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Parametrized Cichoń's diagram and small sets

by

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Abstract. We parametrize Cichoń's diagram and show how cardinals from Cichoń's diagram yield classes of small sets of reals. For instance, we show that there exist subsets N and M of $\omega^\omega \times 2^\omega$ and continuous functions $e, f : \omega^\omega \to \omega^\omega$ such that

- N is $\mathbf{G}_\delta$ and $\{\mathrm{N}_x : x \in \omega^\omega\}$, the collection of all vertical sections of N, is a basis for the ideal of measure zero subsets of 2^ω;
- M is $\mathbf{F}_\sigma$ and $\{\mathrm{M}_x : x \in \omega^\omega\}$ is a basis for the ideal of meager subsets of 2^ω;
- $\forall x, y\ \mathrm{N}_{e(x)} \subseteq \mathrm{N}_y \Rightarrow \mathrm{M}_x \subseteq \mathrm{M}_{f(y)}$.

From this we derive that for a separable metric space X,

- if for all Borel (resp. $\mathbf{G}_\delta$) sets $B \subseteq X \times 2^\omega$ with all vertical sections null, $\bigcup_{x\in X} B_x$ is null, then for all Borel (resp. $\mathbf{F}_\sigma$) sets $B \subseteq X \times 2^\omega$ with all vertical sections meager, $\bigcup_{x\in X} B_x$ is meager;
- if there exists a Borel (resp. a "nice" $\mathbf{G}_\delta$) set $B \subseteq X \times 2^\omega$ such that $\{B_x : x \in X\}$ is a basis for measure zero sets, then there exists a Borel (resp. $\mathbf{F}_\sigma$) set $B \subseteq X \times 2^\omega$ such that $\{B_x : x \in X\}$ is a basis for meager sets.

0. Introduction. Let $\mathcal{S}$ be a family of subsets of the Cantor set 2^ω. The covering number of $\mathcal{S}$ is (by convention, $\min(\emptyset) = \infty$)

$$\mathrm{cov}(\mathcal{S}) = \min\Big\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{S} \ \&\ \bigcup \mathcal{A} = 2^\omega\Big\}.$$

We can say that an abstract set X is "cov($\mathcal{S}$)-small", in a cardinal sense, iff for every choice $\{S_x : x \in X\}$ of sets from $\mathcal{S}$, $\bigcup_{x\in X} S_x$ does not cover 2^ω. Similarly, we can say that a separable metric space X is "cov($\mathcal{S}$)-small", in a continuous (resp. Borel) sense, iff this holds for every "continuous" (resp. "Borel") choice.

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The additivity number of $\mathcal{S}$ is defined by

$$\operatorname{add}(\mathcal{S}) = \min\Big\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{S} \ \&\ \bigcup \mathcal{A} \notin \mathcal{S}\Big\}.$$

As with $\operatorname{cov}(\mathcal{S})$ we can talk about "$\operatorname{add}(\mathcal{S})$-small" spaces.

A study of such "small" spaces may give new insight, as shown by the following results.

(1) Recław [R1] proved that every Lusin set is undetermined in the Point-Open Game, which solved a problem of Galvin [G]. The proof relied on the facts that

- if $X \subseteq 2^\omega$ is a Lusin set, then for every closed set $D \subseteq X \times \omega^\omega$ with all vertical sections D_x ($x \in X$) meager, $\bigcup_{x\in X} D_x \neq \omega^\omega$;
- $X \subseteq 2^\omega$ is undetermined in the Point-Open Game iff for every closed set $D \subseteq X \times \omega^\omega$ with all vertical sections D_x ($x \in X$) meager, $\bigcup_{x\in X} D_x \neq \omega^\omega$.

(2) Pawlikowski's [P2] proof that every Sierpiński set is strongly meager (another problem of Galvin, see [Mi3]) shows, in fact, that if X is a Sierpiński set and $B \subseteq X \times 2^\omega$ is a Borel set with all vertical sections null, then $\bigcup_{x\in X} B_x \neq 2^\omega$ (see [P3]).

(3) A crucial step in Raisonnier's [Ra] proof of Shelah's theorem that Lebesgue measurability of all sets of reals is equiconsistent with the existence of inaccessible cardinals is a construction of a rapid filter. The filter is obtained from a set $X \subseteq 2^\omega$ such that for all $\mathbf{G}_\delta$ sets $G \subseteq 2^\omega \times 2^\omega$ with all vertical sections null, $\bigcup_{x\in X} G_x$ is null.

Let $\mathcal{N}$ and $\mathcal{M}$ be the σ-ideals of null (measure zero) and meager subsets of 2^ω. To have a uniform treatment of cardinal characteristics associated with $\mathcal{N}$ and $\mathcal{M}$ we proceed as follows (see also [F2] and [V]).

For a binary relation ϱ let

$$\mathtt{B}(\varrho) = \{A \subseteq \operatorname{dom}(\varrho) : \exists y \in \operatorname{rng}(\varrho)\ \forall a \in A\ a\varrho y\},$$
$$\mathtt{D}(\varrho) = \{A \subseteq \operatorname{rng}(\varrho) : \neg\forall x \in \operatorname{dom}(\varrho)\ \exists a \in A\ x\varrho a\}.$$

Note that $\mathtt{D}(\varrho) = \mathtt{B}(\neg\varrho^{-1})$. Let $\mathtt{b}(\varrho)$ (resp. $\mathtt{d}(\varrho)$) be the minimal cardinality of a subset of $\operatorname{dom}(\varrho)$ (resp. $\operatorname{rng}(\varrho)$) which is not in $\mathtt{B}(\varrho)$ (resp. $\mathtt{D}(\varrho)$).

If $\mathcal{S}$ is a family of subsets of X, let

$$\operatorname{add}(\mathcal{S}) = \mathtt{b}(\subseteq \cap (\mathcal{S}\times\mathcal{S})), \quad \operatorname{cof}(\mathcal{S}) = \mathtt{d}(\subseteq \cap (\mathcal{S}\times\mathcal{S})),$$
$$\operatorname{non}(\mathcal{S}) = \mathtt{b}(\in \cap (X\times\mathcal{S})), \quad \operatorname{cov}(\mathcal{S}) = \mathtt{d}(\in \cap (X\times\mathcal{S})).$$

Let also, as usual,

$$\mathtt{b} = \mathtt{b}(\preceq), \quad \mathtt{d} = \mathtt{d}(\preceq),$$

where for $x, y \in \omega^\omega$, $x \preceq y$ iff $\forall^\infty n\ x(n) \le y(n)$. (We write "$\forall^\infty$" for "for all but finitely many" and "$\exists^\infty$" for "there exist infinitely many".)

The following diagram is called *Cichoń's diagram* (see [F2], [V]) ($\to$ means that inequality $\le$ is provable in ZFC):

$$\begin{array}{ccccccc}
\mathrm{cov}(\mathcal{N}) & \to & \mathrm{non}(\mathcal{M}) & \to & \mathrm{cof}(\mathcal{M}) & \to & \mathrm{cof}(\mathcal{N}) \\
 & & \uparrow & & \uparrow & & \\
\uparrow & & \mathtt{b} & \to & \mathtt{d} & & \uparrow \\
 & & \uparrow & & \uparrow & & \\
\mathrm{add}(\mathcal{N}) & \to & \mathrm{add}(\mathcal{M}) & \to & \mathrm{cov}(\mathcal{M}) & \to & \mathrm{non}(\mathcal{N})
\end{array}$$

The diagram is complete in the sense that no arrow which is not obtained by composing the old ones can be added to it (see [BJ] for a summary of the necessary consistency results). The proofs of the inequalities in Cichoń's diagram are highly constructive. To show that $\mathrm{add}(\mathcal{M}) \le \mathtt{b}$ Miller [Mi2] takes $A \subseteq \omega^\omega$, $A \notin \mathtt{B}(\preceq)$, constructs for each $a \in A$ a meager set M_a and shows that should $\bigcup_{a\in A} M_a$ be meager, one could construct a function $b \in \omega^\omega$ with $\forall a \in A\ a \preceq b$, thus violating $A \notin \mathtt{B}(\preceq)$. Fremlin [F2] rephrased this as follows: there exist functions $e : \omega^\omega \to \mathcal{M}$ and $f : \mathcal{M} \to \omega^\omega$ such that $\forall x, y\ e(x) \subseteq y \Rightarrow x \preceq f(y)$. Clearly, the existence of such functions implies that $\mathrm{add}(\mathcal{M}) \le \mathtt{b}$ and $\mathtt{d} \le \mathrm{cof}(\mathcal{M})$.

The most difficult inequality in Cichoń's diagram is $\mathrm{add}(\mathcal{N}) \le \mathrm{add}(\mathcal{M})$, proved by Bartoszyński [B1] (independently by Raisonnier and Stern [RaSt]). Fremlin [F1] noted that the arguments of [B1] and [RaSt] lead to functions $e : \omega^\omega \to \mathcal{N}$ and $f : \mathcal{N} \to \prod_n [\omega]^{\le n}$ such that $\forall x, y\ e(x) \subseteq y \Rightarrow x \in^* f(y)$ (where $x \in^* z$ iff $\forall^\infty n\ x(n) \in z(n)$). He also noted that Pawlikowski [P1], in a proof that the Lebesgue measurability of all Σ^1_2 (lightface!) sets implies the Baire property for all such sets, constructed functions $e : \mathcal{M} \to \omega^\omega$ and $f : \prod_n [\omega]^{\le n} \to \mathcal{M}$ such that $\forall x, y\ e(x) \in^* y \Rightarrow x \subseteq f(y)$. Putting this together (see [F2]) we get functions $e : \mathcal{M} \to \mathcal{N}$ and $f : \mathcal{N} \to \mathcal{M}$ such that $\forall x, y\ e(x) \subseteq y \Rightarrow x \subseteq f(y)$. Again, the existence of such functions yields inequalities $\mathrm{add}(\mathcal{N}) \le \mathrm{add}(\mathcal{M})$ and $\mathrm{cof}(\mathcal{M}) \le \mathrm{cof}(\mathcal{N})$.

The picture was completed by Vojtáš [V], who wrote explicitly the remaining inequalities (and some others) in the "e-f" language.

In the present paper we shall show that all the "e-f" functions involved in Cichoń's diagram can be defined so that they are "continuous". This would enable us to convert the inequalities into inclusions of the classes of the corresponding "small" spaces. For instance, we shall prove that "$\mathrm{add}(\mathcal{N})$-small" spaces are "$\mathrm{add}(\mathcal{M})$-small" and "$\mathrm{cof}(\mathcal{M})$-small" spaces are "$\mathrm{cof}(\mathcal{N})$-small". More precisely, we shall show that for a separable metric space X, if for all Borel (resp. $\mathbf{G}_\delta$) sets $B \subseteq X \times 2^\omega$ with all vertical sections null, $\bigcup_{x\in X} B_x$ is null, then for all Borel (resp. $\mathbf{F}_\sigma$) sets $B \subseteq X \times 2^\omega$ with all vertical sections meager, $\bigcup_{x\in X} B_x$ is meager. We shall also prove that if X is "$\mathrm{cof}(\mathcal{N})$-big" in the sense that there exists a Borel set $B \subseteq X \times 2^\omega$ all of whose vertical sections constitute a basis of $\mathcal{N}$ (i.e. all B_x have measure zero and every measure zero set is covered by some B_x), then X is "cof($\mathcal{M}$)-big' in a similar sense.

This paper is an expanded version of [R2]. Recław [R1] undertook a systematic study of small sets defined by "definable" choices of sections. Presenting [R1] at a seminar talk in March 1992, he advocated for Cichoń's diagram for such sets. He gave a mixed "Borel-continuous" version [R2] of it at a meeting in Katowice, October 1992. Shortly after the Katowice meeting Pawlikowski proved that Bartoszyński's inequality has a "continuous" version in the "e-f" language, which together with a folklore fact that the remaining inequalities do have such versions, gave our parametrized diagram.

1. Parametrization. For each n fix an enumeration $\langle N_i^n : i \in \omega\rangle$ of all clopen subsets of 2^ω of measure $\le 2^{-n-4}$ and let $\#(N_i^n, n) = i$. Fix also an enumeration $\langle \tau_i^n : i \in \omega\rangle$ of $\bigcup_{m>n} 2^{[n,m)}$ and let $\#(\tau_i^n) = i$. Let $M_i^n = [\tau_i^n]$, where $[\tau] = \{t \in 2^\omega : \tau \subseteq t\}$. Note that for $A \subseteq 2^\omega$,

$$A \in \mathcal{N} \quad \text{iff} \quad \exists a \in \omega^\omega\ A \subseteq \bigcap_m \bigcup_{n>m} N^n_{a(n)},$$

$$A \in \mathcal{M} \quad \text{iff} \quad \exists a \in \omega^\omega\ A \subseteq 2^\omega \setminus \bigcap_m \bigcup_{n>m} M^n_{a(n)}.$$

This suggests the following definition.

1.1. Definition. Let X be a zero-dimensional separable metric space. For $A \subseteq X \times 2^\omega$ say that $A \in \mathcal{N}^*(X)$, resp. $A \in \mathcal{M}^*(X)$, iff there exists a $*$ function $a : X \to \omega^\omega$ with

$$A_x \subseteq \bigcap_m \bigcup_{n>m} N^n_{a(x)(n)} \qquad (x \in X),$$

resp. with

$$A_x \subseteq 2^\omega \setminus \bigcap_m \bigcup_{n>m} M^n_{a(x)(n)} \qquad (x \in X).$$

We shall consider two cases: Borel functions ($* = \ddagger$) and continuous functions ($* = \dagger$).

1.2. Lemma. *Let $A \subseteq X \times 2^\omega$.*

(a) $A \in \mathcal{N}^\ddagger(X)$ *iff there exists a Borel set $B \subseteq X \times 2^\omega$ with all vertical sections null such that $A \subseteq B$.*

(b) $A \in \mathcal{M}^\ddagger(X)$ *iff there exists a Borel set $B \subseteq X \times 2^\omega$ with all vertical sections meager such that $A \subseteq B$.*

(c) $A \in \mathcal{N}^\dagger(X)$ *iff for any sequence $\varepsilon_n > 0$ ($n \in \omega$) there exist clopen sets $A_n \subseteq X \times 2^\omega$ such that $\mu((A_n)_x) \le \varepsilon_n$ ($x \in X$) and $A \subseteq \bigcup_n A_n$.*

(d) $A \in \mathcal{M}^\dagger(X)$ *iff there exists an* $\mathbf{F}_\sigma$ *set* $B \subseteq X \times 2^\omega$ *with all vertical sections meager such that* $A \subseteq B$.

Proof. The $\Rightarrow$ directions are easy. We concentrate on the $\Leftarrow$ directions. Since (a) and (b) are folklore (see [Ke]), we pass to (c) and (d).

(c) $\Leftarrow$: Suppose that for any sequence $\varepsilon_n > 0$ ($n \in \omega$) there exist clopen sets $A_n \subseteq X \times 2^\omega$ such that $\mu((A_n)_x) \le \varepsilon_n$ ($x \in X$) and $A \subseteq \bigcup_n A_n$. Then for any sequence $\varepsilon_n > 0$ ($n \in \omega$) there exist clopen sets $A_n \subseteq X \times 2^\omega$ such that $\mu((A_n)_x) \le \varepsilon_n$ ($x \in X$) and $A \subseteq \bigcap_m \bigcup_{n>m} A_n$ (split ω into infinitely many infinite sets). Let now $A_n \subseteq X \times 2^\omega$ ($n \in \omega$) be clopen sets such that $A \subseteq \bigcap_m \bigcup_{n>m} A_n$ and $\forall x\ \mu((A_n)_x) \le 2^{-n-4}$. Note that each $(A_n)_x$ is a clopen subset of 2^ω of measure $\le 2^{-n-4}$. Define $a : X \to \omega^\omega$ by

$$a(x)(n) = \#((A_n)_x, n).$$

Then

$$A_x \subseteq \bigcap_m \bigcup_{n>m} (A_n)_x = \bigcap_m \bigcup_{n>m} N^n_{a(x)(n)}.$$

It remains to see that a is continuous. Since 2^ω is compact, the projection of a closed subset of $X \times 2^\omega$ onto X is closed. Thus for any clopen $U \subseteq 2^\omega$, the sets $\{x \in X : U \subseteq (A_n)_x\}$ and $\{x \in X : U \supseteq (A_n)_x\}$ are open (as the complements of the projections of $(X \times U) \setminus A_n$ and $A_n \setminus (X \times U)$). It follows that for any clopen U, $\{x \in X : U = (A_n)_x\}$ is open.

(d) $\Leftarrow$: Suppose that

$$B = \bigcup_n B_n \subseteq X \times 2^\omega,$$

where each B_n is closed and has all vertical sections nowhere dense. For each n fix $\{i(n,k) : k \in \omega\}$ and a cover of X by a family $\{U^n_k : k \in \omega\}$ of pairwise disjoint clopen subsets of X so that

$$U^n_k \times [\tau^n_{i(n,k)}] \cap \bigcup_{m \le n} B_m = \emptyset.$$

Then define $a : X \to \omega^\omega$ by $a(x)(n) = i(n,k) \Leftrightarrow x \in U^n_k$. ∎

Note. It is useful to remember that for any separable metric space X and Borel set $B \subseteq X \times 2^\omega$, the sets $\{x \in X : B_x \in \mathcal{M}\}$ and $\{x \in X : B_x \in \mathcal{N}\}$ are Borel (a theorem of Novikov, see [Ke]).

1.3. Definition. Let

$$\mathrm{N} = \bigcap_m \bigcup_{n>m} \bigcup_{x \in \omega^\omega} \{x\} \times N^n_{x(n)}$$

be the measure master set used above. Clearly $\mathrm{N} \in \mathcal{N}^\dagger(\omega^\omega)$, all vertical sections N_x are in $\mathcal{N}$ and $A \in \mathcal{N}$ iff $\exists x\ A \subseteq \mathrm{N}_x$. (So, the family $\{\mathrm{N}_x : x \in \omega^\omega\}$

is a basis of the ideal $\mathcal{N}$.) Also, for $A \subseteq X \times 2^\omega$, $A \in \mathcal{N}^*(X)$ iff there exists a $*$ function $a : X \to \omega^\omega$ with $A_x \subseteq \mathrm{N}_{a(x)}$ $(x \in X)$.

Similar remarks are true for $\mathcal{M}$ and the meager master set

$$\mathrm{M} = 2^\omega \setminus \bigcap_m \bigcup_{n>m} \bigcup_{x\in\omega^\omega} M^n_{x(n)}.$$

1.4. Definition. Let S, T be binary relations, whose dom's and rng's are equipped with some topologies. Write $S \to T$ iff there are continuous functions

$$e : \mathrm{dom}(T) \to \mathrm{dom}(S), \qquad f : \mathrm{rng}(S) \to \mathrm{rng}(T)$$

such that $f \circ S \circ e \subseteq T$, i.e.,

$$\forall x, y\ \langle e(x), y\rangle \in S \Rightarrow \langle x, f(y)\rangle \in T$$

(equivalently, $\forall x\ f[S_{e(x)}] \subseteq T_x$). Write $S \leftrightarrow T$ iff $S \to T$ and $T \to S$.

We shall use this notion in the following context. Suppose that we have functions $\mathrm{A}, \mathrm{B}, \mathrm{C}, \mathrm{D}$ and relations $\varrho \supseteq \mathrm{rng}(\mathrm{A}) \times \mathrm{rng}(\mathrm{B})$ and $\sigma \supseteq \mathrm{rng}(\mathrm{A}) \times \mathrm{rng}(\mathrm{B})$. Let

$$\varrho^{\mathrm{B}}_{\mathrm{A}} = \{\langle x, y\rangle \in \mathrm{dom}(\mathrm{A}) \times \mathrm{dom}(\mathrm{B}) : \mathrm{A}(x)\varrho\mathrm{B}(y)\},$$
$$\sigma^{\mathrm{D}}_{\mathrm{C}} = \{\langle x, y\rangle \in \mathrm{dom}(\mathrm{C}) \times \mathrm{dom}(\mathrm{D}) : \mathrm{C}(x)\sigma\mathrm{D}(y)\}.$$

Then $\varrho^{\mathrm{B}}_{\mathrm{A}} \to \sigma^{\mathrm{D}}_{\mathrm{C}}$ iff there are continuous functions

$$e : \mathrm{dom}(\mathrm{C}) \to \mathrm{dom}(\mathrm{A}), \qquad f : \mathrm{dom}(\mathrm{B}) \to \mathrm{dom}(\mathrm{D})$$

such that

$$\mathrm{A}(e(x))\varrho\mathrm{B}(y) \Rightarrow \mathrm{C}(x)\sigma\mathrm{D}(f(y)).$$

If the functions e, f above are only Borel, we replace $\to$ with $\Rightarrow$. In such contexts sets like M are treated as functions $x \to \mathrm{M}_x$. Let I (resp. J) be the identity map from ω^ω to ω^ω (resp. from 2^ω to 2^ω).

Let $\mathbf{Q} = \{t \in 2^\omega : \forall^\infty n\ t(n) = 0\}$ and $\mathbf{P} = 2^\omega \setminus \mathbf{Q}$. Since $\mathbf{P}$ is homeomorphic to ω^ω, we shall often identify them. In particular, we can write $\mathrm{I} : \mathbf{P} \to \omega^\omega$, $\omega^\omega \subseteq 2^\omega$, etc.

Note. Note that $\varrho^{\mathrm{B}}_{\mathrm{A}} \to \sigma^{\mathrm{D}}_{\mathrm{C}}$ iff $(\neg\sigma^{-1})^{\mathrm{C}}_{\mathrm{D}} \to (\neg\varrho^{-1})^{\mathrm{A}}_{\mathrm{B}}$.

1.5. Theorem (Parametrized Cichoń's diagram).

$$\begin{array}{ccccccccccc}
\not\ni^{\mathrm{J}}_{\mathrm{N}} & \to & \in^{\mathrm{M}}_{\mathrm{J}} & \substack{\rightarrow \\ \Leftarrow} & \in^{\mathrm{M}}_{\mathrm{I}} & & \to & & \not\supseteq^{\mathrm{M}}_{\mathrm{M}} & \to & \not\supseteq^{\mathrm{N}}_{\mathrm{N}} \\
 & & & & \uparrow & & & \nearrow & & & \\
\uparrow & & \uparrow & & \preceq^{\mathrm{I}}_{\mathrm{I}} & \to & \not\preceq^{\mathrm{I}}_{\mathrm{I}} & & \uparrow & & \uparrow \\
 & & & \nearrow & & & \uparrow & & & & \\
\subseteq^{\mathrm{N}}_{\mathrm{N}} & \to & \subseteq^{\mathrm{M}}_{\mathrm{M}} & & \to & & \not\ni^{\mathrm{I}}_{\mathrm{M}} & \substack{\rightarrow \\ \Leftarrow} & \not\ni^{\mathrm{J}}_{\mathrm{M}} & \to & \in^{\mathrm{N}}_{\mathrm{J}}
\end{array}$$

By the note following Definition 1.4, in order to prove the above theorem we have to deal with half + one arrows (if we know that $\not\ni^{\mathrm{J}}_{\mathrm{M}} \rightarrow \in^{\mathrm{N}}_{\mathrm{J}}$ then we know that $\not\ni^{\mathrm{J}}_{\mathrm{N}} \rightarrow \in^{\mathrm{M}}_{\mathrm{J}}$). The proof is divided into Lemmas 1.7–1.15.

First we need one more definition.

1.6. Definition. For $y \in \omega^\omega$ and $u \in ([\omega]^{<\omega})^\omega$ write $y \in^* u$ and say that u *localizes* y iff $\forall^\infty n\ y(n) \in u(n)$. We extend this notion to sets in the following obvious way: if u localizes each y from some $Y \subseteq \omega^\omega$ we say that u *localizes* Y or that Y is *localizable* by u; if for each $y \in Y$ there is $u \in U$ that localizes y we say that U *localizes* Y.

For each n, let $\langle L^n_i : i \in \omega \rangle$ be a fixed enumeration of of $[\omega]^{\le 2^n}$. Let $\#(L^n_i, n) = i$. Define $\mathrm{L} : \omega^\omega \to \prod_n [\omega]^{\le 2^n}$ by $\mathrm{L}(x) = \langle L^n_{x(n)} : n \in \omega \rangle$.

Note. For technical reasons we use the sequence $\langle 2^n : n \in \omega \rangle$, however, any sequence $a = \langle a_n : n \in \omega \rangle \in \omega^\omega$ with $\lim_n a_n = \infty$ will do ($\in^{*\mathrm{L}_a}_{\mathrm{I}} \leftrightarrow \in^{*\mathrm{L}}_{\mathrm{I}}$, where L_a is defined as L with $\langle a_n : n \in \omega \rangle$ in place of $\langle 2^n : n \in \omega \rangle$).

1.7. Lemma. $\subseteq^{\mathrm{N}}_{\mathrm{N}} \leftrightarrow \in^{*\mathrm{L}}_{\mathrm{I}}$.

Proof. $\leftarrow$: We seek continuous functions $e, f : \omega^\omega \to \omega^\omega$ such that

$$\forall^\infty n\ e(x)(n) \in L^n_{y(n)} \Rightarrow \mathrm{N}_x \subseteq \mathrm{N}_{f(y)}.$$

Define

$$e(x)(n) = \#(N^{2n+1}_{x(2n+1)} \cup N^{2n+2}_{x(2n+2)}, 2n),$$
$$f(y)(n) = \#\Big(\bigcup\{N^{2n}_i : i \in L^n_{y(n)}\}, n\Big).$$

$\rightarrow$: We seek continuous functions $e, f : \omega^\omega \to \omega^\omega$ such that

$$\mathrm{N}_{e(x)} \subseteq \mathrm{N}_y \Rightarrow \forall^\infty n\ x(n) \in L^n_{f(y)(n)}.$$

Let $\{V^n_i : i, n \in \omega\}$ be a matrix of measure independent clopen subsets of 2^ω such that $\mu(V^n_i) = 2^{-n-4}$. Define e by

$$e(x)(n) = \#(V^n_{x(n)}, n).$$

Clearly e is continuous and for each $x \in \omega^\omega$, $\mathrm{N}_{e(x)} = \bigcap_m \bigcup_{n>m} V^n_{x(n)}$.

The definition of f is longer. Fix an enumeration U_k $(k > 0)$ of all clopen subsets of 2^ω. Let $A_\emptyset = \emptyset$ and define inductively, for $\sigma \in \omega^{2k}$ $(k > 0)$,

$$B_\sigma = A_{\sigma|2(k-1)} \cup \bigcup_{n<2k} N^n_{\sigma(n)},$$
$$W_\sigma = \begin{cases} U_k & \text{if } \mu(U_k \setminus B_\sigma) < 2^{-2k}, \\ \emptyset & \text{otherwise,} \end{cases}$$
$$A_\sigma = B_\sigma \cup W_\sigma.$$

Finally, let

$$A = \bigcup_k \bigcup_{\sigma \in \omega^{2k}} [\sigma] \times A_\sigma.$$

Claim 1. (a) $\mathrm{N} \subseteq A$ *and* $\forall y \in \omega^\omega$ $\forall k$ $\mu(A_y \setminus A_{y|2k}) < 2^{-2k-1}$.

(b) *For any clopen set* U, $\{y : U \setminus A_y = \emptyset\}$ *is clopen.*

(c) *If* $U_k \setminus A_y \neq \emptyset$ *then* $\mu(U_k \setminus A_y) > 2^{-2k-1}$.

Proof. (a) We have

$$A_y \setminus A_{y|2k} \subseteq \bigcup_{n \geq 2k} N^n_{y(n)} \cup \bigcup_{n>k} W_{y|2n},$$

so,

$$\mu(A_y \setminus A_{y|2k}) \leq \sum_{n \geq 2k} 2^{-n-4} + \sum_{n>k} 2^{-2n} < 2^{-2k-1}.$$

(b) Let $U = U_k$. Fix $\sigma \in \omega^{2k}$. If $\mu(U_k \setminus B_\sigma) < 2^{-2k}$, then $U_k \subseteq A_\sigma$, so $\forall y \in [\sigma]$ $U_k \subseteq A_y$. If $\mu(U_k \setminus B_\sigma) \geq 2^{-2k}$, then $A_\sigma = B_\sigma$, so $\mu(U_k \setminus A_\sigma) \geq 2^{-2k}$. Since, by (a),

$$\forall y \in [\sigma]\ \mu(A_y \setminus A_\sigma) < 2^{-2k-1},$$

we get

$$\forall y \in [\sigma]\ \mu(U_k \setminus A_y) > 2^{-2k-1}.$$

(c) This is already proved in (b). ■

For $y \in \omega^\omega$ and $k, n \in \omega$ let

$$F(y,k,n) = \{i : V^n_i \cap (U_k \setminus A_y) = \emptyset\}.$$

Note that if $\mathrm{N}_{e(x)} \subseteq \mathrm{N}_y$ then $\mathrm{N}_{e(x)} \subseteq A_y$. So, by Baire's category theorem (applied to $2^\omega \setminus A_y$) there are k and m such that $U_k \setminus A_y \neq \emptyset$ and

$$\forall n > m\ V^n_{x(n)} \cap (U_k \setminus A_y) = \emptyset,$$

i.e.,

$$\forall n > m\ x(n) \in F(y,k,n).$$

Claim 2. *For every* k *and* n *there is a partition of* ω^ω *into clopen sets such that* $y \to F(y,k,n)$ *is constant on each piece of the partition.*

Proof. Let $l, m \in \omega$ be such that $(1 - 2^{-n-4})^l < 2^{-2k-2}$ and $2^{-2m-1} \leq 2^{-2k-2}/l$. For $\tau \in \omega^{2m}$ let

$$G(\tau,k,n) = \{i : \mu(V^n_i \cap (U_k \setminus A_\tau)) < 2^{-2m-1}\}.$$

Note that if $y \in [\tau]$ then $F(y,k,n) \subseteq G(\tau,k,n)$ (remember that $\mu(A_y \setminus A_{y|2m}) < 2^{-2m-1}$).

Subclaim. *Suppose that for some* $y \in [\tau]$, $U_k \setminus A_y \neq \emptyset$. *Then* $|G(\tau,k,n)| < l$.

Proof. Suppose that $|G(\tau,k,n)| \geq l$ and let G consist of the first l elements of $G(\tau,k,n)$. Note that

$$\mu((U_k \setminus A_\tau)) \geq \mu(U_k \setminus A_y) > 2^{-2k-1},$$

and

$$U_k \setminus A_\tau \subseteq \bigcap_{i\in G}(2^\omega \setminus V_i^n) \cup \bigcup_{i\in G}(V_i^n \cap (U_k \setminus A_\tau)).$$

It follows that

$$2^{-2k-1} < (1-2^{-(n+4)})^l + 2^{-2m-1}l < 2^{-2k-1},$$

which is a contradiction. ■

Now use Claim 1(b) with $U = V_i^n \cap U_k$ to see that for every i, the set $\{y : i \in F(y,k,n)\}$ is clopen. The conclusion of Claim 2 follows. ■

For $y \in \omega^\omega$ and $n \in \omega$ let

$$\begin{aligned} F(y,n) = &\{\text{the first } 2^{n-1} \text{ elements from } F(y,0,n)\} \cup \\ &\{\text{the first } 2^{n-2} \text{ elements from } F(y,1,n)\} \cup \ldots \cup \\ &\{\text{the first } 2^{n-n} \text{ elements from } F(y,n-1,n)\}. \end{aligned}$$

Then $|F(y,n)| \le 2^n$ and the function $y \to F(y,n)$ takes each of its values on a clopen set.

Claim 3. $\mathrm{N}_{e(x)} \subseteq \mathrm{N}_y \Rightarrow \forall^\infty n\ x(n) \in F(y,n)$.

Proof. Suppose that $\mathrm{N}_{e(x)} \subseteq \mathrm{N}_y$. Then there are k and m such that $U_k \setminus A_y \ne \emptyset$ and

$$\forall n > m\ x(n) \in F(y,k,n).$$

Also, since V_i^n are independent sets of measure 2^{-n-4} and

$$i \in F(y,k,n) \Rightarrow U_k \setminus A_y \subseteq 2^\omega \setminus V_i^n,$$

we have

$$\prod_{n>m}(1-2^{-n-4})^{|F(y,k,n)|} \ge \mu(U_k \setminus A_y) > 0.$$

So

$$\sum_{n>m} |F(y,k,n)| \cdot 2^{-n-4} < \infty,$$

hence $\forall^\infty n\ |F(y,k,n)| \le 2^{n-k-1}$. Thus $\forall^\infty n\ F(y,k,n) \subseteq F(y,n)$. ■

Now define f by $f(y)(n) = \#(F(y,n),n)$. ■

1.8. Lemma. $\in^{*\mathrm{L}}_{\mathrm{I}} \to \subseteq^{\mathrm{M}}_{\mathrm{M}}$.

Proof. A straightforward modification of the proof from [P1] (see also [F2]). ■

1.9. Lemma. *Let* S *be either* N *or* M. *Then* $\subseteq^{\mathrm{S}}_{\mathrm{S}} \to \not\ni^{\mathrm{I}}_{\mathrm{S}}$.

Proof. We set $e = \mathrm{I}$ and seek a continuous function $f : \omega^\omega \to \mathbf{P}$ such that $\forall y\ f(y) \notin \mathrm{S}_y$.

S = M: Fix $y \in \omega^\omega$. Let

$$a_0 = 0, \qquad a_{n+1} = a_n + |\tau^{a_n+1}_{y(a_n+1)}|.$$

Then define

$$f(y) = \bigcup_n \tau^{a_n+1}_{y(a_n+1)} \cup (\{a_n : n \in \omega\} \times \{1\}).$$

The first summand guarantees $f(y) \notin \mathrm{M}_y$, the second ensures $f(y) \in \mathbf{P}$.

S = N: Let $a : \omega^\omega \to \omega^\omega$ be continuous such that $\forall y\ \mathrm{N}_y \cup \mathbf{Q} \subseteq \mathrm{N}_{a(y)}$ (Lemma 1.2). Fix $y \in \omega^\omega$ and let $z = a(y)$. Write $N(k)$ for $N^0_{z(0)} \cup \ldots \cup N^k_{z(k)}$. Define inductively $t \in 2^\omega$:

$$t(n) = \begin{cases} 0 & \text{if } \mu([(t|n)^\frown\langle 0\rangle] \setminus N(2n+1)) \ge \mu([(t|n)^\frown\langle 1\rangle] \setminus N(2n+1)), \\ 1 & \text{otherwise.} \end{cases}$$

Claim. $t \notin \mathrm{N}_z$.

Proof. Note that

$$\mu([t|1] \setminus N(1)) \ge 2^{-1} - 2^{-1}(2^{-4} + 2^{-5}),$$

$$\mu([t|2] \setminus N(3)) \ge 2^{-2} - 2^{-2}(2^{-4} + 2^{-5}) - 2^{-1}(2^{-6} + 2^{-7}),$$

$$\mu([t|(n+1)] \setminus N(2n+1)) \ge 2^{-(n+1)} - \sum_{i=0}^{n} 2^{i-(n+1)}(2^{-4-2i} + 2^{-4-(2i+1)}) > 0. \ \blacksquare$$

Let $f(y) = t$. Then $f : \omega^\omega \to \mathbf{P}$ is continuous and $\forall y\ f(y) \notin \mathrm{N}_{a(y)} \supseteq \mathrm{N}_y$.

The proof of Lemma 1.9 is complete. $\blacksquare$

1.10. Lemma. *Let* S *be either* N *or* M. *Then* $\subseteq^{\mathrm{S}}_{\mathrm{S}} \to \in^{\mathrm{S}}_{\mathrm{J}}$.

Proof. Let $f = \mathrm{I}$ and let $e : 2^\omega \to \omega^\omega$ be continuous such that $\forall x \in 2^\omega$ $\{x\} \subseteq \mathrm{S}_{e(x)}$ (Lemma 1.2). Then $\mathrm{S}_{e(x)} \subseteq \mathrm{S}_y \Rightarrow x \in \mathrm{S}_{f(y)}$. $\blacksquare$

1.11. Lemma. $\not\ni^{\mathrm{J}}_{\mathrm{N}} \to \in^{\mathrm{M}}_{\mathrm{J}}$.

Proof. Let B be a $\mathbf{G}_\delta$ subset of 2^ω which is null and dense. By Lemma 1.2 there exist continuous functions $e, f : 2^\omega \to \omega^\omega$ such that $\forall x \in 2^\omega$ $B + x \subseteq \mathrm{N}_{e(x)}$ and $\forall y \in 2^\omega$ $(2^\omega \setminus B) + y \subseteq \mathrm{M}_{f(y)}$ (+ here is coordinatewise addition mod 2). Now, if $y \in 2^\omega \setminus \mathrm{N}_{e(x)}$, then $y \notin B + x$. So, $x \in (2^\omega \setminus B) + y$, whence $x \in \mathrm{M}_{f(y)}$. $\blacksquare$

1.12. Lemma. (a) $\in^{\mathrm{N}}_{\mathrm{J}} \leftrightarrow \in^{\mathrm{N}}_{\mathrm{I}}$.

(b) $\in^{\mathrm{M}}_{\mathrm{J}} \to \in^{\mathrm{M}}_{\mathrm{I}}$ *and* $\in^{\mathrm{M}}_{\mathrm{J}} \Leftarrow \in^{\mathrm{M}}_{\mathrm{I}}$.

Proof. (a) $\to$: Let $e = f = \mathrm{I}$. Then $e(x) \in \mathrm{N}_y \Rightarrow x \in \mathrm{N}_{f(y)}$.

$\leftarrow$: Let $e : 2^\omega \to \mathbf{P}$ be a homeomorphic embedding such that $\mu(e[B]) = \mu(B)/2$ for all Borel $B \subseteq 2^\omega$ (it is a standard exercise that for any Polish space X with a nonatomic σ-finite Borel measure λ and for any $0 < \alpha < \lambda(X)$ there exists a continuous embedding $e : 2^\omega \to X$ such that $\lambda(e[B]) =$

$\alpha \cdot \mu(B)$ for all Borel $B \subseteq 2^\omega$). Let $f : \omega^\omega \to \omega^\omega$ be continuous such that $\mathrm{N}_{f(y)} \supseteq e^{-1}[\mathrm{N}_y]$ (Lemma 1.2). Then $e(x) \in \mathrm{N}_y \Rightarrow x \in \mathrm{N}_{f(y)}$.

(b) $\to$: Let $e = f = \mathrm{I}$ as in (a).

$\Leftarrow$: Define $e : 2^\omega \to \omega^\omega$ by $e|\mathbf{P} = \mathrm{I}$ and $e|2^\omega \setminus \mathbf{P} \equiv \langle 0, 0, \ldots\rangle$. Then e is Borel and the preimage of a meager $\mathbf{F}_\sigma$ set is a meager $\mathbf{F}_\sigma$ set. Let $f : \omega^\omega \to \omega^\omega$ be continuous such that $\mathrm{M}_{f(y)} \supseteq e^{-1}[\mathrm{M}_y]$ (Lemma 1.2). Then $e(x) \in \mathrm{M}_y \Rightarrow x \in \mathrm{M}_{f(y)}$. ■

Note. $\in^{\mathrm{M}}_{\mathrm{J}} \leftarrow \in^{\mathrm{M}}_{\mathrm{I}}$ is false. If $e : 2^\omega \to \mathbf{P}$ is continuous, then $e[2^\omega]$ is meager. So, there is z with $e[2^\omega] \subseteq \mathrm{M}_z$. It follows that if $f : \omega^\omega \to \omega^\omega$ is any function such that $e(x) \in \mathrm{M}_y \Rightarrow x \in \mathrm{M}_{f(y)}$, then $2^\omega \subseteq \mathrm{M}_{f(z)}$, a contradiction.

1.13. Lemma. $\subseteq^{\mathrm{M}}_{\mathrm{M}} \to \preceq^{\mathrm{I}}_{\mathrm{I}}$.

Proof. By Lemma 1.2 there exists a continuous $e : \omega^\omega \to \omega^\omega$ such that

$$\forall x \in \omega^\omega \ \{s \in 2^\omega : \forall n \ s(\overline{x}(n)) = 0\} \subseteq \mathrm{M}_{e(x)},$$

where $\overline{x}(n) = n + \max_{m \le n} x(m)$ (to have $\overline{x}$ strictly increasing and $\forall n \ \overline{x}(n) \ge x(n)$).

To define $f : \omega^\omega \to \omega^\omega$ proceed as follows. Fix $y \in \omega^\omega$. Let $\overline{y}(0) = 0$ and

$$\overline{y}(n+1) = \overline{y}(n) + |\tau^{\bar{y}(n)}_{y(\bar{y}(n))}|.$$

Then let $f(y)(n) = \overline{y}(2n)$. Clearly f is continuous. We now show that

$$\mathrm{M}_{e(x)} \subseteq \mathrm{M}_y \Rightarrow \overline{x} \preceq f(y).$$

To this end suppose that $\exists^\infty n \ \overline{x}(n) > f(y)(n)$. Then the set

$$W = \{n : \operatorname{rng}(\overline{x}) \cap [\overline{y}(n), \overline{y}(n+1)) = \emptyset\}$$

is infinite. Let

$$s = \bigcup_{n \in W} \tau^{\overline{y}(n)}_{y(\overline{y}(n))} \cup \Big(\bigcup_{n \notin W} [\overline{y}(n), \overline{y}(n+1)) \times \{0\} \Big).$$

Then $s \in \mathrm{M}_{e(x)} \setminus \mathrm{M}_y$. ■

1.14. Lemma. $\preceq^{\mathrm{I}}_{\mathrm{I}} \to \in^{\mathrm{M}}_{\mathrm{I}}$.

Proof. Let $e = \mathrm{I}$ and $f : \omega^\omega \to \omega^\omega$ be continuous such that $\{t \in \omega^\omega : t \preceq y\} \subseteq \mathrm{M}_{f(y)}$ (Lemma 1.2). Then $e(x) \preceq y \Rightarrow x \in \mathrm{M}_{f(y)}$. ■

1.15. Lemma. $\preceq^{\mathrm{I}}_{\mathrm{I}} \to \not\succeq^{\mathrm{I}}_{\mathrm{I}}$.

Proof. Let $e = \mathrm{I}$ and define $f : \omega^\omega \to \omega^\omega$ by $f(y)(n) = y(n) + 1$. Then $e(x) \preceq y$ implies $\exists^\infty n \ x(n) < y(n)$, hence $f(y) \not\preceq x$. ■

The proof of Theorem 1.5 is complete.

Miller [Mi2] proved that in Cichoń's diagram we also have $\mathrm{add}(\mathcal{M}) = \min\{\mathtt{b}, \mathrm{cov}(\mathcal{M})\}$ and $\mathrm{cof}(\mathcal{M}) = \max\{\mathtt{d}, \mathrm{non}(\mathcal{M})\}$ (see also [F2]). This corresponds to the following theorem (cf. [V] and [Bl]).

1.16. Theorem. *There exist continuous functions*

$$e : \{\langle x, y\rangle \in \omega^\omega \times 2^\omega : y \notin \mathrm{M}_x\} \to \omega^\omega, \qquad f : 2^\omega \times \omega^\omega \to \omega^\omega$$

such that $e(x, y) \preceq z \Rightarrow \mathrm{M}_x \subseteq \mathrm{M}_{f(y,z)}$.

Proof. Suppose that $y \notin \mathrm{M}_x$. Then for every n there is $m \geq n$ with $\tau^m_{x(m)} \subseteq y$. Let

$$e(x, y)(n) = m + |\tau^m_{x(m)}|$$

for the least such m. Let

$$f(y, z)(n) = \#(y|[n, z(n))).$$

We have to show that

$$e(x, y) \preceq z \Rightarrow \mathrm{M}_x \subseteq \mathrm{M}_{f(y,z)}.$$

Suppose that $e(x, y) \preceq z$. If $t \in 2^\omega \setminus \mathrm{M}_{f(y,z)}$, then $\exists^\infty n\ y|[n, z(n)) \subseteq t$. Since

$$\forall^\infty n\ e(x, y)(n) \leq z(n),$$

we have

$$\forall^\infty n\ \exists m \geq n\ \tau^m_{x(m)} \subseteq y|[n, z(n)).$$

It follows that $\exists^\infty m\ \tau^m_{x(m)} \subseteq t$, i.e., $t \notin \mathrm{M}_x$. ∎

2. Small sets. In this section we shall show how cardinals from Cichoń's diagram yield classes of small spaces. We shall restrict ourselves to zero-dimensional separable metric spaces. Each such space is homeomorphic to a subset of 2^ω, so we are really talking about sets of reals.

2.1. Definition. For a relation $\varrho \subseteq V^\omega \times W^\omega$ with $\mathrm{dom}(\varrho) = V^\omega$, $\mathrm{rng}(\varrho) = W^\omega$ $(V, W \in \{2, \omega\})$ and for a zero-dimensional separable metric space X let

$$X \in \mathtt{B}^*(\varrho) \quad \text{iff} \quad \text{for every } * \text{ function } a : X \to V^\omega,\ a[X] \in \mathtt{B}(\varrho),$$

$$X \in \mathtt{D}^*(\varrho) \quad \text{iff} \quad \text{for every } * \text{ function } a : X \to W^\omega,\ a[X] \in \mathtt{D}(\varrho).$$

Let also

$$\mathtt{B}^* = \mathtt{B}^*(\preceq) \quad \text{and} \quad \mathtt{D}^* = \mathtt{D}^*(\preceq).$$

If $\mathcal{S}$ is a family of subsets of 2^ω with a master set $\mathrm{S} \subseteq \omega^\omega \times 2^\omega$ (i.e. all sections S_x $(x \in \omega^\omega)$ are in $\mathcal{S}$ and every set from $\mathcal{S}$ is covered by some S_x), let

$$\mathtt{Add}^*(\mathcal{S}) = \mathtt{B}^*(\subseteq^{\mathrm{S}}_{\mathrm{S}}), \quad \mathtt{Cof}^*(\mathcal{S}) = \mathtt{D}^*(\subseteq^{\mathrm{S}}_{\mathrm{S}}),$$

$$\mathtt{Non}^*(\mathcal{S}) = \mathtt{B}^*(\in^{\mathrm{S}}_{\mathrm{J}}), \quad \mathtt{Cov}^*(\mathcal{S}) = \mathtt{D}^*(\in^{\mathrm{S}}_{\mathrm{J}}),$$

$$\mathtt{Non}^*_{\mathrm{I}}(\mathcal{S}) = \mathtt{B}^*(\in^{\mathrm{S}}_{\mathrm{I}}), \quad \mathtt{Cov}^*_{\mathrm{I}}(\mathcal{S}) = \mathtt{D}^*(\in^{\mathrm{S}}_{\mathrm{I}}).$$

N o t e. Observe that $\mathtt{b}(\varrho) = \mathrm{non}(\mathtt{B}^*(\varrho))$ and $\mathtt{d}(\varrho) = \mathrm{non}(\mathtt{D}^*(\varrho))$. Hence $\mathrm{non}(\mathtt{Zyx}^*(\mathcal{S})) = \mathrm{zyx}(\mathcal{S})$.

With the notation introduced above we have for instance ($\mathcal{S} \in \{\mathcal{M}, \mathcal{N}\}$):

- $X \in \mathtt{Add}^*(\mathcal{S})$ iff $\forall B \in \mathcal{S}^*(X)\ \bigcup_{x\in X} B_x \in \mathcal{S}$;
- $\mathtt{Add}^\ddagger(\mathcal{S}) \subseteq \mathtt{Add}^\dagger(\mathcal{S})$;
- $X \in \mathtt{Add}^\ddagger(\mathcal{S})$ iff every Borel image of X into ω^ω is in $\mathtt{Add}^\dagger(\mathcal{S})$.

Note also that the following are equivalent:

- $X \in \mathtt{Cov}^*(\mathcal{S})$;
- $\forall B \in \mathcal{S}^*(X) \bigcup_{x\in X} B_x \neq 2^\omega$;
- $\forall B \in \mathcal{S}^*(X) \bigcup_{x\in X} B_x \neq \mathbf{P}$;
- $X \in \mathtt{Cov}^*_{\mathrm{I}}(\mathcal{S})$.

This is so because $X \times \mathbf{Q} \in \mathcal{S}^*(X)$, and $\mathcal{S}^*(X)$ is an ideal (see Lemma 1.2). (Another way to see that $\mathtt{Cov}^*(\mathcal{S}) = \mathtt{Cov}^*_{\mathrm{I}}(\mathcal{S})$ is to notice that functions f from Lemma 1.12 are continuous.)

With $\mathtt{Non}^*(\mathcal{S})$ the situation is more complicated. We have (Lemma 1.12) $\mathtt{Non}^*(\mathcal{N}) = \mathtt{Non}^*_{\mathrm{I}}(\mathcal{N})$ and $\mathtt{Non}^\ddagger(\mathcal{M}) = \mathtt{Non}^\ddagger_{\mathrm{I}}(\mathcal{M})$, but $2^\omega \in \mathtt{Non}^\dagger_{\mathrm{I}}(\mathcal{M}) \setminus \mathtt{Non}^\dagger(\mathcal{M})$.

As a consequence of parametrized Cichoń's diagram we get Cichoń's diagrams for small sets ($\to$ means that $\subseteq$ is provable in ZFC).

2.2. THEOREM (Cichoń's diagrams for small sets).

$$\begin{array}{ccccccc}
\mathtt{Cov}^\ddagger(\mathcal{N}) & \to & \mathtt{Non}^\ddagger(\mathcal{M}) & \to & \mathtt{Cof}^\ddagger(\mathcal{M}) & \to & \mathtt{Cof}^\ddagger(\mathcal{N}) \\
 & & \uparrow & & \uparrow & & \\
\uparrow & & \mathtt{B}^\ddagger & \to & \mathtt{D}^\ddagger & & \uparrow \\
 & & \uparrow & & \uparrow & & \\
\mathtt{Add}^\ddagger(\mathcal{N}) & \to & \mathtt{Add}^\ddagger(\mathcal{M}) & \to & \mathtt{Cov}^\ddagger(\mathcal{M}) & \to & \mathtt{Non}^\ddagger(\mathcal{N})
\end{array}$$

$$\begin{array}{ccccccccc}
\mathtt{Cov}^\dagger(\mathcal{N}) & \to & \mathtt{Non}^\dagger(\mathcal{M}) & \to & \mathtt{Non}^\dagger_{\mathrm{I}}(\mathcal{M}) & \to & \mathtt{Cof}^\dagger(\mathcal{M}) & \to & \mathtt{Cof}^\dagger(\mathcal{N}) \\
 & & & & \uparrow & & \uparrow & & \\
\uparrow & & \uparrow & & \mathtt{B}^\dagger & \to & \mathtt{D}^\dagger & & \uparrow \\
 & & & \nearrow & & & \uparrow & & \\
\mathtt{Add}^\dagger(\mathcal{N}) & \to & \mathtt{Add}^\dagger(\mathcal{M}) & & \to & & \mathtt{Cov}^\dagger(\mathcal{M}) & \to & \mathtt{Non}^\dagger(\mathcal{N})
\end{array}$$

P r o o f. We indicate a proof of $\mathtt{Add}^\dagger(\mathcal{N}) \subseteq \mathtt{Add}^\dagger(\mathcal{M})$. Let e and f be the functions establishing $\subseteq^{\mathrm{N}}_{\mathrm{N}} \to \subseteq^{\mathrm{M}}_{\mathrm{M}}$. Suppose that $X \in \mathtt{Add}^\dagger(\mathcal{N})$. Let $a : X \to \omega^\omega$ be any continuous function. Then $b = e \circ a$ is also continuous, so $X \in \mathtt{Add}^\dagger(\mathcal{N})$ implies that there is $y \in \omega^\omega$ with $\bigcup_{x\in X} \mathrm{N}_{b(x)} \subseteq N_y$. Then $\bigcup_{x\in X} \mathrm{M}_{a(x)} \subseteq \mathrm{M}_{f(y)}$. Thus for any continuous function $a : X \to \omega^\omega$, $a[X] \in \mathtt{B}(\subseteq^{\mathrm{M}}_{\mathrm{M}})$, hence $X \in \mathtt{Add}^\dagger(\mathcal{M})$. ∎

Clearly, we could have added arrows from each Borel class to the corresponding continuous class (but not from $\mathtt{Add}^\dagger(\mathcal{N})$ to $\mathtt{Cof}^\ddagger(\mathcal{N})$, see Section 4). It is also easy to see that no arrow which is not obtained by composing the existing ones can be added to the above diagrams. For example, $\mathrm{non}(\mathtt{Cov}^\dagger(\mathcal{N})) = \mathrm{cov}(\mathcal{N})$ and $\mathrm{non}(\mathtt{Add}^\ddagger(\mathcal{M})) = \mathrm{add}(\mathcal{M})$. Since it is consistent with ZFC that $\mathrm{cov}(\mathcal{N}) < \mathrm{add}(\mathcal{M})$, we cannot in ZFC have $\mathtt{Add}^\ddagger(\mathcal{M}) \subseteq \mathtt{Cov}^\dagger(\mathcal{N})$.

However, from Theorem 1.16 we get

2.3. Theorem. (a) $\mathtt{Add}^*(\mathcal{M}) = \mathtt{B}^* \cap \mathtt{Cov}^*(\mathcal{M})$.

(b) $Y \notin \mathtt{Non}^*(\mathcal{M})$ & $Z \notin \mathtt{D}^* \Rightarrow Y \times Z \notin \mathtt{Cof}^*(\mathcal{M})$.

Proof. (a) We shall show the inclusion $\supseteq$ (the opposite one follows from the diagram). Suppose that $X \in \mathtt{B}^* \cap \mathtt{Cov}^*(\mathcal{M})$. Let $a : X \to \omega^\omega$ be a $*$ function. Let e, f be the functions from Theorem 1.16. By $X \in \mathtt{Cov}^*(\mathcal{M})$ there is $y \in 2^\omega \setminus \bigcup_{x\in X} \mathrm{M}_{a(x)}$. Define $b : X \to \omega^\omega$ by $b(x) = e(a(x), y)$. Then b is a $*$ function, so, by $X \in \mathtt{B}^*$, there is $z \in \omega^\omega$ with $\forall x \in X\ b(x) \preceq z$. By Theorem 1.16, $\forall x \in X\ \mathrm{M}_{a(x)} \subseteq \mathrm{M}_{f(y,z)}$. Thus $\bigcup_{x\in X} \mathrm{M}_{a(x)}$ is meager.

(b) Suppose that $Y \notin \mathtt{Non}^*(\mathcal{M})$ and $Z \notin \mathtt{D}^*$. There are $*$ functions $a : Y \to 2^\omega$ and $b : Z \to \omega^\omega$ such that $a[Y] \notin \mathcal{M}$ and $b[Z] \notin \mathtt{D}$. Let e, f be the functions from Theorem 1.16. We shall show that for any $x \in \omega^\omega$ there are $y \in Y$, $z \in Z$ such that $\mathrm{M}_x \subseteq \mathrm{M}_{f(a(y),b(z))}$.

Fix $x \in \omega^\omega$. By $a[Y] \notin \mathcal{M}$ there is $y \in Y$ with $a(y) \notin M_x$. Since $b[Z] \notin \mathtt{D}$ there is $z \in Z$ with $e(x, a(y)) \preceq b(z)$. Then, by Theorem 1.16, $\mathrm{M}_x \subseteq \mathrm{M}_{f(a(y),b(z))}$. ■

3. Characterizations. Lemma 1.7 yields the following characterization.

3.1. Theorem. $\mathtt{Add}^*(\mathcal{N}) = \mathtt{B}^*(\in^*)$ *and* $\mathtt{Cof}^*(\mathcal{N}) = \mathtt{D}^*(\in^*)$.

Thus, for every Borel set $B \subseteq X \times 2^\omega$ whose all vertical sections are null, $\bigcup_{x\in X} B_x$ is null iff every Borel image of X into ω^ω is localizable.

3.2. Definition. We say that $y \in \omega^\omega$ *diagonalizes* $x \in \omega^\omega$ (in symbols: $x =_\infty y$) iff $\exists^\infty n\ x(n) = y(n)$. We extend this notion to sets as we did with "localizes".

Bartoszyński [B2] proved that $\mathrm{cov}(\mathcal{M}) = \mathtt{b}(=_\infty)$ and $\mathrm{non}(\mathcal{M}) = \mathtt{d}(=_\infty)$. This motivates the following theorem.

3.3. Theorem. (a) $\mathtt{Cov}^*(\mathcal{M}) = \mathtt{B}^*(=_\infty)$.

(b) $\mathtt{D}^*(=_\infty) \subseteq \mathtt{Non}^*_{\mathrm{I}}(\mathcal{M})$.

(c) $Y \notin \mathtt{B}^*$ & $Z \notin \mathtt{D}^*(=_\infty) \Rightarrow Y \times Z \notin \mathtt{Non}^*_{\mathrm{I}}(\mathcal{M})$.

Thus, for every Borel (resp. $\mathbf{F}_\sigma$) set $B \subseteq X \times 2^\omega$ with all vertical sections meager, $\bigcup_{x\in X} B_x \neq 2^\omega$ iff for every Borel (resp. continuous) function $a : X \to \omega^\omega$, $a[X]$ is diagonalizable.

For the proof we need two lemmas.

3.4. LEMMA. *There exist continuous functions*

$$g : \omega^\omega \to \omega^\omega,$$
$$e : \{\langle x, y\rangle \in \omega^\omega \times \omega^\omega : \exists^\infty n\ g(x)(n) \le y(n)\} \to \omega^\omega,$$
$$f : \omega^\omega \times \omega^\omega \to \mathbf{P}$$

such that

$$e(x, y) =_\infty z \Rightarrow f(y, z) \notin \mathrm{M}_x.$$

Proof. Define g as follows. Fix $x \in \omega^\omega$. Let $x'(n) = n + |\tau^n_{x(n)}|$ $(n \in \omega)$ and let $\overline{x}(n) = n + \max_{m \le n} x'(m)$ (to have $\overline{x}$ strictly increasing with $\forall n$ $\overline{x}(n) \ge x'(n)$). Let $g(x)(n) = \overline{x}^{2n}(0)$ (the superscript denotes the number of iterates of $\overline{x}$).

Now we define e. Let $y \in \omega^\omega$ be such that $\exists^\infty n\ g(x)(n) \le y(n)$. Let $\overline{y}(n) = n + \max_{m\le n} y(m)$ $(n \in \omega)$ (again $\overline{y}$ is strictly increasing and $\forall n\ y(n) \le \overline{y}(n)$).

CLAIM. $\exists^\infty n\ \overline{y}(n) + |\tau^{\overline{y}(n)}_{x(\overline{y}(n))}| < \overline{y}(n+1)$.

Proof. We have to see that $\exists^\infty n\ x'(\overline{y}(n)) < \overline{y}(n+1)$. We show that $\exists^\infty n\ \overline{x}(\overline{y}(n)) < \overline{y}(n+1)$. Suppose there is k such that

$$\forall m\ \overline{x}(\overline{y}(k+m)) \ge \overline{y}(k+m+1).$$

Let l be such that $\overline{x}^l(0) \ge \overline{y}(k)$. Since $\overline{x}$ is increasing, we get for all m,

$$\begin{aligned}\overline{y}(k+m) &\le \overline{x}(\overline{y}(k+(m-1)))\\ &\le \overline{x}^2(\overline{y}(k+(m-2))) \le \ldots\\ &\le \overline{x}^m(\overline{y}(k)) \le \overline{x}^{l+m}(0),\end{aligned}$$

which violates $\exists^\infty n\ \overline{x}^{2n}(0) \le \overline{y}(n)$. ∎

Let τ_n $(n \in \omega)$ be the $(n+1)$th sequence $\tau^{\overline{y}(m)}_{x(\overline{y}(m))}$ for which

$$\overline{y}(m) + |\tau^{\overline{y}(m)}_{x(\overline{y}(m))}| < \overline{y}(m+1).$$

Let

$$e(x, y)(n) = \#(\langle \tau_0, \ldots, \tau_n\rangle),$$

where # means the number in an enumeration of

$$\{\langle \varrho_0, \ldots, \varrho_n\rangle :$$
$$\exists\ m_0 < m_1 < \ldots < m_n\ \forall i\ \exists k \in (\overline{y}(m_i),\ \overline{y}(m_i+1)]\ \varrho_i \in 2^{[\overline{y}(m_i), k)}\}.$$

(Note that $\overline{y}(m_i+1) - 1 \notin \mathrm{dom}(\varrho_i)$.)

Now we define f. Let $z \in \omega^\omega$. For each $n \in \omega$, find $\sigma_0^n, \ldots, \sigma_n^n$ so that $z(n) = \#(\langle \sigma_0^n, \ldots, \sigma_n^n \rangle)$. Let σ_n be the first of σ_i^n's whose domain is disjoint from the domains of all σ_i^{n-1}'s (σ_0 is the first of all σ_i^0). Then the domains of σ_n's are disjoint. Let

$$f(y,z) = \bigcup_n \sigma_n \cup \Big(\Big(\omega \setminus \bigcup_n \mathrm{dom}(\sigma_n) \Big) \times \{1\} \Big).$$

Note that if $z(n) = e(x,y)(n)$ then σ_n is one of τ_i's for which

$$e(x,y)(n) = \#(\langle \tau_0, \ldots, \tau_n \rangle).$$

Hence, for some $m \geq n$, $\sigma_n = \tau^{\overline{y}(m)}_{x(\overline{y}(m))}$. It follows that if $e(x,y) =_\infty z$ then

$$\exists^\infty m \ \tau^{\overline{y}(m)}_{x(\overline{y}(m))} \subseteq f(y,z),$$

i.e., $f(y,z) \notin \mathrm{M}_x$. To see that $f(y,z) \in \mathbf{P}$ note that $\{\overline{y}(m) - 1 : m > 0\}$ is disjoint from $\bigcup_n \mathrm{dom}(\sigma_n)$. ■

3.5. Lemma. (a) $\not\ni^{\mathrm{I}}_{\mathrm{M}} \to =_\infty{}^{\mathrm{I}}_{\mathrm{I}}$.

(b) $=_\infty{}^{\mathrm{I}}_{\mathrm{I}} \to \not\geq^{\mathrm{I}}_{\mathrm{I}}$.

Proof. (a) Let $e : \omega^\omega \to \omega^\omega$ be continuous such that

$$\{t \in \omega^\omega : \forall^\infty n \ x(n) \neq t(n)\} \subseteq \mathrm{M}_{e(x)}$$

(Lemma 1.2). Let $f = \mathrm{I}$. Then $y \notin \mathrm{M}_{e(x)} \Rightarrow x =_\infty f(y)$.

(b) Let $e = \mathrm{I}$. Define $f : \omega^\omega \to \omega^\omega$ by $f(y)(n) = y(n) + 1$. ■

Proof of 3.3. (a) $\subseteq$: by 3.5(a). $\supseteq$: Suppose that every $*$ image of X into ω^ω is diagonalizable. Let $a : X \to \omega^\omega$ be a $*$ function. We have to show that $\bigcup_{x \in X} \mathrm{M}_{a(x)} \neq 2^\omega$. Let e, f and g be the functions from Lemma 3.4. Since $g \circ a$ is a $*$ function, $(g \circ a)[X]$ is diagonalizable, so there is $y \in \omega^\omega$ such that

$$\forall x \in X \ \exists^\infty n \ g(a(x))(n) \leq y(n).$$

Define $b : X \to \omega^\omega$ by $b(x) = e(a(x), y)$. Since b is a $*$ function, $b[X]$ is diagonalizable, say, by z. Then $\forall x \in X \ f(y,z) \notin \mathrm{M}_{a(x)}$.

(b) Suppose that $X \notin \mathtt{Non}^*_{\mathrm{I}}(\mathcal{M})$. So, there exists a $*$ function $a : X \to \mathbf{P}$ such that $a[X] \notin \mathcal{M}$. Let f be the function from Lemma 3.5(a). Then $f \circ a : X \to \omega^\omega$ is a $*$ function and $(f \circ a)[X]$ diagonalizes ω^ω. So, $(f \circ a)[X] \notin \mathtt{D}(=_\infty)$, and thus $X \notin \mathtt{D}^*(=_\infty)$.

(c) Suppose that $Y \notin \mathtt{B}^*$ and $Z \notin \mathtt{D}^*(=_\infty)$. Fix $*$ functions $a : Y \to \omega^\omega$ and $b : Z \to \omega^\omega$ such that $a[Y] \notin \mathtt{B}$ and $b[Z] \notin \mathtt{D}(=_\infty)$. Let e, f and g be the functions from Lemma 3.4. We show that $f[a[Y] \times b[Z]] \notin \mathcal{M}$. Indeed, suppose $f[a[Y] \times b[Z]] \subseteq \mathrm{M}_x$. Find $y \in Y$ with $\exists^\infty n \ g(x)(n) \leq a(y)(n)$. Next find $z \in Z$ such that $e(x,y) =_\infty b(z)$. Then $f(a(y), b(z)) \notin \mathrm{M}_x$, a contradiction.

Thus there is a $*$ image of $Y \times Z$ into $\mathbf{P}$ which is nonmeager, i.e., $Y \times Z \notin \mathtt{Non}^*_{\mathrm{I}}(\mathcal{M})$. ■

Note. Let $\mathcal{D}$ be the ideal of nowhere-dense subsets of 2^ω. Choose $2^\omega \setminus \bigcup_n \bigcup_{x \in \omega^\omega} M^n_{x(n)}$ as its master set. As in Lemma 1.2, we have $A \in \mathcal{D}^\dagger(X)$ iff there is a closed set $B \subseteq X \times 2^\omega$ with all vertical sections nowhere-dense such that $A \subseteq B$.

The following are equivalent (see [P3]):

• for every $\mathbf{F}_\sigma$ set $B \subseteq X \times 2^\omega$ with all vertical sections meager, $\bigcup_{x \in X} B_x \neq 2^\omega$;

• for every closed set $B \subseteq X \times 2^\omega$ with all vertical sections meager, $\bigcup_{x \in X} B_x \neq 2^\omega$.

This can be incorporated into our scheme as follows. Clearly $\mathtt{Cov}^\dagger(\mathcal{M}) \subseteq \mathtt{Cov}^\dagger(\mathcal{D})$. We shall show the opposite inclusion.

First note that if $X \in \mathtt{Cov}^\dagger(\mathcal{D})$, then for every continuous function $a : X \to \omega^\omega$ there is $y \in \omega^\omega$ with $\forall x \in X\ \exists n\ a(x)(n) = y(n)$. (View each $t \in 2^\omega$ as a sequence of consecutive blocks of 0's separated by 1's. Let $\pi(t) \in \omega^{\leq \omega}$ be such that $\mathrm{dom}(\pi(t))$ is equal to the number of finite blocks and $\pi(t)(n)$ is the number of 0's in the nth block. Then $\pi|\mathbf{P}$ is a natural homeomorphism of $\mathbf{P}$ and ω^ω. Now, given continuous $a : X \to \omega^\omega$, look at $A = \{\langle x, t\rangle \in X \times 2^\omega : \exists n \in \mathrm{dom}(\pi(t))\ a(x)(n) = \pi(t)(n)\}$. Note that $(X \times 2^\omega) \setminus A \in \mathcal{D}^\dagger(X)$. Let $t \in \bigcap_{x \in X} A_x$. Define $y \in \omega^\omega$ by $y(n) = \pi(t)(n)$ if $n \in \mathrm{dom}(\pi(t))$, and $y(n) = 0$ otherwise.)

Next, splitting ω into infinitely many infinite sets, we deduce that for every continuous $a : X \to \omega^\omega$ there is $y \in \omega^\omega$ which diagonalizes $a[X]$, hence $X \in \mathtt{Cov}^\dagger(\mathcal{M})$.

Note, however, that a countable dense subset of $\mathbf{P}$ is in $\mathtt{Non}^\ddagger(\mathcal{M}) \setminus \mathtt{Non}^\dagger(\mathcal{D})$.

4. Relation to other small sets. Some of the classes introduced above have been studied before. The definitions were usually given in the language of covers and subcovers (see [FMi]).

4.1. Definition. Let X be a zero-dimensional separable metric space.

(a) $X \in H$ iff for every family $\mathcal{G}_n$ $(n \in \omega)$ of open covers of X there exist $\mathcal{F}_n \in [\mathcal{G}_n]^{<\omega}$ such that $X \subseteq \bigcup_m \bigcap_{n>m} \bigcup \mathcal{F}_n$.

(b) $X \in M$ iff for every family $\mathcal{G}_n$ $(n \in \omega)$ of open covers of X there exist $\mathcal{F}_n \in [\mathcal{G}_n]^{<\omega}$ such that $X \subseteq \bigcup_n \bigcup \mathcal{F}_n$.

(c) $X \in C''$ iff for every family $\mathcal{G}_n$ $(n \in \omega)$ of open covers of X there exist $G_n \in \mathcal{G}_n$ such that $X \subseteq \bigcup_n G_n$.

(d) $X \in T$ iff for every family $\mathcal{G}_n$ $(n \in \omega)$ of open covers of X there exist $\mathcal{F}_n \in [\mathcal{G}_n]^{\leq 2^n}$ such that $X \subseteq \bigcup_m \bigcap_{n>m} \bigcup \mathcal{F}_n$.

4.2. Theorem. (a) $H = \mathtt{B}^\dagger$.

(b) $M = \mathtt{D}^\dagger$.

(c) $C'' = \mathtt{Cov}^\dagger(\mathcal{M})$.

(d) $T = \mathtt{Add}^\dagger(\mathcal{N})$.

Proof. As noted in [R1], X belongs to H, M, C'', T iff for every continuous function $a : X \to \omega^\omega$, $a[X]$ belongs to $\mathtt{B}$, $\mathtt{D}$, $\mathtt{B}(=_\infty)$, $\mathtt{B}(\in^*)$, respectively. ∎

As mentioned before, every Borel class is contained in the corresponding continuous class. The converse fails strongly. MA (Martin's Axiom) yields a set $X \in \mathtt{Add}^\dagger(\mathcal{N}) \setminus \mathtt{Cof}^\ddagger(\mathcal{N})$. Recław [R3] constructed from MA a γ-set which can be mapped onto ω^ω by a Borel function. A modification of his construction gives a strong γ-set with the same property. Strong γ, on the other hand, easily implies $\mathtt{B}^\dagger(\in^*)$. (See [R1] and [GMi] for definitions.)

MA also implies that there is a set of size $2^{\aleph_0}$ in $\mathtt{Add}^\ddagger(\mathcal{N})$. Todorčević (unpublished) constructed from MA a set X of size $2^{\aleph_0}$ whose Borel images are each strong γ-sets, hence also $X \in \mathtt{Add}^\ddagger(\mathcal{N})$ (see [R1], for this construction under CH). Under MA the classes $\mathtt{Cov}^\ddagger(\mathcal{M}) \setminus \mathtt{Non}^\dagger_{\mathrm{I}}(\mathcal{M})$ and $(\mathtt{Cov}^\ddagger(\mathcal{N}) \cap \mathtt{B}^\ddagger) \setminus \mathtt{Non}^\dagger(\mathcal{N})$ are nonempty. All (λ,κ) Lusin sets for $\kappa \le \mathrm{cov}(\mathcal{M})$ are in $\mathtt{Cov}^\ddagger(\mathcal{M}) \setminus \mathtt{Non}^\dagger_{\mathrm{I}}(\mathcal{M})$ (see [R1]), while all (λ,κ) Sierpiński sets for $\kappa \le \min(\mathrm{cov}(\mathcal{N}), \mathtt{b})$ are in $(\mathtt{Cov}^\ddagger(\mathcal{N}) \cap \mathtt{B}^\ddagger) \setminus \mathtt{Non}^\dagger(\mathcal{N})$ (see [P3]). Recall that for $\lambda \ge \kappa$, a subset of 2^ω is a (λ,κ) Lusin (Sierpiński) set if it has size λ and meets every meager (null) set in a set of size $< \kappa$.

The class $\mathtt{Add}^\dagger(\mathcal{M}) \setminus \mathtt{Cov}^\dagger(\mathcal{N})$ is also nonempty under MA (see [BR] for a construction of a γ-set which is not strongly meager; it is known that $\gamma \Rightarrow H\&C''$, see [FMi], so γ-sets are in $\mathtt{Add}^\dagger(\mathcal{M})$).

Since any continuous image of 2^ω into ω^ω is compact, $2^\omega \in \mathtt{B}^\dagger$. However, consistently $2^{\aleph_0} = \aleph_2$ and $\mathtt{Non}^\dagger(\mathcal{N}) \cup \mathtt{Non}^\dagger(\mathcal{M}) \cup \mathtt{Cof}^\ddagger(\mathcal{N}) \subseteq [\omega^\omega]^{\le\aleph_1}$. (Miller [Mi1] proved that if $\aleph_2$ Sacks reals are added iteratively to a model of CH, then every subset of ω^ω of size $2^{\aleph_0}$ can be continuously mapped onto 2^ω, hence in a Borel way onto ω^ω.)

Recall that a set $X \subseteq 2^\omega$ has strong measure zero iff for every meager set $A \subseteq 2^\omega$, $A + X \ne 2^\omega$ (we take the Galvin–Mycielski–Solovay characterization of strong measure zero sets as our official definition; see [Mi3]). A set $X \subseteq 2^\omega$ is strongly meager iff for every null set $A \subseteq 2^\omega$, $A + X \ne 2^\omega$. The Borel Conjecture says that all strong measure zero sets are countable. The Dual Borel Conjecture says the same about strongly meager sets. Laver [L] showed that the Borel Conjecture is consistent. Carlson [C] did the same for the Dual Borel Conjecture. Now, $\mathtt{Cov}^\dagger(\mathcal{M}) \subseteq [\omega^\omega]^{\le\aleph_0}$ if the Borel Conjecture is true and $\mathtt{Cov}^\dagger(\mathcal{N}) \subseteq [\omega^\omega]^{\le\aleph_0}$ if the Dual Borel Conjecture is true. This is so because of the following theorem ((a) is just a rephrasing of $C'' \Rightarrow$ strong measure zero).

4.3. THEOREM. *Let* $X \subseteq 2^\omega$.

(a) $X \in \mathtt{Cov}^\dagger(\mathcal{M}) \Rightarrow X$ *has strong measure zero.*
(b) $X \in \mathtt{Cov}^\dagger(\mathcal{N}) \Rightarrow X$ *is strongly meager.*

Proof. (a) Suppose that $X \subseteq 2^\omega$ and $X \in \mathtt{Cov}^\dagger(\mathcal{M})$. Let $A \subseteq 2^\omega$ be meager. Let $B = \{\langle x, y\rangle \in 2^\omega \times 2^\omega : y \in A + x\}$. Then $B \in \mathcal{M}^\dagger(X)$. So, there is $y \in 2^\omega \setminus \bigcup_{x\in X} B_x$. Then $y \notin A + X$.

(b) Similar. ■

Consistently $B^\ddagger \subseteq [\omega^\omega]^{\le\aleph_0}$ (Miller [Mi4] showed that consistently every σ-set is countable; easily, $\mathtt{B}^\ddagger$ sets are σ-sets). We do not know whether $\mathtt{Cov}^\ddagger(\mathcal{N}) \cup \mathtt{Cov}^\ddagger(\mathcal{M}) \subseteq [\omega^\omega]^{\le\aleph_0}$ is consistent. We also do not know whether any of the classes $\mathtt{Non}^\ddagger(\mathcal{M})$, $\mathtt{Non}^\ddagger(\mathcal{N})$, $\mathtt{D}^\dagger$ can consistently be a subclass of $[\omega^\omega]^{\le\aleph_0}$. The class $\mathtt{Cof}^\ddagger(\mathcal{M})$, however, cannot.

4.4. THEOREM. $\mathtt{Cof}^\ddagger(\mathcal{M})$ *contains an uncountable set.*

Proof. Either $\mathrm{cof}(\mathcal{M}) > \aleph_1$, and then $[\omega^\omega]^{\le\aleph_1} \subseteq \mathtt{Cof}^\ddagger$, or else $\mathrm{cof}(\mathcal{M}) = \aleph_1$, and then there exists a Lusin set. By [R1] all Lusin sets are in $\mathtt{Cov}^\ddagger(\mathcal{M})$. ■

5. Conclusion. We can go beyond Cichoń's diagram. There are more inequalities between cardinal characteristics which can be parametrized. Some parametrizations are quite straightforward (e.g. parametrizations of inequalities involving the splitting number $\mathtt{s}$ and the reaping number $\mathtt{r}$, see Blass [Bl]), others require two steps as in Theorems 1.16 and 3.3 (e.g. the equality $\mathrm{add}(\mathcal{E}, \mathcal{M}) = \mathrm{cov}(\mathcal{M})$ from [BSh]).

We can also consider small sets $X \subseteq 2^\omega$ defined by "total continuous" functions and/or choices (e.g. the sets X such that $f[X] \in \mathtt{B}(\cdot)$ for all continuous $f : 2^\omega \to \omega^\omega$ and/or the sets X such that $\bigcup_{x\in X} B_x \in \mathtt{B}(\cdot)$ for all closed $B \subseteq 2^\omega \times \omega^\omega$ whose all vertical sections are in $\mathcal{S}$). This leads to some suprising results about strong measure zero sets (see [AR], [P3]).

The reader might wish to consult Vojtáš [V] and Blass [Bl] to get a slightly different perspective onto parametrization.

QUESTIONS. (1) Suppose that for every continuous $f : \omega^\omega \to \omega^\omega$, $f[X]$ can be diagonalized. Does it follow that for every closed $A \subseteq \omega^\omega \times \omega^\omega$ with all sections meager $\bigcup_{x\in X} A_x \ne \omega^\omega$? (OK if the horizontal ω^ω is replaced by a set with property M.)

(2) All the classes from Cichoń's diagram but $\mathtt{Cov}^*(\mathcal{N})$ are easily seen to be σ-additive (for $\mathtt{Cov}^*(\mathcal{M})$ one can use Theorem 3.3(a)). Is $\mathtt{Cov}^*(\mathcal{N})$ additive?

(3) Theorem 2.3(b) cannot be strengthened to $\mathtt{Non}^*(\mathcal{M}) \cup \mathtt{D}^* = \mathtt{Cof}^*(\mathcal{M})$. A modification of a construction from [R1] gives under CH a set $Y \in \mathtt{Non}^\ddagger(\mathcal{M}) \setminus \mathtt{D}^\dagger$. Now, if Z is a Lusin set then $Z \in D^\ddagger \setminus \mathtt{Non}^\dagger(\mathcal{M})$, so $Y \cup Z \in \mathtt{Cof}^\ddagger \setminus (\mathtt{Non}^\dagger(\mathcal{M}) \cup \mathtt{D}^\dagger)$. What about Theorem 3.3(c)? (This is connected with the so-called "half Cohen real" problem.)

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