

Construction of non-subadditive measures and discretization of Borel measures

by

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Abstract. The main result of the paper provides a method for construction of regular non-subadditive measures in compact Hausdorff spaces. This result is followed by several examples. In the last section it is shown that “discretization” of ordinary measures is possible in the following sense. Given a positive regular Borel measure λ , one may construct a sequence of non-subadditive measures μ_n , each of which only takes a finite set of values, and such that μ_n converges to λ in the w^* -topology.

1. Introduction. In this paper we continue the study of non-subadditive measures undertaken in [1], [2] and [5], called there “quasi-measures”. They are set-functions defined on the open and on the closed subsets of a locally compact Hausdorff space X , and represent a genuine generalization of regular Borel measures in such spaces. This paper is devoted to showing how they arise and may be constructed when X is compact, and to giving some applications.

Non-subadditive measures (NSA-measures), as the name indicates, are generally not subadditive. Indeed, if they are, then they turn out to be ordinary regular Borel measures. This lack of subadditivity is what makes NSA-measures different, and in some respects more interesting than ordinary measures. Instead of weighing effects or events on an additive scale, the NSA-measures register a *cumulative* effect of events. To produce a certain result, several other results must occur simultaneously. This is of course a very superficial description, and only future development and applications can substantiate what we indicate here.

Even if NSA-measures by definition are generalizations of ordinary measures, their *existence* is not an obvious matter, and turns out to be closely linked to properties of the underlying topological space. The existence of NSA-measures was first established in the author’s paper [1]. In [2] we gave

1991 *Mathematics Subject Classification*: 28C15.

a description of the basic properties of *extremal* NSA-measures (those taking only the values 0 and 1). In [5] Knudsen gave a procedure for the construction of such extremal NSA-measures in certain spaces. The present paper is devoted to the construction of *general* NSA-measures. Our main result (Theorem 5.1) shows how all NSA-measures arise and may be constructed (in certain spaces). We believe that even when applied to ordinary measures this is a new result of some interest. The main result is followed by several examples of construction. In the last section of the paper we show that “discretization” of ordinary measures is possible in the following sense: Given a positive, regular Borel measure λ (for instance Lebesgue measure on the unit sphere in \mathbb{R}^3), we construct a sequence of NSA-measures μ_n , each of which takes only a finite set of values, and such that μ_n converges to λ in the w^* -topology.

1.1. Notation and basic concepts. Throughout X denotes a compact Hausdorff space and $\mathbf{A} = C(X)$ is the algebra of real-valued continuous functions on X . For $a \in \mathbf{A}$ we let $\mathbf{A}(a)$ denote the smallest uniformly closed subalgebra of \mathbf{A} containing a and 1. A function $\varrho : \mathbf{A} \rightarrow \mathbb{R}$ satisfying $\varrho(1) = 1$, $\varrho(a) \geq 0$ if $a \geq 0$ and such that ϱ is linear on $\mathbf{A}(a)$ for each $a \in \mathbf{A}$ is called a *non-linear state* (previously called a quasi-state).

Let \mathcal{C} denote the collection of closed subsets of X , let \mathcal{O} denote the collection of open subsets of X and put $\mathcal{A} = \mathcal{C} \cup \mathcal{O}$. A real-valued, non-negative function μ on \mathcal{A} is called a *NSA-measure* in X if the following conditions are satisfied:

- (Q0) $\mu(K) + \mu(X \setminus K) = \mu(X)$ for $K \in \mathcal{C}$,
- (Q1) $K_1 \subset K_2 \Rightarrow \mu(K_1) \leq \mu(K_2)$ for $K_1, K_2 \in \mathcal{C}$,
- (Q2) $K_1 \cap K_2 = \emptyset \Rightarrow \mu(K_1 \cup K_2) = \mu(K_1) + \mu(K_2)$ for $K_1, K_2 \in \mathcal{C}$,
- (Q3) $\mu(U) = \sup\{\mu(K) : K \subset U, K \in \mathcal{C}\}$ for $U \in \mathcal{O}$.

μ is *normalized* if $\mu(X) = 1$. For simplicity we shall assume that all NSA-measures in this paper are normalized.

Remark. A NSA-measure μ which is also subadditive is called a *regular content*. In Halmos ([4], §54, Theorem A) it is shown that a regular content has a unique extension to a regular Borel measure in X .

In [1] we established that there is a 1-1 correspondence between non-linear states and normalized NSA-measures. The set of all non-linear states is a convex set, denoted by Q , which is compact in the topology of pointwise convergence on \mathbf{A} .

A subset D of X is *co-connected* if $X \setminus D$ is connected. D is *solid* if it is connected and co-connected.

In what follows a subscript *s* indicates “solid”, and a subscript *c* indicates “connected”, so that for instance \mathcal{A}_s is the collection of all solid sets that

are either open or closed and \mathcal{C}_c is the family of closed connected sets. \mathcal{C}_0 is the family of closed sets with only a finite number of connected components. $\mathcal{O}_0 = \{U \in \mathcal{O} : X \setminus U \in \mathcal{C}_0\}$, and $\mathcal{A}_0 = \mathcal{C}_0 \cup \mathcal{O}_0$.

2. Fundamentals. In this section we introduce the main object of study in this paper: the *solid set-function*. Some preliminaries are needed. From now on we assume that X is a compact Hausdorff space which is *connected* and *locally connected*. By convention the empty set \emptyset is connected, so \emptyset and X both belong to \mathcal{C}_s .

DEFINITION 2.1. A *partition* of X is a collection of mutually disjoint, non-void sets $\{A_i\}_{i \in I} \subseteq \mathcal{A}_s$, where at most finitely many of the A_i are closed, and such that $X = \bigcup_{i \in I} A_i$. The number of closed sets in a partition \mathcal{P} is called the *order* of \mathcal{P} .

Any connected and locally connected space X with more than one point has a partition. For if $x \in X$, then $\{x\}^c$ is open and has a non-void connected component $V \in \mathcal{O}_s$ (cf. Lemma 3.2 in the next section). Hence $C = V^c \in \mathcal{C}_s$ and $\{C, V\}$ is a partition of X of order 1. Accordingly, partitions of order 1 are called *trivial*.

Let $\{A_i\}_{i \in I}$ be a non-trivial partition, and let $I_0 = \{i \in I : A_i \text{ is closed}\}$.

DEFINITION 2.2. $\{A_i\}_{i \in I}$ is *irreducible* if the following two conditions hold:

- (i) $\bigcup_{i \in I_0} A_i$ is *not* co-connected.
- (ii) For any proper subset I' of I_0 , $\bigcup_{i \in I'} A_i$ is co-connected.

Necessarily, any irreducible partition has order ≥ 2 , and any partition of order 2 is irreducible. For a given space X , let n denote the maximal order of any irreducible partition. If n is finite, let $g = n - 1$. If X only permits trivial partitions, put $g = 0$.

DEFINITION 2.3. A function $\mu : \mathcal{A}_s \rightarrow [0, 1]$ satisfying the conditions (A), (B) and (C) below is called a *solid set-function*.

(A) For any finite collection of disjoint sets $\{C_1, \dots, C_n\} \subseteq \mathcal{C}_s$ such that $C_j \subseteq C \in \mathcal{C}_s$ for $j = 1, \dots, n$ we have

$$\sum_{j=1}^n \mu(C_j) \leq \mu(C).$$

(B) For all $U \in \mathcal{O}_s$ we have

$$\mu(U) = \sup\{\mu(C) : C \subseteq U, C \in \mathcal{C}_s\}.$$

(C) For any trivial or irreducible partition $\{A_i\}_{i \in I}$ of X we have

$$\sum_{i \in I} \mu(A_i) = 1.$$

Note. If $g = 0$ then X has no irreducible partitions, so (C) reduces to the condition that $\mu(A) + \mu(A^c) = 1$ for each set $A \in \mathcal{A}_s$.

Remark. It is important to realize that the restriction of a NSA-measure μ to \mathcal{A}_s is always a solid set-function. Indeed, since μ is additive and monotone on \mathcal{C} , (A) is clearly true. If K is closed and contained in a solid open set U , there is a solid closed set C such that $K \subseteq C \subseteq U$ (see Section 3), and (B) follows. Finally, (C) is a consequence of Proposition 2.1 of [1] and Corollary 2.1 of [2].

We next point out that any NSA-measure (or any regular Borel measure) is *uniquely determined by its restriction to \mathcal{C}_s* . For suppose μ is a NSA-measure in X . If the values of μ on \mathcal{C}_s are known, then they are also known on the complements of these sets, i.e. on \mathcal{O}_s . But then, by virtue of Corollary 2.1 quoted above, and Lemma 3.2 of this paper, it follows that μ is determined on the family of closed connected sets, and hence also on \mathcal{C}_0 . Now let K be a closed set contained in an open set U , and let $U = \bigcup_{i \in I} U_i$ be the decomposition of U into its connected components. By compactness of K there is a finite index set $I' \subseteq I$ such that $K \subseteq \bigcup_{i \in I'} U_i$. Let $K_i = K \cap U_i$ ($i \in I'$). By Lemma 3.1 of this paper there are connected closed sets C_i such that $K_i \subseteq C_i \subseteq U_i$ for each $i \in I'$. But then $C = \bigcup C_i$ belongs to \mathcal{C}_0 and $K \subseteq C \subseteq U$. By (Q3) in the definition of NSA-measures it therefore follows that μ is determined on the open sets by the values it takes on the class \mathcal{C}_0 . Taking complements again we see that the uniqueness claim follows.

EXAMPLE. Let $X = S^2$ and let p_1, \dots, p_5 be five distinct points in X . For $C \in \mathcal{C}_s$ define $\mu(C)$ to be 0 if C contains at most one of these points, to be 1/2 if C contains two or three of the points, and to be 1 if C contains four or five points. It is easily seen that (A) and (B) hold, and (C) is true because S^2 only permits trivial partitions.

The main purpose of this paper is to show that a solid set-function has a unique extension to a NSA-measure in X . In the example just given this extension turns out to be an extreme point in the convex set of all normalized NSA-measures in X (cf. [2], Example 2.1).

To begin with, we record some of the basic properties of solid set-functions.

PROPOSITION 2.1. *If μ is a solid set-function in X then*

1. $\mu(\emptyset) = 0$, $\mu(X) = 1$.
2. $A_1 \subseteq A_2 \Rightarrow \mu(A_1) \leq \mu(A_2)$ for all $A_1, A_2 \in \mathcal{A}_s$.

3. If U_1, \dots, U_n belong to \mathcal{O}_s , where n is an arbitrary integer ≥ 1 , $U_i \cap U_j = \emptyset$, $i \neq j$, and $U_i \subseteq U \in \mathcal{O}_s$ for all $i = 1, \dots, n$, then

$$\sum_{i=1}^n \mu(U_i) \leq \mu(U).$$

The proof is left to the reader.

Remark. Let μ be a solid set-function in X , and suppose $\{C_i\}_{i \in I} \subseteq \mathcal{C}_s$ is an arbitrary collection of mutually disjoint sets. From (A) it immediately follows (let $C = X$) that the subfamily $\{C_i : \mu(C_i) > 0\}$ is at most countably infinite. When it comes to summation we may therefore replace the index set I by \mathbb{Z}^+ or \mathbb{N} . This is implicit when we write $\sum_{i \in I} \mu(C_i)$. If $C_i \subseteq C \in \mathcal{C}_s$ for all $i \in I$ we must clearly have $\sum_{i \in I} \mu(C_i) \leq \mu(C)$. Analogously, in view of Proposition 2.1.3 the corresponding statements hold for open sets, and will be utilised without further comments.

For later use we include the following:

LEMMA 2.1. *Let $\mathcal{F} = \{C_j\}_{j \in J} \subseteq \mathcal{C}_s$ ($n \geq 1$) be a finite family of mutually disjoint sets such that $\bigcup_{j \in J} C_j$ is not co-connected. Then \mathcal{F} has a subfamily \mathcal{F}' such that each connected component U_i of $U = (\bigcup\{C_j \in \mathcal{F}'\})^c$ belongs to \mathcal{O}_s ($i \in I =$ some index set) and $\mathcal{F}' \cup \{U_i\}_{i \in I}$ is an irreducible partition of X .*

Proof. \mathcal{F} contains a subfamily $\mathcal{F}' = \{C_j\}_{j \in J'}$ such that $U = (\bigcup_{j \in J'} C_j)^c$ is not connected, but for any proper subset J'' of J' , the set $(\bigcup_{j \in J''} C_j)^c$ is connected. Let $U = \bigcup_{i \in I} U_i$ be the decomposition of U into its connected components. Then $\text{card } I \geq 2$ and each U_i is in \mathcal{O}_c . We claim that each U_i is also co-connected. So pick an arbitrary $U_0 \in \{U_i\}_{i \in I}$, and let $I' = I \setminus \{0\}$. We must show that $U_0^c = (\bigcup_{j \in J'} C_j) \cup (\bigcup_{i \in I'} U_i)$ is connected. Suppose not. Then there are non-void, disjoint, closed sets K_1 and K_2 in X such that $K_1 \cup K_2 = U_0^c$. By connectedness, K_1 (and K_2) must contain all or nothing of each of the sets C_j ($j \in J'$) and U_i ($i \in I'$). Let

$$J_k = \{j \in J' : C_j \subseteq K_k\}, \quad I_k = \{i \in I' : U_i \subseteq K_k\}, \quad k = 1, 2.$$

We must have $J_1 \neq \emptyset$ and $J_2 \neq \emptyset$ and at least one of the sets I_k is non-empty since $\text{card } I \geq 2$. Suppose $I_1 \neq \emptyset$. Since K_1 is closed, its complement $K_1^c = K_2 \cup U_0$ is open. But then also

$$V = (K_2 \cup U_0) \cup \left(\bigcup_{i \in I_1} U_i \right)$$

is open and not connected. But $V^c = \bigcup_{j \in J_1} C_j$, which is a contradiction, since J_1 is a proper subset of J' . Hence U_0^c is connected and the proof is complete. ■

PROPOSITION 2.2. *For a space X we have $g = 0$ if and only if $\bigcup\{C \in \mathcal{F}\}$ is co-connected for any finite family $\mathcal{F} \subseteq \mathcal{C}_s$ of disjoint sets.*

PROOF. This follows from Lemma 2.1. ■

REMARK. In [5] Knudsen has shown that if X is also locally pathwise connected and $H^1(X, \mathbb{Q}) = (0)$ then $g = 0$. Hence S^n (the unit n -sphere) and B^n (the unit n -ball) have $g = 0$, the torus and an annulus in the plane have $g = 1$, etc.

3. Topological preliminaries. The results in this section are more or less known. Some proofs are included for the convenience of the reader. Good references are [3] and [6].

Throughout we assume that X is a compact Hausdorff space which is *connected* and *locally connected*.

LEMMA 3.1. *Let $K \in \mathcal{C}$, $U \in \mathcal{O}$ and $K \subseteq U$. If either K or U is connected, then there is a set $V \in \mathcal{O}_c$ such that $K \subseteq V \subseteq \bar{V} \subseteq U$.*

LEMMA 3.2. *Let $K \in \mathcal{C}_c$. Then each connected component of $V = X \setminus K$ belongs to \mathcal{O}_s .*

LEMMA 3.3. *Let $K \in \mathcal{C}_s$, $U \in \mathcal{O}$ and $K \subseteq U$. Then there is a set $V \in \mathcal{O}_s$ such that $K \subseteq V \subseteq \bar{V} \subseteq U$.*

PROOF OF LEMMA 3.3. By Lemma 3.1 we may assume that U is connected. Let $F = X \setminus U$, so $F \subseteq X \setminus K$, which is open and connected. Hence again by Lemma 3.1 there is an open set $V \in \mathcal{O}_c$ such that $F \subseteq V \subseteq \bar{V}$ and $\bar{V} \cap K = \emptyset$. Let $W = X \setminus \bar{V}$ so $K \subseteq W \subseteq U$. Since $\bar{V} \in \mathcal{C}_c$ it now follows that each connected component of W belongs to \mathcal{O}_s (Lemma 3.2), and K must be contained in exactly one of them. The proof is complete.

3.1. The solid hull of a closed connected set. Let $K \in \mathcal{C}_c$, $V \in \mathcal{O}_s$ and suppose $K \subseteq V$. Let $X \setminus K = W = \bigcup_{i \in I} W_i$ be the decomposition of W into disjoint connected components. Then $C = X \setminus V$ is connected and is contained in one of them, say W_0 , called the *exterior component* of W with respect to V . Each W_i with $i \neq 0$ is then called an *interior component*. Since each W_i is in \mathcal{O}_s the set $\tilde{K} = X \setminus W_0$ belongs to \mathcal{C}_s . We have

$$\tilde{K} = \bigcup\{W_i : i \neq 0\} \cup K$$

and $K \subseteq \tilde{K} \subseteq V$. \tilde{K} is called the *solid hull* of K with respect to V .

LEMMA 3.4. *Suppose C_1, C_2 are disjoint, closed, connected sets, both contained in $V \in \mathcal{O}_s$. Then either*

- (i) $\tilde{C}_1 \cap \tilde{C}_2 = \emptyset$, or
- (ii) $\tilde{C}_1 \subseteq \tilde{C}_2$ or $\tilde{C}_2 \subseteq \tilde{C}_1$ (proper inclusion).

Proof. Let $W^j = X \setminus C_j$ ($j = 1, 2$), and let $W^j = \bigcup_{i \in I_j} W_i^j$ as above, with W_0^j as the exterior component of W^j with respect to V . Now $C_1 \cap C_2 = \emptyset$ so

$$C_1 \subseteq W^2 \quad \text{and} \quad C_2 \subseteq W^1.$$

Since C_1 and C_2 are connected, there are unique components W_i^2 and W_k^1 such that $C_1 \subseteq W_i^2$ and $C_2 \subseteq W_k^1$.

(i) Suppose $C_1 \subseteq W_0^2$ and $C_2 \subseteq W_0^1$. Then $C_2 \subseteq \tilde{C}_2 = X \setminus W_0^2 \subseteq W^1$ by the first inclusion. Since \tilde{C}_2 is connected the second inclusion implies that $\tilde{C}_2 \subseteq W_0^1 = X \setminus \tilde{C}_1$, so $\tilde{C}_1 \cap \tilde{C}_2 = \emptyset$.

(ii) Suppose $C_2 \subseteq W_k^1$ for some $k \neq 0$. Then $W_k^1 \subseteq V$ so $X \setminus V \subseteq X \setminus W_k^1 \subseteq W^2$. Since $X \setminus W_k^1$ is connected and contains $X \setminus V$ we must have $X \setminus W_k^1 \subseteq W_0^2$, so that $\tilde{C}_2 = X \setminus W_0^2 \subseteq W_k^1 \subseteq \tilde{C}_1$. Similarly, if $C_1 \subseteq W_i^2$ for some $i \neq 0$, then $\tilde{C}_1 \subseteq \tilde{C}_2$. The proof is complete.

Remark. We may note that in the above situation one of the sets is always contained in the exterior component associated with the other set.

4. Extension to \mathcal{C}_0 . Let μ be a solid set-function. Our goal in this section is to extend μ to a function on \mathcal{C}_0 to $[0, 1]$ such that

- (Q1)₀ $K_1 \subseteq K_2 \Rightarrow \mu(K_1) \leq \mu(K_2)$ for $K_1, K_2 \in \mathcal{C}_0$.
- (Q2)₀ $K_1 \cap K_2 = \emptyset \Rightarrow \mu(K_1 \cup K_2) = \mu(K_1) + \mu(K_2)$ for $K_1, K_2 \in \mathcal{C}_0$.
- (Q3)₀ For all $C \in \mathcal{C}_0$ and all $\varepsilon > 0$ there is $C' \in \mathcal{C}_0$ such that $C \cap C' = \emptyset$ and $\mu(C) + \mu(C') > 1 - \varepsilon$.

If $K \in \mathcal{C}_c$ we know by Lemma 3.2 that $X \setminus K = V = \bigcup_{i \in I} V_i$, where the sets V_i are open, connected, co-connected and mutually disjoint. From Proposition 2.1.3 it therefore follows that the set $\{V_i : \mu(V_i) > 0\}$ is at most countable, and that $\sum_{i \in I} \mu(V_i) \leq 1$. We may then define

$$(4.1) \quad \mu(K) = 1 - \sum_{i \in I} \mu(V_i).$$

Next, if $K = K_1 \cup \dots \cup K_n \in \mathcal{C}_0$, where the K_j are connected and mutually disjoint, we put $\mu(K) = \sum_{j=1}^n \mu(K_j)$.

PROPOSITION 4.1. *The function μ as defined above maps \mathcal{C}_0 into $[0, 1]$ and satisfies conditions (Q1)₀, (Q2)₀ and (Q3)₀.*

We start with (Q1)₀: If $K_1, K_2 \in \mathcal{C}_0$, then $K_1 \subseteq K_2 \Rightarrow \mu(K_1) \leq \mu(K_2)$.

Suppose first that K_1 is connected, while $K_2 = C_1 \cup \dots \cup C_n$ with the C_j in \mathcal{C}_c and mutually disjoint. Then we must have $K_1 \subseteq C_j$ for some j , say

$K_1 \subseteq C_1$. Let $\bigcup_{i \in I} U_i = X \setminus K_1$ and $\bigcup_{j \in J} V_j = X \setminus C_1$ be the decompositions of these sets into their solid components. Each V_j must be contained in some U_i . Let

$$J_i = \{j \in J : V_j \subseteq U_i\}.$$

Then $\bigcup_{i \in I} J_i = J$, and by Proposition 2.1.3 we get

$$\sum_{j \in J} \mu(V_j) = \sum_{i \in I} \sum_{j \in J_i} \mu(V_j) \leq \sum_{i \in I} \mu(U_i).$$

Hence $\mu(K_1) = 1 - \sum_{i \in I} \mu(U_i) \leq 1 - \sum_{j \in J} \mu(V_j) \leq \mu(C_1)$.

Next, assume that $K_1 = C'_1 \cup \dots \cup C'_m$ is the decomposition of K_1 into its connected components. Since $K_1 \subseteq K_2$ each C'_i is contained in C_j for some $j \in \{1, \dots, n\}$.

Suppose $C'_{i_1}, \dots, C'_{i_k} \subseteq C_j$. We need to show that $\sum_{l=1}^k \mu(C'_{i_l}) \leq \mu(C_j)$. To simplify the notation, assume that C_0, C_1, \dots, C_n are connected, closed, disjoint and contained in $C \in \mathcal{C}_c$. Let $X \setminus C_0 = V = \bigcup_{i \in I} V_i$. Since all the C_j are connected, each C_j for $j \geq 1$ is contained in some V_i , and only finitely many V_i will contain some C_j . Let

$$J_i = \{j : C_j \subseteq V_i\}, \quad i \in I.$$

Let $U = X \setminus C = \bigcup_{k \in K} U_k$. Then $U \subseteq V$ and we put

$$K_i = \{k \in K : U_k \subseteq V_i\}.$$

The U_k are disjoint and obviously also disjoint from all the C_j . We want to show that

$$(4.2) \quad \sum_{j=0}^n \mu(C_j) \leq \mu(C),$$

which is equivalent to

$$1 - \sum_{i \in I} \mu(V_i) + \sum_{j=1}^n \mu(C_j) \leq 1 - \sum_{k \in K} \mu(U_k),$$

or

$$(4.3) \quad \sum_{j=1}^n \mu(C_j) + \sum_{k \in K} \mu(U_k) \leq \sum_{i \in I} \mu(V_i).$$

Now (4.3) will follow if we can show that

$$(4.4) \quad \sum_{j \in J_i} \mu(C_j) + \sum_{k \in K_i} \mu(U_k) \leq \mu(V_i)$$

for all $i \in I$. Therefore, what we really need is the following result, formally stated as a lemma.

LEMMA A. Let $C_1, \dots, C_n \in \mathcal{C}_c$ and $U_1, \dots, U_k, \dots \in \mathcal{O}_s$ be mutually disjoint subsets of $V \in \mathcal{O}_s$. Then

$$(4.5) \quad \sum_{j=1}^n \mu(C_j) + \sum_{k=1}^{\infty} \mu(U_k) \leq \mu(V).$$

Proof. We employ induction on n .

Step 1: $n = 1$. Let $X \setminus C_1 = W = \bigcup_{i \in I} W_i$. Now $X \setminus V$ is connected and contained in W , and is therefore contained in one of W 's components, which we denote by W_0 , i.e. $X \setminus V \subseteq W_0$ and W_0 is the exterior component of W with respect to V (cf. Section 3). Let $I' = I \setminus \{0\}$ and put $\tilde{C}_1 = C_1 \cup (\bigcup_{i \in I'} W_i) = X \setminus W_0$, i.e. \tilde{C}_1 is the solid hull of C_1 with respect to V . We want to show that

$$\mu(C_1) + \sum_{k=1}^{\infty} \mu(U_k) \leq \mu(V),$$

or

$$1 - \sum_{i \in I'} \mu(W_i) - \mu(W_0) + \sum_{k=1}^{\infty} \mu(U_k) \leq \mu(V),$$

i.e.

$$(4.6) \quad \mu(\tilde{C}_1) + \sum_{k=1}^{\infty} \mu(U_k) \leq \mu(V) + \sum_{i \in I'} \mu(W_i).$$

We have $U_k \subseteq W$ for all $k = 1, 2, \dots$ (since all U_k are disjoint from C_1) and we let

$$K_i = \{k \in \mathbb{Z}^+ : U_k \subseteq W_i\}, \quad i \in I.$$

Then we have, using Proposition 2.1.3 again,

$$\begin{aligned} \sum_{k=1}^{\infty} \mu(U_k) &= \sum_{k \in K_0} \mu(U_k) + \sum_{i \in I'} \left[\sum_{k \in K_i} \mu(U_k) \right] \\ &\leq \sum_{k \in K_0} \mu(U_k) + \sum_{i \in I'} \mu(W_i). \end{aligned}$$

Hence (4.6) will follow if we can show that

$$(4.7) \quad \mu(\tilde{C}_1) + \sum_{k \in K_0} \mu(U_k) \leq \mu(V),$$

where we observe that if $k \in K_0$ then U_k is disjoint from \tilde{C}_1 . In turn, (4.7) will follow from the next lemma (which in fact is just a weakened version of Lemma A).

LEMMA B. Let $C_1, \dots, C_m \in \mathcal{C}_s$ and $U_1, \dots, U_k, \dots \in \mathcal{O}_s$ be mutually disjoint subsets of $V \in \mathcal{O}_s$. Then

$$(4.8) \quad \sum_{j=1}^m \mu(C_j) + \sum_{k=1}^{\infty} \mu(U_k) \leq \mu(V).$$

PROOF. Let $\varepsilon > 0$ be arbitrary. Select $N \in \mathbb{Z}^+$ such that $\sum_{k>N} \mu(U_k) < \varepsilon/2$. For $k = 1, \dots, N$ take $C'_k \subseteq U_k$ such that $C'_k \in \mathcal{C}_s$ and $\mu(C'_k) > \mu(U_k) - \varepsilon/2^{k+1}$ by (B). Then

$$\begin{aligned} \sum_{k=1}^{\infty} \mu(U_k) &= \sum_{k=1}^N \mu(U_k) + \sum_{k>N} \mu(U_k) \\ &< \sum_{k=1}^N \mu(C'_k) + \varepsilon \sum_{k=1}^N \frac{1}{2^{k+1}} + \frac{\varepsilon}{2} < \sum_{k=1}^N \mu(C'_k) + \varepsilon. \end{aligned}$$

Let $C = X \setminus V$, so $C \in \mathcal{C}_s$ and $C \cap C_j = C \cap C'_k = \emptyset$ for all j and k . Then by (A) we get

$$\sum_{j=1}^m \mu(C_j) + \sum_{k=1}^N \mu(C'_k) + \mu(C) \leq 1.$$

Hence

$$\sum_{j=1}^m \mu(C_j) + \sum_{k=1}^{\infty} \mu(U_k) \leq \mu(V) + \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, the assertion follows. ■

The inequality (4.7) now follows (taking $m = 1$) since $\tilde{C}_1 \in \mathcal{C}_s$. We have now established Lemma A for $n = 1$.

Step 2. Suppose inductively that Lemma A has been verified for $k = 1, \dots, n - 1$. For $j = 1, \dots, n$ let $X \setminus C_j = W^j = \bigcup \{W_i^j : i \in I_j\}$, and let W_0^j be the exterior component of W^j with respect to V . Let $\tilde{C}_j = X \setminus W_0^j$ denote the solid hull of C_j with respect to V .

Case 1: $\tilde{C}_j \cap \tilde{C}_l = \emptyset$ for all j, l with $j \neq l$. The assertion (4.5) is equivalent to

$$(4.9) \quad \sum_{j=1}^n \mu(\tilde{C}_j) + \sum_{k=1}^{\infty} \mu(U_k) \leq \mu(V) + \sum_{j=1}^n \sum_{i \in I'_j} \mu(W_i^j),$$

where $I'_j = I_j \setminus \{0\}$. Now observe that $X \setminus \bigcup_{j=1}^n \tilde{C}_j = \bigcap_{j=1}^n W_0^j$, so that if

we put

$$K_0 = \left\{ k \in \mathbb{Z}^+ : U_k \subseteq \bigcap_{j=1}^n W_0^j \right\},$$

$$K_j = \{ k \in \mathbb{Z}^+ : U_k \subseteq \tilde{C}_j \}, \quad j = 1, \dots, n,$$

then we exhaust all possibilities for the sets U_k in mutually exclusive cases.

By Lemma B we now get

$$(4.10) \quad \sum_{j=1}^n \mu(\tilde{C}_j) + \sum_{k \in K_0} \mu(U_k) \leq \mu(V).$$

If $U_k \subseteq \tilde{C}_j$ then $U_k \subseteq W_i^j$ for some $i \in I'_j$ so that (using Proposition 2.1.3 again)

$$(4.11) \quad \sum_{k \in K_j} \mu(U_k) \leq \sum_{i \in I'_j} \mu(W_i^j), \quad j = 1, \dots, n.$$

Combining (4.10) and (4.11) we obtain (4.9), which establishes the assertion (4.5) in Case 1.

Case 2: $\tilde{C}_j \cap \tilde{C}_l \neq \emptyset$ for some pair $j \neq l$. By Lemma 3.4 either $\tilde{C}_j \subseteq \tilde{C}_l$ or $\tilde{C}_l \subseteq \tilde{C}_j$ (proper inclusion). We may therefore re-index the sets \tilde{C}_j ($j = 1, \dots, n$) as follows:

1. $\tilde{C}_{10}, \tilde{C}_{20}, \dots, \tilde{C}_{m0}$ ($m < n$) are mutually disjoint.
2. $\tilde{C}_{jl} \subseteq \tilde{C}_{j0}$ for $l = 0, \dots, m_j$ and $1 \leq j \leq m$, where the inclusions are proper if $l \geq 1$.
3. $m + \sum_{j=1}^m m_j = n$.

The verification of this is a simple induction argument based on Lemma 3.4, and is left to the reader. We return to the proof of (4.9), which after the re-indexing takes the form

$$(4.12) \quad \sum_{j=1}^m \sum_{l=0}^{m_j} \mu(\tilde{C}_{jl}) + \sum_{k=1}^{\infty} \mu(U_k) \leq \mu(V) + \sum_{j=1}^m \sum_{l=0}^{m_j} \sum_{i \in I'_{jl}} \mu(W_i^{jl}),$$

where $X \setminus C_{jl} = \bigcup \{ W_i^{jl} : i \in I_{jl} \}$.

Let K_0, K_1, \dots, K_n be defined as in Case 1. Then again by Lemma B we immediately get

$$(4.13) \quad \sum_{j=1}^m \mu(\tilde{C}_{j0}) + \sum_{k \in K_0} \mu(U_k) \leq \mu(V),$$

so to establish (4.12) it remains to show that

$$(4.14) \quad \sum_{j=1}^m \sum_{l=1}^{m_j} \mu(\tilde{C}_{jl}) + \sum_{j=1}^m \sum_{k \in K_j} \mu(U_k) \leq \sum_{j=1}^m \sum_{l=0}^{m_j} \sum_{i \in I'_{jl}} \mu(W_i^{jl}),$$

which will follow if we can show that for each $j = 1, \dots, m$ we have

$$(4.15) \quad \sum_{l=1}^{m_j} \mu(\tilde{C}_{jl}) + \sum_{k \in K_j} \mu(U_k) \leq \sum_{l=0}^{m_j} \sum_{i \in I'_{jl}} \mu(W_i^{jl}).$$

Now $\mu(\tilde{C}_{jl}) = \mu(C_{jl}) + \sum_{i \in I'_{jl}} \mu(W_i^{jl})$, so (4.15) is equivalent to

$$(4.16) \quad \sum_{l=1}^{m_j} \mu(C_{jl}) + \sum_{k \in K_j} \mu(U_k) \leq \sum_{i \in I'_{j0}} \mu(W_i^{j0}).$$

Let

$$J_{ji} = \{l \leq m_j : C_{jl} \subseteq W_i^{j0}\}, \quad i \in I'_{j0},$$

$$K_{ji} = \{k \in K_j : U_k \subseteq W_i^{j0}\}, \quad i \in I'_{j0}.$$

Since $C_{jl} \subseteq W_i^{j0}$ for some $i \in I'_{j0}$ if $1 \leq l \leq m_j$, and $U_k \subseteq W_i^{j0}$ for some $i \in I'_{j0}$ if $k \in K_j$, we exhaust all possibilities for l and k in this manner. Hence (4.16) will follow if we can show that for each $i \in I'_{j0}$,

$$(4.17) \quad \sum_{l \in J_{ji}} \mu(C_{jl}) + \sum_{k \in K_{ji}} \mu(U_k) \leq \mu(W_i^{j0}).$$

The number of elements in J_{ji} is $\leq m_j \leq m < n$ so (4.17) now follows by the induction hypothesis. This concludes the proof of Lemma A. ■

This also concludes the proof of condition (Q1)₀ in Proposition 4.1. As for (Q2)₀, if $K_1, K_2 \in \mathcal{C}_0$ and $K_1 \cap K_2 = \emptyset$ let $K_1 = C_1 \cup \dots \cup C_n$ and $K_2 = C'_1 \cup \dots \cup C'_m$ be their decompositions into connected components. Then $K_1 \cup K_2 = C_1 \cup \dots \cup C_n \cup C'_1 \cup \dots \cup C'_m$ and the right-hand side represents the decomposition of $K_1 \cup K_2$ into connected components. From the definition of μ on \mathcal{C}_0 it now immediately follows that $\mu(K_1 \cup K_2) = \mu(K_1) + \mu(K_2)$, and (Q2)₀ follows.

It remains to verify the regularity property (Q3)₀.

Let $K \in \mathcal{C}_0$ be arbitrary. Then $K = C_0 \cup C_1 \cup \dots \cup C_n$, where $C_j \in \mathcal{C}_c$ and $C_j \cap C_k = \emptyset$ if $j \neq k$. Let $X \setminus C_0 = V = \bigcup_{i \in I} V_i$, where the V_i are mutually disjoint and belong to \mathcal{O}_s . Each C_j ($j = 1, \dots, n$) is contained in some V_i , and we let

$$J_i = \{j \in \{1, \dots, n\} : C_j \subseteq V_i\}, \quad i \in I.$$

From Lemma A it follows that $\sum_{j \in J_i} \mu(C_j) \leq \mu(V_i)$ for all $i \in I$. Let $\varepsilon > 0$ be arbitrary, and choose N such that $\sum_{i > N} \mu(V_i) < \varepsilon/2$ (since

$\{i \in I : \mu(V_i) > 0\}$ is countable, we may assume that $I = \mathbb{Z}^+$). Now suppose for the moment that there are sets $C'_i \in \mathcal{C}_0$ for $i = 1, \dots, N$ such that

$$(4.18) \quad \begin{cases} C'_i \subseteq V_i \text{ and } C'_i \cap C_j = \emptyset \text{ if } j \in J_i, \\ \mu(C'_i) > \mu(V_i) - \sum_{j \in J_i} \mu(C_j) - \varepsilon/2^{i+1}. \end{cases}$$

Then let $C' = \bigcup_{i=1}^N C'_i$, so $C' \in \mathcal{C}_0$ and $C' \cap K = \emptyset$. Moreover (by (Q2)₀),

$$\begin{aligned} \mu(C') &= \sum_{i=1}^N \mu(C'_i) > \sum_{i=1}^N \mu(V_i) - \sum_{i=1}^N \sum_{j \in J_i} \mu(C_j) - \sum_{i=1}^N \frac{\varepsilon}{2^{i+1}} \\ &> \sum_{i=1}^{\infty} \mu(V_i) - \frac{\varepsilon}{2} - \sum_{j=1}^n \mu(C_j) - \frac{\varepsilon}{2} \\ &= 1 - \mu(C_0) - \sum_{j=1}^n \mu(C_j) - \varepsilon = 1 - \mu(K) - \varepsilon, \end{aligned}$$

which establishes (Q3)₀ under the assumption that (4.18) holds. So what we need is the following.

LEMMA C. *Let $C_1, \dots, C_n \in \mathcal{C}_c$ be mutually disjoint and contained in $V \in \mathcal{O}_s$. For each $\varepsilon > 0$ there is a $C' \in \mathcal{C}_0$ such that $C' \cap C_j = \emptyset$ for all j , $C' \subseteq V$ and*

$$\sum_{j=1}^n \mu(C_j) + \mu(C') > \mu(V) - \varepsilon.$$

PROOF. We first consider the special case where all the sets C_1, \dots, C_n belong to \mathcal{C}_s , and $V = X$. The proof in this case goes by induction on n . For $n = 1$ the assertion is covered by (B) in the definition of a solid set-function. Now assume $n \geq 2$ and that the assertion is true for all $k < n$. Let $\varepsilon > 0$ be arbitrary. Combining (B) and Lemma 3.3, we may find a family of mutually disjoint sets V_1, \dots, V_n in \mathcal{O}_s such that $C_j \subseteq V_j$ and

$$\mu(V_j) < \mu(C_j) + \varepsilon/n, \quad j = 1, \dots, n.$$

Next, choose $W_j \in \mathcal{O}_s$ such that

$$C_j \subseteq W_j \subseteq \overline{W_j} \subseteq V_j, \quad j = 1, \dots, n$$

(Lemma 3.3). Let F_j be the solid hull of $\overline{W_j}$ with respect to V_j . Then

$$C_j \subseteq W_j \subseteq F_j \subseteq V_j, \quad F_j \in \mathcal{C}_s,$$

for $j = 1, \dots, n$.

At this point we distinguish between two subcases:

(i) $U = X \setminus \bigcup_{j=1}^n F_j$ is connected. Then $C' = \overline{U}$ is connected and contained in $X \setminus \bigcup_{j=1}^n W_j$, so $C' \cap C_j = \emptyset$ for $j = 1, \dots, n$. On the other hand,

$X \setminus C' \subseteq \bigcup_{j=1}^n V_j$. If $X \setminus C' = \bigcup_{i \in I} O_i$ is the decomposition of $X \setminus C'$ into its disjoint connected components, then each O_i belongs to \mathcal{O}_s and each O_i is contained in some V_j . But then, by Proposition 2.3.3, it follows that $\sum_{i \in I} \mu(O_i) \leq \sum_{j=1}^n \mu(V_j)$. Consequently, we get

$$\mu(C') = 1 - \sum_{i \in I} \mu(O_i) \geq 1 - \sum_{j=1}^n \mu(V_j) > 1 - \sum_{j=1}^n (C_j) - \varepsilon,$$

which yields the assertion in this case.

(ii) $X \setminus \bigcup_{j=1}^n F_j$ is not connected. Let m be the *minimal* number of sets in any subcollection $\mathcal{F}' \subseteq \mathcal{F} = \{F_1, \dots, F_n\}$ such that $\bigcup\{F_j : F_j \in \mathcal{F}'\}$ is *not* co-connected. Let \mathcal{F}_1 be such a subfamily of \mathcal{F} with m elements, $2 \leq m \leq n$. Let $U = (\bigcup\{F_j : F_j \in \mathcal{F}_1\})^c$. By Lemma 2.1 we have $U = \bigcup_{i \in I} U_i$, U_i disjoint, $U_i \in \mathcal{O}_s$. Again we distinguish between two cases:

a) $m = n$. Let N be such that $\sum_{i > N} \mu(U_i) < \varepsilon/2$. Then for $i \leq N$ choose $K_i \subseteq U_i$ such that $K_i \in \mathcal{C}_s$ and

$$\mu(K_i) > \mu(U_i) - \frac{\varepsilon}{2N}.$$

The family $\{F_j, U_i : j = 1, \dots, n, i \in I\}$ is an irreducible partition of X so that by (C),

$$\sum_{j=1}^n \mu(F_j) + \sum_{i \in I} \mu(U_i) = 1,$$

and hence

$$\begin{aligned} \sum_{j=1}^n \mu(C_j) + \sum_{i=1}^N \mu(K_i) &> \sum_{j=1}^n \mu(F_j) - \varepsilon + \sum_{i=1}^N \mu(U_i) - \varepsilon \\ &> \sum_{j=1}^n F_j + \sum_{i \in I} \mu(U_i) - \frac{5\varepsilon}{2} = 1 - \frac{5\varepsilon}{2}, \end{aligned}$$

which proves the assertion when we take $C' = \bigcup_{i=1}^N K_i$.

b) $m < n$. The family $\{F_j, U_i : j = 1, \dots, m, i \in I\}$ is an irreducible partition of X . If $F_j \notin \mathcal{F}_1$ then $F_j \subseteq U_i$ for some $i \in I$. Since $m \geq 2$ we have $n - m < n - 1$, so that for any $i \in I$ the collection

$$\mathcal{F}_i = \{F_k \in \mathcal{F} : F_k \subseteq U_i\} \cup \{X \setminus U_i\}$$

will have at most $n - 1$ elements. By the induction hypothesis it therefore follows that there is a $K_i \in \mathcal{C}_0$ such that $F_k \cap K_i = \emptyset$ for all $F_k \in \mathcal{F}_i$, $K_i \subseteq U_i$ and

$$\sum_{F_k \in \mathcal{F}_i} \mu(F_k) + \mu(K_i) + \mu(X \setminus U_i) > 1 - \frac{\varepsilon}{N},$$

so

$$\sum_{F_k \in \mathcal{F}_i} \mu(F_k) + \mu(K_i) > \mu(U_i) - \frac{\varepsilon}{N}, \quad i = 1, \dots, N.$$

Then we get

$$\begin{aligned} \sum_{j=1}^n \mu(C_j) + \sum_{i=1}^N \mu(K_i) &> \sum_{j=1}^n \mu(F_j) - \varepsilon + \sum_{i=1}^N \mu(K_i) \\ &\geq \sum_{j=1}^m \mu(F_j) + \sum_{i=1}^N \sum_{F_k \in \mathcal{F}_i} \mu(F_k) + \sum_{i=1}^N \mu(K_i) - \varepsilon \\ &> \sum_{j=1}^m \mu(F_j) + \sum_{i=1}^N \mu(U_i) - 2\varepsilon \\ &> \sum_{j=1}^m \mu(F_j) + \sum_{i \in I} \mu(U_i) - \frac{5\varepsilon}{2} = 1 - \frac{5\varepsilon}{2}, \end{aligned}$$

which proves the assertion in case b).

Finally, if $C_1, \dots, C_n \subseteq V \in \mathcal{O}_s$, let $\tilde{C}_0 = X \setminus V$. Applying the argument above to the family C_0, C_1, \dots, C_n now yields Lemma C when all the sets C_1, \dots, C_n are solid.

Next, we turn to the general case, assuming that all the sets C_j ($j = 1, \dots, n$) belong to \mathcal{C}_c . Again we use induction on n .

Step 1: $n = 1$. Let $C \subseteq V \in \mathcal{O}_s$ and $C \in \mathcal{C}_c$. Let $X \setminus C = \bigcup_{i \in I} W_i$ with $W_i \in \mathcal{O}_s$. Let W_0 be the exterior component with respect to V , and put $\tilde{C} = X \setminus W_0$ as usual. We have $\tilde{C} \subseteq V$ and by the first part of the proof, there is a $K_0 \in \mathcal{C}_0$ such that $\tilde{C} \cap K_0 = \emptyset$, $K_0 \subseteq V$ and $\mu(\tilde{C}) + \mu(K_0) > \mu(V) - \varepsilon/2$. Choose N such that $\sum_{i > N} \mu(W_i) < \varepsilon/4$ (restricting to a countable subfamily again). Next, choose $K_i \subseteq W_i$ such that $K_i \in \mathcal{C}_s$ and $\mu(K_i) > \mu(W_i) - \varepsilon/(4 \cdot 2^i)$ for $i = 1, \dots, N$. All the K_i ($i = 0, 1, \dots, N$) are now disjoint from C , mutually disjoint ($K_0 \subseteq W_0 \cap V$) and contained in V . Take $K = \bigcup_{i=0}^N K_i$, so $K \in \mathcal{C}_0$, $K \cap C = \emptyset$ and $K \subseteq V$. Moreover,

$$\begin{aligned} \mu(K) &= \sum_{i=0}^N \mu(K_i) > \mu(V) - \mu(\tilde{C}) - \frac{\varepsilon}{2} + \sum_{i=1}^N \mu(W_i) - \frac{\varepsilon}{4} \\ &> \mu(V) - (1 - \mu(W_0)) + \sum_{i \in I} \mu(W_i) - \varepsilon \\ &= \mu(V) - \mu(C) - \varepsilon, \end{aligned}$$

which establishes the assertion for $n = 1$.

Step 2: Assume inductively that the assertion is true for $k=1, \dots, n-1$.

Case 1: $\tilde{C}_1, \dots, \tilde{C}_n$ are mutually disjoint. By the first part of the proof there is $K_0 \in \mathcal{C}_0$ such that $K_0 \subseteq V, K_0 \cap \tilde{C}_j = \emptyset$ ($j = 1, \dots, n$) and $\mu(K_0) > \mu(V) - \sum_{j=1}^n \mu(\tilde{C}_j) - \varepsilon/2$.

Let $X \setminus C_j = W^j = \bigcup_{i \in I_j} W_i^j$, where W_0^j is the exterior component of W^j with respect to V (i.e. $\tilde{C}_j = X \setminus W_0^j$). Let $N_j \in \mathbb{Z}^+$ be such that $\sum_{i > N_j} \mu(W_i^j) < \varepsilon/(4n)$. For $i = 1, \dots, N_j$ pick $K_i^j \subseteq W_i^j$ such that $K_i^j \in \mathcal{C}_s$ and

$$\mu(K_i^j) > \mu(W_i^j) - \frac{\varepsilon}{4n} \cdot \frac{1}{2^i}, \quad j = 1, \dots, n.$$

Since the \tilde{C}_j are mutually disjoint, all the sets K_i^j are mutually disjoint and also disjoint from $K_0 \subseteq \bigcap_{j=1}^n W_0^j$. Let $K = K_0 \cup \bigcup \{K_i^j : 1 \leq i \leq N_j, j = 1, \dots, n\}$ so $K \in \mathcal{C}_0, K \cap \tilde{C}_j = \emptyset$ ($j = 1, \dots, n$) and $K \subseteq V$. We now obtain

$$\begin{aligned} \mu(K) &= \mu(K_0) + \sum_{j=1}^n \sum_{i=1}^{N_j} \mu(K_i^j) \\ &> \mu(V) - \sum_{j=1}^n \mu(\tilde{C}_j) - \frac{\varepsilon}{2} + \sum_{j=1}^n \sum_{i=1}^{N_j} \left[\mu(W_i^j) - \frac{\varepsilon}{4n} \cdot \frac{1}{2^i} \right] \\ &> \mu(V) - \sum_{j=1}^n \left[\mu(\tilde{C}_j) - \sum_{i \in I_j'} \mu(W_i^j) \right] - \varepsilon \\ &= \mu(V) - \sum_{j=1}^n \mu(C_j) - \varepsilon, \end{aligned}$$

which proves the assertion in Case 1.

Case 2: $\tilde{C}_j \cap \tilde{C}_k \neq \emptyset$ for some pair $j \neq k$. We proceed as in the proof of Lemma A, and re-index the sets C_1, \dots, C_n so that

1. $\tilde{C}_{10}, \tilde{C}_{20}, \dots, \tilde{C}_{m0}$ ($m < n$) are mutually disjoint.
2. $\tilde{C}_{jl} \subseteq \tilde{C}_{j0}$ for $l = 0, \dots, m_j$ and $1 \leq j \leq m$, where the inclusions are proper if $l \geq 1$.
3. $m + \sum_{j=1}^m m_j = n$.

As before, let $X \setminus C_{jl} = \bigcup \{W_i^{jl} : i \in I_{jl}\} = W^{jl}$, and let W_0^{jl} denote the exterior component of W^{jl} with respect to V . Again, by the first part of the proof, there is $K_0 \in \mathcal{C}_0$ such that $K_0 \subseteq V, K_0 \cap \tilde{C}_{j0} = \emptyset$ ($j = 1, \dots, m$), and

$$(4.19) \quad \mu(K_0) > \mu(V) - \sum_{j=1}^m \mu(\tilde{C}_{j0}) - \frac{n-m}{n} \varepsilon.$$

For each $j \in \{1, \dots, m\}$ choose $N_j \in \mathbb{Z}^+$ such that $\sum_{i > N_j} \mu(W_i^{j0}) < \varepsilon/(2n)$.

When $l \geq 1$ each set C_{jl} is contained in some interior component W_i^{j0} of W^{j0} , i.e. $i \neq 0$. We let

$$J_{ji} = \{l : 1 \leq l \leq m_j, C_{jl} \subseteq W_i^{j0}\}, \quad 1 \leq i \leq N_j.$$

If $J_{ji} = \emptyset$ there is by (B) a set $K_i^j \subseteq W_i^{j0}$ such that $K_i^j \in \mathcal{C}_s$ and

$$(4.20) \quad \mu(K_i^j) > \mu(W_i^{j0}) - \frac{\varepsilon}{2N_j n}.$$

If $J_{ji} \neq \emptyset$ there is, by the induction hypothesis, since $m_j < n$, a set $K_i^j \subseteq W_i^{j0}$ such that $K_i^j \in \mathcal{C}_0$ and

$$(4.21) \quad \mu(K_i^j) > \mu(W_i^{j0}) - \sum_{l \in J_{ji}} \mu(C_{jl}) - \frac{\varepsilon}{2N_j n}.$$

Let $K_j = \bigcup_{i=1}^{N_j} K_i^j \in \mathcal{C}_0$ for $j = 1, \dots, m$. Then, adding the inequalities (4.20) and (4.21), we get

$$\begin{aligned} \mu(K_j) &= \sum_{i=1}^{N_j} \mu(K_i^j) \\ &> \sum_{i=1}^{N_j} \mu(W_i^{j0}) - \sum_{i=1}^{N_j} \sum_{l \in J_{ji}} \mu(C_{jl}) - \frac{\varepsilon}{2n} \\ &> \sum_{i=1}^{\infty} \mu(W_i^{j0}) - \sum_{l=1}^{m_j} \mu(C_{jl}) - \frac{\varepsilon}{n} \\ &= \mu(\tilde{C}_{j0}) - \mu(C_{j0}) - \sum_{l=1}^{m_j} \mu(C_{jl}) - \frac{\varepsilon}{n}, \end{aligned}$$

so

$$(4.22) \quad \mu(K_j) > \mu(\tilde{C}_{j0}) - \sum_{l=0}^{m_j} \mu(C_{jl}) - \frac{\varepsilon}{n}, \quad j = 1, \dots, m.$$

Now let $K = \bigcup_{j=0}^m K_j \in \mathcal{C}_0$. By addition of (4.19) and the equations (4.22) we get

$$\begin{aligned} \mu(K) &= \sum_{j=0}^m \mu(K_j) \\ &> \mu(V) - \sum_{j=1}^m \mu(C_{j0}) - (n-m) \frac{\varepsilon}{n} \\ &\quad + \sum_{j=1}^m \mu(\tilde{C}_{j0}) - \sum_{j=1}^m \sum_{l=0}^{m_j} \mu(C_{jl}) - \frac{m\varepsilon}{n} \\ &= \mu(V) - \sum_{j=1}^m \sum_{l=0}^{m_j} \mu(C_{jl}) - \varepsilon, \end{aligned}$$

which concludes the proof of the lemma since the double sum above comprises all the sets C_1, \dots, C_n given initially. ■

Lemma C was what was needed to establish (4.18) and the proof of Proposition 4.1 is therefore also complete. ■

5. Extension to a NSA-measure. Let X be a compact Hausdorff space which is *connected* and *locally connected*. Suppose $\mu : \mathcal{C}_0 \rightarrow [0, 1]$ satisfies (Q1)₀, (Q2)₀ and (Q3)₀ of the preceding section. If $V \in \mathcal{O}_0$ we may define

$$(5.1) \quad \mu(V) = 1 - \mu(X \setminus V),$$

thereby obtaining a function $\mu : \mathcal{A}_0 \rightarrow [0, 1]$ which satisfies the conditions (Q0), (Q1), (Q2) and (Q3) for a NSA-measure, as given in the introduction. In particular, we may note that (Q3) is an immediate consequence of (Q3)₀ and (5.1), i.e. we have, for $V \in \mathcal{O}_0$,

$$(5.2) \quad \mu(V) = \sup\{\mu(K) : K \subseteq V, K \in \mathcal{C}_0\}.$$

We now use (5.2) to extend μ to all of \mathcal{O} , and then define

$$(5.3) \quad \mu(C) = 1 - \mu(X \setminus C)$$

for an arbitrary set $C \in \mathcal{C}$.

PROPOSITION 5.1. *The function $\mu : \mathcal{A} \rightarrow [0, 1]$, as defined by (5.2) for $V \in \mathcal{O}$ and by (5.3) for $C \in \mathcal{C}$, is a NSA-measure in X .*

Proof. Our task is to verify that (Q0)–(Q3) hold on \mathcal{A} . (Q0) is automatic by (5.3). From (5.2) it follows that $\mu(V_1) \leq \mu(V_2)$ if $V_1 \subseteq V_2$ and $V_1, V_2 \in \mathcal{O}$. (Q1) now follows from (5.3). To prove (Q3) we first observe that if $K \subseteq U$, $K \in \mathcal{C}$, $U \in \mathcal{O}$, then there is a set $C \in \mathcal{C}_0$ such that $K \subseteq C \subseteq U$. This follows as in the remark subsequent to the definition of a solid set-function. (Q3) now follows from (5.2) and (Q1).

To prove (Q2) we first show that μ is additive on \mathcal{O} . Let U_1 and U_2 be disjoint open sets, and let $\varepsilon > 0$ be arbitrary. Choose $K \in \mathcal{C}_0$ such that $K \subseteq U_1 \cup U_2$ and $\mu(K) > \mu(U_1 \cup U_2) - \varepsilon$.

Next, choose $K_1, K_2 \in \mathcal{C}_0$ such that $K_i \subseteq U_i$, $\mu(K_i) > \mu(U_i) - \varepsilon$ ($i = 1, 2$). Then we get, since $K_1 \cap K_2 = \emptyset$,

$$\begin{aligned} \mu(U_1) + \mu(U_2) &< \mu(K_1) + \mu(K_2) + 2\varepsilon = \mu(K_1 \cup K_2) + 2\varepsilon \\ &\leq \mu(U_1 \cup U_2) + 2\varepsilon < \mu(K) + 3\varepsilon \\ &= \mu(K'_1 \cup K'_2) + 3\varepsilon = \mu(K'_1) + \mu(K'_2) + 3\varepsilon \\ &\leq \mu(U_1) + \mu(U_2) + 3\varepsilon, \end{aligned}$$

where $K'_i = K \cap U_i \in \mathcal{C}_0$ ($i = 1, 2$). Since $\varepsilon > 0$ was arbitrary it follows that

if $U_1 \cap U_2 = \emptyset$ and $U_1, U_2 \in \mathcal{O}$, then

$$(5.4) \quad \mu(U_1 \cup U_2) = \mu(U_1) + \mu(U_2).$$

We are now ready to show that μ is additive on \mathcal{C} . Let $K_1 \cap K_2 = \emptyset$ and $K_1, K_2 \in \mathcal{C}$. Combining (5.2) and (5.3) we find, for $\varepsilon > 0$ arbitrary, sets U, U_1, U_2 in \mathcal{O} such that

$$\begin{aligned} K_1 \cup K_2 &\subseteq U, & \mu(U) &< \mu(K_1 \cup K_2) + \varepsilon, \\ K_i &\subseteq U_i, & \mu(U_i) &< \mu(K_i) + \varepsilon, \quad i = 1, 2. \end{aligned}$$

Since μ is monotone on \mathcal{O} we may assume that $U_1 \cap U_2 = \emptyset$. Then we get

$$\begin{aligned} \mu(K_1) + \mu(K_2) &> \mu(U_1) + \mu(U_2) - 2\varepsilon = \mu(U_1 \cup U_2) - 2\varepsilon \\ &\geq \mu(K_1 \cup K_2) - 2\varepsilon && \text{(by (Q3))} \\ &> \mu(U) - 3\varepsilon \\ &\geq \mu((U \cap U_1) \cup (U \cap U_2)) - 3\varepsilon \\ &= \mu(U \cap U_1) + \mu(U \cap U_2) - 3\varepsilon \\ &\geq \mu(K_1) + \mu(K_2) - 3\varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ was arbitrary it follows that if $K_1, K_2 \in \mathcal{C}$ and $K_1 \cap K_2 = \emptyset$, then

$$(5.5) \quad \mu(K_1) + \mu(K_2) = \mu(K_1 \cup K_2),$$

which is (Q2).

Remark. In the author's paper [1] another version of the extension from \mathcal{C}_0 to \mathcal{C} is given, which avoids the connectedness assumptions on X ([1], Proposition 5.1). However, a different assumption has to be made, and the present treatment is closer in spirit to the rest of this paper.

Combining the last result with the results of Section 4 we have our main result:

THEOREM 5.1. *Let X be a compact Hausdorff space which is connected and locally connected.*

(i) *Any solid set-function $\mu : \mathcal{A}_s \rightarrow [0, 1]$ has a unique extension to a NSA-measure in X .*

(ii) *The arising NSA-measure is extremal if and only if $\mu : \mathcal{A}_s \rightarrow \{0, 1\}$.*

(iii) *If μ is subadditive on \mathcal{C}_c , i.e. if $\mu(C_1 \cup C_2) \leq \mu(C_1) + \mu(C_2)$ for all $C_1, C_2 \in \mathcal{C}_c$, then μ extends to a uniquely determined regular Borel measure in X .*

Part (ii) is obvious and the proof of the third part is left to the reader. One has to show that μ is subadditive on \mathcal{C} and then apply a standard result of measure theory (cf. the remark in the introduction).

As a first application of the theorem above we show that the construction of extremal NSA-measures as established in [5] now follows as a special case. We assume that $g = 0$.

A subfamily \mathcal{S} of \mathcal{C}_s is called a *co-basis* if the following four conditions hold ([5], Definition 3.6):

- (S1) $C_1 \subseteq C_2 \in \mathcal{S} \Rightarrow C_1 \in \mathcal{S}$ if $C_1 \in \mathcal{C}_s$.
- (S2) $C_1, C_2 \in \mathcal{C}_s \setminus \mathcal{S} \Rightarrow C_1 \cap C_2 \neq \emptyset$.
- (S3) $C_1, C_2 \in \mathcal{S} \Rightarrow C_1 \cup C_2 \neq X$.
- (S4) For all $C_1 \in \mathcal{S}$ there is a $C_2 \in \mathcal{S}$ such that $C_1 \subseteq C_2^0$.

Suppose \mathcal{S} is a co-basis and define, for $C \in \mathcal{C}_s$,

$$(5.6) \quad \mu(C) = \begin{cases} 0 & \text{if } C \in \mathcal{S}, \\ 1 & \text{if } C \notin \mathcal{S}. \end{cases}$$

Let us verify that μ is a solid set-function. By (S1) it is clear that $\emptyset \in \mathcal{S}$, and by (S3) it follows that $X \notin \mathcal{S}$. Hence $\mu(X) = 1$. Now suppose $C_1, C_2 \in \mathcal{C}_s$, $C_1 \cap C_2 = \emptyset$. By (S2) at most one of these sets is not in \mathcal{S} . Combining this fact with (S1) we get (A). To get (B) it suffices to show that if $C_1 \in \mathcal{S}$ then there is a set $C'_1 \in \mathcal{C}_s \setminus \mathcal{S}$ such that $C_1 \cap C'_1 = \emptyset$. By (S4) we get $C_2 \in \mathcal{S}$ such that $C_1 \subseteq C_2^0$. By Lemma 3.3 there is a set $W \in \mathcal{O}_s$ such that $C_1 \subseteq W \subseteq C_2^0$. Then $C'_1 = X \setminus W \in \mathcal{C}_s$ and $C'_1 \cup C_2 = X$ so by (S3) we must have $C'_1 \notin \mathcal{S}$. So μ is a solid set-function, and we get the following (Corollary 3.7 of [5]):

PROPOSITION 5.2. *μ as defined by (5.6) has a unique extension to an extremal NSA-measure in X .*

A more general example may be given as follows: Let $P = \{p_1, \dots, p_q\}$ be a set of q distinct points in X , where

$$q = 2^{n+1} + 1, \quad n \in \mathbb{Z}^+.$$

For any set $C \in \mathcal{C}_s$ let $\#C$ denote the number of elements in $P \cap C$. Define, for $k = 0, 1, \dots, 2^n$,

$$\mu_n(C) = k/2^n \quad \text{if } \#C = 2k \vee 2k + 1.$$

We leave it to the reader to verify that μ_n is a solid set-function. It is easily seen that μ_n is not subadditive, so that the resulting NSA-measure is not (the restriction of) a regular Borel measure. However, one may show (using the same technique as in Example 2.1 of [2]) that each μ_n is an extreme point in the convex set Q of all normalized NSA-measures in X .

We close this section with a “continuous example”, i.e. a NSA-measure μ such that $\text{Sp } \mu$ is all of $[0,1]$. Let $X = S^2 =$ the unit sphere in \mathbb{R}^3 , and let λ denote normalized Lebesgue measure in X , i.e. $\lambda(X) = 1$. Let $E = \{(x, y, z) \in S^2 : z = 0, \text{ i.e. } x^2 + y^2 = 1\} =$ the equator on S^2 .

Define μ on \mathcal{C}_s as follows:

- (i) $\mu(C) = 1$ if $E \subseteq C$.
- (ii) $\mu(C) = \lambda(C)$ if $E \cap C \neq \emptyset$ and $E \cap C^c \neq \emptyset$.
- (iii) $\mu(C) = 0$ if $E \subseteq C^c$.

Let us verify (A) and (B).

(A) Let C_1, \dots, C_n be a disjoint family in \mathcal{C}_s , all contained in $C \in \mathcal{C}_s$. We may assume that all the C_i fall in the category (ii), otherwise there is nothing to prove. Then $\sum_{i=1}^n \mu(C_i) = \sum_{i=1}^n \lambda(C_i) \leq \lambda(C)$. Since $C \cap E \neq \emptyset$ the assertion follows.

(B) Let $C \in \mathcal{C}_s$ and $\varepsilon > 0$ be given. If $E \subseteq C$ there is nothing to prove, and if $E \subseteq C^c$ we just take $C' = E$. So assume that $E \cap C \neq \emptyset$ and $E \cap C^c \neq \emptyset$. Since $C^c \in \mathcal{O}_s$, by Lemma 3.3 it follows that there is a set $C' \in \mathcal{C}_s$ such that $\lambda(C') + \lambda(C) > 1 - \varepsilon$, and $C' \subseteq C^c$, $C' \cap E \neq \emptyset$. This establishes (B).

The NSA-measure constructed in this way is not a measure since it lacks subadditivity.

6. Discretization of measures. In this section we provide an affirmative solution to the following problem: *Given a regular Borel measure λ with $\text{Sp } \lambda = [0, 1]$, is it possible to find NSA-measures μ with finite spectra which approximate λ arbitrarily well?*

We shall assume that X is a compact Hausdorff space which is *connected*, *locally connected* and have $g = 0$. Since there is no need to assume that the initial measure λ is really a measure, we just assume that λ is a *normalized NSA-measure*.

DEFINITION. Two disjoint sets $C_1, C_2 \in \mathcal{C}_s$ with non-zero λ -measure are called a *splitting pair* for λ if $\lambda(C_1) + \lambda(C_2) = 1$. λ is *non-splitting* if there is no splitting pair for λ .

Now suppose λ is a non-splitting NSA-measure satisfying $\lambda(X) = 1$. Let n be an arbitrary non-negative integer and put $q = 2^n + 1$. We are going to construct a NSA-measure μ_n such that $\text{Sp } \mu_n = \{k/2^n : k = 0, 1, \dots, 2^n\}$. Define intervals I_k as follows:

$$I_k = [k/q, (k + 1)/q), \quad k = 0, 1, \dots, 2^n - 1,$$

$$I_{2^n} = [2^n/q, 1]$$

We define, for $C \in \mathcal{C}_s$,

$$(D) \quad \mu_n(C) = k/2^n \quad \text{if } \lambda(C) \in I_k, \quad \mu_n(X \setminus C) = 1 - \mu_n(C),$$

Note that $\mu_n(C) \in I_k$ for all k ,

$$\begin{aligned} \mu_n(C) &= 0 && \text{if } \lambda(C) < 1/q \text{ and} \\ \mu_n(C) &= 1 && \text{if } \lambda(C) \geq 2^n/q. \end{aligned}$$

PROPOSITION 6.1. μ_n is a solid set-function in X .

Proof. Let $C_1, \dots, C_m \subseteq C \in \mathcal{C}_s$, $C_i \in \mathcal{C}_s$ for $i = 1, \dots, m$ and $C_i \cap C_j = \emptyset$ if $i \neq j$.

Case 1: $\lambda(C) \in I_k$ for some $k < 2^n$. Then $\sum_{i=1}^m \lambda(C_i) \leq \lambda(C) < (k + 1)/q$. For $j = 0, 1, \dots, 2^n$ let

$$J_j = \{i : 1 \leq i \leq m, \lambda(C_i) \in I_j\}$$

and let n_j denote the number of elements in J_j . Since $C_i \subseteq C$ for all $i = 1, \dots, m$ we have $n_j = 0$ if $j > k$. Clearly $\sum_{j=0}^k n_j = m$. Now, if $\lambda(C_i) \in I_j$ then $\lambda(C_i) \geq j/q$, so we must have

$$\frac{k + 1}{q} > \sum_{j=1}^m \lambda(C_j) \geq \sum_{j=0}^k n_j \frac{j}{q}$$

and so

$$(6.1) \quad \sum_{j=1}^k j n_j \leq k.$$

Now $\mu_n(C_i) = j/2^n$ if $i \in J_j$ so we get

$$\begin{aligned} \sum_{i=1}^m \mu_n(C_i) &= \sum_{j=1}^k \sum_{i \in J_j} \mu_n(C_i) = \sum_{j=1}^k n_j \frac{j}{2^n} \\ &= \frac{1}{2^n} \sum_{j=1}^n n_j j \leq \frac{k}{2^n} = \mu(C). \end{aligned}$$

Case 2: $\lambda(C) \in I_{2^n}$. With notation as above we now get

$$(6.2) \quad \sum_{j=1}^{2^n} j n_j \leq 2^n + 1.$$

Suppose equality holds in (6.2). Then

$$1 \geq \sum_{i=1}^m \lambda(C_i) = \sum_{j=1}^{2^n} \sum_{i \in J_j} \lambda(C_i) \geq \sum_{j=1}^{2^n} \frac{j}{2^n + 1} n_j = 1.$$

Hence we must have $\lambda(C_i) = j/q$ if $i \in J_j$. This violates the non-splitting of λ . We must therefore have

$$(6.3) \quad \sum_{j=1}^{2^n} jn_j \leq 2^n.$$

and the assertion $\sum_{i=1}^m \mu_n(C_i) \leq \mu_n(C)$ now follows as in Case 1. We have established (A).

Now let $C \in \mathcal{C}_s$ be arbitrary, and assume that $\lambda(C) \in I_k$ for some $k < 2^n$. (If $k = 2^n$ then $\mu_n(C) = 1$ and there is nothing to prove). Let $U = X \setminus C$ so $\lambda(U) \in ((q - (k + 1))/q, (q - k)/q]$. By the regularity of λ and Lemma 3.1 there is a set $C' \subseteq U$ such that $C' \in \mathcal{C}_s$ and $\lambda(C') \in$ same interval. Then $\mu_n(C') = (q - (k + 1))/(q - 1)$, since we cannot have $\lambda(C') = (q - k)/q$ because of the non-splitting of λ . But then

$$\mu_n(C) + \mu_n(C') = \frac{k}{q - 1} + \frac{q - (k + 1)}{q - 1} = 1,$$

which shows that (B) is true. The proof is complete. ■

PROPOSITION 6.2. *Let λ be a normalized non-splitting NSA-measure (or regular Borel measure) in X . For $n = 1, 2, \dots$ let μ_n be defined by (D). Then the resulting sequence $\{\mu_n\}$ of NSA-measures converges to λ in the topology of Q .*

PROOF. By definition it follows that $|\lambda(C) - \mu_n(C)| < 1/2^n$ for all $C \in \mathcal{C}_s$. Let $\{\mu_{n_i}\}$ be an arbitrary subsequence of $\{\mu_n\}$. Since Q is compact, $\{\mu_{n_i}\}$ must have a subnet $\{\mu_{n_{i_j}}\}$ which converges to some element $\lambda' \in Q$ (we identify Q with the set of non-linear states). For $C \in \mathcal{C}_s$ we then have

$$\lambda'(C) = \lim_j \mu_{n_{i_j}}(C) = \lambda(C)$$

so λ and λ' agree on \mathcal{C}_s . But then, by uniqueness, we must have $\lambda = \lambda'$. We have therefore shown that any subsequence of $\{\mu_n\}$ has a subnet which converges to λ . But then μ_n converges to λ . The proof is complete. ■

We do not know whether $\mu_n(K) \rightarrow \lambda(K)$ for all closed sets K in X . What follows is what we can show, and what will suffice in most cases.

PROPOSITION 6.3. *If $C \in \mathcal{C}_0$ then $\mu_n(C) \rightarrow \lambda(C)$.*

PROOF. By definition we have $\mu_n(C) = k/2^n$ if $\lambda(C) \in I_k$ ($k = 0, 1, \dots, \dots, 2^n$) if $C \in \mathcal{C}_s$. Let $I'_k = (k/q, (k + 1)/q]$ for $k = 1, \dots, 2^n$ and put $I'_0 = [0, 1/q]$. It is then easily verified that $\mu_n(V) = k/2^n$ if $\lambda(V) \in I'_k$ ($k = 0, 1, \dots, 2^n$) and $V \in \mathcal{O}_s$. To prove the assertion it suffices to consider $C \in \mathcal{C}_c$. Let such a $C \in \mathcal{C}_c$ be given. We have $X \setminus C = V = \bigcup_{i \in I} V_i$, where the V_i belong to \mathcal{O}_s and are disjoint.

We now claim that for arbitrary $n \in \mathbb{Z}^+$ we have (with $q_n = 2^n + 1$)

$$(6.4) \quad \sum_{i \in I} \mu_n(V_i) < \sum_{i \in I} \lambda(V_i) + \frac{1}{q_n}.$$

Indeed, let $J_j = \{i \in I : \lambda(V_i) \in I'_j\}$ for $j = 1, \dots, 2^n$. For each $j \geq 1$ the number of elements in J_j is finite and is denoted by n_j . If $\lambda(V) \in I'_k$ we have $n_j = 0$ for $j > k$. Since $\lambda(V_i) > j/q$ if $i \in J_j$ we now get

$$\sum_{j=1}^k n_j \frac{j}{q} < \sum_{j=1}^k \sum_{i \in J_j} \lambda(V_i) \leq \sum_{i \in I} \lambda(V_i) = \lambda(V) \leq \frac{k+1}{q}$$

so that $\sum n_j j \leq k$.

If $\lambda(V_i) \in I'_j$ then $\mu_n(V_i) = j/2^n$ so we get

$$\sum_{i \in I} \mu_n(V_i) = \sum_{j=1}^k \sum_{i \in J_j} \mu_n(V_i) = \sum_{j=1}^k n_j \frac{j}{2^n} \leq \frac{k}{2^n} < \frac{k+1}{q}.$$

Since $\sum_{i \in I} \lambda(V_i) \in I'_k$ the claim (6.4) follows.

Now let $\varepsilon > 0$ be arbitrary. Choose N such that $\sum_{i > N} \lambda(V_i) < \varepsilon$. Then choose N' such that $1/(2N') < \varepsilon/N$. For $n \geq N'$ we then obtain

$$\sum_{i=1}^N |\lambda(V_i) - \mu_n(V_i)| < \frac{N}{2^n} < \varepsilon.$$

Fix an arbitrary $n \geq N'$. By (6.4) (applied to the family $\{V_i\}_{i > N}$) we get

$$\sum_{i > N} \mu_n(V_i) < \sum_{i > N} \lambda(V_i) + \frac{1}{q_n} < \varepsilon + \frac{1}{q_n} < 2\varepsilon.$$

Consequently, for any $n \geq N'$,

$$\begin{aligned} |\lambda(C) - \mu_n(C)| &\leq \sum_{i \in I} |\lambda(V_i) - \mu_n(V_i)| \\ &\leq \sum_{i=1}^N |\lambda(V_i) - \mu_n(V_i)| + \sum_{i > N} \lambda(V_i) + \sum_{i > N} \mu_n(V_i) \\ &< \varepsilon + \varepsilon + 2\varepsilon = 4\varepsilon. \end{aligned}$$

Hence $\mu_n(C) \rightarrow \lambda(C)$ if $C \in \mathcal{C}_c$. The proof is complete. ■

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Received 28 January 1994;
in revised form 27 January 1995