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Characterization of knot complements in the n-sphere

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Abstract. Knot complements in the n-sphere are characterized. A connected open subset W of S^n is homeomorphic with the complement of a locally flat $(n-2)$-sphere in S^n, $n \geq 4$, if and only if the first homology group of W is infinite cyclic, W has one end, and the homotopy groups of the end of W are isomorphic to those of S^1 in dimensions less than $n/2$. This result generalizes earlier theorems of Daverman, Liem, and Liem and Venema.

1. Introduction. In this note we characterize those subsets of the n-sphere that are homeomorphic to complements of locally flat knots. We find conditions on an open subset W of S^n under which W is homeomorphic to $S^n - h(S^{n-2})$ for some locally flat topological embedding $h : S^{n-2} \to S^n$. Our main theorem is the following.

THEOREM. *Let W be a connected open subset of S^n, $n \geq 4$, such that W has one end ε. Then $W \cong S^n - h(S^{n-2})$ for some locally flat topological embedding $h : S^{n-2} \to S^n$ if and only if*

(1.1) $H_1(W) \cong \mathbb{Z}$,
(1.2) $\pi_1(\varepsilon)$ *is stable and* $\pi_1(\varepsilon) \cong \mathbb{Z}$, *and*
(1.3) $\pi_i(\varepsilon) = 0$ *for* $1 < i < n/2$.

The first theorem of this type was proved by Daverman [4, Theorem 4]. Daverman's theorem is very similar to ours, but he adds the extra hypothesis that W has the homotopy type of a finite complex. The main point of the present paper is the fact that conditions (1.1)–(1.3) imply that W automatically satisfies this finiteness condition. Our theorem also generalizes a theorem of Liem [6]. Liem's theorem is a variation on Daverman's: he drops the finiteness condition but adds the additional hypothesis that $\pi_i(\varepsilon) = 0$

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for $i = n/2$. The hypotheses in our theorem are the intersection of those in the theorems of Daverman and Liem. It should be noted that each of the remaining hypotheses is essential; if any one of them is dropped, the other two are not strong enough to imply the conclusion of the theorem.

The 4-dimensional case of Theorem 1 is already known [7]. At the time that we proved the earlier theorem we believed that the 4-dimensional case was special because of the low dimensions involved. Since that time we have developed stronger techniques for dealing with the algebraic questions that arise in the middle dimensions (see [7], [8], [13], and [14]). We now realize that the 4-dimensional theorem is not special; instead we can improve the high-dimensional theorem.

The main improvement in our theorem over Daverman's is the fact that the finiteness condition has been dropped. But we also improve Daverman's theorem in that we can get by with a weaker version of condition (1.1). Daverman assumes that all the homology groups of W match those of S^1 while we only assume that $H_1(W)$ is infinite cyclic. Since W is a subset of S^n, the strong conditions (1.2) and (1.3) on the homotopy groups of the end of W combine with (1.1) to imply that all the higher homology groups of W vanish. We do not prove this directly, but it becomes apparent as the proof develops.

A final way in which our theorem differs from that of Daverman is the fact that it covers the cases $n = 4$ and $n = 5$. As noted above, the 4-dimensional case of the theorem is proved in [7] and does require some specifically 4-dimensional techniques. Daverman needs $n \geq 6$ because he applies the main result of Siebenmann's thesis [10]. Since Siebenmann's result is now known to hold in dimension 5 (at least for certain fundamental groups—see [9]), Daverman's proof actually does cover the 5-dimensional case. A theorem similar to ours could be stated in dimension 3, but it would have to take into account the fact that the expected $\pi_1(\varepsilon)$ in that case would be $\mathbb{Z} \oplus \mathbb{Z}$. Another difference in dimension 3 is the fact that it might be W which is knotted rather than its complement. The statement of the theorem would have to account for this possibility.

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2. The construction of W_k. For the remainder of this paper we will assume that $n \geq 5$ and use k to denote the greatest integer in $(n-1)/2$. Thus $k = (n-1)/2$ if n is odd and $k = n/2 - 1$ if n is even. Notice that condition (1.3) can be restated as $\pi_i(\varepsilon) = 0$ for $2 \leq i \leq k$.

In this section we perform a sequence of modifications to W to produce a finite sequence of manifolds $W_1, \ldots, W_k$.

Since $\pi_1(\varepsilon)$ is finitely generated, $\pi_1(W)$ is finitely generated as well. It follows that the kernel of the Hurewicz homomorphism $\pi_1(W) \to H_1(W)$ is normally generated by a finite set. We do a finite number of 1-surgeries to kill this kernel. In other words, we find an embedded circle representing each normal generator of $\ker[\pi_1(W) \to H_1(W)]$. A regular neighborhood of such a curve is homeomorphic with $S^1 \times B^{n-1}$. We remove the interior of the regular neighborhood and paste in a copy of $B^2 \times S^{n-2}$. We use W_1 to denote the manifold which results from doing all the 1-surgeries. Notice that $\pi_1(W_1) \cong \mathbb{Z}$.

We use $\mathbb{J}$ to denote $\pi_1(W_1)$ and Λ to denote the integral group ring $\Lambda = \mathbb{Z}[\mathbb{J}]$. We also use $p : \widetilde{W}_1 \to W_1$ to denote the universal cover. Notice that the homology groups $H_i(\widetilde{W}_1)$ have the structure of modules over the ring Λ. By [7, Lemma 1.4], $\widetilde{W}_1$ has one simply connected end.

Now consider $\pi_2(W_1)$. Of course $\pi_2(W_1) \cong \pi_2(\widetilde{W}_1) \cong H_2(\widetilde{W}_1)$. Let U be a manifold neighborhood of ε such that $\pi_2(U) \to \pi_2(W_1)$ is the trivial homomorphism. Since the end of $\widetilde{W}_1$ is simply connected, we may assume that $H_2(p^{-1}(U)) \to H_2(\widetilde{W}_1)$ is trivial as well. In the Mayer–Vietoris sequence

$$H_2(\widetilde{W}_1 - p^{-1}(\operatorname{Int} U)) \oplus H_2(p^{-1}(U)) \xrightarrow{\alpha} H_2(\widetilde{W}_1) \xrightarrow{\beta} H_1(p^{-1}(\partial U))$$

both $\operatorname{im}\alpha$ and $\operatorname{im}\beta$ are finitely generated over Λ and so, using [15, Lemma 1.5], we see that $H_2(\widetilde{W}_1)$ is finitely generated over Λ. Thus we can do a finite number of 2-surgeries to produce a manifold W_2 with $\pi_2(W_2) = 0$ (and with $\pi_1(W_2)$ still equal to $\mathbb{J}$).

This process is continued inductively through dimension k. In that way we see that it is possible to produce from W a new manifold W_k which satisfies $\pi_1(W_k) = \mathbb{J}$ and $\pi_i(W_k) = 0$ for $2 \le i \le k$. Since we perform only a finite number of surgeries to W, the new manifold has the same end as the original did. In other words, there are compact sets $C \subset W$ and $C' \subset W_k$ such that $W - C = W_k - C'$. Another important property of W_k is the fact that W_k is contained in a compact manifold M with $M - W_k = S^n - W$. The reason for this is the fact that we can think of the surgeries we have done to W as being done to $S^n \supset W$. Thus M is just the compact manifold which results from doing our finite sequence of surgeries to S^n. We record the properties of W_k in a lemma so that they are available for future use.

LEMMA 2. *It is possible to do a finite number of surgeries to W to produce a new n-manifold W_k having the following properties*:

(2.1) $\pi_1(W_k) = \mathbb{J}$.

(2.2) $\pi_i(W_k) = 0$ *for* $2 \le i \le k$.

(2.3) *There exist compact sets $C \subset W$ and $C' \subset W_k$ and a compact manifold $M \supset W_k$ such that $S^n - C = M - C'$.*

3. The homotopy dimension of W_k. In this section we prove the following lemma.

LEMMA 3. *W_k has the homotopy type of a* (*possibly infinite*) *complex of dimension $n-k-1$.*

Proof. By Chapter 3 of Siebenmann's thesis [10, Theorem 3.10], the end of W_k has arbitrarily close 1-neighborhoods. This means that for every compact set $C \subset W_k$ there exists a neighborhood U of the end of W_k such that $U \subset W_k - C$ and the inclusion induced homomorphism $\pi_1(\partial U) \to \pi_1(U) \cong \pi_1(\varepsilon) \cong \mathbb{Z}$ is an isomorphism. (In this setting it is obvious that ∂U must carry a generator of $\pi_1(\varepsilon)$, so the proof is accomplished by trading 2-handles to kill the kernel of the homomorphism $\pi_1(\partial U) \to \pi_1(U)$. It is not necessary to trade any 1-handles as in the general case.) It should also be observed that, by [7, Lemma 1.1], the inclusion induced map $\pi_1(U) \to \pi_1(W_k)$ is an isomorphism.

Once we have arbitrarily close 1-neighborhoods of the end, we can proceed to construct k-neighborhoods of the end. For each compact subset C of W_k there exists a neighborhood U of ε such that the inclusion induced homomorphism $\pi_2(U) \to \pi_2(W_k - C)$ is trivial. Thus we can attach 3-handles to U to kill $\pi_2(U)$. This can be accomplished as in the proof of [2] or [10] and results in a 2-neighborhood of the end. It should be noted that, in contrast with the surgery done in the previous section, the surgery being done here is ambient surgery; the handle is added to U by subtracting it from $W_k - U$. Using induction we construct arbitrarily close k-neighborhoods of the end. Embedding the handles we need is relatively easy because $k < n/2$. (Only the case $n = 5$ requires special care.)

Let C be a PL embedded circle in W_k which represents a generator of the fundamental group and let N be a regular neighborhood of C in W_k. Define $U_0 = \overline{W_k - N}$. By the previous paragraph we can inductively find a sequence $U_0, U_1, \ldots$ of closed neighborhoods of the end so that $U_{i+1} \subset \operatorname{Int} U_i$ for each i, $\bigcap_{i=0}^{\infty} U_i = \emptyset$, $\pi_1(U_i) \to \pi_1(U_0)$ is an isomorphism for every i, and $\pi_j(U_i) = 0$ for $2 \le j \le k$ and for every i. Let $V_i = U_{i-1} - \operatorname{Int} U_i$. Then $H_j(\widetilde{V}_i, \partial\widetilde{U}_i) = H_j(\widetilde{U}_{i-1}, \widetilde{U}_i)$ by excision. The exact sequence

$$0 = H_j(\widetilde{U}_{i-1}) \to H_j(\widetilde{U}_{i-1}, \widetilde{U}_i) \to H_{j-1}(\widetilde{U}_i) = 0$$

shows that $H_j(\widetilde{V}_i, \partial\widetilde{U}_i) = 0$ for every $j \le k$ and for every i.

We can think of V_i as a cobordism based on ∂U_i. As in the proof of the s-cobordism theorem, the observation in the previous paragraph allows us to trade handles to eliminate all handles of dimensions $\le k$. If we think dually of V_i as a cobordism based on ∂U_{i-1}, then we have cancelled all handles of index $\geq n-k$. Thus each V_i collapses to $\partial U_{i-1} \cup K_i$ where K_i is a polyhedron of dimension $n-k-1$. Since $\partial U_0 = \partial N$ and N collapses to the circle C, it follows that W_k has the homotopy type of an $(n-k-1)$-dimensional complex. ∎

4. The finiteness of W_k in case n is odd. Suppose n is odd. Then $k = (n-1)/2$, so $n-k-1 = k$.

LEMMA 4. *If n is odd, then W_k has the homotopy type of S^1.*

Proof. By Lemmas 2 and 3, W_k has the homotopy type of a k-dimensional complex K with $\pi_1(K) \cong \mathbb{Z}$ and $\pi_i(K) = 0$ for $2 \leq i \leq k$. By the Hurewicz theorem, the universal cover of K is contractible. Thus all the higher homotopy groups of K vanish and K has the homotopy type of S^1. ∎

5. The finiteness of W_k in case n is even. In case n is even, we will show that W_k has the homotopy type of the wedge of one copy of S^1 together with a finite number of copies of S^{k+1}. Notice that, in case n is even, $k = n/2 - 1$, so $n-k-1 = k+1$. It follows from Lemmas 2 and 3 that W_k has the homotopy type of a $(k+1)$-dimensional polyhedron L such that $\pi_1(L) \cong \mathbb{Z}$ and $\pi_i(L) = 0$ for $2 \leq i \leq k$. We must examine $\pi_{k+1}(L) \cong H_{k+1}(\widetilde{W}_k)$.

LEMMA 5.1. *$H_{k+1}(\widetilde{W}_k)$ is a free Λ-module.*

Proof. First we observe that $H_{k+1}(\widetilde{W}_k)$ is a projective module over Λ by [15, Lemma 2.1]. But every projective module over $\mathbb{Z}[\mathbb{J}]$ is free ([12] and [1]). ∎

LEMMA 5.2. *$H_{k+1}(W_k)$ is finitely generated over $\mathbb{Z}$.*

Proof. Let $X = S^n - W = M - W_k$. We begin by showing that the kth Čech cohomology group of X is finitely generated. Let U_i be one of the submanifolds of W_k constructed in the proof of Lemma 3. Then $U_i \cup X$ is a neighborhood of X in M and X has arbitrarily close neighborhoods of this kind. In the Mayer–Vietoris sequence

$$0 = H^{k-1}(U_i) \to H^k(M) \to H^k(U_i \cup X) \oplus H^k(W_k) \to H^k(U_i) = 0,$$

$H^k(W_k) = 0$, so there is a natural isomorphism from $H^k(U_i \cup X)$ to $H^k(M)$. Thus

$$\check{H}^k(X) = \varinjlim H^k(U_i \cup X) \cong H^k(M)$$

and $\check{H}^k(X)$ is finitely generated.

By Alexander Duality we have $H_{k+2}(M, W_k) \cong \check{H}^k(X)$. This means that both the first and the last terms in the exact sequence

$$H_{k+2}(M, W_k) \to H_{k+1}(W_k) \to H_{k+1}(M)$$

are finitely generated, so the middle term is as well. ∎

LEMMA 5.3. *$H_{k+1}(\widetilde{W}_k)$ is finitely generated over Λ.*

Proof. Let t denote a generator of the group of deck transformations of $\widetilde{W}_k$. The exact sequence

$$0 \to C_*(\widetilde{W}_k) \xrightarrow{t-1} C_*(\widetilde{W}_k) \xrightarrow{p_*} C_*(W_k) \to 0$$

of chain complexes gives rise to an exact sequence

$$\ldots \to H_{k+1}(\widetilde{W}_k) \xrightarrow{t-1} H_{k+1}(\widetilde{W}_k) \xrightarrow{p_*} H_{k+1}(W_k) \to 0$$

of homology groups. Thus

$$H_{k+1}(\widetilde{W}_k)/\operatorname{im}(t-1) \cong H_{k+1}(W_k).$$

Now $H_{k+1}(\widetilde{W}_k)$ is free over Λ by Lemma 5.1. Hence

$$H_{k+1}(\widetilde{W}_k) = \bigoplus_{i\in I} \Lambda_i$$

where each Λ_i is a copy of Λ and I is some countable indexing set. The homomorphism $(t-1)$ respects this decomposition, so we must look at $\Lambda/\operatorname{im}(t-1)$. Now Λ consists of all Laurent polynomials in t with coefficients in $\mathbb{Z}$. It is easy to see that two such polynomials p_1 and p_2 are equivalent over $\operatorname{im}(t-1)$ if and only if $p_1(1) = p_2(1)$. Thus $\Lambda/\operatorname{im}(t-1) \cong \mathbb{Z}$ and so

$$\begin{aligned} H_{k+1}(\widetilde{W}_k)/\operatorname{im}(t-1) &\cong \Big(\bigoplus_{i\in I} \Lambda_i\Big)\Big/\operatorname{im}(t-1) \\ &= \bigoplus_{i\in I}(\Lambda_i/\operatorname{im}(t-1)) \cong \bigoplus_{i\in I} \mathbb{Z}. \end{aligned}$$

On the other hand, $H_{k+1}(\widetilde{W}_k)/\operatorname{im}(t-1) \cong H_{k+1}(W_k)$, and so Lemma 5.2 implies that the indexing set I must be finite. Let us say that $I = \{1, \ldots, m\}$. ∎

LEMMA 5.4. *W_k has the homotopy type of $S^1 \vee (\bigvee_{i=1}^m S_i^{k+1})$.*

Proof. We know that W_k has the homotopy type of a $(k+1)$-dimensional polyhedron L with $\pi_1(L) \cong \mathbb{Z}$, $\pi_i(L) = 0$ for $2 \le i \le k$ and $\pi_{k+1}(L)$ a finitely generated free module over $\mathbb{Z}[\pi_1(L)]$. This is enough information to construct a natural map $S^1 \vee (\bigvee_{i=1}^m S_i^{k+1}) \to L$ which induces isomorphisms on π_i for $i \le k+1$. This map is a homotopy equivalence by the Whitehead Theorem [16, Theorem 1]. ∎

6. The proof of the Theorem. In this section we complete the proof of the Theorem. First we apply Siebenmann's thesis [10] to conclude that the end of W_k is collared. (We need some help from Quinn [9] in dimension 5.) Since W has the same end, this means that the end of W is collared as well. Thus there is a compact manifold $P \subset W$ such that $\overline{W - P} \cong \partial P \times [0, 1)$. The hypotheses (1.1)–(1.3) imply that $\pi_1(P) \cong \pi_1(\partial P) \cong \mathbb{Z}$ and $\pi_i(\partial P) = 0$ for $2 \le i \le k$.

Let $Q = \overline{S^n - P}$. Since S^n is simply connected, there must be a disk $(D, \partial D) \subset (Q, \partial Q)$ such that ∂D represents a generator of $\pi_1(\partial P)$. (See [4, p. 370].) Let B be an n-cell in Q, $B \cong D \times I^{n-2}$, such that $B \cap \partial P$ is a neighborhood of ∂D in ∂P. Define $S = \partial(P \cup B)$. Notice that S is obtained from ∂P by doing 1-surgery on a generator of $\pi_1(\partial P)$, so S is an $(n-1)$-manifold with $\pi_i(S) = 0$ for $i \le (n-1)/2$. The Poincaré Conjecture in dimensions ≥ 4 ([5] and [11]) tells us that S is an $(n-1)$-sphere. Then the Schoenflies Theorem [3] tells us that S bounds an n-cell in S^n. Hence Q consists of an n-cell with an $(n-2)$-handle attached. There is therefore a homeomorphism $h : S^{n-2} \times I^2 \to Q$. Finally, we see that $W \cong P \cup (\partial P \times [0, 1)) \cong S^n - h(S^{n-2} \times \{\text{point}\})$ and the proof is complete.

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