B}\mathbf{N}(T_{\mathcal{X}})$ is called the *normalizer* of $T_{\mathcal{X}}$ (in $\mathcal{X}$). The *p-normalizer* of $T_{\mathcal{X}}$, denoted by $\mathbf{N}_p(T_{\mathcal{X}})$, is the p-compact group which is the inverse image in $\mathbf{N}(T_{\mathcal{X}})$ of a p-Sylow subgroup of $\pi_0\mathbf{N}(T_{\mathcal{X}}) = W_{\mathcal{X}}$.

2.5. *p-discrete approximations*. A *p-discrete torus* is by definition a (discrete) group isomorphic to $(\mathbb{Z}/p^\infty)^r$ for some r; more generally, a *p-discrete toral group* is a group which lies in a short exact sequence in which the kernel is a p-discrete torus and the quotient is a finite p-group. If G is a p-compact toral group then there exists an essentially unique homomorphism $f : \breve{G} \to G$ such that $\breve{G}$ is a p-discrete toral group and $\mathrm{B}f$ is an $\mathbb{F}_p$-equivalence; the $\mathbb{F}_p$-completion of $\mathrm{B}\breve{G}$ is equivalent to $\mathrm{B}G$. One says that $\breve{G}$ is a *p-discrete approximation* to G or that G is the *p-closure* of $\breve{G}$. For example, if T is a p-compact torus then $\breve{T}$ is the group $\mathbb{Z}/p^\infty \otimes \pi_1 T$. If $\mathcal{X}$ is a p-compact group, G is a p-compact toral group, and $i : \breve{G} \to G$ is a p-discrete approximation, then the assignment $f \mapsto f \cdot i$ gives a bijection between homotopy (resp. conjugacy) classes of homomorphisms $G \to \mathcal{X}$ and homotopy (resp. conjugacy) classes of homomorphisms $\breve{G} \to \mathcal{X}$; moreover, if $f : G \to \mathcal{X}$ is a homomorphism, then the natural map $\mathbf{C}_{\mathcal{X}}(f(G)) \to \mathbf{C}_{\mathcal{X}}(fi(\breve{G}))$ is an equivalence.

If L is a finitely generated free $\mathbb{Z}_p$-module we denote $\mathbb{Z}/p^\infty \otimes_{\mathbb{Z}} L = \mathbb{Z}/p^\infty \otimes_{\mathbb{Z}_p} L$ by $\breve{L}$, so that for instance if $\mathcal{X}$ is p-compact group there is an isomorphism $\breve{T}_{\mathcal{X}} \cong \breve{L}_{\mathcal{X}}$. The formula $L = \mathrm{Hom}(\mathbb{Z}/p^\infty, \breve{L})$ allows L be recovered from $\breve{L}$.

3. Double coset spaces. Suppose that $f:\mathcal{Y}\to\mathcal{X}$ and $g:\mathcal{Z}\to\mathcal{X}$ are homomorphisms of loop spaces.

3.1. Definition. The *double coset space* $f(\mathcal{Y})\backslash\mathcal{X}/g(\mathcal{Z})$ (or $\mathcal{Y}\backslash\mathcal{X}/\mathcal{Z}$ for short) is the homotopy pullback of the diagram

$$\mathrm{B}\mathcal{Y}\xrightarrow{\mathrm{B}f}\mathrm{B}\mathcal{X}\xleftarrow{\mathrm{B}g}\mathrm{B}\mathcal{Z}.$$

For example, if $\mathcal{Y}=*$ is trivial, then $\mathcal{Y}\backslash\mathcal{X}/\mathcal{Z}$ is the coset space $\mathcal{X}/\mathcal{Z}$. The following proposition is interesting even in this case: it implies that a homomorphism $g:\mathcal{Z}\to\mathcal{X}$ of p-compact groups is a monomorphism (see 2.2) if and only if every nontrivial homomorphism $\mathbb{Z}/p\to\mathcal{Z}$ remains nontrivial when composed with g.

3.2. Proposition. *Suppose that $f:\mathcal{Y}\to\mathcal{X}$ and $g:\mathcal{Z}\to\mathcal{X}$ are homomorphisms of p-compact groups. Then the following conditions are equivalent*:

(1) *the space $\mathcal{Y}\backslash\mathcal{X}/\mathcal{Z}$ is $\mathbb{F}_p$-finite*, *and*

(2) *whenever $i:\mathbb{Z}/p\to\mathcal{Y}$ and $j:\mathbb{Z}/p\to\mathcal{Z}$ are homomorphisms with fi conjugate to gj*, *then i and j are trivial.*

Proof. Suppose first that i and j are homomorphisms as described with fi conjugate to gj and i, say, nontrivial. It follows from the basic property of homotopy pullbacks that there exists a map $h:\mathrm{B}\mathbb{Z}/p\to\mathcal{Y}\backslash\mathcal{X}/\mathcal{Z}$ which lifts the map $\mathrm{B}i:\mathrm{B}\mathbb{Z}/p\to\mathrm{B}\mathcal{Y}$. Clearly h is not homotopic to a constant map. By Lannes [7, 3.1.1] (and the fact that $\mathrm{H}^*\mathrm{B}\mathbb{Z}/p$ is an exterior algebra on a one-dimensional class tensored with a polynomial algebra on its Bockstein) the map h induces a nontrivial map on mod p cohomology in infinitely many dimensions, and so $\mathcal{Y}\backslash\mathcal{X}/\mathcal{Z}$ is not $\mathbb{F}_p$-finite.

To prove the implication in the other direction, we first work in the case $\mathcal{Y}=*$. The hypothesis is now that if $j:\mathbb{Z}/p\to\mathcal{Z}$ is a nontrivial homomorphism then gj is also nontrivial. As in [5, 3.4], this hypothesis is equivalent to the statement that the homotopy fixed point set of the action of $gj(\mathbb{Z}/p)$ on $\mathcal{X}/*=\mathcal{X}$ is empty for each such homomorphism j. There is a fibration sequence

$$\mathcal{Z}/\mathbf{N}_p(T_{\mathcal{Z}})\to\mathcal{X}/\mathbf{N}_p(T_{\mathcal{Z}})\to\mathcal{X}/\mathcal{Z}$$

in which the fibre has mod p Euler characteristic prime to p [5, proof of 2.4], so that in order to show that $*\backslash\mathcal{X}/\mathcal{Z}=\mathcal{X}/\mathcal{Z}$ is $\mathbb{F}_p$-finite it is enough by a transfer argument [5, 9.13] to show that $\mathcal{X}/\mathbf{N}_p(T_{\mathcal{Z}})$ is $\mathbb{F}_p$-finite. For this it is enough, as in the proof of [5, 7.3], to show that $\mathcal{X}\breve{\mathbf{N}}_p(T_{\mathcal{Z}})$ is $\mathbb{F}_p$-finite (see 2.5) or even that $\mathcal{X}/G$ is $\mathbb{F}_p$-finite for each finite subgroup G of $\breve{\mathbf{N}}_p(T_{\mathcal{Z}})$. However, $\mathcal{X}/G$ can be interpreted as the homotopy orbit space $\mathcal{X}_{\mathrm{h}G}$ for a suitable "action" of G on $\mathcal{X}$ [5, §10] and so the fact that $\mathcal{X}/G$ is $\mathbb{F}_p$-finite follows by [5, 7.4] from the fact that $\mathcal{X}^{\mathrm{h}K}$ is empty for each order p cyclic subgroup K of G.

Now consider the general case in which $\mathcal{Y}$ is not necessarily contractible. The assumption is that whenever $i : \mathbb{Z}/p \to \mathcal{Y}$ and $j : \mathbb{Z}/p \to \mathcal{Z}$ are homomorphisms with fi is conjugate to gj, then i and j are trivial. By the above considerations this implies first of all that $\mathcal{X}/\mathcal{Z}$ is $\mathbb{F}_p$-finite, and secondly that the homotopy fixed point set of the action of $fi(\mathbb{Z}/p)$ on $\mathcal{X}/\mathcal{Z}$ is empty for each nontrivial homomorphism $i : \mathbb{Z}/p \to \mathcal{Y}$. Now copy the above argument. There is a fibration sequence

$$\mathcal{Y}/\mathbf{N}_p(T_{\mathcal{Y}}) \to \mathbf{N}_p(T_{\mathcal{Y}})\backslash\mathcal{X}/\mathcal{Z} \to \mathcal{Y}\backslash\mathcal{X}/\mathcal{Z}$$

in which the fibre has Euler characteristic prime to p, so that in order to show that $\mathcal{Y}\backslash\mathcal{X}/\mathcal{Z}$ is $\mathbb{F}_p$-finite, it is enough by a transfer argument to show that $\mathbf{N}_p(T_{\mathcal{Y}})\backslash\mathcal{X}/\mathcal{Z}$ is $\mathbb{F}_p$-finite. For this it is enough to show that $\breve{\mathbf{N}}_p(T_{\mathcal{Y}})\backslash\mathcal{X}/\mathcal{Z}$ is $\mathbb{F}_p$-finite or even that $G\backslash\mathcal{X}/\mathcal{Z}$ is $\mathbb{F}_p$-finite for each finite subgroup G of $\breve{\mathbf{N}}_p(T_{\mathcal{Y}})$. This, however, follows by [5, 7.4] from the fact that $(\mathcal{X}/\mathcal{Z})^{\mathrm{h}K}$ is empty for each order p cyclic subgroup K of G. ■

3.3. Corollary. *Suppose that $f : \mathcal{Y} \to \mathcal{X}$ and $g : \mathcal{Z} \to \mathcal{X}$ are homomorphisms of connected p-compact groups. Then $\mathcal{Y}\backslash\mathcal{X}/\mathcal{Z}$ is $\mathbb{F}_p$-finite if and only if $T_{\mathcal{Y}}\backslash\mathcal{X}/T_{\mathcal{Z}}$ is.*

Proof. This follows immediately from 3.2 and the fact that if $\mathcal{U}$ is a connected p-compact group, then any homomorphism $\mathbb{Z}/p \to \mathcal{U}$ lifts up to homotopy to a homomorphism $\mathbb{Z}/p \to T_{\mathcal{U}}$ [5, proof of 9.1]. ■

3.4. Proposition. *Let $\mathcal{X}$ be p-compact group with maximal torus $i : T_{\mathcal{X}} \to \mathcal{X}$, and let $j : T_{\mathcal{X}} \to \mathbf{N}(T_{\mathcal{X}})$ be the usual homomorphism. Suppose that A is an abelian toral group and $f, g : A \to T_{\mathcal{X}}$ a pair of monomorphisms such that if is conjugate to ig. Then f and g differ by the action of an element of $W_{\mathcal{X}}$, and so jf is conjugate to jg.*

Proof. We can assume after adjusting the homomorphisms up to conjugacy that $\mathrm{B}(if) = \mathrm{B}(ig) = \mathrm{B}k$. Let $\mathcal{Y}$ be the identity component of $\mathbf{C}_{\mathcal{X}}(k(A))$. Since A and $T = T_{\mathcal{X}}$ are abelian (see 2.3) and T is connected there are homomorphisms

$$A \xrightarrow{f'} \mathbf{C}_T(f(A)) \simeq T \xrightarrow{u} \mathcal{Y} \xrightarrow{e} \mathcal{X}, \qquad A \xrightarrow{g'} \mathbf{C}_T(g(A)) \simeq T \xrightarrow{v} \mathcal{Y} \xrightarrow{e} \mathcal{X}$$

such that both horizontal composites are conjugate to k, f' is conjugate to f, g' is conjugate to g, uf' is conjugate to vg', and the composites eu and ev are conjugate to i. The homomorphisms u and v are both maximal tori for $\mathcal{Y}$ [6, 4.2]. Since maximal tori in $\mathcal{Y}$ are unique up to conjugacy [5, 9.4] there is a homomorphism $h : T \to T$ such that vh is conjugate to u. This implies that $evh \sim ih$ is conjugate to $eu \sim i$, so that h represents an element in $W_{\mathcal{X}}$ (see 2.4). As above, the homomorphism $vhf' \sim uf'$ is conjugate to

vg', and so by [6, 6.5], hf' is conjugate to g' (this is a homotopical form of the observation that inner automorphisms of a group induce the identity on a central subgroup). Thus, as an element of $W_{\mathcal{X}}$, the map h carries f to g and so provides a conjugacy in $\mathbf{N}(T_{\mathcal{X}})$ between jf and jg. ∎

3.5. Remark. Let $f : \mathcal{Y} \to \mathcal{X}$ be a homomorphism of connected p-compact groups. It follows from [5, 8.11] and 3.4 that there is an induced homomorphism $f_T : T_{\mathcal{Y}} \to T_{\mathcal{X}}$, unique up to the action of $W_{\mathcal{X}}$ on the set of homotopy classes of homomorphisms $T_{\mathcal{Y}} \to T_{\mathcal{X}}$, such that the diagram

$$\begin{array}{ccc} T_{\mathcal{Y}} & \xrightarrow{f_T} & T_{\mathcal{X}} \\ \big\downarrow & & \big\downarrow \\ \mathcal{Y} & \longrightarrow & \mathcal{X} \end{array}$$

commutes up to homotopy. For convenience we will refer to such a homomorphism f_T as "the homomorphism induced by f"; it is unique, however, only up to the action of $W_{\mathcal{X}}$. The corresponding map $\pi_1(f_T) : L_{\mathcal{Y}} \to L_{\mathcal{X}}$ is denoted by f_L.

3.6. Proposition. *Suppose that $\mathcal{X}, \mathcal{Y}$, and $\mathcal{Z}$ are connected p-compact groups and that $f : \mathcal{Y} \to \mathcal{X}$ and $g : \mathcal{Z} \to \mathcal{X}$ are homomorphisms. Let $\overline{f}_L : \overline{L}_{\mathcal{Y}} \to \overline{L}_{\mathcal{X}}$ and $\overline{g}_L : \overline{L}_{\mathcal{Z}} \to \overline{L}_{\mathcal{X}}$ be the maps obtained by tensoring f_L and g_L (3.5) with $\mathbb{Z}/p$. Then $\mathcal{Y}\backslash\mathcal{X}/\mathcal{Z}$ is $\mathbb{F}_p$-finite if and only if*

(1) *$\overline{f}_L$ and $\overline{g}_L$ are monomorphisms*, and

(2) *for each element $w \in W_{\mathcal{X}}$ the groups $\overline{f}_L(\overline{L}_{\mathcal{Y}})$ and $w \cdot g_L(\overline{L}_{\mathcal{Z}})$ intersect trivially in $\overline{L}_{\mathcal{X}}$.*

Proof. By 3.3, $\mathcal{Y}\backslash\mathcal{X}/\mathcal{Z}$ is $\mathbb{F}_p$-finite if and only if $T_{\mathcal{Y}}\backslash\mathcal{X}/T_{\mathcal{Z}}$ is $\mathbb{F}_p$-finite. By 3.2, this latter space is *not* $\mathbb{F}_p$-finite if and only if there are homomorphisms $i : \mathbb{Z}/p \to T_{\mathcal{Y}}$ and $j : \mathbb{Z}/p \to T_{\mathcal{Z}}$, at least one of which is nontrivial, such that fi is conjugate to gj. By 3.4, this last is the case if and only if there are homomorphisms $i : \mathbb{Z}/p \to T_{\mathcal{Y}}$ and $j : \mathbb{Z}/p \to T_{\mathcal{Z}}$, at least one of which is nontrivial, and some element $w \in W_{\mathcal{X}}$ which carries $f_T \cdot i$ to $g_T \cdot j$. The proof is finished by observing that if T is a p-compact torus, the set of conjugacy classes of homomorphisms $\mathbb{Z}/p \to T$ corresponds bijectively in a natural way to $\mathbb{Z}/p \otimes \pi_1 T$. The correspondence assigns to a map $\mathrm{B}i : \mathrm{B}\mathbb{Z}/p \to \mathrm{B}T$ the image under the induced homology map of a chosen generator of $\mathrm{H}_2\mathrm{B}\mathbb{Z}/p$. ∎

Recall that if $\mathcal{X}$ is a space with suitable finiteness properties, the (cohomological) dimension $\mathrm{cd}_{\mathbb{F}_p}(\mathcal{X})$ (resp. $\mathrm{cd}_{\mathbb{Q}_p}(\mathcal{X})$) is defined to be the largest degree in which $\mathrm{H}^*\mathcal{X}$ (resp. $\mathrm{H}^*_{\mathbb{Q}_p}\mathcal{X}$) is nonzero. We end this section with a short discussion of the relationship between the dimension of a double coset space and the dimensions of the associated p-compact groups.

3.7. PROPOSITION. *Suppose that* $f:\mathcal{Y}\to\mathcal{X}$ *and* $g:\mathcal{Z}\to\mathcal{X}$ *are homomorphisms of p-compact groups, and assume that the associated double coset space* $\mathcal{Y}\backslash\mathcal{X}/\mathcal{Z}$ *is* $\mathbb{F}_p$*-finite. Then*

$$\mathrm{cd}_{\mathbb{F}_p}(\mathcal{Y}\backslash\mathcal{X}/\mathcal{Z})=\mathrm{cd}_{\mathbb{F}_p}(\mathcal{X})-(\mathrm{cd}_{\mathbb{F}_p}(\mathcal{Y})+\mathrm{cd}_{\mathbb{F}_p}(\mathcal{Z})).$$

Proof. This follows from applying the argument of [5, proof of 6.14] to the principal fibration $\mathcal{Y}\times\mathcal{Z}\to\mathcal{X}\to\mathcal{Y}\backslash\mathcal{X}/\mathcal{Z}$. The fundamental group of the base acts nilpotently on the homology of the fibre here, because $\pi_0(\mathcal{Y}\times\mathcal{Z})$ is a finite p-group [5, 11.6]. ∎

In applying 3.7, it is useful to have a way of calculating the dimension of p-compact group $\mathcal{X}$ in terms of $W_{\mathcal{X}}$. Suppose that M is a vector space over $\mathbb{Q}_p$ of rank r and that W is a finite subgroup of $\mathrm{Aut}_{\mathbb{Q}_p}(M)$ generated by reflections (see 2.4). Let $M^\#$ denote the $\mathbb{Q}_p$-dual of M, so that the symmetric algebra $\mathrm{Sym}(M^\#)$ is the algebra of polynomial functions on M. We regard $\mathrm{Sym}(M^\#)$ as a graded algebra in which $M^\#$ itself has degree 2. The action of W on M induces an action of W on $M^\#$ and a degree-preserving action of W on $\mathrm{Sym}(M^\#)$. It is known [3] that the fixed point subalgebra $\mathrm{Sym}(M^\#)^W$ is isomorphic to a graded polynomial algebra on r generators of degree, say, $d_1,\ldots,d_r$. The sequence of integers $d_1,\ldots,d_r$ is independent, up to permutation, of the choice of polynomial generators for $\mathrm{Sym}(M^\#)^W$. We define the *dimension* associated to W, denoted by $\delta(W)$, to be

$$\delta(W)=\sum_{i=1}^{r}(d_i-1)=\Big(\sum_{i=1}^{r}d_i\Big)-r.$$

3.8. LEMMA. *Suppose that* $\mathcal{X}$ *is a connected p-compact group with Weyl group* $W_{\mathcal{X}}\subset\mathrm{Aut}_{\mathbb{Q}_p}(\mathbb{Q}\otimes L_{\mathcal{X}})$ (*see* 2.4). *Then* $\mathrm{cd}_{\mathbb{F}_p}(\mathcal{X})=\delta(W_{\mathcal{X}})$.

Proof. As in [6, 4.5], $\mathrm{cd}_{\mathbb{F}_p}(\mathcal{X})=\mathrm{cd}_{\mathbb{Q}_p}(\mathcal{X})$. Set $M=\mathbb{Q}\otimes L_{\mathcal{X}}$ and let $r=\mathrm{rank}_{\mathbb{Q}_p}(M)$. The graded algebras $\mathrm{H}^*_{\mathbb{Q}_p}(\mathrm{B}T_{\mathcal{X}})$ and $\mathrm{Sym}(M^\#)$ are isomorphic in a way which respects the actions of $W_{\mathcal{X}}$ on the two of them. By [5, 9.7] there is an isomorphism $\mathrm{H}^*_{\mathbb{Q}_p}(\mathrm{B}\mathcal{X})\cong\mathrm{Sym}(M^\#)^{W_{\mathcal{X}}}$, and so $\mathrm{H}^*_{\mathbb{Q}_p}(\mathrm{B}\mathcal{X})$ is isomorphic to a polynomial algebra on generators of dimension $d_1,\ldots,d_r$, where these integers are the ones associated to the reflection group $W_{\mathcal{X}}$ above. It follows from the Eilenberg–Moore spectral sequence that $\mathrm{H}^*_{\mathbb{Q}_p}(\mathcal{X})$ is an exterior algebra on generators of dimension $d_1-1,\ldots,d_r-1$. Thus

$$\mathrm{cd}_{\mathbb{F}_p}(\mathcal{X})=\mathrm{cd}_{\mathbb{Q}_p}(\mathcal{X})=\sum_{i=1}^{r}(d_i-1)=\delta(W_{\mathcal{X}}).\ ∎$$

3.9. LEMMA. *Suppose that* M_1 *and* M_2 *are finite-dimensional vector spaces over* $\mathbb{Q}_p$, *and that* W_i $(i=1,2)$ *is a finite subgroup of* $\mathrm{Aut}_{\mathbb{Q}_p}(M_i)$ *generated by reflections. Let* $W=W_1\times W_2\subset\mathrm{Aut}_{\mathbb{Q}_p}(M_1\times M_2)$. *Then* W *is generated by reflections* (see 2.4), *and* $\delta(W)=\delta(W_1)+\delta(W_2)$.

Proof. It is obvious that W is generated by reflections. The formula for $\delta(W)$ follows from the calculation

$$\mathrm{Sym}((M_1\times M_2)^{\#})^W=\mathrm{Sym}(M_1^{\#})^{W_1}\otimes_{\mathbb{Q}_p}\mathrm{Sym}(M_2^{\#})^{W_2}. \blacksquare$$

4. Normalizers. Suppose that $f:\mathcal{Y}\to\mathcal{X}$ is a homomorphism of p-compact groups. We can assume after making an adjustment up to homotopy that $\mathrm{B}f:\mathrm{B}\mathcal{Y}\to\mathrm{B}\mathcal{X}$ is a fibration.

4.1. Definition. The *Weyl space* of f, denoted by $\mathbf{W}_{\mathcal{X}}(f(\mathcal{Y}))$ or $\mathbf{W}_{\mathcal{X}}(\mathcal{Y})$, is the space of maps $\mathrm{B}\mathcal{Y}\to\mathrm{B}\mathcal{Y}$ over $\mathrm{B}\mathcal{X}$.

4.2. Remark. The Weyl space $\mathbf{W}_{\mathcal{X}}(\mathcal{Y})$ is a topological monoid under composition of maps, and acts in a natural way on $\mathrm{B}\mathcal{Y}$. The above definition of $\mathbf{W}_{\mathcal{X}}(\mathcal{Y})$ generalizes the one given previously (see 2.4) in the special case $\mathcal{Y}=T_{\mathcal{X}}$.

Let $q:f(\mathcal{Y})\backslash\mathcal{X}/f(\mathcal{Y})\to\mathrm{B}\mathcal{Y}$ be the natural fibration (see 3.1). It is clear that $\mathbf{W}_{\mathcal{X}}(\mathcal{Y})$ can be interpreted as the space $\Gamma(q)$ of sections of q. In the language of [5, 10.8] it is natural to write $\mathbf{W}_{\mathcal{X}}(\mathcal{Y})=(\mathcal{X}/f(\mathcal{Y}))^{\mathrm{h}f\mathcal{Y}}$ or simply $\mathbf{W}_{\mathcal{X}}(\mathcal{Y})=(\mathcal{X}/\mathcal{Y})^{\mathrm{h}\mathcal{Y}}$. Evaluation at the basepoint of $\mathrm{B}\mathcal{Y}$ gives a natural map

$$\mathbf{W}_{\mathcal{X}}(\mathcal{Y})=(\mathcal{X}/\mathcal{Y})^{\mathrm{h}\mathcal{Y}}\to\mathcal{X}/\mathcal{Y}.$$

The Weyl space $\mathbf{W}_{\mathcal{X}}(f(\mathcal{Y}))$ is particularly useful when f is a monomorphism. One reason is the following.

4.3. Proposition. *Suppose $f:\mathcal{Y}\to\mathcal{X}$ is a monomorphism of p-compact groups. Then $\pi_0\mathbf{W}_{\mathcal{X}}(\mathcal{Y})$ is a group. Equivalently, every self map of $\mathrm{B}\mathcal{Y}$ over $\mathrm{B}\mathcal{X}$ is an equivalence.*

Proof. Let $i:\mathrm{B}\mathcal{Y}\to\mathrm{B}\mathcal{Y}$ be a self map of $\mathrm{B}\mathcal{Y}$ over $\mathrm{B}\mathcal{X}$. By 3.2 i is a monomorphism. By dimension counting (see 3.7) the coset space $\mathcal{Y}/i(\mathcal{Y})$ is homotopically discrete, and so i is an equivalence on identity components [5, 6.16]. A fibration argument now shows that the homotopy fibre of the map $\mathrm{B}\pi_0\mathcal{Y}\to\mathrm{B}\pi_0\mathcal{Y}$ induced by i is the same as $\mathcal{Y}/i(\mathcal{Y})$ and in particular is homotopically discrete. This implies that $\pi_0(i)$ is injective, and hence an isomorphism because $\pi_0\mathcal{Y}$ is finite. $\blacksquare$

4.4. Definition. Suppose that $f:\mathcal{Y}\to\mathcal{X}$ is a monomorphism of p-compact groups. The *normalizer* of $f(\mathcal{Y})$ in $\mathcal{X}$, denoted by $\mathbf{N}_{\mathcal{X}}(f(\mathcal{Y}))$ or $\mathbf{N}_{\mathcal{X}}(\mathcal{Y})$, is the loop space whose classifying space is the Borel construction of the action (see 4.2) of $\mathbf{W}_{\mathcal{X}}(\mathcal{Y})$ on $\mathrm{B}\mathcal{Y}$.

4.5. Remark. This definition of $\mathbf{N}_{\mathcal{X}}(\mathcal{Y})$ generalizes one given previously (see 2.4) in the special case $\mathcal{Y}=T_{\mathcal{X}}$. In general (even if $\mathcal{Y}=T_{\mathcal{X}}$), $\mathbf{N}_{\mathcal{X}}(\mathcal{Y})$ is not a p-compact group. Since the action of $\mathbf{W}_{\mathcal{X}}(\mathcal{Y})$ on $\mathrm{B}\mathcal{Y}$ commutes with the projection $\mathrm{B}f : \mathrm{B}\mathcal{Y} \to \mathrm{B}\mathcal{X}$, there is a natural map $\mathrm{B}\mathbf{N}_{\mathcal{X}}(\mathcal{Y}) \to \mathrm{B}\mathcal{X}$ or equivalently a natural loop space homomorphism $\mathbf{N}_{\mathcal{X}}(\mathcal{Y}) \to \mathcal{X}$.

4.6. Remark. There is a fibration sequence of spaces (corresponding to a short exact sequence [5, 3.2] of loop spaces)

$$\mathrm{B}\mathcal{Y} \to \mathrm{B}\mathbf{N}_{\mathcal{X}}(\mathcal{Y}) \to \mathrm{B}\mathbf{W}_{\mathcal{X}}(\mathcal{Y}).$$

The space $\mathbf{W}_{\mathcal{X}}(\mathcal{Y})$ should therefore be interpreted as a quotient $\mathbf{N}_{\mathcal{X}}(\mathcal{Y})/\mathcal{Y}$. In fact, the map $\mathbf{W}_{\mathcal{X}}(\mathcal{Y}) \to \mathcal{X}/\mathcal{Y}$ of 4.2 corresponds to the usual map $\mathbf{N}_{\mathcal{X}}(\mathcal{Y})/\mathcal{Y} \to \mathcal{X}/\mathcal{Y}$ if, say, the p-compact groups involved are finite p-groups. Observe that the restriction to $\mathcal{Y}$ of the homomorphism $\mathbf{N}_{\mathcal{X}}(\mathcal{Y}) \to \mathcal{X}$ is the original homomorphism $f : \mathcal{Y} \to \mathcal{X}$.

4.7. Definition. Let $f : \mathcal{Y} \to \mathcal{X}$ be a monomorphism of p-compact groups. The monomorphism f is said to be *normal* (or $\mathcal{Y}$ is said to be a *normal subgroup* of $\mathcal{X}$) if the natural homomorphism (see 4.5) $\mathbf{N}_{\mathcal{X}}(\mathcal{Y}) \to \mathcal{X}$ is an equivalence.

4.8. Proposition. *If $f : \mathcal{Y} \to \mathcal{X}$ is a monomorphism of p-compact groups, the following three conditions are equivalent*:

(1) *$f(\mathcal{Y})$ is a normal subgroup of $\mathcal{X}$.*
(2) *The standard map* (see 4.2) *$\mathbf{W}_{\mathcal{X}}(\mathcal{Y}) \to \mathcal{X}/\mathcal{Y}$ is an equivalence.*
(3) *The fibration $\mathcal{Y}\backslash\mathcal{X}/\mathcal{Y} \to \mathrm{B}\mathcal{Y}$ with fibre $\mathcal{X}/\mathcal{Y}$ is trivial.*

4.9. Remark. As usual, a fibration $\mathcal{E} \to \mathcal{B}$ with fibre $\mathcal{F}$ is said to be *trivial* if it is fibre homotopy trivial, that is, if there is a map $\mathcal{E} \to \mathcal{F}$ which restricts on $\mathcal{F} \subset \mathcal{E}$ to the identity map.

Proof of 4.8. The equivalence between the first two conditions results from constructing a commutative diagram

$$\begin{array}{ccccc}
\mathbf{W}_{\mathcal{X}}(\mathcal{Y}) & \longrightarrow & \mathrm{B}\mathcal{Y} & \longrightarrow & \mathrm{B}\mathbf{N}_{\mathcal{X}}(\mathcal{Y}) \\
\big\downarrow & & {\scriptstyle =}\big\downarrow & & \big\downarrow \\
\mathcal{Y}/\mathcal{X} & \longrightarrow & \mathrm{B}\mathcal{Y} & \longrightarrow & \mathrm{B}\mathcal{X}
\end{array}$$

and comparing the upper fibration sequence to the lower one. The equivalence between the second two follows from [6, 9.3]; in general, a fibration q over $\mathrm{B}\mathcal{Y}$ with fibre a space $\mathcal{F}$ which is both $\mathbb{F}_p$-complete and $\mathbb{F}_p$-finite is trivial if and only if evaluation at the basepoint of $\mathrm{B}\mathcal{Y}$ gives an equivalence $\Gamma(q) \to \mathcal{F}$. ∎

4.10. Proposition. *If $f : A \to \mathcal{X}$ is a central monomorphism of p-compact groups, then $f(A)$ is a normal subgroup of $\mathcal{X}$.*

Proof. The map $\mathrm{B}f : \mathrm{B}A \to \mathrm{B}\mathcal{X}$ is up to homotopy the principal fibration constructed by pulling back to $\mathrm{B}\mathcal{X}$ the path fibration over $\mathrm{B}(\mathcal{X}/A)$. The fibration $q : A\backslash\mathcal{X}/A \to \mathrm{B}A$ is thus also principle, since it is a further pullback of $\mathrm{B}f$. Since the fibration q has an obvious section given by the identity map of $\mathrm{B}A$, it follows that q is a trivial fibration, and the desired result is a consequence of 4.8. ∎

5. Bundles over $\mathrm{B}\mathcal{X}$**.** In this section we study bundles over the classifying space of a p-compact group, with the goal of finding situations in which 4.8(3) can be used to show that some given monomorphism is normal. The basic idea in this section is due to Notbohm [11]. The main theorem is the following one.

5.1. THEOREM. *Let $\mathcal{X}$ be a p-compact group, and $q:\mathcal{E}\to\mathrm{B}\mathcal{X}$ a fibration in which the fibre $\mathcal{F}$ is $\mathbb{F}_p$-finite and $\mathbb{F}_p$-complete. Then q is a trivial fibration* (see 4.9) *if and only if its pullback over $\mathrm{B}\mathbf{N}_p(T_{\mathcal{X}})$ is trivial. If $\mathcal{X}$ is connected, then q is a trivial fibration if and only if its pullback over $\mathrm{B}T_{\mathcal{X}}$ is trivial.*

By Theorem 4.8, this has an immediate corollary.

5.2. COROLLARY. *Suppose that $f:\mathcal{Y}\to\mathcal{X}$ is a monomorphism of p-compact groups. Then f is normal* (see 4.7) *if and only if the fibration $\mathbf{N}_p(T_{\mathcal{Y}})\backslash\mathcal{X}/\mathcal{Y}\to\mathrm{B}\mathbf{N}_p(T_{\mathcal{Y}})$ is trivial. If $\mathcal{Y}$ is connected, then f is normal if and only if the fibration $T_{\mathcal{Y}}\backslash\mathcal{X}/\mathcal{Y}\to\mathrm{B}T_{\mathcal{Y}}$ is trivial.*

The rest of this section is devoted to proving 5.1.

5.3. DEFINITION. A space $\mathcal{Z}$ is said to be $\mathrm{B}\mathbb{Z}/p$-*null* if evaluation at the basepoint of $\mathrm{B}\mathbb{Z}/p$ gives an equivalence

$$\mathrm{Map}(\mathrm{B}\mathbb{Z}/p,\mathcal{Z})\xrightarrow{\cong}\mathcal{Z}.$$

5.4. EXAMPLE. By Miller's theorem [8], if $\mathcal{F}$ is a space which is $\mathbb{F}_p$-finite and $\mathbb{F}_p$-complete, then $\mathcal{F}$ is $\mathrm{B}\mathbb{Z}/p$-null.

If G is a finite p-group and $f:G\to\mathcal{Z}$ is a homomorphism of loop spaces, define the *kernel* $\ker(f)$ of f to be the set of all elements $g\in G$ such that the restriction of f to the cyclic subgroup $\langle g\rangle$ of G is trivial (see 2.2). In general $\ker(f)$ is not necessarily a subgroup of G, but there is one key situation in which the kernel is well behaved.

5.5. PROPOSITION. *Suppose that G is a finite p-group, $\mathcal{Z}$ is a loop space which is $\mathrm{B}\mathbb{Z}/p$-null, and $f:G\to\mathcal{Z}$ is a homomorphism. Then $\ker(f)$ is a normal subgroup of G, and f factors uniquely up to homotopy through a homomorphism $G/\ker(f)\to\mathcal{Z}$. In particular, f is trivial if and only if $\ker(f)=G$.*

Proof. This is essentially the same as the proof given in [5, §7] for the special case in which $\mathcal{Z}$ is a p-compact group. It is only necessary to replace the reference to [5, 5.4] in the proof of [5, 7.2] by the assumption that $\mathcal{Z}$ is

$\mathrm{B}\mathbb{Z}/p$-null, and observe that the cyclic group K in the proof of [5, 7.7] can be assumed to have order p. ∎

Recall that a space $\mathcal{U}$ is $\mathbb{F}_p$-*local* (in the sense of Bousfield [1]) if any $\mathbb{F}_p$-equivalence $\mathcal{A}\to\mathcal{B}$ induces an equivalence $\mathrm{Map}(\mathcal{B},\mathcal{U})\to\mathrm{Map}(\mathcal{A},\mathcal{U})$. Any $\mathbb{F}_p$-complete space is $\mathbb{F}_p$-local [1].

5.6. Theorem. *Suppose that $\mathcal{X}$ is p-compact group and $\mathcal{Z}$ is a loop space such that $\mathcal{Z}$ is $\mathrm{B}\mathbb{Z}/p$-null and $\mathrm{B}\mathcal{Z}$ is $\mathbb{F}_p$-local. Then a homomorphism $f:\mathcal{X}\to\mathcal{Z}$ is trivial* (*see* 2.2) *if and only if the restriction of f to $\mathbf{N}_p(T_{\mathcal{X}})$ is trivial. If $\mathcal{X}$ is connected*, *a homomorphism $f:\mathcal{X}\to\mathcal{Z}$ is trivial if and only if the restriction of f to $T_{\mathcal{X}}$ is trivial.*

Proof. The statement involving $\mathbf{N}_p(T_{\mathcal{X}})$ appears in a slightly different form in [11, 1.4]. One way to prove this statement is to copy the inductive argument of [4, §5] (compare [4, 5.4]), being careful to rely on the homology decomposition theorem for p-compact groups from [6, §8] instead of on the theorem of Jackowski and McClure used in [4]. The effect of this line of reasoning is to show that $f:\mathcal{X}\to\mathcal{Z}$ is trivial if and only if it is *null on finite p-groups*, i.e., if and only if the composite $f\varrho$ is trivial for every homomorphism $\varrho:G\to\mathcal{X}$ with G a finite p-group. The desired result now follows from the fact that every such homomorphism $\varrho:G\to\mathcal{X}$ lifts up to conjugacy to a homomorphism $G\to\mathbf{N}_p(T_{\mathcal{X}})$ [6, 2.14], so that $f\varrho$ is trivial if the restriction of f to $\mathbf{N}_p(T_{\mathcal{X}})$ is.

Suppose now that $\mathcal{X}$ is connected. We will prove the remaining statement in the theorem by using the idea in the proof of [9, 5.7]. As noted above, we have proved that $f:\mathcal{X}\to\mathcal{Z}$ is trivial if and only if the composite $f\varrho$ is trivial for every homomorphism $\varrho:G\to\mathcal{X}$ with G a finite p-group. For such a homomorphism ϱ, the composite $f\varrho$ is trivial if and only if $\ker(f\varrho)=G$ (see 5.5). However, any homomorphism from a cyclic p-group $\langle g\rangle$ to $\mathcal{X}$ lifts up to conjugacy to a homomorphism $\langle g\rangle\to T_{\mathcal{X}}$ (see the proof of [6, 2.14]), and so $\ker(f\varrho)=G$ if the restriction of f to $T_{\mathcal{X}}$ is trivial. ∎

5.7. Proposition. *Suppose that $\mathcal{F}$ is a space which is $\mathbb{F}_p$-finite and $\mathbb{F}_p$-complete. Let $\mathrm{Aut}^{\mathrm{h}}(\mathcal{F})$ denote the the topological monoid of self homotopy equivalences of $\mathcal{F}$*, *and $\mathcal{Z}$ some union of components of $\mathrm{Aut}^{\mathrm{h}}(\mathcal{F})$ such that $\mathcal{Z}$ is closed under composition and $\pi_0\mathcal{Z}$ is a finite p-group. Then $\mathcal{Z}$ is $\mathrm{B}\mathbb{Z}/p$-null and $\mathrm{B}\mathcal{Z}$ is $\mathbb{F}_p$-local.*

Remark. Since $\mathcal{Z}$ is a topological monoid such that $\pi_0\mathcal{Z}$ is a group, $\mathcal{Z}$ is equivalent to $\Omega\mathrm{B}\mathcal{Z}$, i.e., $\mathcal{Z}$ is a loop space.

Proof of 5.7. If $g:\mathcal{A}\to\mathcal{B}$ is an $\mathbb{F}_p$-equivalence, it is clear that the induced map

$$\mathrm{Map}(\mathcal{F},\mathrm{Map}(\mathcal{B},\mathcal{F}))\xrightarrow{\mathrm{Map}(\mathrm{Id},\mathrm{Map}(g,\mathrm{Id}))}\mathrm{Map}(\mathcal{F},\mathrm{Map}(\mathcal{A},\mathcal{F}))$$

is an equivalence, and hence from an adjointness argument that the induced map

$$\mathrm{Map}(\mathcal{B}, \mathrm{Map}(\mathcal{F},\mathcal{F})_{\mathrm{Id}}) \xrightarrow{\mathrm{Map}(g,\mathrm{Id})} \mathrm{Map}(\mathcal{A}, \mathrm{Map}(\mathcal{F},\mathcal{F})_{\mathrm{Id}})$$

is also an equivalence. This in effect shows that $\mathcal{Z}$ is $\mathbb{F}_p$-local, since a space is $\mathbb{F}_p$-local if and only if each of its components is. The space $\mathcal{Z}$ is $\mathbb{F}_p$-good [5, 1.3] because it is a loop space, and therefore [5, 11.3] $\mathbb{F}_p$-complete. An application of the fibre lemma [5, 11.7] to the path fibration over $\mathrm{B}\mathcal{Z}$ now shows that $\mathrm{B}\mathcal{Z}$ is $\mathbb{F}_p$-complete and hence $\mathbb{F}_p$-local.

In order to show that $\mathcal{Z}$ is $\mathrm{B}\mathbb{Z}/p$-null, consider the map

$$\mathrm{Map}(\mathrm{B}\mathbb{Z}/p, \mathrm{Map}(\mathcal{F},\mathcal{F})) \cong \mathrm{Map}(\mathcal{F}, \mathrm{Map}(\mathrm{B}\mathbb{Z}/p, \mathcal{F})) \to \mathrm{Map}(\mathcal{F},\mathcal{F}),$$

given by evaluation at the basepoint of $\mathrm{B}\mathbb{Z}/p$. This map is an equivalence by 5.4. The fact that $\mathcal{Z}$ is $\mathrm{B}\mathbb{Z}/p$-null follows from the fact that a space is $\mathrm{B}\mathbb{Z}/p$-null if and only if each of its components is. ■

Proof of 5.1. Let $\mathrm{B}f : \mathrm{B}\mathcal{X} \to \mathrm{B}\,\mathrm{Aut}^{\mathrm{h}}(\mathcal{F})$ be the classifying map for the fibration q, and $f = \Omega\mathrm{B}f : \mathcal{X} \to \mathrm{Aut}^{\mathrm{h}}(\mathcal{F})$ the corresponding loop space homomorphism. By bundle theory the fibration q is trivial if and only if the homomorphism f is trivial. Let $\mathcal{Z} \subset \mathrm{Aut}^{\mathrm{h}}(\mathcal{F})$ be the union of those components of $\mathrm{Aut}^{\mathrm{h}}(\mathcal{F})$ which intersect the image of f nontrivially. Then $\mathcal{Z}$ is closed under composition, and $\pi_0\mathcal{Z}$, which is a quotient of $\pi_0\mathcal{X}$, is a finite p-group. By 5.7, $\mathcal{Z}$ is $\mathrm{B}\mathbb{Z}/p$-null and $\mathrm{B}\mathcal{Z}$ is $\mathbb{F}_p$-local. The homomorphism f clearly lifts to a homomorphism $\widetilde{f} : \mathcal{X} \to \mathcal{Z}$, and $\widetilde{f}$ is trivial if and only if f is. The desired result now follows from 5.6. ■

6. Splittings of p-compact groups. In this section we will prove Theorem 1.4. By induction it is enough to treat the case in which there are only two factors in the product decomposition of the dual weight lattice. The goal of this section is thus to prove the following.

6.1. THEOREM. *If $\mathcal{X}$ is a connected p-compact group, then any product decomposition $L_{\mathcal{X}} = L_1 \times L_2$ of modules over $W_{\mathcal{X}}$ can be realized by a p-compact group splitting $\mathcal{X} \simeq \mathcal{X}_1 \times \mathcal{X}_2$.*

We will now set up some notation which will be used for the rest of the section. Let $\mathcal{X}$ be a p-compact group, and $L_{\mathcal{X}} = L_1 \times L_2$ a product splitting of modules over $W_{\mathcal{X}}$. For $i = 1, 2$ denote by $k_i : T_i \to T_{\mathcal{X}}$ the central subgroup (see 2.3) of $T_{\mathcal{X}}$ obtained by taking the p-closure (see 2.5) of $\breve{L}_i \to \breve{L}_{\mathcal{X}} = \breve{T}_{\mathcal{X}}$. Let $\mathcal{Y}' = \mathbf{C}_{\mathcal{X}}(T_2)$ and $\mathcal{Z}' = \mathbf{C}_{\mathcal{X}}(T_1)$. By [6, 4.3] the maximal torus $T_{\mathcal{X}} \to \mathcal{X}$ lifts to maximal tori $T_{\mathcal{X}} \to \mathcal{Y}'$ and $T_{\mathcal{X}} \to \mathcal{Z}'$. Denote the quotients $\mathcal{Y}'/T_2$ and $\mathcal{Z}'/T_1$ by $\mathcal{Y}$ and $\mathcal{Z}$ respectively. These p-compact groups can then be arranged into a commutative diagram

$$
\begin{tikzcd}
T_2 \arrow[r, "k_2"] \arrow[d, "="] & T_{\mathcal{X}} \arrow[r, "="] \arrow[d, "i_{\mathcal{Y}'}"'] & T_{\mathcal{X}} \arrow[d, "i_{\mathcal{X}}"'] & T_{\mathcal{X}} \arrow[l, "="'] \arrow[d, "i_{\mathcal{Z}'}"'] & T_1 \arrow[l, "k_1"'] \arrow[d, "="'] \\
T_2 \arrow[r, "k_2'"] & \mathcal{Y}' \arrow[r] \arrow[d, "q_{\mathcal{Y}}'"'] & \mathcal{X} & \mathcal{Z}' \arrow[l] \arrow[d, "q_{\mathcal{Z}}'"'] & T_1 \arrow[l, "k_1'"'] \\
 & \mathcal{Y} = \mathcal{Y}'/T_2 & & \mathcal{Z} = \mathcal{Z}'/T_1 &
\end{tikzcd}
\tag{6.2}
$$

in which the upper vertical maps are maximal torus monomorphisms and the lower vertical maps are quotient projections. The fact that the outer squares commute, where k_2', for instance, is the natural map (see 2.3) from T_2 to its centralizer, follows either from inspection or from [6, 6.5]. All of these p-compact groups are connected, since the centralizer of a torus inside a connected p-compact group is connected [6, 7.8], and any central quotient of a connected p-compact group is connected.

Let $W_1 \subset W_{\mathcal{X}}$ be the subgroup which pointwise fixes L_2, and $W_2 \subset W_{\mathcal{X}}$ the subgroup which pointwise fixes L_1.

6.3. Lemma. *The subgroups W_1 and W_2 of $W_{\mathcal{X}}$ commute, and the product map $W_1 \times W_2 \to W_{\mathcal{X}}$ is an isomorphism.*

Proof. It is clear from the definitions that if $x \in W_1$ and $y \in W_2$ then the commutator $[x, y]$ belongs to $W_1 \cap W_2$. Since $\mathcal{X}$ is connected the action of $W_{\mathcal{X}}$ on $L_{\mathcal{X}}$ is faithful [5, 9.7]. This shows that $W_1 \cap W_2$ is trivial, thus that x commutes with y, and thus that the product map $W_1 \times W_2 \to W_{\mathcal{X}}$ is a monomorphism. By elementary rank considerations, if $s \in W_{\mathcal{X}}$ is a reflection (see 2.4) then either $s \in W_1$ or $s \in W_2$. Since $W_{\mathcal{X}}$ is generated by reflections, the indicated product map is onto. ■

We will now concentrate briefly on $\mathcal{Y}'$ and $\mathcal{Y}$. By [6, 7.6], the action of $W_{\mathcal{Y}'}$ on $\pi_1 T_{\mathcal{Y}'} = \pi_1 T_{\mathcal{X}} = L_{\mathcal{X}}$ (see above) gives an isomorphism between $W_{\mathcal{Y}'}$ and W_1. The calculation $(\mathcal{Y}'/T_2)/(T_{\mathcal{X}}/T_2) \simeq \mathcal{Y}'/T_{\mathcal{X}}$ shows that the quotient map $T_{\mathcal{X}}/T_2 \to \mathcal{Y}$ is a maximal torus for $\mathcal{Y}$ (see the proof of [6, 6.3]). The composite map

$$
T_1 \to T_{\mathcal{X}} \to T_{\mathcal{X}}/T_2 \tag{6.4}
$$

is an equivalence because on fundamental groups it induces the quotient isomorphism $L_{\mathcal{X}}/L_2 \cong L_1$. It follows that we can choose a maximal torus $T_{\mathcal{Y}}$ by setting $T_{\mathcal{Y}} = T_1$ and letting $i_{\mathcal{Y}} : T_{\mathcal{Y}} \to \mathcal{Y}$ be the result of following the composite (see 6.4) by the monomorphism $T_{\mathcal{X}}/T_2 \to \mathcal{Y}$. The group $W_{\mathcal{Y}}$ is then isomorphic in its action on $L_{\mathcal{Y}} = \pi_2 T_1 = L_1$ to W_1, so from now on we denote L_1 by $L_{\mathcal{Y}}$ and W_1 by $W_{\mathcal{Y}}$. The map k_1' above (see (6.2)) thus gives a commutative diagram

$$\begin{array}{ccc} T_{\mathcal{Y}} & \xrightarrow{k_2'} & \mathcal{Y}' \\ {\scriptstyle =}\big\downarrow & & \big\downarrow{\scriptstyle q_{\mathcal{Y}}'} \\ T_{\mathcal{Y}} & \xrightarrow{i_{\mathcal{Y}}} & \mathcal{Y} \end{array}$$

6.5. Lemma. *There exists up to homotopy a unique homomorphism* $s_{\mathcal{Y}}:\mathcal{Y}\to\mathcal{Y}'$ *such that* $s_{\mathcal{Y}}$ *is a section for* $q_{\mathcal{Y}}'$ *and agrees on* $T_{\mathcal{Y}}$ *with* k_2', *i.e., such that* $q_{\mathcal{Y}}'\cdot s_{\mathcal{Y}}$ *is homotopic to* $\mathrm{Id}_{\mathcal{Y}}$ *and* $s_{\mathcal{Y}}\cdot i_{\mathcal{Y}}$ *is homotopic to* k_2'.

The proof of this will be given below.

6.6. Remark. By construction the map $\mathrm{B}(q_{\mathcal{Y}}'):\mathrm{B}\mathcal{Y}'\to\mathrm{B}(\mathcal{Y}'/T_{\mathcal{Z}})=\mathrm{B}\mathcal{Y}$ is a principal fibration [5, proof of 8.3]. The existence of the section $\mathrm{B}(s_{\mathcal{Y}})$ implies that this fibration is trivial, or in other words that $\mathcal{Y}'$ is equivalent as p-compact group to the product $T_{\mathcal{Z}}\times\mathcal{Y}$.

The above considerations and their "$\mathcal{Z}$" analogues lead to a diagram

$$\mathcal{Y}\xrightarrow{s_{\mathcal{Y}}}\mathcal{Y}'\to\mathcal{X}\longleftarrow\mathcal{Z}'\xleftarrow{s_{\mathcal{Z}}}\mathcal{Z}.$$

6.7. Proposition. *The double coset space* (*see* 3.1) $\mathcal{Y}\backslash\mathcal{X}/\mathcal{Z}$ *is contractible*, *i.e.*, *the loop space multiplication map* $\mathcal{Y}\times\mathcal{Z}\to\mathcal{X}$ *is an equivalence of spaces.*

Proof. Since $\mathbb{Z}/p\otimes L_{\mathcal{X}}$ is isomorphic to $(\mathbb{Z}/p\otimes L_1)\times(\mathbb{Z}/p\otimes L_2)=(\mathbb{Z}/p\otimes L_{\mathcal{Y}})\times(\mathbb{Z}/p\otimes L_{\mathcal{Z}})$ as a module over $W_{\mathcal{X}}$, Proposition 3.6 implies that $\mathcal{Y}\backslash\mathcal{X}/\mathcal{Z}$ is $\mathbb{F}_p$-finite. By 6.3, $W_{\mathcal{X}}$ is isomorphic to $W_{\mathcal{Y}}\times W_{\mathcal{Z}}$ as a reflection group, so by 3.9 and 3.8, $\mathrm{cd}_{\mathbb{F}_p}(\mathcal{X})=\mathrm{cd}_{\mathbb{F}_p}(\mathcal{Y})+\mathrm{cd}_{\mathbb{F}_p}(\mathcal{Z})$. Therefore (see 3.7), $\mathrm{cd}_{\mathbb{F}_p}(\mathcal{Y}\backslash\mathcal{X}/\mathcal{Z})=0$. As a $\mathbb{F}_p$-complete space which has the mod p cohomology of a point, the double coset space is contractible. ∎

6.8. Remark. Exactly the same argument as in the proof of 6.7 shows that the double coset space $\mathcal{Y}\backslash\mathcal{Y}'/T_{\mathcal{Z}}$ is contractible. This is clear in any case from 6.6.

6.9. Proposition. *The square*

$$\begin{array}{ccc} \mathrm{B}T_{\mathcal{Z}} & \longrightarrow & \mathrm{B}\mathcal{Z} \\ \big\downarrow & & \big\downarrow \\ \mathrm{B}\mathcal{Y}' & \longrightarrow & \mathrm{B}\mathcal{X} \end{array}$$

is a homotopy fibre square.

Proof. By 6.7 and 6.8 the natural map

$$\mathcal{Y}\backslash\mathcal{Y}'/T_{\mathcal{Z}}\to\mathcal{Y}\backslash\mathcal{X}/\mathcal{Z}$$

is an equivalence; in fact, both spaces are contractible. It follows that the natural map $\mathcal{Y}'/T_{\mathcal{Z}}\to\mathcal{X}/\mathcal{Z}$ is also an equivalence; this, however, is the induced map on vertical fibres in the above square. ∎

Proof of 6.1. By 4.10 and 4.8(3), the fibration $T_{\mathcal{Z}}\backslash\mathcal{Y}'/T_{\mathcal{Z}} \to \mathrm{B}T_{\mathcal{Z}}$ is trivial. By 6.9, this fibration is the same up to homotopy as the fibration $T_{\mathcal{Z}}\backslash\mathcal{X}/\mathcal{Z} \to \mathrm{B}T_{\mathcal{Z}}$, and so this second fibration is also trivial. According to 5.2, then, the monomorphism $\mathcal{Z} \to \mathcal{X}$ is normal. By 6.7, the sum map

$$\pi_*(\mathrm{B}\mathcal{Y}) \times \pi_*(\mathrm{B}\mathcal{Z}) \to \pi_*(\mathrm{B}\mathcal{X}) \tag{6.10}$$

is an isomorphism. Now consider the diagram

$$\begin{tikzcd} & \mathrm{B}\mathcal{Y} \arrow[d] & \\ \mathrm{B}\mathcal{Z} \arrow[r] & \mathrm{B}\mathcal{X} \simeq \mathrm{B}\mathbf{N}_{\mathcal{X}}(\mathcal{Z}) \arrow[r] & \mathrm{B}\mathbf{W}_{\mathcal{X}}(\mathcal{Z}) \end{tikzcd}$$

in which the lower line is a fibration sequence. Calculating with homotopy groups shows that the composite map $\mathrm{B}\mathcal{Y} \to \mathrm{B}\mathbf{W}_{\mathcal{X}}(\mathcal{Z})$ is an equivalence and that, under this equivalence, the induced map

$$\pi_*(\mathrm{B}\mathcal{Y}) \times \pi_*(\mathrm{B}\mathcal{Z}) \cong \pi_*(\mathrm{B}\mathcal{X}) \to \pi_*\mathrm{B}\mathbf{W}_{\mathcal{X}}(\mathcal{Z}) \cong \pi_*\mathrm{B}\mathcal{Y}$$

amounts to projection on the second factor. Repeating the construction with $\mathcal{Y}$ and $\mathcal{Z}$ interchanged gives an analogous map

$$\mathrm{B}\mathcal{X} \simeq \mathrm{B}\mathbf{N}_{\mathcal{X}}(\mathcal{Y}) \to \mathrm{B}\mathbf{W}_{\mathcal{X}}(\mathcal{Y}) \simeq \mathrm{B}\mathcal{Z}$$

which on homotopy groups has the effect of projecting onto the second summand (see (6.10)) of $\pi_*\mathrm{B}\mathcal{X}$. It is now easy to see that the product map

$$\mathrm{B}\mathcal{X} \to \mathrm{B}\mathbf{W}_{\mathcal{X}}(\mathcal{Z}) \times \mathrm{B}\mathbf{W}_{\mathcal{X}}(\mathcal{Y}) \simeq \mathrm{B}\mathcal{Y} \times \mathrm{B}\mathcal{Z}$$

is an equivalence which realizes the product decomposition $L_{\mathcal{X}} \cong L_1 \times L_2 = L_{\mathcal{Y}} \times L_{\mathcal{Z}}$. ■

All that remains is to give a proof of 6.5. This depends on a lemma.

6.11. Lemma. *For any connected p-compact group $\mathcal{U}$ the natural map*

$$\mathrm{H}_0(W_{\mathcal{U}}, L_{\mathcal{U}}) = \mathrm{H}_0(W_{\mathcal{U}}, \mathrm{H}_2(\mathrm{B}T_{\mathcal{U}})) \to \mathrm{H}_2(\mathrm{B}\mathcal{U})$$

is an epimorphism with finite kernel.

Proof. To prove surjectivity, it is enough by reducing mod p and dualizing to show that the restriction map $\mathrm{H}^2(\mathrm{B}\mathcal{U}) \to \mathrm{H}^2(\mathrm{B}T_{\mathcal{U}})$ is injective. Pick a nonzero element $k \in \mathrm{H}^2(\mathrm{B}\mathcal{U})$. The principal fibration sequence

$$\mathrm{B}\mathbb{Z}/p \to \mathrm{B}\mathcal{V} \to \mathrm{B}\mathcal{U} \tag{6.12}$$

classified by k expresses $\mathcal{U}$ as the quotient of a connected p-compact group $\mathcal{V}$ by a central subgroup equivalent to $\mathbb{Z}/p$. Parallel to (6.12) is another sequence

$$\mathrm{B}\mathbb{Z}/p \to \mathrm{B}T_{\mathcal{V}} \to \mathrm{B}T_{\mathcal{U}}$$

in which the middle space $\mathrm{B}T_{\mathcal{V}}$ is of course simply connected (see (6.14)); this second fibration sequence is classified by the image of k in $\mathrm{H}^2(\mathrm{B}T_{\mathcal{V}})$

under the restriction map. This image of k must be nonzero because the second sequence is not a product.

Finiteness of the kernel is equivalent to the statement that the natural map $\mathrm{H}_0(W_{\mathcal{U}}, \mathbb{Q} \otimes \mathrm{H}_2(\mathrm{B}T_{\mathcal{U}})) \to \mathbb{Q} \otimes \mathrm{H}_2(\mathrm{B}\mathcal{U})$ is injective. This injectivity follows from the fact that the dual map $\mathrm{H}^2_{\mathbb{Q}_p}(\mathrm{B}\mathcal{U}) \to \mathrm{H}^0(W_{\mathcal{U}}, \mathrm{H}^2_{\mathbb{Q}_p}(\mathrm{B}T_{\mathcal{U}}))$ is an isomorphism [5, 9.7]. ∎

Proof of 6.5. Consider the map of exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L_{\mathcal{Z}} & \longrightarrow & \mathrm{H}_0(W_{\mathcal{Y}}, L_{\mathcal{X}}) & \longrightarrow & \mathrm{H}_0(W_{\mathcal{Y}}, L_{\mathcal{Y}}) & \longrightarrow & 0 \\ & & \big\downarrow{=} & & \big\downarrow & & \big\downarrow{u} & & \\ 0 & \longrightarrow & L_{\mathcal{Z}} & \longrightarrow & \mathrm{H}_2\mathrm{B}\mathcal{Y}' & \longrightarrow & \mathrm{H}_2\mathrm{B}\mathcal{Y} & \longrightarrow & 0 \end{array} \tag{6.13}$$

The upper sequence here comes from applying $\mathrm{H}_*(W_{\mathcal{Y}}, -)$ to the exact sequence

$$0 \to L_{\mathcal{Z}} \to L_{\mathcal{X}} \to L_{\mathcal{Y}} \to 0$$

and noticing that the appropriate connecting homomorphism is zero because the group $\mathrm{H}_1(W_{\mathcal{X}}, L_C)$ is torsion. The lower line in (6.13) is derived by looking at the Serre spectral sequence of (6.14) and using the fact that the differential terminating at $E^2_{0,2}$ is trivial because $\pi_3\mathrm{B}\mathcal{Y}$ and hence $\mathrm{H}_3\mathrm{B}\mathcal{Y}$ are torsion groups [5, 9.7]. The vertical maps in (6.14) are surjective with finite kernel (see 6.11). The Yoneda class of the lower exact sequence determines an element k in the group

$$\mathrm{Ext}^1_{\mathbb{Z}_p}(\mathrm{H}_2\mathrm{B}\mathcal{Y}, L_{\mathcal{Z}}) \cong \mathrm{H}^3(\mathrm{B}\mathcal{Y}, L_{\mathcal{Z}})$$

and by a naturality argument k can be identified with the classifying map

$$\mathrm{B}\mathcal{Y} \to \mathrm{B}^2 T_{\mathcal{Z}} \simeq K(L_{\mathcal{Z}}, 3)$$

of the fibration sequence

$$\mathrm{B}T_{\mathcal{Z}} \to \mathrm{B}\mathcal{Y}' \xrightarrow{\mathrm{B}(q'_{\mathcal{Y}})} \mathrm{B}\mathcal{Y}. \tag{6.14}$$

Let j be the Yoneda class of the upper exact sequence in (6.13), so that j is the image of k under the map

$$u^* : \mathrm{Ext}^1_{\mathbb{Z}_p}(\mathrm{H}_2\mathrm{B}\mathcal{Y}, L_{\mathcal{Z}}) \to \mathrm{Ext}^1_{\mathbb{Z}_p}(\mathrm{H}_0(W_{\mathcal{Y}}, L_{\mathcal{Y}}), L_{\mathcal{Z}}).$$

Since $L_{\mathcal{X}}$ splits as the product $L_{\mathcal{Z}} \times L_{\mathcal{Y}}$ of $W_{\mathcal{Y}}$-modules, it is clear that $\mathrm{H}_0(W_{\mathcal{Y}}, L_{\mathcal{X}})$ splits as a product $L_{\mathcal{Z}} \times \mathrm{H}_0(W_{\mathcal{Y}}, L_{\mathcal{Y}})$ of abelian groups, and so the extension class j is zero. Since the abelian group $L_{\mathcal{Z}}$ is torsion free and the kernel of u is finite, the map u^* is a monomorphism and hence the extension class k is also zero. This shows that the fibration (6.14) is trivial, and that the map $\mathrm{B}(q'_{\mathcal{Y}})$ has a right inverse up to homotopy. Now any two sections of $\mathrm{B}(q'_{\mathcal{Y}})$ differ up to homotopy by an element of $\mathrm{H}^2(\mathrm{B}\mathcal{Y}, L_{\mathcal{Z}})$, whereas the map $L_{\mathcal{Y}} \to L_{\mathcal{X}}$ induced (see 3.5) by such a section differs from the one induced by the original map $T_{\mathcal{Y}} \to T_{\mathcal{X}}$ by an element of $\mathrm{Hom}_{W_{\mathcal{Y}}}(L_{\mathcal{Y}}, L_{\mathcal{Z}})$. Since $L_{\mathcal{Z}}$ is torsion free and has a trivial action of $W_{\mathcal{Y}}$, Lemma 6.11 implies that the restriction map

$$\mathrm{H}^2(\mathrm{B}\mathcal{Y}, L_{\mathcal{Z}}) \to \mathrm{Hom}_{W_{\mathcal{Y}}}(L_{\mathcal{Y}}, L_{\mathcal{Z}})$$

is an isomorphism. Thus any section of $\mathrm{B}q'_{\mathcal{Y}}$ can be adjusted uniquely up to homotopy to have the desired restriction to $\mathrm{B}T_{\mathcal{Y}}$. ∎

7. p-compact goups with trivial center. In this section we will prove 1.5.

7.1. Proposition. *Suppose that $\mathcal{X}$ is a connected p-compact group and that*

$$\mathbb{Q} \otimes L_{\mathcal{X}} \cong M_1 \oplus \ldots \oplus M_n$$

is a direct sum decomposition of $L_{\mathcal{X}}$ as a module over $W_{\mathcal{X}}$. Let W_i ($i = 1, \ldots, n$) be the subgroup of $W_{\mathcal{X}}$ given by the elements which act trivially on $M_1 \oplus \ldots \oplus \widehat{M_i} \oplus \ldots \oplus M_n$. Then

(1) *W_i commutes with W_j if $i \neq j$, and*

(2) *the product map $W_1 \times \ldots \times W_n \to W$ is a group isomorphism.*

Proof. This is essentially the same as the proof of 6.3. ∎

7.2. *Calculating the center*. We now recall from [6, §7] how the center of a p-compact group $\mathcal{X}$ is computed. The loop space $\mathbf{N}(T_{\mathcal{X}})$ has in its own right a "discrete approximation"; this is a group $\breve{\mathbf{N}}(T_{\mathcal{X}})$ which lies in a short exact sequence

$$1 \to \breve{T}_{\mathcal{X}} \to \breve{\mathbf{N}}(T_{\mathcal{X}}) \to W_{\mathcal{X}} \to 1.$$

The conjugation action of $W_{\mathcal{X}}$ on $\breve{T}_{\mathcal{X}} = \breve{L}_{\mathcal{X}}$ derived from this short exact sequence is the same as the obvious one induced by the action of $W_{\mathcal{X}}$ on $L_{\mathcal{X}}$. If $\mathcal{X}$ is connected the p-discrete approximation $\breve{C}$ to the center C of $\mathcal{X}$ is a subgroup of $\breve{T}_{\mathcal{X}} = \breve{L}_{\mathcal{X}}$ which can be described in the following way. Let $\mathrm{rank}(\mathcal{X})$ denote the rank of $L_{\mathcal{X}}$ over $\mathbb{Z}_p$. For each reflection (see 2.4) $s \in W_{\mathcal{X}}$,

(1) the *fixed point set* $F(s)$ of s is the fixed point set of the conjugation action of s on $\breve{T}_{\mathcal{X}}$,

(2) the *singular hyperplane* $H(s)$ of s is the maximal divisible subgroup of $F(s)$ (so that $H(s) \cong (\mathbb{Z}/p^\infty)^{\mathrm{rank}(\mathcal{X})-1}$),

(3) the *singular coset* $K(s)$ of s is the subset of $\breve{T}_{\mathcal{X}}$ given by elements of the form $x^{\mathrm{ord}(s)}$, as x runs through elements of $\breve{\mathbf{N}}(T_{\mathcal{X}})$ which project to s in $W_{\mathcal{X}}$, and

(4) the *singular set* $\sigma(s)$ of s is the union $\sigma(s) = H(s) \cup K(s)$.

Then $\breve{C}$ is the intersection $\bigcap_s \sigma(s)$, where s runs through the reflections in $W_{\mathcal{X}}$. Note that if the prime p is odd then this intersection is just the fixed point set of the action of $W_{\mathcal{X}}$ on $\breve{T}_{\mathcal{X}}$; if $p = 2$, the intersection has finite index in the fixed point set.

Proof of 1.5. By definition, the map α is a map of modules over $W_{\mathcal{X}}$. Since α fits into a commutative diagram

$$\begin{array}{ccc} L_1 \times \ldots \times L_n & \longrightarrow & M_1 \times \ldots \times M_n \\ {\scriptstyle\alpha}\big\downarrow & & \cong\big\downarrow \\ L_{\mathcal{X}} & \longrightarrow & \mathbb{Q} \otimes L_{\mathcal{X}} \end{array}$$

in which the other maps are injective, it is clear that α is injective. Let D denote coker(α). If x is any element of $\mathbb{Q} \otimes L_{\mathcal{X}}$, there exists an integer k such that $p^k x \in L_{\mathcal{X}}$; this implies that each quotient group M_i/L_i is a p-primary torsion group, hence that their direct sum is torsion, and hence that D is torsion. Homological algebra now gives a short exact sequence

$$0 \to \mathrm{Tor}(\mathbb{Z}/p^\infty, D) \to \oplus_i(\mathbb{Z}/p^\infty \otimes L_i) \xrightarrow{\mathbb{Z}/p^\infty \otimes \alpha} \mathbb{Z}/p^\infty \otimes L_{\mathcal{X}} \to 0$$

in which the kernel $\mathrm{Tor}(\mathbb{Z}/p^\infty, D)$ is isomorphic to D, so in order to show that D is zero it is enough to show that $\mathbb{Z}/p^\infty \otimes \alpha$ is injective. Note that, by the definition of L_i, the cokernel of the inclusion $L_i \to L_{\mathcal{X}}$ is torsion free; this implies that the map $\mathbb{Z}/p^\infty \otimes L_i \to \mathbb{Z}/p^\infty \otimes L_{\mathcal{X}}$ is a monomorphism and allows $\mathbb{Z}/p^\infty \otimes L_i$ to be treated as a subgroup of $\mathbb{Z}/p^\infty \otimes L_{\mathcal{X}}$.

Let W_i $(i = 1, \ldots, n)$ be as in 7.1, so that each reflection in $W_{\mathcal{X}}$ lies in W_i for some i. Choose $x \in \ker(\mathbb{Z}/p^\infty \otimes \alpha)$, so that $x = (x_1, \ldots, x_n)$ with $x_i \in \mathbb{Z}/p^\infty \otimes L_i$ and $\sum x_i = 0$. We will be done if we can prove that x is zero. Consider some component x_k of x, and pick a reflection $s \in W_{\mathcal{X}}$ with $s \in W_i$. If $i \neq k$, then $x_k \in \mathbb{Z}/p^\infty \otimes L_k$, which is a divisible subgroup of $\breve{L}_{\mathcal{X}} = \mathbb{Z}/p^\infty \otimes L_{\mathcal{X}}$ fixed by s, and so x_k belongs to $H(s)$ (see 7.2). If $i = k$, then $x_k = -\sum_{j\neq k} x_j$ belongs to $\sum_{j\neq k} \mathbb{Z}/p^\infty \otimes L_k$, which is a divisible subgroup of $\breve{L}_{\mathcal{X}}$ fixed by s, and so in this case $x_k \in H(s)$ as well. The conclusion is that $x_k \in \bigcap_s H(s)$, where the intersection is taken over all reflections in $W_{\mathcal{X}}$. This intersection is contained in the p-discrete approximation $\breve{C}$ to the center C of $\mathcal{X}$, and is trivial because C is. Thus each component x_k of x is zero, and so x itself is zero. ∎

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UNIVERSITY OF NOTRE DAME
NOTRE DAME, INDIANA 46556
U.S.A.
E-mail: DWYER.1@ND.EDU

PURDUE UNIVERSITY
WEST LAFAYETTE, INDIANA 47907
U.S.A.

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