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Large families of dense pseudocompact subgroups of compact groups

by

Gerald Itzkowitz (New York)
and **Dmitri Shakhmatov** (Matsuyama)

Abstract. We prove that every nonmetrizable compact connected Abelian group G has a family $\mathcal{H}$ of size $|G|$, the maximal size possible, consisting of proper dense pseudocompact subgroups of G such that $H \cap H' = \{0\}$ for distinct $H, H' \in \mathcal{H}$. An easy example shows that connectedness of G is essential in the above result. In the general case we establish that every nonmetrizable compact Abelian group G has a family $\mathcal{H}$ of size $|G|$ consisting of proper dense pseudocompact subgroups of G such that each intersection $H \cap H'$ of different members of $\mathcal{H}$ is nowhere dense in G. Some results in the non-Abelian case are also given.

1. Results. All topological groups considered in this paper are assumed to be T_0 (and therefore, completely regular). The cardinality of a set X is denoted by $|X|$, and $w(X)$ denotes the weight of a topological space X. A (completely regular) space X is *pseudocompact* if every real-valued continuous function defined on it is bounded [14].

Every nonmetrizable compact Abelian group contains a proper dense pseudocompact subgroup [1]. This result brings into consideration the following vague question:

1.1. Question. Given a nonmetrizable compact group G, is it possible to find a "large" family $\mathcal{H} = \{H_\alpha : \alpha < \tau\}$ consisting of "distinct" proper dense pseudocompact subgroups of G?

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[1] This result in some particular cases has appeared in [16, 5.5 and 5.6] and [11, Theorems 4.3 and 5.3], but in full generality it was first proved in 1983 by Comfort in the almost inaccessible article [3]. A different proof of the result was later published in [9, Theorem 4.10(c)].

Both words enclosed in quotes in Question 1.1 require additional discussion. We start with the second word. The obvious way to interpret "distinct" is simply to require $H_\alpha \neq H_\beta$ for different $\alpha, \beta < \tau$. A slightly stronger condition was considered by Comfort [4] who added the requirement that the intersection $H_\alpha \cap H_\beta$ *not* be dense in G for different $\alpha, \beta < \tau$ (thus trivially $H_\alpha \neq H_\beta$ for $\alpha \neq \beta$). One can get an even stronger measure of distinctiveness by requiring each intersection $H_\alpha \cap H_\beta$ with $\alpha \neq \beta$ to be nowhere dense in G. Finally, a much stronger interpretation of "distinct" would be to insist that $H_\alpha \cap H_\beta = \{e\}$ for $\alpha \neq \beta$, where e is the identity element of G. The next definition pushes the last two conditions even further.

1.2. Definition. Let $\mathcal{H} = \{H_\alpha : \alpha < \tau\}$ be a family of subgroups of a topological group G, and let $\langle X\rangle$ denote the smallest subgroup of G that contains $X \subseteq G$. We say that $\mathcal{H}$ is

(i) *weakly almost disjoint* if the intersection $H_\alpha \cap \langle \bigcup\{H_\beta : \beta \neq \alpha\}\rangle$ is nowhere dense in G for each $\alpha < \tau$, and

(ii) *almost disjoint* if $H_\alpha \cap \langle \bigcup\{H_\beta : \beta \neq \alpha\}\rangle = \{e\}$ for every $\alpha < \tau$.

If G is not discrete (for example, compact and infinite), then every almost disjoint family of subgroups of G is weakly almost disjoint, which justifies the terminology.

To help understand the word "large", the following lemma gives a constraint on the size of the family $\mathcal{H}$ in Question 1.1:

1.3. Lemma. *Let G be an infinite compact group and let $\mathcal{H}=\{H_\alpha : \alpha < \tau\}$ be a family of dense subsets of G such that the intersection $H_\alpha \cap H_\beta$ is not dense in G for $\alpha \neq \beta$. Then $|\mathcal{H}| = \tau \leq |G|$.*

Proof. Indeed, let $w(G) = \kappa$, and let $\mathcal{B}$ be a base for G with $|\mathcal{B}| \leq \kappa$. For each $\alpha < \tau$ choose a dense set $D_\alpha \subseteq H_\alpha$ with $|D_\alpha| \leq \kappa$. Since H_α is dense in G, so is each D_α. Let $[G]^{\leq\kappa}$ be the family of all subsets of G of cardinality $\leq \kappa$, and let $\mathcal{F}$ be the family of all functions $f : \mathcal{B} \to [G]^{\leq\kappa}$. We claim that $|\mathcal{F}| \leq |G|$. Indeed, by Lemma 2.2 below, $|G| = 2^\kappa$, and so $|[G]^{\leq\kappa}| = 2^\kappa$. Since we also have $|\mathcal{B}| \leq \kappa$, the desired inequality follows. For each $\alpha < \tau$ define $f_\alpha \in \mathcal{F}$ by $f_\alpha(U) = U \cap D_\alpha$ for $U \in \mathcal{B}$. Since all D_α's are dense in G, while each intersection $D_\alpha \cap D_\beta \subseteq H_\alpha \cap H_\beta$ for $\alpha \neq \beta$ is *not* dense in G, one may easily deduce that $f_\alpha \neq f_\beta$ for $\alpha \neq \beta$. Since $\{f_\alpha : \alpha < \tau\} \subseteq \mathcal{F}$, this yields $\tau \leq |\mathcal{F}| \leq |G|$. ∎

Comfort [4] was apparently the first to consider a special case of Question 1.1. The definition of an almost disjoint family of subgroups (Definition 1.2(ii)) is also due to Comfort [6]. The notion was later developed in [7].

In this paper we establish a number of results on the existence of (weakly) almost disjoint families $\mathcal{H}$ consisting of dense pseudocompact subgroups in compact groups. Since the bigger the size of the family $\mathcal{H}$ the better, it is natural to maximize its size $|\mathcal{H}|$. Lemma 1.3 gives us an upper bound for the cardinality of such an $\mathcal{H}$:

1.4. Corollary. *If $\mathcal{H}$ is a weakly almost disjoint family of dense subgroups of an infinite compact group G, then $|\mathcal{H}| \le |G|$.*

We start with the special case when G is Abelian.

1.5. Theorem. *For every nonmetrizable compact Abelian group G there exists a weakly almost disjoint family of size $|G|$ consisting of dense pseudocompact subgroups of G.*

Theorem 1.5 improves substantially the result of Comfort [4] who showed the existence of a family $\{H_\alpha : \alpha < \mathfrak{c}\}$ of dense pseudocompact subgroups of G such that $H_\alpha \cap H_\beta$ is not dense in G for distinct $\alpha, \beta < \mathfrak{c}$ (here $\mathfrak{c}$ denotes the cardinality of the continuum).

For connected groups we can significantly strengthen Theorem 1.5:

1.6. Theorem. *Let G be a nonmetrizable compact connected Abelian group and K any closed, totally disconnected subgroup of G. Then there exists an almost disjoint family $\mathcal{H}$ consisting of dense pseudocompact subgroups of G such that $|\mathcal{H}| = |G|$ and $\langle \bigcup \mathcal{H} \rangle \cap K \subseteq \{0\}$. In addition, each $H \in \mathcal{H}$ is algebraically isomorphic to the free Abelian group of size $|G|$.*

It turns out that connectedness of G is essential in Theorem 1.6, and therefore one cannot replace weak almost disjointness by almost disjointness in Theorem 1.5. The next example, essentially due to Wilcox, demonstrates this:

1.7. Example. Let p and q be different primes. For a prime number p we use $\mathbb{Z}(p)$ to denote the quotient group $\mathbb{Z}/p\mathbb{Z}$, where $\mathbb{Z}$ is the group of integers. *If G is a dense subgroup of $\mathbb{Z}(p)^\tau \times \mathbb{Z}(q)$, then $G = H \times \mathbb{Z}(q)$ for some dense subgroup H of $\mathbb{Z}(p)^\tau$.* In particular, *if G and G' are dense subgroups of $\mathbb{Z}(p)^\tau \times \mathbb{Z}(q)$, then $G \cap G' \supseteq \{0\} \times \mathbb{Z}(q) \neq \{0\} \times \{0\}$.*

A particular case of this example was considered by Wilcox [24, Example 2.5], but he proved the analogous statement only in the special case when G is pseudocompact. The analysis of his proof shows though that pseudocompactness of G is superfluous and can be omitted. For completeness we present a proof of the italicized statement. Let $g \in \mathbb{Z}(q) \setminus \{0\}$. Since $U_g = \mathbb{Z}(p)^\tau \times \{g\}$ is an open subset of $\mathbb{Z}(p)^\tau \times \mathbb{Z}(q)$ and G is dense in $\mathbb{Z}(p)^\tau \times \mathbb{Z}(q)$, there is $(g', g) \in U_g \cap G$. Since G is a subgroup of $\mathbb{Z}(p)^\tau \times \mathbb{Z}(q)$, $(0, pg) = (pg', pg) = p \cdot (g', g) \in G$. Observe that, since p and q are different primes and $g \in \mathbb{Z}(q) \setminus \{0\}$, $h = pg \neq 0$. Since $\langle h \rangle = \mathbb{Z}(q)$, $(0, h) \in G$ and G is a subgroup of $\mathbb{Z}(p)^\tau \times \mathbb{Z}(q)$, we conclude that $\{0\} \times \mathbb{Z}(q) \subseteq G$. Let $H \subseteq \mathbb{Z}(p)^\tau$ be the image of G under the projection from $\mathbb{Z}(p)^\tau \times \mathbb{Z}(q)$

onto $\mathbb{Z}(p)^\tau$. Obviously H is a dense subgroup of $\mathbb{Z}(p)^\tau$. From the inclusion $\{0\} \times \mathbb{Z}(q) \subseteq G$ it follows that $G = H \times \mathbb{Z}(q)$. ∎

All groups in Example 1.7 are torsion. The next theorem and its corollaries show that Theorem 1.6 remains valid also for some special nontorsion Abelian groups. (Recall that a subset of an Abelian group G is *independent* if the smallest subgroup of G generated by it is isomorphic to a free Abelian group.)

1.8. Theorem. *Let G be an infinite compact Abelian group such that every closed G_δ-subgroup of G contains an independent set of size $|G|$. Then there exists an almost disjoint family $\mathcal{H}$ of dense pseudocompact subgroups of G such that $|\mathcal{H}| = |G|$ and each $H \in \mathcal{H}$ is algebraically isomorphic to the free Abelian group of size $|G|$.*

Theorem 1.8 is applicable to a variety of groups given by Cartesian products:

1.9. Corollary. *Let $G = \prod\{G_\alpha : \alpha < \tau\}$ be an uncountable product of compact Abelian groups each of which has cardinality $\le 2^\tau$. Suppose that τ many G_α's contain a subgroup algebraically isomorphic to $\mathbb{Z}$. Then there exists an almost disjoint family $\mathcal{H}$ of dense pseudocompact subgroups of G such that $|\mathcal{H}| = |G|$ and each $H \in \mathcal{H}$ is algebraically isomorphic to the free Abelian group of size $|G|$.*

Proof. Let H be a closed G_δ-subgroup of G. Then there is a countable set $A \subseteq \tau$ such that $N = \{0_A\} \times G' \subseteq H$, where 0_A is the zero element of the group $\prod\{G_\alpha : \alpha \in A\}$ and $G' = \prod\{G_\alpha : \alpha \in \tau \setminus A\}$. Since $\tau \ge \omega_1$, $|B \setminus A| = \tau$, where $B = \{\alpha \in \tau : G_\alpha$ contains a subgroup algebraically isomorphic to $\mathbb{Z}\}$. Since each G_α with $\alpha \in B$ contains a subgroup algebraically isomorphic to $\mathbb{Z}$, $G' = \prod\{G_\alpha : \alpha \in B \setminus A\} \subseteq H$ has an independent subset Y of size $2^{|B \setminus A|} = 2^\tau$. Then $X = \{0_A\} \times Y \subseteq N \subseteq H$ is an independent set of size 2^τ. Now observe that $|G| = 2^\tau$ and apply Theorem 1.8. ∎

1.10. Corollary. *Let $G = \prod\{G_\alpha : \alpha < \tau\}$ be an uncountable product of compact metric Abelian groups. If τ many G_α's contain a subgroup algebraically isomorphic to $\mathbb{Z}$, then there exists an almost disjoint family $\mathcal{H}$ of size $|G|$ consisting of dense pseudocompact subgroups of G. Moreover, each $H \in \mathcal{H}$ is algebraically isomorphic to the free Abelian group of size $|G|$.*

Proof. Since each G_α is a compact metric space, we have $|G_\alpha| \le \mathfrak{c} \le 2^\tau$, and Corollary 1.9 can be applied. ∎

Theorem 1.6 remains also valid for some torsion groups G such as, for example, $\mathbb{Z}(p)^\tau$ for $\tau \ge \omega_1$; see Lemma 3.4 (compare this with Example 1.7).

The situation in the non-Abelian case is much more complicated. In particular, the authors do not know the answer to the following

1.11. Question. Can one drop the word "Abelian" in Theorem 1.5 (2)?

However, we can give a positive answer to Question 1.11 in the connected case:

1.12. Theorem. *For every nonmetrizable compact connected group G there exists a weakly almost disjoint family of size $|G|$ consisting of dense pseudocompact subgroups of G.*

Quite surprisingly, Theorem 1.6 is not valid in the non-Abelian case, so the common generalization of both Theorems 1.6 and 1.12 which could have been obtained via removing the word "weakly" in Theorem 1.12 is impossible. This is the substance of our next example:

1.13. Example. *Let N be a compact Abelian group, K a non-Abelian compact metric group and $C \neq \{e\}$ the commutator subgroup of K, i.e. the smallest subgroup of K generated by the set $\{xyx^{-1}y^{-1} : x, y \in K\}$. Then every dense pseudocompact group $H \subseteq G = N \times K$ contains $\{0\} \times C \neq \{0\} \times \{e\}$.* Therefore, there is no pair of dense pseudocompact subgroups of G with a trivial intersection. *If both N and K are connected* (one may take the group $SO(3, \mathbb{R})$ of all 3×3 matrices with determinant 1 as K), *then G provides an example of a compact connected group without any pair of distinct dense pseudocompact subgroups with a trivial intersection.*

Indeed, it suffices to show that $ghg^{-1}h^{-1} \in H$ whenever $g = (g_0, g_1) \in H \subseteq N \times K$ and $h = (h_0, h_1) \in H \subseteq N \times K$. Since K is metric, both $F_g = N \times \{g_1\}$ and $F_h = N \times \{h_1\}$ are closed G_δ-subsets of G. Since H is pseudocompact and dense in G, H must meet these two sets ([10]; see also [5, Theorem 6.4]). So let $\overline{g} = (\overline{g}_0, g_1) \in H \cap F_g$ and $\overline{h} = (\overline{h}_0, h_1) \in H \cap F_h$. Then $ghg^{-1}h^{-1} = (0, g_1h_1g_1^{-1}h_1^{-1}) = \overline{g}\overline{h}\overline{g}^{-1}\overline{h}^{-1} \in H$ because H is a subgroup of G. Since g_1, h_1 are arbitrary elements of K, $\{0\} \times C \subseteq H$.

A careful analysis of Example 1.13 shows that the main reason for the pathology is that the "non-Abelian part" of G is small. If this part is big enough, we do get a positive result:

1.14. Theorem. *Let G be a nonmetrizable connected group with center Z. If $w(G/Z) = w(G)$, then there exists an almost disjoint family $\mathcal{H}$ consisting of dense pseudocompact subgroups of G such that $|\mathcal{H}| = |G|$ and each $H \in \mathcal{H}$ is algebraically isomorphic to the free group of size $|G|$.*

(2) In fact, it is not even known in ZFC whether every nonmetrizable compact group has a proper dense pseudocompact subgroup. As was mentioned in the paragraph preceding Question 1.1, the answer is positive in the Abelian case (this also follows from Theorem 1.5). Theorem 1.12 below shows that the answer is positive for connected groups. The authors recently proved that the answer is also positive under $2^\omega < 2^{\omega_1}$ [18]. We note in passing that the situation is different for locally compact groups, since there exists a locally compact Abelian group without a proper dense subgroup [21].

Since a compact metric space does not contain proper dense pseudocompact subspaces, "nonmetrizable" is essential in Theorems 1.5, 1.6, 1.12 and 1.14, as well as in Corollaries 1.9 and 1.10.

1.15. Question. Is it possible to choose the families $\mathcal{H}$ in Theorems 1.6 and 1.14 satisfying the additional restriction that each $H \in \mathcal{H}$ is a (algebraically) normal subgroup of G?

Recall that a space X is *countably compact* if every infinite subset of X has an accumulation point, and X is *ω-bounded* if the closure of every countable subset of X is compact. One can easily see that ω-bounded $\Rightarrow$ countably compact $\Rightarrow$ pseudocompact. It is therefore natural to ask the following question: Which of our results (if any) can be improved via replacing "pseudocompact" by "countably compact" (or even, "ω-bounded") in their conclusions? To motivate this question, it should be mentioned that Comfort [3] proved that every compact Abelian group contains a proper dense ω-bounded (hence countably compact) subgroup, and the authors recently showed that the same is true for every compact connected group [19]. Furthermore, under $2^\omega < 2^{\omega_1}$, each compact group has a proper dense countably compact subgroup [18].

The following example, which is a particular case of a recent result by Dikranjan [12, Lemma 2.4], demonstrates that at least Theorem 1.8 and Corollaries 1.9, 1.10 are sharp in the sense that in their conclusions "pseudocompact" cannot be strengthened even to "countably compact".

1.16. Example. Let $p > 1$ be any prime number, and let $\mathbb{Z}_p$ be the (compact, totally disconnected) group of p-adic numbers (note that $\mathbb{Z}_p$ contains a copy of $\mathbb{Z}$). Define $G = \mathbb{T} \times \mathbb{Z}_p^{\omega_1}$, and let $C = \mathbb{T} \times \{\mathbf{0}\}$ be the connected component of G, where $\mathbf{0}$ is the zero element of $\mathbb{Z}_p^{\omega_1}$. Then *$C \subseteq H$ for every dense, countably compact subgroup H of G*. In particular, *$H \cap H' \supseteq C \neq \{0\}$ whenever H and H' are dense, countably compact subgroups of G.*

However, the above example leaves open the following

1.17. Question. In Theorems 1.5, 1.6, 1.12 and 1.14, can one replace "pseudocompact" by "countably compact" (or even, "ω-bounded")?

Some results of this paper were announced in [17].

2. Preliminaries. For Abelian groups we use additive notation. In particular, 0 always denotes the zero element of an Abelian group G. The symbol e denotes the identity element of a non-Abelian group G. We use $\langle X\rangle$ to denote the smallest subgroup of G that contains $X \subseteq G$. For $x \in G$ we write $\langle x\rangle$ instead of $\langle\{x\}\rangle$. A subset $X \subseteq G$ of an Abelian group G is *independent* if $\langle X\rangle$ is the free Abelian group over X, or equivalently, if, whenever

$x_0, \ldots, x_n \in X$ are distinct, $k_0, \ldots, k_n \in \mathbb{Z}$ and $k_0x_0 + \ldots + k_nx_n = 0$, then $k_0 = \ldots = k_n = 0$. We use $\mathbb{Z}(p)$ to denote the quotient group $\mathbb{Z}/p\mathbb{Z}$.

In this section we collect some "folklore" results that we need for future references. Their proofs will be mostly omitted.

2.1. Lemma. *If $\pi : G \to H$ is a continuous group homomorphism of a compact group G onto a topological group H, then $\pi(F)$ is a G_δ-subset of H for every closed G_δ-set $F \subseteq G$.*

2.2. Lemma [5, Theorem 3.1(i)]. $|G| = 2^{w(G)}$ *for every infinite compact group G.*

2.3. Lemma ([10]; see also [5, Theorem 6.4]). *Let H be a subgroup of a compact group G such that $H \cap F \neq \emptyset$ for every nonempty, closed G_δ-subset of G. Then H is pseudocompact* (*and dense in G*).

2.4. Lemma. *If $\mathcal{F}$ is the family of all closed G_δ-subsets of an infinite compact group G, then $|\mathcal{F}| \le |G|$.*

2.5. Lemma (the Disjoint Refinement Lemma). *Let τ be an infinite cardinal and let $\mathcal{F}$ be a family of subsets of a set Z with $|\mathcal{F}| \le \tau$. Suppose also that X is a subset of Z such that $|X \cap F| = \tau$ for every $F \in \mathcal{F}$. Then there is a partition $X = \bigcup\{X_\alpha : \alpha < \tau\}$ of X into pairwise disjoint sets of size τ so that $F \cap X_\alpha \neq \emptyset$ whenever $\alpha < \tau$ and $F \in \mathcal{F}$.*

We finish this section with the lemma which will be our technical tool for constructing weakly almost disjoint families in Theorems 1.5 and 1.12.

2.6. Lemma. *Let $\pi : G \to H$ be a continuous group homomorphism of a compact group G onto an infinite group H. If $\{H_\alpha : \alpha < \tau\}$ is a weakly almost disjoint family of dense pseudocompact subgroups of H, then $\{\pi^{-1}(H_\alpha) : \alpha < \tau\}$ is a weakly almost disjoint family of dense pseudocompact subgroups of G.*

Proof. Each $\pi^{-1}(H_\alpha)$ is a dense pseudocompact subgroup of G [11, Lemma 4.1(b)]. Being a continuous group homomorphism defined on a compact group, the map π is both open and closed, so $\pi^{-1}(E)$ is nowhere dense in G for every set E nowhere dense in H, and the result follows. ∎

3. Proofs of Theorems 1.5, 1.6 and 1.8. We first prove Theorem 1.8, then Theorem 1.6, and finally, Theorem 1.5. Such an unusual order is due to the fact that both Theorem 1.6 and Theorem 1.8 are required for the proof of Theorem 1.5, and some features of the proof of Theorem 1.8 are also used in that of Theorem 1.6.

Proof of Theorem 1.8. Let $\kappa = |G|$. We are going to find a collection $\{H_\alpha : \alpha < \kappa\}$ of subgroups of G such that:

(i) each H_α is a free Abelian group of size κ,
(ii) $H_\alpha \cap \langle \bigcup\{H_\beta : \beta \neq \alpha\}\rangle = \{0\}$ for all $\alpha < \kappa$, and
(iii) if $\alpha < \kappa$ and $F \subseteq G$ is a nonempty G_δ-set, then $H_\alpha \cap F \neq \emptyset$.

We get the combination of (i) and (ii) once we construct a disjoint family $\{I_\alpha : \alpha < \kappa\}$ of independent sets, each of size κ, such that the union $\bigcup\{I_\alpha : \alpha < \kappa\}$ is independent as well. We may then set $H_\alpha = \langle I_\alpha \rangle$ for each α. Lemma 2.3 would give us pseudocompactness of all H_α's as soon as we ensure property (iii). To get it we simply insist that $I_\alpha \cap F \neq \emptyset$ for every $\alpha < \kappa$ and each nonempty G_δ-set F in G.

By Lemma 2.4, the set of nonempty G_δ-subsets of G may be enumerated as $\{F_\beta : \beta < \kappa\}$. By the assumptions of our theorem we can choose, for each β, an independent set $D_\beta \subseteq F_\beta$ of size κ. Without loss of generality we may assume that D_β's are pairwise disjoint. Now by a straightforward induction one can construct an independent set D such that $|D \cap D_\beta| = \kappa$ for all β. Let $D \cap D_\beta = \{x_{\beta\alpha} : \alpha < \kappa\}$ (without repetition). Finally, set $I_\alpha = \{x_{\beta\alpha} : \beta < \kappa\}$. The collection $\{I_\alpha : \alpha < \kappa\}$ is as required. ∎

Proof of Theorem 1.6. Observe that to prove our theorem, it suffices to modify the proof of Theorem 1.8 to make sure that the set D chosen there would satisfy the condition $\langle D \rangle \cap K \subseteq \{0\}$. One can easily verify that the inductive construction of D with this additional condition can be carried out provided that we can choose all $D_\beta \subseteq F_\beta$ in such a way that $\langle D_\beta \rangle \cap K \subseteq \{0\}$. Therefore, Lemma 3.1 below completes the proof of Theorem 1.6. ∎

3.1. Lemma. *Let G be a nonmetrizable*, *compact*, *connected Abelian group*, *K its closed*, *totally disconnected subgroup and F a nonempty closed G_δ-subset of G. Then there is an independent set $X \subseteq F$ such that $|X| = |G|$ and $\langle X \rangle \cap K = \{0\}$.*

Proof. Let $\tau = w(G)$. First we will check the following

Claim. *There exists a continuous*, *surjective group homomorphism $\pi : G \to \mathbb{T}^\tau$ with $K \subseteq \ker \pi$.*

Proof. There exists a continuous, surjective group homomorphism $\varphi : G \to \mathbb{T}^\tau$ (see, for example, [11, Lemma 5.2]). Note that $L = \varphi(K)$ is a totally disconnected subgroup of $\mathbb{T}^\tau$, and so is each image $N_\alpha = \pi_\alpha(L) \subseteq \mathbb{T}$ of L under the αth projection $\pi_\alpha : \mathbb{T}^\tau \to \mathbb{T}$. Being closed in $\mathbb{T}$, N_α must be finite. Observe that

$$L \subseteq N = \prod\{N_\alpha : \alpha < \tau\} \subseteq \mathbb{T}^\tau.$$

Since every $\mathbb{T}/N_\alpha$ is topologically isomorphic to $\mathbb{T}$, it follows that $\mathbb{T}^\tau/N$ is topologically isomorphic to $\mathbb{T}^\tau$. Let $\psi : \mathbb{T}^\tau \to \mathbb{T}^\tau$ be the natural quotient homomorphism with the kernel N. Finally, $\pi = \psi \circ \varphi : G \to \mathbb{T}^\tau$ is a continuous, surjective homomorphism and

$$K \subseteq \varphi^{-1}(\varphi(K)) = \varphi^{-1}(L) \subseteq \varphi^{-1}(N) \subseteq \varphi^{-1}(\psi^{-1}(0)) = \pi^{-1}(0) = \ker \pi. \ \blacksquare$$

Returning back to the proof of our lemma, note that $\pi(F)$ is a G_δ-subset of T^τ (Lemma 2.1), so $\pi(F)$ contains an independent subset Y of size 2^τ. For every $y \in Y$ choose $x_y \in F$ so that $\pi(x_y) = y$, and let $X = \{x_y : y \in Y\}$. One can easily check that X is an independent subset of F with $|X| = |Y| = 2^\tau$ and $\langle X\rangle \cap K \subseteq \langle X\rangle \cap \ker \pi \subseteq \{0\}$. Now it remains only to note that $|G| = 2^\tau$ by Lemma 2.2. ∎

3.2. Lemma. *Let G be a compact Abelian group of weight $\kappa \geq \omega_1$. Then there exists a continuous group homomorphism $\pi : G \to H$ of G onto a* (*compact Abelian*) *group H which contains an almost disjoint family $\{H_\alpha : \alpha < 2^\kappa\}$ of dense pseudocompact subgroups of H.*

Proof of Theorem 1.5. Since G is not metrizable, its weight κ is uncountable. Let H be as in the conclusion of Lemma 3.2. Then, according to Lemma 2.6, G has a weakly almost disjoint family $\{G_\alpha : \alpha < 2^\kappa\}$ consisting of dense pseudocompact subgroups of G. Now it remains only to note that $|G| = 2^\kappa$ by Lemma 2.2. ∎

The rest of this section is devoted to proving Lemma 3.2. To do this we need two auxillary lemmas first. In what follows we will use $\mathbb{P}$ to denote the set of all prime numbers bigger than 1.

3.3. Lemma. *Suppose that $\{\kappa_p : p \in \mathbb{P}\}$ is a set of cardinals*, $\kappa = \sup\{\kappa_p : p \in \mathbb{P}\} \geq \omega_1$ *and $2^{\kappa_p} < 2^\kappa$ for all $p \in \mathbb{P}$. Then every closed G_δ-subgroup of $H = \prod\{\mathbb{Z}(p)^{\kappa_p} : p \in \mathbb{P}\}$ contains an independent set of size $|H| = 2^\kappa$.*

Proof. Let H' be a closed G_δ-subgroup of H. Then for every $p \in \mathbb{P}$ there is a countable set $A_p \subseteq \kappa_p$ such that

$$N = \prod\{\{0_{A_p}\} \times \mathbb{Z}(p)^{\kappa_p \setminus A_p} : p \in \mathbb{P}\} \subseteq H',$$

where 0_A denotes the zero element of the group $\mathbb{Z}(p)^A$. Define $\kappa'_p = |\kappa_p \setminus A_p|$ and observe that the group N is isomorphic to $\prod\{\mathbb{Z}(p)^{\kappa'_p} : p \in \mathbb{P}\}$, $\kappa = \sup\{\kappa'_p : p \in \mathbb{P}\} \geq \omega_1$ and $2^{\kappa'_p} < 2^\kappa$ for all $p \in \mathbb{P}$. This argument shows that to prove our lemma, it suffices only to check that H itself contains an independent set of size $|H| = 2^\kappa$.

Let $t(H)$ be the torsion subgroup of H. We have $t(H) = \bigoplus\{H_p : p \in \mathbb{P}\}$, where $H_p = \{h \in H : ph = 0\}$ is the p-torsion part of H [15, Theorem A.3]. Now observe that $H_p = \mathbb{Z}(p)^{\kappa_p}$, and so $|H_p| = 2^{\kappa_p}$. Since $2^{\kappa_p} < 2^\kappa$ for all

$p \in \mathbb{P}$, by König's theorem [8, Theorem 1.19] we have

$$\begin{aligned}|t(H)| &= \Big|\bigoplus\{H_p : p \in \mathbb{P}\}\Big| \\ &= \sum\{|H_p| : p \in \mathbb{P}\} = \sum\{2^{\kappa_p} : p \in \mathbb{P}\} < (2^\kappa)^\omega = 2^\kappa.\end{aligned}$$

Since $|t(H)| < |H|$, we conclude that H contains an independent set of size $|H| = 2^\kappa$. ∎

3.4. Lemma. *If $p \in \mathbb{P}$ and $\kappa \ge \omega_1$, then there is an almost disjoint family of size 2^κ consisting of dense pseudocompact subgroups of $\mathbb{Z}(p)^\kappa$.*

Proof. Let $\tau = 2^\kappa = |\mathbb{Z}(p)^\kappa|$, and let $\mathcal{F}$ be the family of all closed, nonempty G_δ-subsets of $\mathbb{Z}(p)^\kappa$. We have $|\mathcal{F}| \le \tau$ by Lemma 2.4. Also, $|F| = \tau$ for every $F \in \mathcal{F}$. This permits us to use a straightforward transfinite induction to construct a set $X = \{x_\alpha : \alpha < \tau\}$ such that

(i) $x_\alpha \notin \langle\{x_\beta : \beta < \alpha\}\rangle$ for $\alpha < \tau$, and
(ii) $|X \cap F| = \tau$ for every $F \in \mathcal{F}$.

Let $X = \{X_\alpha : \alpha < \tau\}$ be a partition of X of the form given in Lemma 2.5. Then each $H_\alpha = \langle X_\alpha\rangle$ is a dense pseudocompact subgroup of $\mathbb{Z}(p)^\kappa$ (Lemma 2.3), and from (i) it follows that $\{H_\alpha : \alpha < \tau\}$ is an almost disjoint family of subgroups of $\mathbb{Z}(p)^\kappa$. ∎

Proof of Lemma 3.2. Let $\widehat{G}$ be the dual group of G. Then $|\widehat{G}| = w(G) = \kappa$ (see [15, Theorem 24.15(i)]). Since $\widehat{G}$ is uncountable, $|\widehat{G}| = \sup\{r_p(\widehat{G}) : p \in \mathbb{P} \cup \{0\}\}$, where $r_0(H) = \sup\{|X| : X$ is an independent subset of $H\}$ is the free rank of H and $r_p(H) = \sup\{\kappa : H$ contains a subgroup algebraically isomorphic to $\mathbb{Z}(p)^{(\kappa)}\}$ is the p-rank of H for $p \in \mathbb{P}$. (As usual, for an Abelian group G we use $G^{(\kappa)}$ to denote the weak direct product (= free sum) of κ many copies of G.) If $r_0(\widehat{G}) = \kappa$, then there exists a continuous surjective homomorphism $\pi : G \to \mathbb{T}^\kappa$ (see [11, Proof of Lemma 5.2]), and $H = \mathbb{T}^\kappa$ will work by Theorem 1.6 (proved by this time). So, without loss of generality, we will assume in the future that $r_0(\widehat{G}) < \kappa$. For typographical reasons set $\kappa_p = r_p(\widehat{G})$. By our assumption we have $\kappa = |\widehat{G}| = \sup\{\kappa_p : p \in \mathbb{P}\}$, and we need to consider two cases.

Case 1: $2^{\kappa_p} = 2^\kappa$ *for some* $p \in \mathbb{P}$. In this case $\widehat{G}$ contains a subgroup algebraically isomorphic to $\mathbb{Z}(p)^{(\kappa_p)}$, and so there is a continuous surjective homomorphism $\pi : G \to \mathbb{Z}(p)^{\kappa_p}$ (see [11, Proof of Theorem 4.3]). Since $2^{\kappa_p} = 2^\kappa$, $H = \mathbb{Z}(p)^{\kappa_p}$ satisfies the conclusion of our theorem by Lemma 3.4.

Case 2: $2^{\kappa_p} < 2^\kappa$ *for all* $p \in \mathbb{P}$. In this case $\widehat{G}$ contains a subgroup isomorphic to $\bigoplus\{\mathbb{Z}(p)^{(\kappa_p)} : p \in \mathbb{P}\}$, and therefore there is a continuous surjective homomorphism

$$\pi : G \to H = \Big(\bigoplus\{\mathbb{Z}(p)^{(\kappa_p)} : p \in \mathbb{P}\}\Big)^{\widehat{}} = \prod\{\mathbb{Z}(p)^{\kappa_p} : p \in \mathbb{P}\}.$$

Now note that $\sup\{\kappa_p : p \in \mathbb{P}\} = \kappa \geq \omega_1$, so H satisfies the hypothesis of Lemma 3.3, and Theorem 1.8 (also proved by this time) yields the desired family of subgroups of H. ∎

4. Proofs of Theorems 1.12 and 1.14. We need the following classical structure theorem for compact connected groups:

4.1. Lemma [20, Theorem 6.5.6]. *Let G be a compact connected group and C the connected component of the center of G. Then there exist a family $\{L_\alpha : \alpha < \lambda\}$ consisting of compact, simply connected, simple Lie groups L_α and a continuous surjective homomorphism $\pi : C \times L \to G$ with the totally disconnected kernel* $\ker \pi \subseteq C \times Z$, *where Z is the center of $L = \prod\{L_\alpha : \alpha < \lambda\}$.*

Our next lemma is the key to our proofs.

4.2. Lemma. *Let G, C, L, Z and L_α for $\alpha < \lambda$ be as in Lemma* 4.1. *Suppose also that $\lambda = w(G) \geq \omega_1$. Then there exists an almost disjoint family $\mathcal{H}$ of size 2^λ consisting of dense pseudocompact subgroups of G such that each $H \in \mathcal{H}$ is algebraically isomorphic to the free (non-Abelian) group of size 2^λ.*

We will prove the last lemma later, but first we show how to deduce our theorems from it.

Proof of Theorem 1.12. Let G be a compact connected group of weight $\kappa \geq \omega_1$. Then $|G| = 2^\kappa = \tau$ (Lemma 2.2). Let $\{L_\lambda : \alpha < \lambda\}$ and π be as in the conclusion of Lemma 4.1.

Claim. $\lambda \leq \kappa$.

Proof. Let Z_α be the center of L_α. Observe that, since $\ker \pi \subseteq C \times Z$, the group $L/Z = \prod\{L_\alpha/Z_\alpha : \alpha < \lambda\}$ is a quotient group of G. Since the last group is compact, so is L/Z. Applying [13, Theorem 3.1.22], we conclude that $w(\prod\{L_\alpha/Z_\alpha : \alpha < \lambda\}) \leq w(G) = \kappa$, which implies $\lambda \leq \kappa$. ∎

If $\lambda = \kappa$, then Lemma 4.2 yields the desired (even stronger) conclusion. So it remains only to consider the case $\lambda < \kappa$. Let $K = \{0\} \times L \subseteq C \times L$ and $N = \pi(K)$. Since K is a (closed) normal subgroup of $C \times L$, N is a (closed) normal subgroup of G. Let $H = G/N$ be the quotient group of G and $\psi : G \to H$ the quotient group homomorphism. Since N is compact as a continuous image of the compact space K, $w(N) \leq w(K) = \omega \cdot \lambda < \kappa$ by [13, Theorem 3.1.22]. Since $\lambda = w(G) = \max\{w(N), w(G/N)\}$ and $\kappa \geq \omega_1$, it follows that $w(H) = w(G) = \kappa$. Now note that H is a nonmetrizable compact connected Abelian group, so Theorem 1.6 applied to H permits us to find an almost disjoint family $\{\widetilde{H}_\alpha : \alpha < \tau\}$ of dense pseudocompact subgroups of $\widetilde{H}$. If one defines $H_\alpha = \psi^{-1}(\widetilde{H}_\alpha)$, then $\{H_\alpha : \alpha < \tau\}$ would be the desired family (Lemma 2.6). ∎

Proof of Theorem 1.14. Since we are going to use Lemmas 4.1 and 4.2, to avoid mixing of notations let us agree to denote the center of G by Z^*. So let $\kappa = w(G) = w(G/Z^*) \geq \omega_1$. Then $|G| = 2^{w(G)} = 2^\kappa = \tau$. Let $\{L_\lambda : \alpha < \lambda\}$ and π be as in the conclusion of Lemma 4.1. Let Z_α be the center of L_α. Observe that $G/Z^* = L/Z = \prod\{L_\alpha/Z_\alpha : \alpha < \lambda\}$ and each L_α/Z_α is again a compact, simply connected, simple Lie group [23]. Since $w(G/Z^*) = \kappa \geq \omega_1$, it follows that $\kappa = \lambda$. Now an application of Lemma 4.2 finishes the proof. ∎

The rest of this section will be devoted to the (quite technical) proof of Lemma 4.2. In its turn, it will be split into a sequence of lemmas.

4.3. Lemma. *Suppose that τ is an infinite cardinal, Z' and Z'' are nonempty sets, $\mathcal{F}$, $\mathcal{F}'$ and $\mathcal{F}''$ are families of nonempty subsets of $Z' \times Z''$, Z' and Z'' respectively, such that*

(a) $|\mathcal{F}'| \leq \tau$, $|\mathcal{F}''| \leq \tau$, *and*
(b) *if $F \in \mathcal{F}$, then $F' \times F'' \subseteq F$ for some $F' \in \mathcal{F}'$ and $F'' \in \mathcal{F}''$.*

Let $p : Z' \times Z'' \to Z'$ be the natural projection. Assume also that $X \subseteq Z'$ and

(c) $|X \cap F'| = \tau$ *for every $F' \in \mathcal{F}'$.*

Then there exists $Y \subseteq Z' \times Z''$ such that:

(i) $p(Y) \subseteq X$,
(ii) $p{\restriction}Y : Y \to X$ *is one-to-one*, *and*
(iii) $|Y \cap F| = \tau$ *for every $F \in \mathcal{F}$.*

Proof. Applying Lemma 2.5 one can get a pairwise disjoint family $\{X_{F'} : F' \in \mathcal{F}'\}$ of sets of size τ such that $X_{F'} \subseteq X \cap F'$ for each $F' \in \mathcal{F}'$. Applying Lemma 2.5 once more, one can split each $X_{F'}$ into a (pairwise disjoint) family $\{X_{F',F''} : F'' \in \mathcal{F}''\}$ of sets of size τ. Now we can take for Y the graph of any function $f : Z' \to Z''$ with the property that $f(x) \in F''$ whenever $x \in X_{F',F''}$. ∎

Since we are considering non-Abelian groups in this section, for the rest of this paper let us agree to call a subset X of a group G *independent* if $\langle X\rangle$ is a free (non-Abelian) group over X.

4.4. Lemma. *Suppose that τ is an uncountable cardinal*, *L_α is a compact*, *simply connected*, *simple Lie group for each $\alpha < \tau$, and Z is the center of $L = \prod\{L_\alpha : \alpha < \tau\}$. Then there is an independent set $X \subseteq L$ such that*

(i) $|X \cap F| = 2^\tau$ *for every* (*closed*) *nonempty* G_δ-*set* $F \subseteq L$, *and*
(ii) $\langle X\rangle \cap Z \subseteq \{e\}$.

Proof. Let $\kappa = 2^\tau$. Since $\tau \geq \omega_1$, in the rest of our proof we will assume, without loss of generality, that

$$L = \prod\{L_{\alpha\beta} : (\alpha,\beta) \in \tau \times \omega_1\},$$

where each $L_{\alpha\beta}$ is a compact, simply connected, simple Lie group. (This assumption may be easily achieved by splitting the index set if necessary.) For $(\alpha,\beta) \in \tau \times \omega_1$, the center $Z_{\alpha\beta}$ of $L_{\alpha\beta}$ is a finite group, so the quotient group $H_{\alpha\beta} = L_{\alpha\beta}/Z_{\alpha\beta}$ is again a compact, simply connected, simple Lie group (see, for example, [2]). We have

$$Z = \prod\{Z_{\alpha\beta} : (\alpha,\beta) \in \tau \times \omega_1\} \quad \text{and} \quad L/Z = \prod\{H_{\alpha\beta} : (\alpha,\beta) \in \tau \times \omega_1\}.$$

For each $\beta < \omega_1$ define

$$G_\beta = \prod\{L_{\alpha\beta} : \alpha < \tau\}, \quad H_\beta = \prod\{H_{\alpha\beta} : \alpha < \tau\} \quad \text{and} \quad Z_\beta = \prod\{Z_{\alpha\beta} : \alpha < \tau\}.$$

We need the following facts about these groups:

(1) $$|H_\beta| = \kappa \quad \text{for all } \beta < \omega_1,$$

(2) each H_β contains an independent subset $I_\beta = \{h^\mu_\beta : \mu < \kappa\}$,

(3) $$Z = \prod\{Z_\beta : \beta < \omega_1\} \quad \text{and} \quad L/Z = \prod\{H_\beta : \beta < \omega_1\}.$$

(1) easily follows from the fact that $|H_{\alpha\beta}| = \mathfrak{c}$ for $(\alpha,\beta) \in \tau \times \omega_1$. To prove (2) observe that each $H_{\alpha\beta}$ contains an independent subset of size $\mathfrak{c}$ [1, Lemma 2], and then apply [22, Lemma 2.16]. Finally, (3) trivially follows from the definitions of Z_β and H_β.

Let $\pi : L \to L/Z$ and $\pi_\beta : G_\beta \to H_\beta$ for $\beta < \omega_1$ be the natural homomorphisms. Let $\mathcal{F}$ be the family of all nonempty, closed G_δ-subsets of L. Observe that $|\mathcal{F}| \leq |L| = 2^\tau = \kappa$ by Lemma 2.4. For $F \in \mathcal{F}$ given, $\pi(F)$ is a nonempty, closed G_δ-subset of L/Z (Lemma 2.1), so by (3) there are $\delta_F < \omega_1$ and $g_F \in \prod\{H_\beta : \beta < \delta_F\}$ so that

(4) $$Q_F = \{g_F\} \times \prod\{H_\beta : \delta_F \leq \beta < \omega_1\} \subseteq \pi(F).$$

For each $\gamma < \omega_1$ use (1) to enumerate the subproduct $\prod\{H_\beta : \beta < \gamma\}$ as $\{g^\xi_\gamma : \xi i < \kappa\}$. Since $\kappa \geq \omega_1$, there are maps $\varphi : \kappa \to \kappa$ and $\psi : \kappa \to \omega_1$ such that

(5) $$|\{\mu < \kappa : \varphi(\mu) = \eta \text{ and } \psi(\mu) = \delta\}| = \kappa \quad \text{for all } \eta < \kappa \text{ and } \delta < \omega_1.$$

Finally, for every $\mu < \kappa$ define a point $y_\mu \in \prod\{H_\beta : \beta < \omega_1\}$ by

$$y_\mu(\beta) = \begin{cases} g^{\varphi(\mu)}_{\psi(\mu)}(\beta) & \text{if } \beta < \psi(\mu), \\ h^\mu_\beta & \text{if } \beta \geq \psi(\mu), \end{cases} \tag{6}$$

and set $Y = \{y_\mu : \mu < \kappa\}$.

CLAIM 1. $Y = \{y_\mu : \mu < \kappa\}$ *is a* (*faithfully indexed*) *independent subset of* $\prod\{H_\beta : \beta < \omega_1\}$.

Proof. It suffices to prove that, for a given finite subset F of κ, the set $\{y_\mu : \mu \in F\}$ is independent in $\prod\{H_\beta : \beta < \omega_1\}$. Take any $\beta > \max\{\psi(\mu) : \mu \in F\}$. Then $y_\mu(\beta) = h^\mu_\beta$ by (6). Since $\{h^\mu_\beta : \mu \in F\}$ is a faithfully indexed subset of I_β, the result follows from (2). ∎

CLAIM 2. $|Y \cap \pi(F)| = \kappa$ *for all* $F \in \mathcal{F}$.

Proof. Since $g_F \in \prod\{H_\beta : \beta < \delta_F\}$, there is $\eta < \kappa$ with $g_F = g^\eta_{\delta_F}$. From (4) and (6) it follows that $y_\mu \in Q_F \subseteq \pi(F)$ for all $\mu \in M_F = \{\mu < \kappa : \varphi(\mu) = \eta \text{ and } \psi(\mu) = \delta_F\}$. Since $|M_F| = \kappa$ according to (5), it remains only to note that $y_\mu \neq y_{\mu'}$ for different $\mu, \mu' < \kappa$. ∎

As was noted above, $|\mathcal{F}| \leq \kappa$, so we can use Claim 2 and Lemma 2.5 to split κ into a family $\{A_F : F \in \mathcal{F}\}$ of pairwise disjoint sets, each of size κ, such that $y_\mu \in \pi(F)$ if $\mu \in A_F$. Finally, we choose $x_\mu \in F$ with $\pi(x_\mu) = y_\mu$ whenever $\mu \in A_F$. It is now easy to see that $X = \{x_\mu : \mu < \kappa\} \subseteq L$ satisfies the conclusion of Lemma 4.4. ∎

Proof of Lemma 4.2. Let $\tau = 2^\lambda$. Let $\mathcal{F}$, $\mathcal{F}'$ and $\mathcal{F}''$ be the families of all nonempty closed G_δ-subsets of $C \times L$, L and C respectively. Note that $|\mathcal{F}'| \leq |L| = 2^\lambda = \tau$ by Lemma 2.4. Since $C \subseteq G$, $w(C) \leq w(G) = \lambda$, and so $|\mathcal{F}''| \leq |C| = 2^{w(C)} \leq 2^\lambda = \tau$ by Lemmas 2.2 and 2.4. Let $p : C \times L \to L$ be the projection onto the second coordinate. Let X be the independent subset of $L = \prod\{L_\alpha : \alpha < \kappa\}$ constructed in Lemma 4.4.

Now we can use Lemma 4.3 with $Z' = L$ and $Z'' = C$ to obtain a set $Y \subseteq C \times L$ such that

(i) $p(Y) \subseteq X$,
(ii) $p{\restriction}Y : Y \to X$ is one-to-one, and
(iii) $|Y \cap F| = \tau$ for every $F \in \mathcal{F}$.

CLAIM. Y *is an independent subset of* $C \times L$ *with* $\langle Y \rangle \cap (C \times Z) \subseteq \{0\} \times \{e\}$.

Proof. Since X is an independent subset of L, from (i) and (ii) it follows that Y is an independent subset of $C \times L$. Since $\langle X \rangle \cap Z \subseteq \{e\}$, (i) and (ii) imply also the second statement of our claim. ∎

CLAIM. *$\pi{\restriction}Y : Y \to G$ is a one-to-one map, and $\pi(Y)$ is an independent subset of G such that $|\pi(Y) \cap \Phi| = \tau$ for every nonempty closed G_δ-subset Φ of G.*

Proof. Since $\ker \pi \subseteq C \times Z$ and $\langle Y\rangle \cap (C \times Z) \subseteq \{0\} \times \{e\}$, the first two statements of our claim follow. To prove the last one, observe that $F = \pi^{-1}(\Phi) \in \mathcal{F}$, and so $|F \cap Y| = \tau$. Since $\pi{\restriction}Y : Y \to G$ is one-to-one, we conclude that $|\Phi \cap \pi(Y)| = \tau$. ∎

Since the cardinality of the family $\mathcal{F}^*$ of all nonempty closed G_δ-subsets of G does not exceed $|G| = 2^\lambda = \tau$ (Lemmas 2.2 and 2.4), we may apply Lemma 2.5 (with $X = \pi(Y)$ and $\mathcal{F} = \mathcal{F}^*$) to find a partition $\pi(Y) = \bigcup\{Y_\alpha : \alpha < \tau\}$ of $\pi(Y)$ into pairwise disjoint sets Y_α of size τ such that $F \cap Y_\alpha \neq \emptyset$ for all $F \in \mathcal{F}^*$ and $\alpha < \tau$. Define $H_\alpha = \langle Y_\alpha\rangle$ for $\alpha < \tau$. Since $F \cap H_\alpha \supseteq F \cap Y_\alpha \neq \emptyset$ for every $F \in \mathcal{F}^*$, each H_α is a dense pseudocompact subgroup of G (Lemma 2.3). Since $\pi(Y)$ is independent, $\{H_\alpha : \alpha < \tau\}$ is an almost disjoint family. And since $Y_\alpha \subseteq \pi(Y)$ and $|Y_\alpha| = \tau$, each H_α is the free group of size τ. ∎

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DEPARTMENT OF MATHEMATICS
QUEENS COLLEGE
CITY UNIVERSITY OF NEW YORK
65-30 KISSENA BLVD.
FLUSHING, NEW YORK 11367
U.S.A.
E-mail: ZEV@QCVAXA.ACC.QC.EDU

DEPARTMENT OF MATHEMATICS
FACULTY OF SCIENCE
EHIME UNIVERSITY
MATSUYAMA 790
JAPAN
E-mail: DMITRI@EHIMEGW.DPC.EHIME-U.AC.JP

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