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Hausdorff dimension and measures on Julia sets of some meromorphic maps

by

Krzysztof Barański (Warszawa)

Abstract. We study the Julia sets for some periodic meromorphic maps, namely the maps of the form $f(z) = h(\exp \frac{2\pi i}{T} z)$ where h is a rational function or, equivalently, the maps $\widetilde{f}(z) = \exp(\frac{2\pi i}{T} h(z))$. When the closure of the forward orbits of all critical and asymptotic values is disjoint from the Julia set, then it is hyperbolic and it is possible to construct the Gibbs states on $J(\widetilde{f})$ for $-\alpha \log |\widetilde{f}'|$. For $\widetilde{\alpha} = \mathrm{HD}(J(\widetilde{f}))$ this state is equivalent to the $\widetilde{\alpha}$-Hausdorff measure or to the $\widetilde{\alpha}$-packing measure provided $\widetilde{\alpha}$ is greater or smaller than 1. From this we obtain some lower bound for $\mathrm{HD}(J(f))$ and real-analyticity of $\mathrm{HD}(J(f))$ with respect to f. As an example the family $f_\lambda(z) = \lambda \tan z$ is studied. We estimate $\mathrm{HD}(J(f_\lambda))$ near $\lambda = 0$ and show it is a monotone function of real λ.

1. Introduction. In the recent years some work was done in the dynamics of meromorphic (non-rational) maps (see [BKL], [DK], [K]). One of the basic differences which appear in this case compared with the rational one is that not all points have well defined forward orbits—the preimages of the poles after some time reach infinity, where the function is not defined. However, there are many similarities to the dynamics of rational and entire functions.

Let $f : \mathbb{C} \to \widehat{\mathbb{C}}$ be a meromorphic map with an infinite number of poles. Denote by $N(f)$ the set of points $z \in \mathbb{C}$ such that for some neighbourhood U of z the sequence $\{f^n_{|U}\}_{n\in\mathbb{N}}$ is defined, meromorphic and forms a normal family. The *Julia set* $J(f)$ is the complement of $N(f)$ in $\widehat{\mathbb{C}}$. We assume $\infty \in J(f)$. $J(f)$ is a compact perfect set and $z \in J(f)$ iff $f(z) \in J(f)$ or $z = \infty$. As in the rational case, $J(f) \subset \mathrm{cl} \bigcup_{n\geq 0} f^{-n}(z)$ for all $z \in \widehat{\mathbb{C}}$ except for at most two singular values in $\mathbb{C}$ with finite inverse orbits. Moreover, the Julia set is the closure of the periodic repelling points (see [BKL.I]).

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In this paper we shall deal with meromorphic maps of the form

$$f(z) = h(e^{\frac{2\pi i}{T}z}),$$

where h is a rational function (not a polynomial) with all poles in $\mathbb{C} \setminus \{0\}$ and $T \neq 0$ is a complex constant. These maps are periodic with period T. Our basic assumption is that the closure in $\widehat{\mathbb{C}}$ of the forward orbits of all critical and asymptotic values of f is disjoint from $J(f)$ (we do not treat infinity as a critical value). Note that the critical values of f are critical values of h (and vice versa, except for 0), so there are a finite number of them. There are at most two asymptotic values $h(0)$, $h(\infty)$. From the Fatou–Sullivan classification the complement of $J(f)$ is the union of a finite number of basins of sinks. Here are some examples of maps which satisfy our assumptions.

Remark 1.1. Consider a particular case when all critical and asymptotic values lie in the immediate basin of attraction of one sink. Then there exists a topological disk U containing $J(f) \cap \mathbb{C}$ such that for all k all branches of f^{-k} are defined on it and for some n, $f^{-n}(U) \subset U$. Hence the Julia set is a Cantor set and $f^n_{|J(f)}$ is conjugate to the shift on the space of one-sided sequences with infinitely many symbols supplemented with finite sequences corresponding to the preimages of the poles. Such a situation occurs in Examples 1.2 and 1.3.

Example 1.2. Suppose h is a homography. Then f has constant Schwarzian derivative. Such maps were studied in [DK]. They have two asymptotic values and no critical values. Take

$$h(z) = -\lambda i\,\frac{z-1}{z+1}$$

and $T = \pi$. Then $f(z) = \lambda \tan z$. For $\lambda \in \mathbb{R}$ with $0 < |\lambda| < 1$ both asymptotic values lie on the imaginary axis, f maps this axis into itself and all points from it are attracted to 0. More generally, let

$$h(z) = ai\,\frac{z+b}{z+c}$$

with $a, T \in \mathbb{R} \setminus \{0\}$, $b \in \mathbb{R}$, $c > 0$, $b \neq c$ and $\pi|a||b-c| < 2|T|c$. Then the picture is the same. For the tangent family, if $\lambda \in \mathbb{C}$ with $0 < |\lambda| < 1$ then 0 is still a fixed sink and both asymptotic values are attracted to it. For these λ all the maps are quasiconformally conjugate (see [DK]). This family will be studied in Section 4.

Example 1.3. It is easy to check that the dynamics described in the previous example is the same when $h(z) = ip(z)/q(z)$, $\deg p \leq \deg q$, p and q have real coefficients, q has no non-negatives roots, all critical points of h

are real and T is real and sufficiently large. For instance, take

$$h(z) = i\frac{(z+a)^m}{(z+b)^l}$$

with $m \le l$, $a \in \mathbb{R}$, $b > 0$. In particular, if

$$h(z) = \left(-\lambda i\,\frac{z-1}{z+1}\right)^p,$$

$|\lambda| < 1$, p some positive integer, $T = \pi$ then $f(z) = (\lambda \tan z)^p$.

EXAMPLE 1.4. In the case when f has more than one sink, the Julia set is no longer a Cantor set. For $f(z) = \lambda \tan z$ with $\lambda \in \mathbb{R}$, $|\lambda| > 1$, we have $J(f) = \mathbb{R} \cup \{\infty\}$ and there are two sinks on the imaginary axis attracting the upper and lower half-planes. The same is true for $h(z) = ai\frac{z+b}{z+c}$ with $a, T \in \mathbb{R} \setminus \{0\}$, $b \in \mathbb{R}$, $c > 0$, $b \ne c$ if there is a fixed source on the imaginary axis, and for some parameters in the families from Example 1.3. Like previously, for $\lambda \tan z$, $\lambda \in \mathbb{R}$, $|\lambda| > 1$ for small perturbation of λ there is a quasiconformal conjugation between the maps. The Julia set is then a quasi-circle.

Changing variables we assume $T = 2\pi i$ to simplify notation. We are going to study measures on the Julia set using the thermodynamic formalism. However, because of the periodicity of f the Perron–Frobenius–Ruelle operator for the function $-\alpha \log |f'|$ is infinite. To avoid this we shall consider the map $\widetilde{f}(z) = \exp(h(z))$. It is holomorphic on $\widehat{\mathbb{C}}$ except for the poles of h where it has essential singularities. The definition of the Julia set of $\widetilde{f}$ is the same as for meromorphic maps. Let $J = J(f)$ and $\widetilde{J} = J(\widetilde{f})$. We have

$$\exp(f^n(z)) = \widetilde{f}^n(\exp(z)) \tag{1}$$

so $\exp(J \cap \mathbb{C}) \subset \widetilde{J} \subset \exp(J \cap \mathbb{C}) \cup \{0, \infty\}$. By assumption $h(0), h(\infty) \notin J$, hence $0, \infty \notin \widetilde{J}$. Consequently,

$$\widetilde{J} = \exp(J \cap \mathbb{C}). \tag{2}$$

Thus the local geometric properties of $\widetilde{J}$ and J are the same.

The plan of the paper is as follows: in the next section we show that $\widetilde{f}$ and thus f is expanding on the Julia set and prove the "bounded distortion" lemma. In Section 3 we study the Perron–Frobenius–Ruelle operator $\mathcal{L}_\varphi$ on $C(\widetilde{J})$ for $\varphi = -\alpha \log |\widetilde{f}'|$. It is well defined for α greater than $\alpha_0 = p/(p+1)$, where p is the highest degree of the poles of h. The topological pressure P is finite if and only if $\alpha > \alpha_0$. The Gibbs state μ_α on $\widetilde{J}$ is constructed in the standard way. The Hausdorff dimension $\mathrm{HD}(\widetilde{J}) = \mathrm{HD}(J)$ is equal to $\widetilde{\alpha}$ which is the unique root of P. Consequently, it is greater than α_0. From the perturbation theory for $\mathcal{L}_\varphi$ we obtain the real-analyticity of $\mathrm{HD}(\widetilde{J})$ with respect to $\widetilde{f}$. Moreover, $\mathrm{HD}(\widetilde{J})$ equals the packing and box dimensions of $\widetilde{J}$.

However, the $\widetilde{\alpha}$-Hausdorff measure $\Lambda_{\widetilde{\alpha}}$ and the $\widetilde{\alpha}$-packing measure $\Pi_{\widetilde{\alpha}}$ on $\widetilde{J}$ are not always equivalent to $\mu_{\widetilde{\alpha}}$. For $\mathrm{HD}(\widetilde{J}) < 1$ we have $\Pi_{\widetilde{\alpha}} \sim \mu_{\widetilde{\alpha}}$ and $\Lambda_{\widetilde{\alpha}} = 0$, for $\mathrm{HD}(\widetilde{J}) = 1$ all three measures are equivalent, and for $\mathrm{HD}(\widetilde{J}) > 1$, $\Lambda_{\widetilde{\alpha}} \sim \mu_{\widetilde{\alpha}}$ and $\Pi_{\widetilde{\alpha}} = \infty$. In Section 4 these results are applied to the family $f_\lambda(z) = \lambda \tan z$ for $\lambda \in \mathbb{C}$, $0 < |\lambda| < 1$, and λ in the neighbourhood of $\mathbb{R} \setminus [-1, 1]$. We show that $\mathrm{HD}(J(f_\lambda)) = 1/2 + O(\sqrt{|\lambda|})$ for $\lambda \to 0$ and that it is strictly increasing with respect to $\lambda \in \mathbb{R}$ for $0 < \lambda < \lambda_0$.

Recently R. D. Mauldin and M. Urbański ([MU]) have studied the limit sets for so-called conformal iterated function systems (c.i.f.s.), i.e. countable families of conformal contractions on some compact subsets of $\mathbb{R}^n$. Such objects appear in various situations, e.g. the limit sets of geometrically finite Kleinian groups and the Julia sets for parabolic rational maps. Using the thermodynamical formalism they prove existence of an invariant probability measure, give a formula for the Hausdorff dimension and conditions for the behaviour of the Hausdorff and packing measures. For the meromorphic maps considered in this paper the situation when the Julia set is a Cantor set is a particular case of the general theory of [MU].

The family $\lambda \tan z$ and other families of meromorphic maps were studied by J. Kotus in [K]. She considered a measure on $J(f)$ which is the limit of a sequence of measures μ_n distributed on the sets of the nth generation A_n from the construction of the Cantor set $J(f)$ proportionally to $(\operatorname{diam} A_n)^\alpha$. Then she used the Frostman lemma to estimate $\mathrm{HD}(J(f))$. Some results from that paper and from the present one are similar.

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2. Preliminary lemmas. Recall that we study the maps $f(z) = h(e^z)$ for which $J \cap S = \emptyset$, where S is the closure in $\widehat{\mathbb{C}}$ of the forward orbits of the critical and asymptotic values of f. From (1) and (2), $\widetilde{J}$ is disjoint from $\widetilde{S}$ which is the closure of the forward orbits of the critical and asymptotic values of $\widetilde{f}$. Note that if $|\mathrm{Re}(z)|$ is large, then $f(z)$ lies in a small neighbourhood of one of the asymptotic values which are outside J. Thus $J \cap \mathbb{C}$ is contained in a strip $\{|\mathrm{Re}(z)| \le a\}$ for some a and $\widetilde{J}$ is contained in an annulus $\{b^{-1} \le |z| \le b\}$. As in the rational case, our assumption implies expanding on the Julia set.

Lemma 2.1. *There exist $n \ge 1$ and $q > 1$ such that $|(f^n)'(x)| > q$ for all $x \in J$ for which f^n is defined. The same is true for $\widetilde{f}$ and $\widetilde{J}$.*

Proof. For f the proof is the same as for a rational function. The derivative of f^n at the point z differs by a bounded factor from the derivative of $\widetilde{f}^n$ at the point $\exp z$. This gives the proof for $\widetilde{f}$. ∎

Now we estimate the distortion of the branches of $\widetilde{f}^{-n}$. Let us remark that constants c, c_1 etc. have different values depending on the context.

Recall the Koebe distortion lemma (see e.g. [G]).

LEMMA 2.2. *For every* $0 < \delta < 1$ *there is a constant* $c_\delta > 0$ *such that for every univalent function* f *on the open unit disk*,

$$\frac{1}{c_\delta} < \left|\frac{f'(x)}{f'(y)}\right| < c_\delta$$

for every x, y *such that* $|x|, |y| < \delta$. *Moreover*, $c_\delta \to 1$ *as* $\delta \to 0$. ∎

LEMMA 2.3. *Let* $\alpha > 0$. *Then there exists a constant* $c > 0$ *such that for every* $x, y \in \widetilde{J}$ *and every* $n > 0$,

$$\frac{1}{c} < \frac{\sum_{z\in\tilde{f}^{-n}(x)} |(\widetilde{f}^n)'(z)|^{-\alpha}}{\sum_{z\in\tilde{f}^{-n}(y)} |(\widetilde{f}^n)'(z)|^{-\alpha}} < c$$

if one of these sums is finite. Otherwise they are both infinite.

Proof. Recall that $\widehat{\mathbb{C}} \setminus \widetilde{J}$ is a finite union of the basins of sinks, so $\widehat{\mathbb{C}} \setminus \widetilde{S}$ is connected. Hence we can assume that x, y are some points from a small ball $B_r(\xi)$ such that all branches of $\widetilde{f}^{-n}$ are defined on $B_{2r}(\xi)$. Then it is sufficient to use the Koebe lemma. ∎

LEMMA 2.4. *The series* $S(x) = \sum_{z\in\tilde{f}^{-1}(x)} |\widetilde{f}'(z)|^{-\alpha}$ *is* (*uniformly*) *convergent for* $x \in \widetilde{J}$ *if and only if* $\alpha > \alpha_0$, *where* $\alpha_0 = p/(p+1)$ *with* p *denoting the highest degree of the poles of* h.

Proof. By definition

$$S(x) = \sum_{k\in\mathbb{Z}} \sum_{z\in h^{-1}(\log x + kT)} |\widetilde{f}(z)h'(z)|^{-\alpha},$$

where we choose $\log x$ lying in the strip $\{z : 0 \le \mathrm{Im}(z) < 2\pi i\}$. Denote by $b_1, \ldots, b_s$ the poles of h and let p_j be the degree of b_j. If $|k|$ is large, then the set $h^{-1}(\log x + kT)$ consists of $d = \deg h$ points lying at small distances to the poles. For $z \in h^{-1}(\log x + kT)$ which is close to b_j we have

$$c_1^{-1}|k|^{(p_j+1)/p_j} \le |h'(z)| \le c_1|k|^{(p_j+1)/p_j}.$$

Moreover, $b^{-1} \le |\widetilde{f}(z)| \le b$. Therefore, if p is the maximum of the p_j, then

$$c_2^{-1}|k|^{-(p+1)\alpha/p} \le \sum_{z\in h^{-1}(\log x + kT)} |\widetilde{f}'(z)|^{-\alpha} \le c_2|k|^{-(p+1)\alpha/p}.$$

Hence, the convergence of $S(x)$ is equivalent to the convergence of $\sum_{k=1}^{\infty} k^{-(p+1)\alpha/p}$. The uniform convergence follows from Lemma 2.3. ∎

Remark. These estimates are similar to the ones for the jump map for parabolic rational maps (see [DU]). Both cases are geometrically very similar so one can obtain the same results for the Hausdorff and packing measures on J (see Section 3).

3. Thermodynamic formalism and measures on the Julia set. Let $\varphi = -\alpha \log|\widetilde{f}'|$ for some $\alpha > 0$. We are going to study the Perron–Frobenius–Ruelle operator $\mathcal{L}_\alpha = \mathcal{L}_\varphi$ on the space $C(\widetilde{J})$ of real continuous functions on $\widetilde{J}$. By definition,

$$\mathcal{L}_\alpha \psi(x) = \sum_{z \in \tilde{f}^{-1}(x)} \psi(z)|\widetilde{f}'(z)|^{-\alpha}. \tag{3}$$

Note that φ has singularities in $\widetilde{J}$. However, Lemma 2.4 shows that the operator is well defined on $C(\widetilde{J})$ for $\alpha > \alpha_0$.

Let $\alpha > \alpha_0$. The proof of the existence of the Gibbs state on $\widetilde{J}$ for φ is the same as in the case of the hyperbolic Julia set of a rational function. It is sufficient to make use of the lemmas from the previous section. The details are in e.g. [B1], [P], [PP]. The topological pressure equals

$$P(\alpha) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{z \in \tilde{f}^{-n}(x)} |(\widetilde{f}^n)'(z)|^{-\alpha} \tag{4}$$

for any $x \in \widetilde{J}$. Denote the Gibbs state for the function φ by μ_α. From the construction we have

PROPOSITION 3.1. *Fix some small numbers $r_0 < r_1$. There exists a constant $c > 0$ such that for every $B = B_r(y)$, $y \in \widetilde{J}$, $r_0 < r < r_1$, every $n > 0$ and every branch $\widetilde{f}_\nu^{-n}$ on B,*

$$\frac{1}{c} < \frac{\mu_\alpha(\widetilde{f}_\nu^{-n}(B))e^{nP(\alpha)}}{|(\widetilde{f}_\nu^{-n})'(x)|^\alpha} < c$$

for every $x \in B$. ■

It is easy to see that for $\alpha > \alpha_0$ the function $\alpha \mapsto P(\alpha)$ is strictly decreasing, convex and $P(\alpha) \to \infty$ as $\alpha \to \alpha_0^+$. Knowing that P is differentiable (we shall prove it later) one can easily check that $P'(\alpha) < c < 0$. Hence there is a unique $\widetilde{\alpha} > \alpha_0$ such that $P(\widetilde{\alpha}) = 0$.

Now we compare the invariant measure $\mu_{\widetilde{\alpha}}$ with some measures on $\widetilde{J}$ which are defined in a geometric way. We shall consider the $\widetilde{\alpha}$-Hausdorff measure $\Lambda_{\widetilde{\alpha}}$ and the $\widetilde{\alpha}$-packing measure $\Pi_{\widetilde{\alpha}}$. In the case of rational functions $\Lambda_{\widetilde{\alpha}}$ is equivalent to the Gibbs measure so $\mathrm{HD}(J) = \widetilde{\alpha}$ (see [R]). In our situation $\Lambda_{\widetilde{\alpha}}$ is equivalent to $\mu_{\widetilde{\alpha}}$ only if $\mathrm{HD}(\widetilde{J}) \geq 1$. Else $\Lambda_{\widetilde{\alpha}}(\widetilde{J}) = 0$. Examples of both cases will be given in Section 4. On the other hand, the $\widetilde{\alpha}$-packing measure is equivalent to the Gibbs state for $\mathrm{HD}(\widetilde{J}) \leq 1$ and is infinite in the other case. In particular, all three measures are equivalent when $\mathrm{HD}(\widetilde{J}) = 1$. However, the Hausdorff dimension of $\widetilde{J}$ (of course it equals $\mathrm{HD}(J)$) is always equal to $\widetilde{\alpha}$. Moreover, it coincides with the packing and box dimensions.

Recall the definitions. Let A be a subset of a metric space. Let $\alpha > 0$. The *α-Hausdorff measure* of A is defined as

$$\Lambda_\alpha(A) = \lim_{\varepsilon\to 0} \inf_{\mathcal{U}} \Big\{ \sum_{U\in\mathcal{U}} (\operatorname{diam} U)^\alpha \Big\},$$

where the infimum is taken over all countable coverings $\mathcal{U}$ of A by open sets of diameters less than ε.

A countable family of open balls $\{B_{r_i}(x_i)\}$ is called a *packing* of A if $x_i \in A$ for every i and $\operatorname{dist}(x_i, x_j) \ge r_i + r_j$ for any $i \ne j$. For $\alpha > 0$ we define

$$\Pi^*_\alpha(A) = \lim_{\varepsilon\to 0} \sup \Big\{ \sum_i r_i^\alpha \Big\},$$

where the supremum is taken over all packings of A by balls $B_{r_i}(x_i)$ with radii less than ε. The *α-packing measure* of A is defined by

$$\Pi_\alpha(A) = \inf_{\bigcup_i A_i = A} \Big\{ \sum_i \Pi^*_\alpha(A_i) \Big\},$$

where A_i are arbitrary subsets of A.

The *Hausdorff dimension* of A is

$$\mathrm{HD}(A) = \sup\{\alpha > 0 : \Lambda_\alpha(A) = \infty\} = \inf\{\alpha > 0 : \Lambda_\alpha(A) = 0\}.$$

The *packing dimension* $\mathrm{PD}(A)$ is defined in the same way.

Let $N_\varepsilon(A)$ be the minimal number of balls of diameters ε needed to cover A. Define the *lower* and *upper box dimensions* as

$$\underline{\mathrm{BD}}(A) = \liminf_{\varepsilon\to 0} \frac{\log N_\varepsilon(A)}{-\log \varepsilon}, \qquad \overline{\mathrm{BD}}(A) = \limsup_{\varepsilon\to 0} \frac{\log N_\varepsilon(A)}{-\log \varepsilon}.$$

In the above definitions $N_\varepsilon(A)$ can be replaced by $N'_\varepsilon(A)$ equal to the maximal number of elements of any packing of A by balls of diameter ε.

If the lower and upper box dimensions are equal, their common value is the *box dimension* $\mathrm{BD}(A)$.

Now we recall the theorem which enables us to compare the Hausdorff and packing measures with other measures on compact sets in $\mathbb{R}^n$, i.e. the stronger version of the Frostman lemma and its analogue for the packing measure. This theorem follows from the definition of the Hausdorff and packing measures and the Besicovitch covering theorem.

THEOREM 3.2. *Let A be a compact set in $\mathbb{R}^n$ and let μ be a finite Borel measure on A. Then there exists a constant C depending only on n such that for every Borel subset E of A*:

• *If for every* $x \in E$,

$$\limsup_{r\to 0} \mu(B_r(x))/r^\alpha \ge a$$

then for every Borel subset D *of* E,

$$\Lambda_\alpha(D) \le 2^\alpha C a^{-1}\mu(D).$$

• *If for every* $x \in E$,

$$\limsup_{r\to 0} \mu(B_r(x))/r^\alpha \le a$$

then for every Borel subset D *of* E,

$$\Lambda_\alpha(D) \ge 2^\alpha a^{-1}\mu(D).$$

• *If for every* $x \in E$,

$$\liminf_{r\to 0} \mu(B_r(x))/r^\alpha \ge a$$

then for every Borel subset D *of* E,

$$\Pi_\alpha(D) \le a^{-1}\mu(D).$$

• *If for every* $x \in E$,

$$\liminf_{r\to 0} \mu(B_r(x))/r^\alpha \le a$$

then for every Borel subset D *of* E,

$$\Pi_\alpha(D) \ge C^{-1}a^{-1}\mu(D). \blacksquare$$

Since the measure $\mu_{\tilde\alpha}$ is invariant, we have $\mu_{\tilde\alpha}(\bigcup_n \widetilde{f}^{-n}(h^{-1}(\infty))) = 0$ so it is sufficient to compare the measures on $\widetilde{J}\setminus\bigcup_n \widetilde{f}^{-n}(h^{-1}(\infty))$. (The points from this set have well-defined forward orbits.) Take a point x from this set and a ball $B = B_r(\widetilde{f}^n(x))$, $r_0 < r < r_1$ (see Proposition 3.1). Taking the branch of $\widetilde{f}^{-n}$ specified by the trajectory of x and using Proposition 3.1 and the fact that the distortion is universally bounded we can find a ball centered at x whose μ_α measure is comparable up to a universal (i.e. independent of x, n) constant to its radius. As $n \to \infty$, by expanding the radii of these balls tend to 0. Hence using Theorem 3.2 we obtain

Proposition 3.3. *There exist constants* $c_1, c_2 > 0$ *such that*

$$\Lambda_{\tilde\alpha} \le c_1\mu_{\tilde\alpha} \quad \textit{and} \quad \Pi_{\tilde\alpha} \ge c_2\mu_{\tilde\alpha}. \blacksquare$$

Now we show when the Hausdorff and packing measures are equivalent to the Gibbs state. First we prove a useful lemma.

Lemma 3.4.

$$\mu_{\tilde\alpha}(\{x \in \widetilde{J} : \limsup_{n\to\infty} |h(\widetilde{f}^n(x))| < \infty\}) = 0.$$

Proof. It is sufficient to prove the lemma with ∞ replaced by k_0T for a fixed k_0. Let $D_n = \{x : |h(\widetilde{f}^n(x))| < k_0T\}$. We prove that $\mu_{\tilde\alpha}(\bigcap_{i\ge n_0} D_i) = 0$ for every n_0. From compactness of $\widetilde{J}$ and Proposition 3.1,

$$\mu_{\tilde\alpha}(\widetilde{J}) \ge c_1 \sum_{z\in\tilde f^{-n}(x)} |(\widetilde{f}^n)'(z)|^{-\tilde\alpha},$$

$$\mu_{\tilde\alpha}\Big(\bigcap_{i=n_0}^{n_0+j} D_i\Big) \le c_2 \sum_{\substack{z\in\tilde f^{-n}(x)\\ |h(\tilde f^i(z))|\le (k_0+1)T\\ i=n_0,\dots,n_0+j}} |(\widetilde{f}^n)'(z)|^{-\tilde\alpha},$$

where c_1, c_2 are independent of x, j. Using the chain rule and Lemmas 2.3 and 2.4 we get

$$\mu_{\tilde\alpha}\Big(\bigcap_{i=n_0}^{n_0+j} D_i\Big) \le c_3\bigg(\frac{\sum_{|k|\le(k_0+1)} k^{-(p+1)\tilde\alpha/p}}{\sum_{k\in\mathbb{Z}} k^{-(p+1)\tilde\alpha/p}}\bigg)^j < c_3q^j$$

for some $q < 1$. Hence $\mu_{\tilde\alpha}(\bigcap_{i\ge n_0} D_i) = 0$. ■

We shall also need the following:

Lemma 3.5. *For every a_i, $\alpha > 0$,*

$$\Big(\sum_i a_i\Big)^\alpha \le \sum_i a_i^\alpha \quad \textit{if and only if} \quad \alpha \le 1.$$

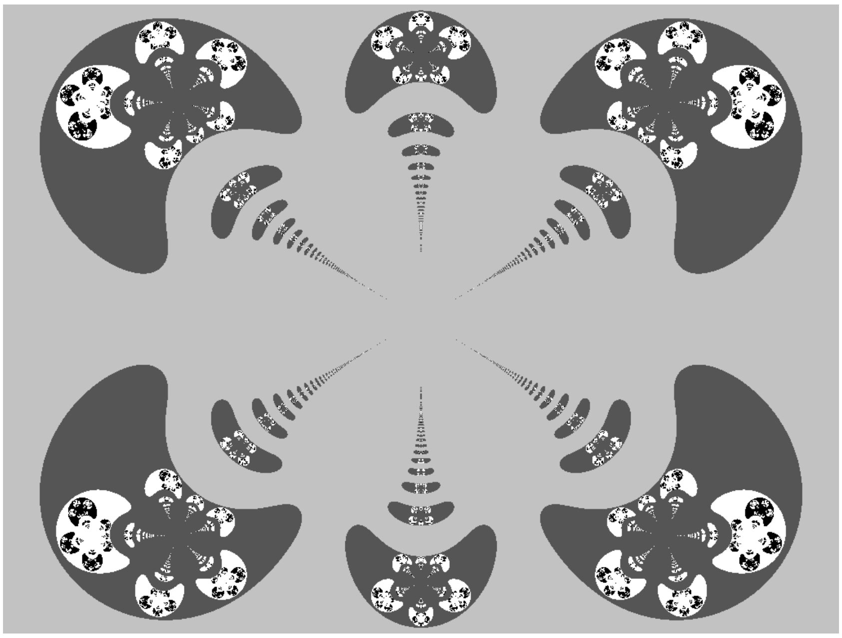
Proof. If $\alpha \le 1$ then

$$\frac{\sum_i a_i^\alpha}{(\sum_i a_i)^\alpha} = \sum_i \bigg(\frac{a_i}{\sum_j a_j}\bigg)^\alpha \ge \sum_i \frac{a_i}{\sum_j a_j} = 1.$$

For $\alpha \ge 1$ we proceed analogously. ■

Theorem 3.6. *$\Lambda_{\tilde\alpha}$ is equivalent to $\mu_{\tilde\alpha}$ if and only if* $\mathrm{HD}(\widetilde{J}) \ge 1$. *Otherwise $\Lambda_{\tilde\alpha}(\widetilde{J}) = 0$.*

Proof. To prove the theorem we must look closer at the geometric structure of the Julia set. First consider J. Fix $a > 0$ such that $J\cap\mathbb{C} \subset \{z : |\mathrm{Re}(z)| < a\}$. Let $E_M = \{z : |\mathrm{Re}(z)| < a, \mathrm{Im}(z) > M$ (resp. $< M$)$\}$ provided $M > 0$ (resp. $M < 0$) and let $J_k = J \cap \{z : kT \le \mathrm{Im}(z) < (k+1)T\}$. For sufficiently large $|M|$ the branches of f^{-1} are defined on E_M. For the branch f_ν^{-1} on E_M going to the neighbourhood of some pole b the modulus of its derivative on J_k is comparable with $|k|^{-(p+1)/p}$, where p is the degree of b (see Lemma 2.4). There are $2p$ such branches going to the neighbourhood of b. Thus J close to b consists of $2p$ sequences of sets $f_\nu^{-1}(J_k)$ of diameters comparable with $|k|^{-(p+1)/p}$ and converging to b like $\{|k|^{-1/p}\}$ as $k \to \infty$



A detail of the Julia set for $f(z) = 0.7(\tan z)^3$

or $-\infty$. The ratio between the diameters of the sets and their distances to b tends to 0 and the sequences form $2p$ equal angles around b.

By (2) the Julia set $\widetilde{J}$ in the neighbourhood of the poles of h consists of $2p$ sequences of sets $\widetilde{J}_{s,k} = \exp(f_s^{-1}(J_k))$ for $s = 1, \dots, p$, $k \in \mathbb{Z}$, with the same asymptotic behaviour. This picture is transferred with bounded distortion by $\widetilde{f}^{-n}$ to the neighbourhood of the preimages of the poles of h. Note that by Proposition 3.1, the measure $\mu_{\tilde{\alpha}}$ of the set $\widetilde{f}_\nu^{-n}(\widetilde{J}_{s,k})$ is comparable with its diameter to the power $\widetilde{\alpha}$.

Assume $\widetilde{\alpha} < 1$. We want to show $\Lambda_{\tilde{\alpha}}(\widetilde{J}) = 0$. From Proposition 3.3, $\Lambda_{\tilde{\alpha}}$ is absolutely continuous with respect to $\mu_{\tilde{\alpha}}$. Hence by Lemma 3.4 it suffices to prove that $\Lambda_{\tilde{\alpha}}(K) = 0$, where $K = \{x \in \widetilde{J} : \limsup_{n\to\infty} |h(\widetilde{f}^n(x))| = \infty\}$. Take x from K. There exists a sequence $k_n \to \infty$ such that $\widetilde{f}^{k_n}(x)$ lies in a small neighbourhood of some pole b of h and belongs to $\exp(f^{-1}(J_{l_n}))$ with $|l_n| \to \infty$. Let $\widetilde{f}_\nu^{-k_n}$ be the branch on $B_1 = B_{|b-\tilde{f}^{k_n}(x)|}(\widetilde{f}^{k_n}(x))$ specified by the trajectory of x. Let $B = B_{\operatorname{diam} \tilde{f}_\nu^{-k_n}(B_1)}(x)$. By bounded distortion and Proposition 3.1,

$$\frac{\mu_{\tilde{\alpha}}(B)}{r^{\tilde{\alpha}}} \ge c_1 \frac{\sum_{k=|l_n|}^{\infty} k^{-(p+1)\tilde{\alpha}/p}}{(\sum_{k=|l_n|}^{\infty} k^{-(p+1)/p})^{\tilde{\alpha}}} \ge c_2 |l_n|^{1-\tilde{\alpha}}$$

for some universal constants c_1, c_2. Thus $\limsup_{r\to 0} \mu_{\tilde{\alpha}}(B_r(x))/r^{\tilde{\alpha}} = \infty$ and from Theorem 3.2, $\Lambda_{\tilde{\alpha}}(K)=0$.

Now assume $\widetilde{\alpha}\geq 1$. We show $\Lambda_{\tilde{\alpha}}\geq c\mu_{\tilde{\alpha}}$ (Proposition 3.3 gives the inverse inequality). Fix some small $\varrho>0$ and large $C,M>0$. Take $x\in\widetilde{J}$ and $B=B_r(x)$ for a small r. We prove

$$\mu_{\tilde{\alpha}}(B)\leq c_1 r^{\tilde{\alpha}} \tag{5}$$

for some universal constant c_1. Like previously we can assume that x has a well-defined forward orbit. Set $G_n=\widetilde{f}_\nu^{-n}(B_\varrho(\widetilde{f}^n(x)))$ for the branch $\widetilde{f}_\nu^{-n}$ specified by the trajectory of x. Let $B'=B_{Cr}(x)$ and let n_0 be the smallest integer such that $G_{n_0}\subset B'$ (it exists by expanding). This implies

$$r\geq\frac{1}{2C}\operatorname{diam}G_{n_0}. \tag{6}$$

If we take C large enough then $B\subset G_{n_0-1}$ by bounded distortion. Now there are two possibilities:

(i) $|h(\widetilde{f}^{n_0-1}(x))|\leq M$. Then $|\widetilde{f}'(\widetilde{f}^{n_0-1}(x))|\leq c_2$ for a universal constant c_2 so $\operatorname{diam}G_{n_0-1}\leq c_3\operatorname{diam}G_{n_0}$. Hence $\mu_{\tilde{\alpha}}(B)\leq\mu_{\tilde{\alpha}}(G_{n_0-1})\leq c_4(\operatorname{diam}G_{n_0})^{\tilde{\alpha}}$. From (6) we have (5).

(ii) $|h(\widetilde{f}^{n_0-1}(x))|>M$. Then $\widetilde{f}^{n_0-1}(x)$ lies close to some pole of h and $\widetilde{J}\cap G_{n_0-1}$ has the structure described at the beginning of the proof. Let $L=\{|k|:\widetilde{J}_{s,k}$ intersects $B_\varrho(\widetilde{f}^{n_0-1}(x))$ for some $s\}$ and let $i=\inf L$, $j=\sup L$ (j may be equal to infinity). One can check that the diameters of sets $\widetilde{J}_{s,k}$ intersecting $B_\varrho(\widetilde{f}^{n_0-1}(x))$ are less than $c_5\varrho$. Then again using bounded distortion and Proposition 3.1 we get

$$\frac{\mu_{\tilde{\alpha}}(B)}{r^{\tilde{\alpha}}}\leq c_6\frac{\sum_{k=i}^{j}k^{-(p+1)\tilde{\alpha}/p}}{(\sum_{k=i}^{j}k^{-(p+1)/p})^{\tilde{\alpha}}}.$$

Now (5) holds by Lemma 3.5.

Having (5) we can use Theorem 3.2 to end the proof. ∎

In the very same way one can prove the dual theorem about the packing measure:

THEOREM 3.7. *$\Pi_{\tilde{\alpha}}$ is equivalent to $\mu_{\tilde{\alpha}}$ if and only if* $\mathrm{HD}(\widetilde{J})\leq 1$. *Otherwise $\Lambda_{\tilde{\alpha}}(\widetilde{J})=\infty$. In particular, if* $\mathrm{HD}(\widetilde{J})=1$ *then all three measures are equivalent.* ∎

Remark. Theorems 3.6 and 3.7 hold in various situations, e.g. for parabolic Julia sets of rational maps ([DU]) and limit sets of geometrically finite Kleinian groups. The following lemma shows the geometric assumptions needed to prove the theorems. The proof is the same as previously.

LEMMA 3.8. *Let J be a bounded Borel subset of $\mathbb{C}$ and let μ be a finite Borel measure on J. Fix $c,\delta>0$ and some positive integer p. Assume* *that for every* $x \in J$ *there exist a sequence* $r_n \to 0$, $p_n \in \{1, \ldots, p\}$ *and homeomorphisms* $h_n : B_n = B_{r_n}(x) \to h_n(B_n) \subset B_\delta(0)$ *such that*

$$|h_n(x)|^{p_n+1} \le c \operatorname{diam} h_n(B_n) \frac{r_{n+1}}{r_n} \quad \text{and}$$

$$c^{-1} \le \frac{|h_n(x_1) - h_n(y_1)|}{|x_1 - y_1|} \bigg/ \frac{|h_n(x_2) - h_n(y_2)|}{|x_2 - y_2|} \le c$$

for every x_1, x_2, y_1, y_2. *Set* $g_n(y) = 1/(h_n(y))^{p_n}$. *Suppose that the following holds*:

(1) *Fix some* $a, b, T > 0$ *and let* $J_k = \{z : |\mathrm{Re}(z)| \le a,\ kT \le \mathrm{Im}(z) \le (k+1)T\}$. *Then* $\operatorname{diam} g_n(B_n) \ge b$, $g_n(J) \subset \bigcup_k J_k$, *and* $g_n(J) \cap J_k \ne \emptyset$ *for every* k.

(2) *For every ball* $B = B_\varrho(z) \subset g_n(B_n)$ *with* $z \in g_n(J)$ *and* $\varrho_0 < \varrho < \varrho_1$ *for some fixed small* ϱ_0, ϱ_1 *and for every component* U *of* $g_n^{-1}(B)$,

$$c_1^{-1} \le \frac{\mu(U)}{(\operatorname{diam} U)^\alpha} \le c_1$$

for some fixed α, c_1.

(3) *For* μ*-almost every* $x \in J$ *there exists a subsequence* n_k *such that*

$$\lim_{k\to\infty} |g_{n_k}(x)| = \infty \quad \text{and} \quad \infty \in g_{n_k}(B_{n_k}).$$

Then Theorems 3.6 *and* 3.7 *hold for the measure* μ *and for the* α*-Hausdorff and* α*-packing measures on* J. ∎

From Proposition 3.3 we have $\mathrm{HD}(J) = \mathrm{HD}(\widetilde{J}) \le \widetilde{\alpha}$. Theorem 3.6 gives the inverse inequality when $\widetilde{\alpha} \ge 1$. Now we shall prove it for $\widetilde{\alpha} < 1$. We shall consider the map f and the Julia set J. Note that in this case the complement of J is connected, because if there are two different components of $\widehat{\mathbb{C}} \setminus J$ then the Julia set contains their borders and has Hausdorff dimension at least 1. Thus all the critical and asymptotic values of f are in one immediate basin of a sink s which is a fixed point (cf. Remark 1.1). Join them to s by disjoint non-self-intersecting curves and take one more curve $\gamma : [0, 1) \to \mathbb{C}$ disjoint from the Julia set such that $\gamma(0) = s$ and $\mathrm{Re}(\gamma(t)) \to \infty$ as $t \to 1^-$ (it exists since $J \subset \{|\mathrm{Re}(z)| \le a\}$). The complement of the union of all these curves in $\mathbb{C}$ is an open topological disk containing $J \cap \mathbb{C}$ and the branches of f^{-1} are defined on it. Hence $J \cap \mathbb{C}$ consists of compact and pairwise disjoint sets J_k such that $J_k = J_0 + kT$.

Now we define a sequence A_m of subsets of J such that $\mathrm{HD}(A_m) \to \widetilde{\alpha}$ as $m \to \infty$. Let $C_m = \bigcup_{|k|\le m} J_k$ and $A_m = \bigcap_{n\ge 0} f^{-n}(C_m)$. The sets A_m are compact, forward-invariant and do not contain poles. Set $V_m = \{z : \mathrm{dist}(z, A_m) < \varepsilon\}$, with ε so small that all branches f_ν^{-n} are defined on disks contained in V_m and $|(f_\nu^{-n})'| < cq^{-n}$, $q > 1$. The map $f_{|V_m} : V_m \to \mathbb{C}$ is holomorphic.

LEMMA 3.9. *For large m the set A_m is a mixing repeller for $f_{|V_m}$, i.e.*

(i) *there exist constants $c > 0$ and $q > 1$ such that $|(f^n)'(z)| > cq^n$ for every $z \in A_m$,*

(ii) *$A_m = \{z \in V_m : f^n(z) \in V_m$ for every $n > 0\}$,*

(iii) *$f_{|V_m}$ is mixing, i.e. for every non-empty open set U intersecting A_m there exists $n > 0$ such that $(f_{|V_m})^n(U) \supset A_m$.*

Proof. It is sufficient to use expanding, the density of the preimages of the points from J and the fact that $\operatorname{dist}(C_m, J \setminus C_m) > 0$. ∎

Thus the sets A_m form a sequence of mixing repellers contained in J. Fix some $x \in J_0$ with bounded trajectory. The topological pressure P_m for A_m equals

$$P_m(\alpha) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{\substack{z\in f^{-n}(x)\\ z\in A_m}} |(f^n)'(z)|^{-\alpha}$$

and the Hausdorff dimension of A_m is equal to the unique root of P_m (see [B1], [B2], [R]). By periodicity of f,

$$P_m(\alpha) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{\substack{z\in f^{-n}(x)\\ z\in A_m\cap J_0}} |(f^n)'(z)|^{-\alpha}$$

and from (3) and (1),

$$P(\alpha) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{\substack{z\in f^{-n}(x)\\ z\in J_0}} |(f^n)'(z)|^{-\alpha}.$$

Hence $P_m(\alpha) \le P_{m+1}(\alpha) \le P(\alpha)$.

PROPOSITION 3.10. *For $\alpha > \alpha_0$, $P(\alpha) = \lim_{m\to\infty} P_m(\alpha)$.*

Proof. Let $\varepsilon > 0$. Define

$$S(k,x,z) = \sum_{k\in\mathbb{Z}} \sum_{\substack{z\in f^{-1}(x+kT)\\ z\in J_0}} |f'(z)|^{-\alpha},$$

$$S_m(k,x,z) = \sum_{|k|\le m} \sum_{\substack{z\in f^{-1}(x+kT)\\ z\in J_0}} |f'(z)|^{-\alpha}.$$

By Lemma 2.3 there is c such that $1/c < S(k,x,z)/S(k,y,z) < c$ for every $x, y \in J_0$. Hence we can take m so large that $S_m(k,x,z) > (1-\varepsilon)S(k,x,z)$ for every $x \in J_0$. We have

$$P(\alpha) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{\substack{z_1\in f^{-1}(x)\\ z_1\in J_0}} |f'(z_1)|^{-\alpha} S(k_1,z_1,z_2)\ldots S(k_{n-1},z_{n-1},z_n),$$

and similarly for $P_m(\alpha)$ with S_m in place of S. Let

$$Z(z_j) = S(k_j, z_j, z_{j+1}) \ldots S(k_{n-1}, z_{n-1}, z_n)$$

and define $Z_m(z_j)$ in the same way. We want to compare $Z_m(z_1)$ with $Z(z_1)$. From Lemma 2.3 we can assume $1/c < Z(z_j)/Z(z_j') < c$ for every $z_j, z_j' \in J_0$. We know $Z_m(z_{n-1}) > (1-\varepsilon)Z(z_{n-1})$. We proceed by induction. Suppose $Z_m(z_j) > q_j Z(z_j)$ for some q_j. Then

$$\begin{aligned}\frac{Z_m(z_{j-1})}{Z(z_{j-1})} &> q_j \frac{S_m(k_{j-1}, z_{j-1}, z_j) Z(z_j)}{S(k_{j-1}, z_{j-1}, z_j) Z(z_j)} \\ &= q_j \left(1 - \frac{(S - S_m)(k_{j-1}, z_{j-1}, z_j) Z(z_j)}{S(k_{j-1}, z_{j-1}, z_j) Z(z_j)}\right) \\ &> q_j \left(1 - c^2 \frac{(S - S_m)(k_{j-1}, z_{j-1}, z_j)}{S(k_{j-1}, z_{j-1}, z_j)}\right) > q_j(1 - c^2\varepsilon).\end{aligned}$$

Hence $Z_m(z_1) > (1 - c^2\varepsilon)^{n-1} Z(z_1)$ and $P_m(\alpha) \geq \log(1 - c^2\varepsilon) + P(\alpha)$ for arbitrary ε, which ends the proof. ∎

From this proposition $\mathrm{HD}(J) \geq \sup_m \mathrm{HD}(A_m) = \widetilde{\alpha}$. Hence we obtain

THEOREM 3.11. $\mathrm{HD}(J) = \mathrm{HD}(\widetilde{J}) = \widetilde{\alpha}$. *In particular*, $\mathrm{HD}(J) > p/(p+1) > 1/2$, *where p is the greatest degree of the pole of f.* ∎

Remark. The above lower bound on $\mathrm{HD}(J)$ is valid under weaker assumptions about f. It is sufficient to assume that the closure of the forward orbits of the critical and asymptotic values is bounded and disjoint from the set of poles. Then instead of J we can study its subset J' consisting of points with forward orbits contained in a small neighbourhood of infinity. In the same way we obtain the formula for the Hausdorff dimension of J' and the inequality also holds for J.

Now we show that the Hausdorff dimension of the Julia set is a real-analytic function of f. This is so for hyperbolic Julia sets of rational functions ([R]). We use the perturbation theory for $\mathcal{L}_\alpha$. Recall the suitable theorem (see [PP]).

THEOREM 3.12. *Let $\mathcal{B}$ be a complex Banach space and $\mathcal{L} : \mathcal{B} \to \mathcal{B}$ be a bounded operator. If λ is an isolated simple eigenvalue for $\mathcal{L}$ then for $\mathcal{L}'$ close to $\mathcal{L}$ there is an isolated simple eigenvalue λ'. Moreover*, *the function $\mathcal{L}' \mapsto \lambda'$ is analytic.* ∎

In our case f and $\widetilde{f}$ are parametrized in $\mathbb{C}^{2d+1} = \mathbb{R}^{4d+2}$. The map $\mathbb{R}^{4d+2} \ni t \mapsto -\alpha \log |\widetilde{f}_t'|$ is real-analytic. We extend it for complex $t \in \mathbb{C}^{4d+2}$ in the neighbourhood of $\widetilde{\alpha}$ to an analytic map $\varphi(\alpha, t)$. Regard $\mathcal{L}_\varphi$ as an operator on the space of continuous complex functions on $\widetilde{J}$. Then $(\alpha, t) \mapsto \mathcal{L}_{\varphi(\alpha,t)}$ is analytic. For real φ the spectrum of the operator on the space of complex functions is the same as on the space of real functions. $\lambda = \exp P$ is an isolated simple eigenvalue for $\mathcal{L}_\varphi$. Hence by Theorem 3.12, $(\alpha, t) \mapsto P(\alpha, t)$ is analytic for α in the neighbourhood of $\widetilde{\alpha}$, so for real α and t it is real-analytic. Recall that $\partial P/\partial\alpha < 0$. From the implicit function theorem the function $\mathbb{R}^{4d+2} \ni t \mapsto \widetilde{\alpha}$ is real-analytic.

Corollary 3.13. HD(J) *depends real-analytically on* f.

Recall that from the definitions of the Hausdorff, packing and box dimensions it follows that

$$\mathrm{HD}(\widetilde{J}) \le \mathrm{PD}(\widetilde{J}) \le \overline{\mathrm{BD}}(\widetilde{J}). \tag{7}$$

Now we are going to show that all three dimensions coincide.

Theorem 3.14. $\overline{\mathrm{BD}}(\widetilde{J}) = \widetilde{\alpha}$.

Proof. From (7), $\overline{\mathrm{BD}}(\widetilde{J}) \ge \widetilde{\alpha}$. To prove the inverse consider two cases:

(i) $\widetilde{\alpha} \le 1$. In the same way as in the proof of Theorem 3.6 we find that there exist $c, \varepsilon_0 > 0$ such that for every $x \in \widetilde{J}$ and every $\varepsilon < \varepsilon_0$ we have $\mu_{\tilde{\alpha}}(B_\varepsilon(x))/\varepsilon^{\tilde{\alpha}} \ge c$. Hence if $N'_\varepsilon(\widetilde{J})$ equals the maximal number of elements of packings of $\widetilde{J}$ by balls of diameter ε then $\mu_{\tilde{\alpha}}(\widetilde{J}) \ge c\varepsilon^{\tilde{\alpha}} N'_\varepsilon(\widetilde{J})$. Thus

$$\overline{\mathrm{BD}}(\widetilde{J}) = \limsup_{\varepsilon\to 0} \frac{\log N'_\varepsilon(\widetilde{J})}{-\log\varepsilon} \le \widetilde{\alpha}.$$

(ii) $\widetilde{\alpha} > 1$. Since $\overline{\mathrm{BD}}(A) = \overline{\mathrm{BD}}(\mathrm{cl}\, A)$ it is sufficient to compute the box dimension of $H = \bigcup_n \widetilde{f}^{-n}(\text{poles of } h)$. Let B_i, $i = 1, \ldots, s$, be small disjoint balls centered at the poles of h. Let $\varepsilon > 0$. We estimate the number N_ε of balls of diameter ε needed to cover H. Note that it is sufficient to cover only $H' = \bigcup_{n\le n_0} \widetilde{f}^{-n}(\text{poles of } h)$, $n_0 = c_1 \log\varepsilon^{-1}$ for some constant $c_1 > 0$, because by expanding all points in $\widetilde{J}$ are at a distance less than ε to H'. Consider the branches $\widetilde{f}_\nu^{-n}$ on B_i. Let

$$\mathcal{V}_n = \{\widetilde{f}_\nu^{-n}(B_i) : \mathrm{diam}\, \widetilde{f}_\nu^{-n}(B_i) < \varepsilon,\ i = 1, \ldots, s\},$$
$$\mathcal{W}_n = \{\widetilde{f}_\nu^{-n}(B_i) : \mathrm{diam}\, \widetilde{f}_\nu^{-n}(B_i) \ge \varepsilon,\ i = 1, \ldots, s\},$$

and let $\mathcal{V} = \bigcup_{n\le n_0} \mathcal{V}_n$, $\mathcal{W} = \bigcup_{n\le n_0} \mathcal{W}_n$. Take $W \in \mathcal{W}_n$. The sets $V \in \mathcal{V}_{n+1}$ contained in W form $2p$ sequences converging to $z \in W$ which is the nth preimage of some pole of degree p as was described in the proof of Theorem 3.6. Hence it is possible to cover the point z and all sets $V \in \mathcal{V}_{n+1}$ which are contained in W by less than $c_2\varepsilon^{-1}\,\mathrm{diam}\, W$ balls of diameter ε, where c_2 is independent of n, W, ε. Denote by $\mathcal{U}$ the family of covering balls contained in all $W \in \mathcal{W}$. Now we prove that

$$\mathrm{dist}\Big(H', \bigcup_{U\in\mathcal{U}} U\Big) < c_3\varepsilon \tag{8}$$

for some constant c_3. Let $x \in H'$. Then $x \in \widetilde{f}^{-n}(\text{poles of } h)$ for some $n \le n_0$. If $x \in W \in \mathcal{W}_n$ then from the construction it is contained in some ball from $\mathcal{U}$. It remains to consider the case $x \in V \in \mathcal{V}_n$. For $j = 0, \ldots, n$ let i_j be such that B_{i_j} contains the pole of h nearest to $\widetilde{f}^j(x)$. Take the branch $\widetilde{f}_\nu^{-j}$ specified by the trajectory of x defined on an open connected set containing $\widetilde{f}^j(x)$ and B_{i_j}. We also require that $\widetilde{f}^j(x)$ and B_{i_j} can be connected in this set by a curve of universally bounded length and that the distortion is universally bounded. (It is possible to find such sets since all the critical and asymptotic values are attracted by sinks.) Set

$$D_j = \widetilde{f}_\nu^{-j}(B_{i_j}).$$

We have $D_0 \in \mathcal{W}$ and $D_n \in \mathcal{V}$. Let j_0 be the smallest integer such that $D_{j_0} \in \mathcal{V}$. As in the proof of Theorem 3.6 we have two possibilities:

(a) $|h(\widetilde{f}^{j_0-1}(x))| \le M$ for a large fixed M. Then we have $\operatorname{diam} D_{j_0-1} \le c_4 \operatorname{diam} D_{j_0}$ for a universal constant c_4. Thus by bounded distortion,

$$\operatorname{dist}(x, D_{j_0-1}) \le c_5 \operatorname{diam} D_{j_0-1} \le c_4 c_5 \operatorname{diam} D_{j_0} \le c_4 c_5 \varepsilon.$$

From the construction there exists a ball from $\mathcal{U}$ contained in D_{j_0-1} and $\operatorname{diam} D_{j_0-1} \le c_4 \varepsilon$ so (8) holds.

(b) $|h(\widetilde{f}^{j_0-1}(x))| > M$. Then $f^{j_0-1}(x)$ lies close to the pole from B_{i_j} and $\mathcal{W}_{j_0} \ni D_{j_0} \subset D_{j_0-1} \in \mathcal{W}_{j_0-1}$. Thus the balls from $\mathcal{U}$ cover D_{j_0} and $\operatorname{dist}(x, D_{j_0}) \le c_5 \operatorname{diam} D_{j_0} \le c_5 \varepsilon$ so we have (8).

By (8),

$$N_\varepsilon \le c_6 \varepsilon^{-1} \sum_{W \in \mathcal{W}} \operatorname{diam} W. \tag{9}$$

Note that for every n the sets $W \in \mathcal{W}_n$ are pairwise disjoint. Hence using Proposition 3.1 we obtain

$$\sum_{W \in \mathcal{W}} (\operatorname{diam} W)^{\tilde{\alpha}} \le c_7 \sum_{W \in \mathcal{W}} \mu_{\tilde{\alpha}}(W) = c_7 \sum_{n=0}^{n_0} \sum_{W \in \mathcal{W}_n} \mu_{\tilde{\alpha}}(W) \le c_8 n_0.$$

Therefore

$$c_8 n_0 \ge \sum_{W \in \mathcal{W}} (\operatorname{diam} W)^{\tilde{\alpha}-1} \operatorname{diam} W \ge \varepsilon^{\tilde{\alpha}-1} \sum_{W \in \mathcal{W}} \operatorname{diam} W. \tag{10}$$

From (9) and (10), $N_\varepsilon \le c_1 c_8 \varepsilon^{-\widetilde{\alpha}} \log \varepsilon^{-1}$ and from the definition we get $\overline{\mathrm{BD}}(\widetilde{J}) \le \widetilde{\alpha}$. ∎

Corollary 3.15. $\mathrm{HD}(\widetilde{J}) = \mathrm{PD}(\widetilde{J}) = \mathrm{BD}(\widetilde{J})$. ∎

4. The family $\lambda \tan z$. Let us consider the family of maps

$$f_\lambda(z) = \lambda \tan z, \quad \lambda \in \mathbb{C} \setminus \{0\}.$$

The dynamics of this family was studied in [DK]. We recall some basic properties:

• f_λ has no critical values, there are two asymptotic values $\pm\lambda i$, if they do not belong to the Julia set then all branches f_ν^{-1} are defined on the whole $J(f_\lambda)$, $(f_\nu^{-1})'(z) = \lambda/(\lambda^2 + z^2)$, $J(f_\lambda)$ is symmetric with respect to the origin and $J(f_{-\lambda}) = J(f_\lambda)$.

• If $\lambda \in \mathbb{R} \setminus 0$ then $J(f_\lambda) \cap \mathbb{C} \subset \mathbb{R}$, since $J(f) = \mathrm{cl} \bigcup_{n\ge 0} f^{-n}(\infty)$.

• If $\lambda \in \mathbb{R}$ and $|\lambda| \ge 1$, then $J(f_\lambda) = \mathbb{R} \cup \{\infty\}$ and all points from the upper and lower half-planes are attracted respectively to two sinks on the imaginary axis.

• If $\lambda \in \mathbb{R}$ and $0 < |\lambda| < 1$, then $J(f_\lambda) \cap \mathbb{C}$ is contained in pairwise disjoint closed intervals J_k, $k \in \mathbb{Z}$ (see Fig. 1).

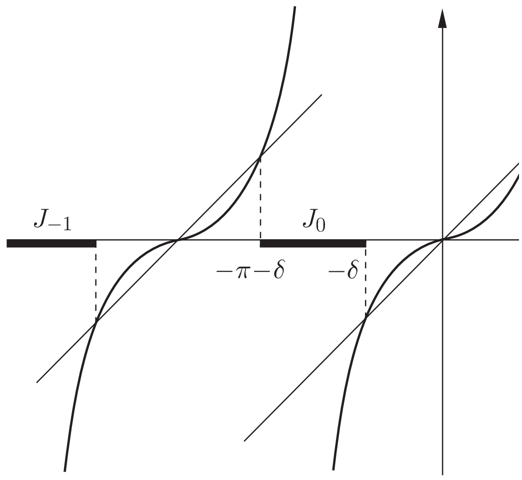


Fig. 1

• For $\lambda \in \mathbb{C}$ with $0 < |\lambda| < 1$, the Julia set $J(f_\lambda)$ is a Cantor set and all f_λ are quasiconformally conjugate.

• For $\lambda = \pm 1$ the origin is a neutral point and $J(f_\lambda) = \mathbb{R} \cup \{\infty\}$.

Recall that $f_\lambda(z) = h(e^{2iz})$ for $h(z) = -\lambda i \frac{z-1}{z+1}$. Note that $\deg h = 1$. For $0 < |\lambda| < 1$, $\lambda \in \mathbb{C}$, both asymptotic values lie in the immediate basin of attraction of the origin. $J(f_\lambda) \cap \mathbb{C}$ is divided into compact disjoint sets J_k, $J_k = J_0 + k\pi$. Denote by f_k^{-1} the branch of f^{-1} leading to J_k for $k \in \mathbb{Z}$ and define $J_{k_1,\ldots,k_n} = f_{k_1}^{-1} \circ \ldots \circ f_{k_{n-1}}^{-1}(J_{k_n})$. Then $J_{k_1,\ldots,k_{n+1}} \subset J_{k_1,\ldots,k_n}$ and $f(J_{k_1,\ldots,k_n}) = J_{k_2,\ldots,k_n}$. Every point $x \in J(f_\lambda) \cap \mathbb{C}$ can be regarded as the sequence $(k_1, \ldots, k_n, \ldots)$ where $x = \bigcap_{n=1}^{\infty} J_{k_1,\ldots,k_n}$ or as $(k_1, \ldots, k_n)$ when $x = f_{k_1}^{-1} \circ \ldots \circ f_{k_n}^{-1}(\infty)$. The dynamics of f_λ on $J(f_\lambda)$ is equivalent to the shift on the space of one-sided sequences with infinitely many symbols supplemented with finite sequences corresponding to the preimages of infinity.

Applying the results of the previous section we conclude that $\mathrm{HD}(J(f_\lambda)) > 1/2$ for $0 < |\lambda| < 1$, $\lambda \in \mathbb{C}$. Moreover, $\mathrm{HD}(J(f_\lambda))$ is a real-analytic function of λ for $0 < |\lambda| < 1$, $\lambda \in \mathbb{C}$, and in the neighbourhood of $|\lambda| > 1$, $\lambda \in \mathbb{R}$. We can compute how $\mathrm{HD}(J(f_\lambda))$ behaves when λ is near 0.

Proposition 4.1. *If* $0 < |\lambda| < b$, $\lambda \in \mathbb{C}$, *then there exists a constant* $c > 0$ *such that*

$$\frac{1}{2} + c^{-1}|\lambda|^{1/2} \le \mathrm{HD}(J(f_\lambda)) \le \frac{1}{2} + c|\lambda|^{1/2}.$$

Proof. We have

$$P(\alpha) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{k_1,\ldots,k_{n-1}\in\mathbb{Z}} |(f_0^{-1} \circ f_{k_1}^{-1} \circ \ldots \circ f_{k_{n-1}}^{-1})'(x)|^\alpha.$$

Since $(f_k^{-1})'(z) = \lambda/(\lambda^2 + z^2)$, there exists $c_1 > 0$ such that for every $z \in J(f_\lambda) \cap \mathbb{C}$ and $\alpha > 1/2$,

$$\frac{1}{c_1^\alpha}|\lambda|^\alpha \sum_{k=1}^\infty \frac{1}{k^{2\alpha}} \le \sum_{k\in\mathbb{Z}} |(f_0^{-1})'(x + k\pi)|^\alpha \le c_1^\alpha |\lambda|^\alpha \sum_{k=1}^\infty \frac{1}{k^{2\alpha}}$$

for λ in the neighbourhood of 0. Hence

$$\log\left(c_1^{-\alpha}|\lambda|^\alpha \sum_{k=1}^\infty \frac{1}{k^{2\alpha}}\right) \le P(\alpha) \le \log\left(c_1^\alpha |\lambda|^\alpha \sum_{k=1}^\infty \frac{1}{k^{2\alpha}}\right)$$

so

$$\log \frac{c_2^{-\alpha}|\lambda|^\alpha}{2\alpha - 1} \le P(\alpha) \le \log \frac{c_2^\alpha |\lambda|^\alpha}{2\alpha - 1}$$

for $\alpha > 1/2$. From this one can easily get the desired estimates. ∎

Note that $\mathrm{HD}(J(f_\lambda)) = \widetilde{\alpha} < 1$ for λ near 0, and so by Theorem 3.6 the $\widetilde{\alpha}$-Hausdorff measure of $J(f_\lambda)$ is 0.

Assume now $\lambda \in \mathbb{R}$. Note that $\mathrm{HD}(J(f_{-\lambda})) = \mathrm{HD}(J(f_\lambda))$. Now we prove the following:

Proposition 4.2. *For* $0 < \lambda < \lambda_0$ *the function* $\lambda \mapsto \mathrm{HD}(J(f_\lambda))$ *is strictly increasing.*

Proof. We will show that for $0 < \lambda < \lambda_0$ for every $x \in J(f_\lambda) \cap \mathbb{C}$, every $n > 0$ and every branch f_λ^{-n},

$$\frac{\partial}{\partial\lambda}(f_\lambda^{-n})'(x) > 0. \tag{11}$$

Then $\lambda \mapsto P_\lambda(\alpha)$ is strictly increasing and so is $\lambda \mapsto \mathrm{HD}(J(f_\lambda))$.

Take a branch f_λ^{-n} and let $g_i = f_\lambda^{-i}(x)$ for $i = 0, \ldots, n$. Then

$$\frac{\partial}{\partial\lambda}\frac{\partial}{\partial x} f_\lambda^{-n}(x) = \frac{\lambda^{n-1}}{(\lambda^2+g_0^2)\ldots(\lambda^2+g_{n-1}^2)} \sum_{i=0}^{n-1} \frac{g_i^2 - 2\lambda g_i \frac{\partial g_i}{\partial\lambda} - \lambda^2}{\lambda^2+g_i^2}.$$

It is sufficient to show that for every i we have $g_i^2 - 2\lambda g_i \frac{\partial g_i}{\partial\lambda} - \lambda^2 > 0$. Since

$$\frac{\partial g_i}{\partial\lambda} = \frac{\partial}{\partial\lambda} \arctan \frac{g_{i-1}}{\lambda}$$

we get by induction

$$\frac{\partial g_i}{\partial\lambda} = -\sum_{j=0}^{i-1} \frac{\lambda^{i-1-j} g_j}{(\lambda^2+g_j^2)\ldots(\lambda^2+g_{i-1}^2)} \qquad \text{for } i \ge 1.$$

We have $J(f_\lambda) \subset \mathbb{R} \setminus (-\delta,\delta)$ where $\delta\cot\delta = \lambda$ (see Fig. 1) so $|g_i| > \delta$. Hence it is sufficient to have

$$g_i^2 - 2\lambda g_i \sum_{j=0}^{i-1} \frac{\lambda^{i-1-j}\delta}{(\lambda^2+\delta^2)^{i-j}} - \lambda^2 > 0.$$

It is easy to compute that this inequality holds for $0 < \lambda < \lambda_0$ where λ_0 is $0.402\ldots$ Consequently, (10) holds for such λ. This ends the proof. ∎

For $\lambda \in \mathbb{R} \setminus \{-1, 0, 1\}$ the Hausdorff dimension of the Julia set is a real-analytic function of λ. For $|\lambda| \ge 1$, $J(f_\lambda) = \mathbb{R} \cup \{\infty\}$ so $\mathrm{HD}(J(f_\lambda))$ is equal to 1. At the bifurcation point $\lambda = 1$, $\mathrm{HD}(J(f_\lambda))$ cannot be analytic since it is not constant. However, it is continuous for $\lambda \in \mathbb{R}$, i.e. $\mathrm{HD}(J(f_\lambda)) \to 1$ as $|\lambda| \to 1^-$, $\lambda \in \mathbb{R}$. Let $\lambda \in (-1,1)$. Denote by I an interval of nth generation in the Cantor set $J(f_\lambda)$ and by I_k, $k \in \mathbb{Z}$, the intervals of $(n+1)$th generation contained in I (see Fig. 2).

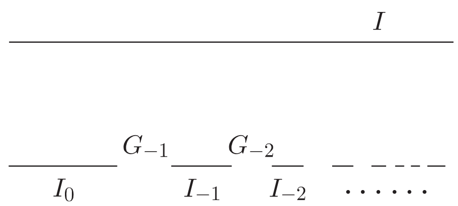


Fig. 2

(G_k are the gaps between the intervals I_k). Denote the length of I by $|I|$.

Lemma 4.3. *Let $\lambda \in (-1,1)\setminus\{0\}$. Suppose that for some α, $\sum_{k\in\mathbb{Z}} |I_k|^\alpha \ge |I|^\alpha$ for all intervals I of nth generation and all n. Then* $\mathrm{HD}(J(f_\lambda)) \ge \alpha$.

Proof. In fact, this lemma is geometric but in our situation we can prove it immediately:

$$P(\alpha) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{\substack{I \text{ of } n\text{th} \\ \text{gen. in } J_0}} |I|^\alpha \ge \lim_{n\to\infty} \frac{1}{n} \log |J_0|^\alpha = 0. \ \blacksquare$$

Now we can use the fact that in $I\setminus(I_0\cup I_1)$ the geometry of the intervals and gaps is universally bounded. More precisely, we have

LEMMA 4.4. *There is a constant $c>0$ such that for every $0<|\lambda|<1$, every n, every interval I of nth generation and for all $k\in\mathbb{Z}\setminus\{0,1\}$,*

$$c\,\frac{\pi-2\delta}{2\delta} < \frac{|I_k|}{|G_k|} < c^{-1}\frac{\pi-2\delta}{2\delta},$$

where $\delta=\operatorname{dist}(J(f_\lambda),0)$ (see Fig. 1).

Proof. We have $I_k = f_\lambda^{-n}(J_{k'})$ and $G_k = f_\lambda^{-n}(\widetilde{G}_{k'})$ for some branch of f_λ^{-n}. Here $J_{k'}$ is an interval of length $\pi-2\delta$ and $k'\ne 0,1$, and $\widetilde{G}_{k'}$ is a gap of length 2δ. The trajectories of asymptotic values of f_λ are contained in the imaginary axis. The distance between the imaginary axis and $J_{k'}\cup\widetilde{G}_{k'}$ is at least $\pi/2$. Therefore from the Koebe lemma the distortion of f_ν^{-n} is universally bounded on $J_{k'}\cup\widetilde{G}_{k'}$. ■

PROPOSITION 4.5. $\mathrm{HD}(J(f_\lambda))\to 1$ *as* $\lambda\to 1^-$.

Proof. Let $\alpha<1$. We want to use Lemma 4.3. From Lemma 3.5 the assumption of Lemma 4.3 will be satisfied if for every I,

$$\sum_{k\in\mathbb{Z}\setminus\{0,1\}} |I_k|^\alpha \ge |I\setminus(I_0\cup I_1)|^\alpha. \tag{12}$$

By bounded geometry, $|I_k| < b\sum_{k\in\mathbb{Z}\setminus\{0,1\}}|I_k|$, where $b<1$ is independent of k and λ. Thus

$$\begin{aligned}\sum_{k\in\mathbb{Z}\setminus\{0,1\}} |I_k|^\alpha &= \sum_{k\in\mathbb{Z}\setminus\{0,1\}} |I_k|^{\alpha-1}|I_k| \\ &> \Big(b\sum_{k\in\mathbb{Z}\setminus\{0,1\}}|I_k|\Big)^{\alpha-1}\sum_{k\in\mathbb{Z}\setminus\{0,1\}}|I_k| = b^{\alpha-1}\Big(\sum_{k\in\mathbb{Z}\setminus\{0,1\}}|I_k|\Big)^\alpha.\end{aligned}$$

From this and Lemma 4.4,

$$\sum_{k\in\mathbb{Z}\setminus\{0,1\}} |I_k|^\alpha \ge \frac{b^{\alpha-1}}{1+\big(c\frac{\pi-2\delta}{2\delta}\big)^\alpha}|I\setminus(I_0\cup I_1)|^\alpha.$$

Let α_1 be such that

$$\frac{b^{\alpha_1-1}}{1+\big(c\frac{\pi-2\delta}{2\delta}\big)^{\alpha_1}} = 1.$$

Then (11) holds for α_1 and from Lemma 4.3, $\mathrm{HD}(J(f_\lambda)) \geq \alpha_1$. It is easy to compute that $\alpha_1 \geq 1 - c_1\delta$. We know that $\delta \to 0$ as $\lambda \to 1^-$. This completes the proof. ■

COROLLARY 4.6. $\mathbb{R} \ni \lambda \mapsto \mathrm{HD}(J(f_\lambda))$ *is analytic for* $\lambda \neq 0, \pm 1$ *and continuous for* $\lambda \neq 0$. ■

We end this section by giving an example of maps for which the Julia set is a Cantor set and the Hausdorff dimension is greater than 1, so according to Theorem 3.6 the $\widetilde{\alpha}$-Hausdorff measure is equivalent to the Gibbs measure. Fix $0 < |\lambda| < 1$, $\lambda \in \mathbb{C}$. Consider the map

$$f_{\lambda,p}(z) = (\lambda \tan z)^p$$

for some large positive integer p. It is easy to see that $f_{\lambda,p}$ has the same dynamics as $\lambda \tan z$, expect that 0 is now a critical point. The set $\bigcup_{k\in\mathbb{Z}} f_{\lambda,p}^{-1}(x+k\pi) \cap J_0$ consists of $2p$ sequences converging to the pole of degree p with asymptotic behaviour like $\{|k|^{-1/p}\}$. For each branch $f_{\lambda,p}^{-1}$,

$$|(f_{\lambda,p}^{-1})'(y)| \geq c(\lambda)p^{-1}|y|^{-(p+1)/p}.$$

Moreover, for every p the Julia sets $J(f_{\lambda,p})$ are contained in the same strip $\{|\mathrm{Im}(z)| \leq a\}$. Therefore

$$\sum_{k\in\mathbb{Z}} \sum_{\substack{z\in f^{-1}(x+k\pi)\\ z\in J_0}} |(f_{\lambda,p}^{-1})'(x+k\pi)| \geq c_1(\lambda) \sum_{k=k_0}^{\infty} k^{-(p+1)/p} \geq c_2(\lambda)pk_0^{-1/p}$$

for some k_0 independent of x, p. For large p the expression is greater than $q > 1$. This implies $P(1) > 0$, so $\widetilde{\alpha} > 1$. Hence we obtain

COROLLARY 4.7. *For given* $0 < |\lambda| < 1$, $\lambda \in \mathbb{C}$, *there exists* p_0 *such that* $\mathrm{HD}(J(f_{\lambda,p})) > 1$ *for* $p > p_0$. ■

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INSTITUTE OF MATHEMATICS
WARSAW UNIVERSITY
BANACHA 2
02-097 WARSZAWA, POLAND
E-mail: BARANSKI@MIMUW.EDU.PL

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