FUNDAMENTA
MATHEMATICAE
147 (1995)

A remark concerning random walks with random potentials

by

Yakov G. Sinai (Princeton, N.J., and Moscow)

Abstract. We consider random walks where each path is equipped with a random weight which is stationary and independent in space and time. We show that under some assumptions the arising probability distributions are in a sense uniformly absolutely continuous with respect to the usual probability distribution for symmetric random walks.

We consider random walks on the d-dimensional lattice $\mathbb{Z}^d$ with each path having a random statistical weight. Paths starting at (x,k) and ending at (y,n) will be denoted by $\omega_{x,k}^{y,n}$, i.e. $\omega_{x,k}^{y,n} = \{\omega(t) \in \mathbb{Z}^d,\ k \le t \le n,\ \omega(k) = x,\ \omega(n) = y,\ \|\omega(t+1) - \omega(t)\| = 1\}$. To define a random weight introduce a sequence of iid rv $F = \{F(x,t)\}$, $x \in \mathbb{Z}^d$, $t \in \mathbb{Z}$. Without any loss of generality we may assume that the $F(x,t)$ are given for all $x \in \mathbb{Z}^d$, $t \in \mathbb{Z}$. The space of all possible realizations of F is denoted by Φ. The measure corresponding to F is denoted by Q, the expectation with respect to Q is denoted by M. We do not use any special notation for the natural σ-algebra in Φ. Our main assumption concerning the distribution of $F(x,t)$ is

$$M \exp(2F(x,t)) < \infty.$$

The natural group of space-time translations acting in Φ is denoted by $\{T^{x,t}\}$. It preserves the measure Q.

We shall consider the statistical weight of $\omega_{x,k}^{y,n}$ equal to

$$\pi(\omega_{x,k}^{y,n}) = \exp\Big\{\sum_{t=k}^{n} F(t,\omega(t))\Big\}\frac{1}{(2d)^{n-k}}.$$

Introduce partition functions

$$Z_{x,k}^{y,n} = \sum_{\omega_{x,k}^{y,n}} \pi(\omega_{x,k}^{y,n}), \qquad Z_{x,k}^{n} = \sum_{y} Z_{x,k}^{y,n}.$$

1991 *Mathematics Subject Classification*: Primary 34F05.

Now we may define the "random" probability distribution $P^n_{F;\,x,k}$ defined on paths $\omega^{y,n}_{x,k}$ by the formula

$$p(\omega^{y,n}_{x,k}) = \frac{\pi(\omega^{y,n}_{x,k})}{Z^n_{x,k}}.$$

The induced probability distribution of $y = \omega(n)$ is

$$p^{y,n}_{x,k} = \sum_{\omega^{y,n}_{x,k}} p(\omega^{y,n}_{x,k}) = \frac{Z^{y,n}_{x,k}}{Z^n_{x,k}}.$$

We shall also need the usual transition probabilities

$$q^{(n-k)}(y-x) = \sum_{\omega^{y,n}_{x,k}} \frac{1}{(2d)^{n-k}}.$$

It is well known that for any $A > 0$ and y for which $\|y - x\| \le A\sqrt{n}$,

$$q^{(n-k)}(y-x) = \frac{1}{(2\pi(n-k)/d)^{d/2}} \exp\left\{-\frac{d\|y-x\|^2}{2(n-k)}\right\}(1 + \gamma^{(n-k)}(y-x)),$$

where $\|\cdot\|$ is the Euclidean norm and $\gamma^{(n-k)}(z)$ tends to zero uniformly in z satisfying the above-mentioned restrictions.

Our purpose in this note is to study the behavior of the distribution of the normalized displacement

$$\frac{\omega(n) - \omega(k)}{\sqrt{(n-k)/d}} = \frac{y - x}{\sqrt{(n-k)/d}}$$

with respect to $P^n_{F;\,x,k}$ as $n \to \infty$. The problem was considered by J. Imbrie and T. Spencer [3] and later by E. Bolthausen [1]. In [1] and [3], it was shown that if the $F(x,t)$ are small enough in appropriate sense, and $d \ge 3$, then the limiting distribution of the displacement is Gaussian and for typical F the mean of the square of displacement grows proportionally to time. Recently these results were extended to some random processes with continuous time by J. Conlon and P. Olsen [2]. All these results can be formulated also in terms of diffusion of directed polymers in random environments.

We show below that some of the results of [1] and [3] remain valid under weaker assumptions on the distribution of F and the dimension d. Define

$$\alpha_d = \sum_{n>0} \sum_z (q^{(n)}(z))^2.$$

This is finite if $d \ge 3$. Put

$$\Lambda = M \exp\{F(x,t)\} \quad \text{and} \quad \lambda = \frac{M \exp\{2F(x,t)\} - \Lambda^2}{\Lambda^2}.$$

Our main assumption is

$$\lambda \alpha_d < 1. \tag{1}$$

It is easy to see that (1) is valid for $d \geq 3$ if λ is small enough. If $F(x,t)$ takes two values $\pm c$ with probability $1/2$, then (1) is valid for those d for which $\alpha_d < 1$, and does not require the smallness of c. Indeed, in this case always $\lambda < 1$, i.e.,

$$M \exp\{2F(x,t)\} \leq 2\Lambda^2,$$

because this is equivalent to the obvious inequality

$$\frac{1}{2}(e^{2c} + e^{-2c}) \leq 2\left(\frac{e^c + e^{-c}}{2}\right)^2.$$

If the $F(x,t)$ have Gaussian distribution $N(0,\sigma)$, then (1) is valid for small enough σ.

Put

$$h(x,t) = \frac{\exp\{F(x,t)\} - \Lambda}{\Lambda}$$

and introduce the series

$$\varphi(x,k) = \sum_{r\geq 1} \sum_{k\leq k_1<\ldots<k_r} \sum_{z_1,\ldots,z_r} q^{(k_1-k)}(z_1 - x) q^{(k_2-k_1)}(z_2 - z_1) \ldots$$
$$\ldots q^{(k_r-k_{r-1})}(z_r - z_{r-1}) h(z_1,k_1) h(z_2,k_2) \ldots h(z_r,k_r),$$
$$\psi(y,n) = \sum_{r\geq 1} \sum_{k_1<\ldots<k_r\leq n} \sum_{z_1,\ldots,z_r} q^{(k_2-k_1)}(z_2 - z_1) \ldots$$
$$\ldots q^{(k_r-k_{r-1})}(z_r - z_{r-1}) q^{(n-k_r)}(y - z_r) h(z_1,k_1) \ldots h(z_r,k_r).$$

It is clear that $\varphi(x,t)$ and $\psi(y,t)$ constitute stationary (with respect to space-time translations) random fields, i.e. $\varphi(x,k) = T^{x,k}\varphi(0,0)$ and $\psi(y,n) = T^{y,n}\psi(0,0)$. Also they are transformed into each other by reversal of time in random walks. This implies, in particular, that the distributions of $\varphi(x,t)$ and $\psi(x,t)$ coincide.

Below we prove the following theorems.

Theorem 1. *If* (1) *is valid then the series giving* $\varphi(x,k)$ *and* $\psi(y,n)$ *converge in the space* $L^2(\Phi, Q)$.

Theorem 2. *If* (1) *is valid and* $\|y - x\| \leq A\sqrt{n-k}$ *where* A *is any constant, then the partition function* $Z^{y,n}_{x,k}$ *has the representation*

$$Z^{y,n}_{x,k} = \Lambda^{n-k+1} q^{(n-k)}(y - x)[(1 + \varphi(x,k))(1 + \psi(y,n)) + \delta^{(y,n)}_{(x,k)}],$$

where $M\delta^{y,n}_{x,k} = 0$ *and* $M(\delta^{y,n}_{x,k})^2 \to 0$ *as* $n \to \infty$, x,k *remain fixed and* y *satisfies the above-mentioned restriction.*

Proof of Theorem 1. It is clear that φ and ψ are represented as sums of orthogonal vectors in the space $L^2(\Phi, Q)$. Therefore

$$\begin{aligned} M\varphi^2(x,k) &= \sum_{r\ge 1} \lambda^r \sum_{k<k_1<\ldots<k_r} \sum_{z_1,\ldots,z_r} (q^{(k_1-k)}(z_1-x))^2 \\ &\quad \times (q^{(k_2-k_1)}(z_2-z_1))^2 \ldots (q^{(k_r-k_{r-1})}(z_r-z_{r-1}))^2 \\ &= \sum_{r\ge 1} (\lambda\alpha_d)^r < \infty. \end{aligned}$$

The same is true for $\psi(x,t)$. We also have $M\varphi(x,k) = M\psi(y,n) = 0$. ∎

Theorem 2 is proven in Appendix 1.

THEOREM 3. *If* (1) *holds then* $1+\varphi(x,t) > 0$ *and* $1+\psi(y,t) > 0$ *for* Q*-a.e.* F.

Proof. We already showed that $M\varphi(x,t) = M\psi(x,t) = 0$, $M\varphi^2(x,t) > 0$ and $M\psi^2(x,t) > 0$. It is enough to consider $\varphi(x,k)$ since $\varphi(x,k)$ and $\psi(y,n)$ have the same distribution. By Theorem 2,

$$\frac{Z_{x,k}^{y,n}}{\Lambda^{n-k+1}} - q^{(n-k)}(y-x)[(1+\varphi(x,k))(1+\psi(y,n))] = \delta_{(x,k)}^{(y,n)} q^{(n-k)}(y-x).$$

Take a continuous non-negative function f with compact support on $\mathbb{R}^d$, and write

$$\begin{aligned} &\sum_y \frac{Z_{x,k}^{y,n}}{\Lambda^{n-k+1}} f\left(\frac{x-y}{\sqrt{n-k}}\sqrt{d}\right) \\ &\qquad = (1+\varphi(x,k)) \sum_y q^{(n-k)}(y-x) f\left(\frac{y-x}{\sqrt{n-k}}\sqrt{d}\right)(1+\psi(y,n)) \\ &\qquad\quad + \sum_y q^{(n-k)}(y-x) f\left(\frac{y-x}{\sqrt{n-k}}\sqrt{d}\right) \delta_{(x,k)}^{(y,n)}. \end{aligned}$$

Theorem 2 immediately implies that the last term tends to zero in $L^2(\Phi, Q)$ for any fixed x, k and $n \to \infty$. Since $M\psi(y,n) = 0$ the sum

$$\sum_y q^{(n-k)}(y-x) f\left(\frac{y-x}{\sqrt{n-k}}\sqrt{d}\right)(1+\psi(y,n))$$

converges in $L^2(\Phi, Q)$ to $C = \int e^{-\|z\|^2/2} f(z)\, dz/(2\pi)^{d/2} > 0$. Thus

$$\operatorname*{l.i.m.}_{n\to\infty} \frac{1}{C} \sum_y \frac{Z_{x,k}^{y,n}}{\Lambda^{n-k+1}} f\left(\frac{y-x}{\sqrt{n-k}}\sqrt{d}\right) = 1 + \varphi(x,k).$$

Now we can use the obvious inequality

$$Z^{y,n}_{x,k-2} \geq \sum_{\langle x,x'\rangle} \left(\frac{1}{2d}\right)^2 e^{F(x,k-2)+F(x',k-1)} Z^{y,n}_{x,k} = g(x,k-2) Z^{y,n}_{x,k},$$

where the last expression gives also the definition of $g(x,k-2)$ which is positive a.e., and the sum is taken over x' such that $\|x-x'\|=1$. We use the notation $\langle x,x'\rangle$ for the nearest neighbors on the lattice. Thus we have

$$1+\varphi(x,k-2) \geq g(x,k-2)(1+\varphi(x,k)). \tag{2}$$

Assume that $1+\varphi(x,k-2)=0$ with positive probability. Take x and consider the set $\mathcal{H}^+$ of those numbers $2k$ such that $1+\varphi(x,2k)>0$. It follows from (2) that if $2k \in \mathcal{H}^+$ then $2k-2 \in \mathcal{H}^+$. Therefore $\mathcal{H}^+ = 2\mathbb{Z}^1$ for a.e. F. The ergodicity of $T^{0,2}$ implies that $Q(\{F : 1+\varphi(x,k)=0\})=0$. ■

Let the conditions of Theorem 2 be valid. As in the proof of Theorem 3 take a continuous function f on $\mathbb{R}^d$ with compact support. Using Theorem 2 we can write

$$\begin{aligned}
&\sum_y f\left(\frac{y-x}{\sqrt{n-k}}\sqrt{d}\right) \frac{Z^{y,n}_{x,k}}{Z^n_{x,k}} \\
&\quad = \frac{(1+\varphi(x,k))\Lambda^{n-k+1}}{Z^n_{x,k}} \\
&\qquad \times \left[\sum_y f\left(\frac{y-x}{\sqrt{n-k}}\sqrt{d}\right) q^{(n-k)}(y-x)(1+\psi(y,n)) \right.\\
&\qquad \left. + \sum_y f\left(\frac{y-x}{\sqrt{n-k}}\sqrt{d}\right) \delta^{(y,n)}_{(x,k)} q^{(n-k)}(y-x)\right].
\end{aligned} \tag{3}$$

Our estimations during the proof of Theorem 2 in the Appendix give

$$\operatorname*{l.i.m.}_{n\to\infty} \frac{Z^n_{x,k}}{\Lambda^{n-k+1}} = 1+\varphi(x,k).$$

Also the last sum in (3) tends to zero in $L^2(\Phi,Q)$ as $n\to\infty$. Therefore, we have the following theorem.

Theorem 4.

$$\operatorname*{l.i.m.}_{n\to\infty} \frac{1}{Z^n_{x,k}} \sum_y f\left(\frac{y-x}{\sqrt{n-k}}\sqrt{d}\right) Z^{y,n}_{x,k} = \int f(z) e^{-\|z\|^2/2} \frac{dz}{(2\pi)^{d/2}}.$$

This theorem shows in what sense the normalized displacement $(\omega(n)-\omega(k))\sqrt{d}/\sqrt{n-k}$ has the limiting Gaussian distribution. Its variance is the same as for the usual random walk.

Appendix

Proof of Theorem 2. We have

$$Z_{x,k}^{y,n} = \sum_{\omega_{x,k}^{y,n}} \exp\Big\{\sum_{t=k}^{n} F(t,\omega(t))\Big\} \frac{1}{(2d)^{n-k}}$$
$$= \sum_{\omega_{x,k}^{y,n}} \prod_{t=k}^{n} (\Lambda + \exp\{F(t,\omega(t))\} - \Lambda) \frac{1}{(2d)^{n-k}}$$
$$= \Lambda^{n-k+1} \sum_{\omega_{x,k}^{y,n}} \prod_{t=k}^{n} (1 + h(\omega(t),t)) \frac{1}{(2d)^{n-k}}$$
$$= \Lambda^{n-k+1} \Big[q^{(n-k)}(y-x) + \sum_{r\geq 1} \sum_{k\leq k_1 < \ldots < k_r < n} \sum_{z_1,\ldots,z_r} q^{(k_1-k)}(z_1 - x)$$
$$\times\, q^{(k_2-k_1)}(z_2 - z_1) \ldots q^{(k_r - k_{r-1})}(z_r - z_{r-1})$$
$$\times\, q^{(n-k_r)}(y - z_r) h(z_1,k_1) \ldots h(z_r,k_r) h(y,n) \Big].$$

In what follows we only deal with the finite sum

$$\widetilde{Z}_{x,k}^{y,n} = \sum_{r\geq 1} \sum_{k\leq k_1 < \ldots < k_r \leq n} \sum_{z_1,\ldots,z_r} q^{(k_1-k)}(z_1 - x) q^{(k_2-k_1)}(z_2 - z_1) \ldots$$
$$\ldots q^{(k_r-k_{r-1})}(z_r - z_{r-1}) q^{(n-k_r)}(y - z_r) h(z_1,k_1) \ldots h(z_r,k_r).$$

It is clear that $M\widetilde{Z}_{x,k}^{y,n} = 0$ and

$$M(\widetilde{Z}_{x,k}^{y,n})^2 = \sum_{r\geq 1} \lambda^r \sum_{k\leq k_1 < \ldots < k_r \leq n} \sum_{z_1,\ldots,z_r} (q^{(k_1-k)}(z_1 - x))^2$$
$$\times\, (q^{(k_2-k_1)}(z_2 - z_1))^2 \ldots (q^{(n-k_r)}(y - z_r))^2.$$

Fix some constant B whose value will be chosen later and consider

$$\widetilde{Z}_{x,k}^{y,n}(1) = \sum_{r\leq B\ln n} \sum_{k\leq k_1 < \ldots < k_r \leq n} \sum_{z_1,\ldots,z_r} q^{(k_1)}(z_1 - x)$$
$$\times\, q^{(k_2-k_1)}(z_2 - z_1) \ldots q^{(k_r-k_{r-1})}(z_r - z_{r-1})$$
$$\times\, q^{(n-k_r)}(y - z_r) h(z_1,k_1) \ldots h(z_r,k_r).$$

Let $\widetilde{Z}_{x,k}^{y,n}(2)$ be a similar sum where $r > B\ln n$. Then the trivial estimation gives

$$M(\widetilde{Z}_{x,k}^{y,n}(2))^2 \leq \sum_{r > B\ln n} (\lambda\alpha_d)^r = \frac{(\lambda\alpha_d)^{B\ln n}}{1 - \lambda\alpha_d}.$$

Take B so large that

$$\frac{(\lambda\alpha_d)^{B\ln n}}{1-\lambda\alpha_d} \le \frac{1}{n^{2d}} \quad \text{for all large enough } n.$$

We can write

$$\frac{Z_{x,k}^{y,n}}{\Lambda^{n-k+1}} = q^{(n-k)}(y-x)(1+\widetilde{Z}_{x,k}^{y,n}(1)+\widetilde{Z}_{x,k}^{y,n}(2)).$$

From our estimations it follows that

(i) for all y with $\|y-x\| \le A\sqrt{n-k}$ the ratio $\widetilde{Z}_{x,k}^{y,n}(2)/q^{(n-k)}(y-x)$ tends to zero in $L^2(\Phi,Q)$ uniformly in y;

(ii) for any continuous function f with compact support, the sum

$$\sum_y f\left(\frac{y-x}{\sqrt{(n-k)/d}}\right)\widetilde{Z}_{x,k}^{y,n}(2)$$

converges to zero in $L^2(\Phi,Q)$.

Thus it remains to study $\widetilde{Z}_{x,k}^{y,n}(1)$ assuming $\|y-x\| \le A\sqrt{n-k}$. Let us call an interval (k_{j-1},k_j) *large* if $k_j-k_{j-1} \ge n^\beta$ for some β with $1/2<\beta<1$. Here $k_0=k$, $k_{r+1}=n$. If $r \le B\ln n$ then at least one large interval in the sequence $(0,k_1,k_2,\dots,k_r,n)$ is present. We shall show that the main contribution to $\widetilde{Z}_{x,k}^{y,n}(1)$ comes from r-tuples $(k_1,k_2,\dots,k_r)$ with only one large interval. Write

$$\begin{aligned}
&\widetilde{Z}_{x,k}^{y,n}(1,1)\\
&= \sum_{\substack{0\le r_1\le B\ln n\\ 0\le r_2\le B\ln n\\ 1\le r=r_1+r_2\le B\ln n}} \sum_{\substack{k\le k_1<\dots<k_r\le n\\ (k_{r_1},k_{r_1+1}) \text{ is the unique}\\ \text{large interval}}} \sum_{z_1,\dots,z_r} q^{(k_1)}(z_1-x)\\
&\quad \times q^{(k_2-k_1)}(z_2-z_1)\dots q^{(k_{r_1+1}-k_{r_1})}(z_{r_1+1}-z_{r_1})\\
&\quad \times q^{(k_{r_1+2}-k_{r_1+1})}(z_{r_1+2}-z_{r_1+1})\dots q^{(k_r-k_{r-1})}(z_r-z_{r-1})q^{(n-k_r)}(y-z_r)\\
&\quad \times [h(z_1,k_1)\dots h(z_{r_1},k_{r_1})]\cdot[h(z_{r_1+1},k_{r_1+1})\dots h(z_r,k_r)].
\end{aligned}$$

We can write

$$\frac{\widetilde{Z}_{x,k}^{y,n}(1,1)}{q^{(n)}(y-x)} = (1+\varphi(x,k))(1+\psi(y,n))-1+\delta_{(x,k)}^{(y,n)}(2).$$

The last formula also implies the definition of $\delta_{(x,k)}^{(y,n)}(2)$. Since we can restrict ourselves by summation over those $(z_1,\dots,z_r)$ where $\|z_{r_1}-x\| \le n^{2\beta}$, $\|z_{r_1+1}-y\| \le n^{2\beta}$, the summation over all other z is exceedingly small. Thus $M(\delta_{x,k}^{y,n}(2))^2 \to 0$ as $n\to\infty$ uniformly over all y under consideration.

The rest of our argument is to show that the contribution of r-tuples where the number of large intervals is greater than 1 is relatively small. Again we write down the square of the norm of the corresponding sum:

$$S_{x,k}^{y,n} = \alpha_d \sum_{r\ge 1} (\alpha_d \lambda)^r \sum_{k\le k_1<\ldots<k_r\le n} \sum_{z_1,\ldots,z_r} p^{(k_1-k)}(z_1-x) \times p^{(k_2-k_1)}(z_2-z_1)\ldots p^{(k_r-k_{r-1})}(z_2-z_1)\ldots \ldots p^{(k_r-k_{r-1})}(z_r-z_{r-1})p^{(n-k_r)}(y-z_r),$$

where $p^{(i)}(z) = (q^{(i)}(z))^2/\alpha_d$. The last double sum can again be considered as the probability that the sum $\vec{\eta}_1+\ldots+\vec{\eta}_r$ takes the values $y-x$, $n-k$, where $\vec{\eta}_j = (z_j - z_{j-1}, k_j - k_{j-1})$. It is easy to show that the distribution of the time component of η_j decays as $\mathrm{const}/t^{d/2}$. Direct probabilistic arguments show that the probability to have at least two values of j for which the value of the "time" component is greater than n^β decays as $1/n^{(\beta+1)d}$. This shows that the contribution of terms with two large increments $(k_j - k_{j-1})$ to $S_{x,k}^{y,n}(1)$ is small in $L^2(\Phi, Q)$ compared with the norm of $q^{(n-k)}(y-x)$. We omit the details.

I thank K. M. Khanin and Yu. I. Kifer for useful discussions.

The financial support from NSF (grant DMS-9404437) and from the Russian Foundation of Fundamental Research (grant N93-01-16090) are highly appreciated.

References

[1] E. Bolthausen, *A note on the diffusion of directed polymers in a random environment*, Comm. Math. Phys. 123 (1989), 529–534.

[2] J. G. Conlon and P. A. Olson, *A Brownian motion version of directed polymer problem*, preprint, University of Michigan, 1994.

[3] J. Z. Imbrie and T. Spencer, *Diffusion of directed polymers in a random environment*, J. Statist. Phys. 52 (1988), 609–626.

MATHEMATICS DEPARTMENT
PRINCETON UNIVERSITY
PRINCETON, NEW JERSEY 08540
U.S.A.

LANDAU INSTITUTE OF THEORETICAL PHYSICS
MOSCOW, RUSSIA

Received 5 December 1994;
in revised form 30 January 1995