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Self homotopy equivalences of classifying spaces of compact connected Lie groups

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Abstract. We describe, for any compact connected Lie group G and any prime p, the monoid of self maps $BG\hat{_p} \to BG\hat{_p}$ which are rational equivalences. Here, $BG\hat{_p}$ denotes the p-adic completion of the classifying space of G. Among other things, we show that two such maps are homotopic if and only if they induce the same homomorphism in rational cohomology, if and only if their restrictions to the classifying space of the maximal torus of G are homotopic.

In an earlier paper [JMO], we gave a complete description of all homotopy classes of self maps of the classifying space BG, when G is any compact connected simple Lie group. In this paper, we extend those results to the case where G is any compact connected Lie group, but only considering self maps of BG which are rational equivalences. Most of the paper deals with self maps of the p-adic completions $BG\hat{_p}$; and the results are extended to global maps only at the end.

The first complete description of $[BG, BG]$ for any nonabelian connected Lie group G was given by Mislin [Ms], for the group $G = S^3$. More recently, in [JMO] (and based on earlier work by Hubbuck [Hu] and Ishiguro [Is]), we extended Mislin's result to a description of $[BG, BG]$ for an arbitrary compact connected simple Lie group G. The assumption that G be simple was, however, crucial: examples were given in [JMO, §7] to show that a similar, simple description of all self maps is unlikely for arbitrary connected G.

When G is simple, any $f : BG\hat{_p} \to BG\hat{_p}$ is either a $\mathbb{Q}$-equivalence or nullhomotopic. The most natural setting for obtaining similar strong results for semisimple or connected groups seems to be to restrict attention to the $\mathbb{Q}$-equivalences. For example, we will see in Corollary 2.6 below that for

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connected G, two $\mathbb{Q}$-equivalences $f, f' : BG^\wedge_p \to BG^\wedge_p$ are homotopic if and only if $H^*(f;\mathbb{Q}) = H^*(f';\mathbb{Q})$. This is not the case for arbitrary self maps, as was shown in [JMO, Example 7.1].

Throughout this paper, p will be a fixed prime, G a fixed compact connected Lie group, $T \subseteq G$ a maximal torus, and $W = N(T)/T$ the Weyl group. The monoid of homotopy classes of $\mathbb{Q}$-equivalences $BG^\wedge_p \to BG^\wedge_p$ will be denoted by $[BG^\wedge_p, BG^\wedge_p]_{\mathbb{Q}}$, and the subgroup of homotopy equivalences by $[BG^\wedge_p, BG^\wedge_p]_{\mathrm{h}}$. Let $T_\infty \subseteq T$ be the subgroup of elements of p-power order. A homomorphism $\phi : T_\infty \to T_\infty$ will be called *admissible* [AM, p. 5] if it is equivariant with respect to some endomorphism of W; i.e. if there is some $\bar{\phi} : W \to W$ such that $\phi(wt) = \bar{\phi}(w)\phi(t)$ for all $w \in W$ and $t \in T$. The monoid of admissible epimorphisms from T_∞ to itself will be denoted by $\mathrm{AdmEpi}(T_\infty, T_\infty)$.

Our first step (Propositions 1.2 and 1.4) is to construct a homomorphism of monoids

$$\Theta : [BG^\wedge_p, BG^\wedge_p]_{\mathbb{Q}} \to \mathrm{AdmEpi}(T_\infty, T_\infty)/W,$$

where W acts on the admissible maps by composition. This is just the p-adic version of the construction of Adams & Mahmud [AM, Corollary 1.8] (in the special case of self maps). The map Θ is characterized by the property that $\Theta(f) = \phi \cdot W$ for any $f \in [BG^\wedge_p, BG^\wedge_p]_{\mathbb{Q}}$ and any $\phi : T_\infty \twoheadrightarrow T_\infty$ such that $f|BT \simeq B\phi$. And in this situation, f is a homotopy equivalence if and only if ϕ is an isomorphism.

The main result in Section 2 (Theorem 2.5) is that Θ is an injection, and its image is closed in the p-adic topology. The precise image of Θ is then described in Theorem 3.4. The formulation of Theorem 3.4 is somewhat technical, but it leads to a very simple result about homotopy equivalences (Corollary 3.5): Θ restricts to a group monomorphism

$$\Theta_{\mathrm{h}} : [BG^\wedge_p, BG^\wedge_p]_{\mathrm{h}} \to N_{\mathrm{Aut}(T_\infty)}(W)/W,$$

which is an isomorphism except when $p = 2$ and G contains a direct factor of the form $SO(2n+1) \times Sp(n)$. We also show that all components of $\mathrm{map}(BG^\wedge_p, BG^\wedge_p)_{\mathbb{Q}}$ have the homotopy type of $BZ(G)^\wedge_p$.

At the end of the paper, Sullivan's arithmetic square for completions and localizations is applied to obtain the analogous results for $\mathbb{Q}$-equivalences from BG to itself.

Maps $f : BG^\wedge_p \to BG^\wedge_p$ (or, equivalently, $BG \to BG^\wedge_p$) are studied here using the p-local approximation of BG constructed in [JMO, §1]. More precisely, we showed there that BG is $\mathbb{F}_p$-homology equivalent to the homotopy direct limit of classifying spaces BP for certain p-toral subgroups $P \subseteq G$. The details of this approximation will be recalled in Section 2 below.

Our general reference for completion techniques and results is Bousfield & Kan [BK]. For any space X, we let $X^\wedge_p$ denote the $\mathbb{F}_p$-completion of X.

Then $\mathrm{map}(Y, X\hat{_p}) \simeq \mathrm{map}(Y', X\hat{_p})$ for any $\mathbb{F}_p$-homology equivalence $Y \to Y'$ [BK, II.2.8]. Also, if X has finite type and $\pi_1(X)$ is a finite p-group, then $\pi_1(X\hat{_p}) \cong \pi_1(X)$, and $\pi_i(X\hat{_p}) \cong \pi_i(X)\hat{_p} \cong \pi_i(X) \otimes \hat{\mathbb{Z}}_p$ for each $i \geq 2$. This last statement follows from [BK, VI.5.2] when X is 1-connected; and holds in the general case since the sequence $(\widetilde{X})\hat{_p} \to X\hat{_p} \to K(\pi_1 X, 1)$ is a homotopy fibration (see [BK, II.5.2(iv)]). Here, $\widetilde{X}$ denotes the universal cover of X.

For any pair X, Y of spaces, and any map $f : X \to Y$, $\mathrm{map}(X, Y)_f$ will denote the space of maps $X \to Y$ which are homotopic to f.

Partial results similar to those given here have also been obtained by Dietrich Notbohm [No2] and Jesper Møller [Mø].

We would like to thank Bill Dwyer for suggesting the formulation of some of the results in Section 3.

1. We first recall the description by Dwyer–Zabrodsky [DZ] and Notbohm [No1] of the mapping spaces $\mathrm{map}(BP, BG)$, when P is p-toral and G is an arbitrary compact Lie group.

For any pair G and G' of compact Lie groups, define

$$\mathrm{Rep}(G, G') := \mathrm{Hom}(G, G')/\,\mathrm{Inn}(G'),$$

the set of G'-conjugacy classes of homomorphisms from G to G'. For any $\varrho : G \to G'$, $C_{G'}(\varrho)$ will denote the centralizer in G' of $\mathrm{Im}(\varrho)$.

THEOREM 1.1. *Fix a p-toral group P and a compact connected Lie group G. Let $T \subseteq P$ denote the identity component of P, and let $T_\infty \subseteq T$ be the set of elements of p-power order. Then there exists a dense subgroup $P_\infty \subseteq P$ such that $P_\infty \cap T = T_\infty$; and any two such subgroups are conjugate by some element of T. Furthermore, if we regard P_∞ as a discrete group*:

(i) *The maps*

$$\mathrm{Rep}(P_\infty, G) \xrightarrow[\cong]{B} [BP_\infty, BG\hat{_p}] \xleftarrow[\cong]{\mathrm{res}} [BP, BG\hat{_p}]$$

are bijections.

(ii) *For any $\varrho : P_\infty \to G$, the pairing $BC_G(\varrho) \times BP_\infty \to BG$ induces a homotopy equivalence*

$$\hat{e}_\varrho : BC_G(\varrho)\hat{_p} \xrightarrow{\simeq} \mathrm{map}(BP_\infty, BG\hat{_p})_{B\varrho} \quad (\simeq \mathrm{map}(BP, BG\hat{_p})_{B\varrho}).$$

Proof. This theorem is implicit in the work of Notbohm [No1]. In fact, it is much more elementary than his results, which involve the analogous description of maps from BP to BG (without completion).

A choice of P_∞ is equivalent to a choice of splitting map for the extension $1 \to T/T_\infty \to P/T_\infty \to P/T \to 1$. Hence the existence of P_∞ will follow if $H^2(P/T; T/T_\infty) = 0$, and its uniqueness will follow if $H^1(P/T; T/T_\infty) = 0$.

And these cohomology groups vanish because P/T is a finite p-group and T/T_∞ is uniquely p-divisible.

By construction, P_∞ is the union of an increasing sequence $P_1 \subseteq P_2 \subseteq P_3 \subseteq \ldots$ of finite p-groups, and hence $BP_\infty \simeq \underrightarrow{\mathrm{hocolim}}_n(BP_n)$. For each $n \geq 1$, $[BP_n, BG\hat{_p}] \cong \mathrm{Rep}(P_n, G)$, and

$$\mathrm{map}(BP_n, BG\hat{_p})_{B\varrho} \simeq BC_G(\varrho)\hat{_p} \tag{1}$$

for each $\varrho : P_n \to G$, by a theorem of Dwyer and Zabrodsky [DZ]. In particular, each component of $\mathrm{map}(BP_n, BG\hat{_p})$ has finite fundamental group, so the appropriate $\underleftarrow{\lim}^1 \pi_1(-)$ all vanish, and

$$[BP_\infty, BG\hat{_p}] \cong \underleftarrow{\lim}[BP_n, BG\hat{_p}] \cong \underleftarrow{\lim}\,\mathrm{Rep}(P_n, G) \cong \mathrm{Rep}(P_\infty, G).$$

Also, $\mathrm{map}(BP_\infty, BG\hat{_p})$ is the homotopy inverse limit of the spaces $\mathrm{map}(BP_n, BG\hat{_p})$; and so by (1),

$$\mathrm{map}(BP_\infty, BG\hat{_p})_{B\varrho} \simeq (\mathrm{map}(BP_\infty, BG)_{B\varrho})\hat{_p} \simeq BC_G(\varrho)\hat{_p}$$

for any $\varrho : P_\infty \to G$. And $\mathrm{map}(BP_\infty, BG\hat{_p}) \simeq \mathrm{map}(BP, BG\hat{_p})$, since the inclusion $BP_\infty \hookrightarrow BP$ is a $\mathbb{F}_p$-homology equivalence (cf. Feshbach [Fe, Proposition 2.3]). ∎

By Theorem 1.1, for any p-toral P and any compact Lie group G, any two choices $P_\infty, P'_\infty \subseteq P$ are conjugate by some element $t \in T$ (the identity component of P). Also, if $tP_\infty t^{-1} = P_\infty$, then t has p-power order in $T/(Z(P) \cap T)$; and so conjugation by t is an inner automorphism of P_∞. It follows that there is a unique natural way to identify the sets $\mathrm{Rep}(P_\infty, G)$ and $\mathrm{Rep}(P'_\infty, G)$ (independently of the choice of t); or in other words that we can regard $\mathrm{Rep}(P_\infty, G)$ as depending only on P and not on P_∞.

An element of $\mathrm{Rep}(P_\infty, G)$—i. e., a homomorphism from P_∞ to G defined up to conjugacy in G—will be called a *quasirepresentation* from P to G. We let $\mathrm{QRep}(P, G)$ denote the set of all quasirepresentations $\varphi : P_\infty \to G$ (note that the prime p is implicit in this definition). For example, if T is an n-dimensional torus, then $\mathrm{QRep}(T, T) \cong M_n(\hat{\mathbb{Z}}_p)$.

In these terms, Theorem 1.1(i) says that $[BP, BG\hat{_p}] \cong \mathrm{QRep}(P, G)$ for any p-toral P and any compact Lie group G. This is implicit in [No1], where Notbohm also proves the (much harder) corresponding global result that $[BP, BG] \cong \mathrm{Rep}(P, G)$.

The next proposition (except for part (iii)) is analogous to a theorem of Adams & Mahmud [AM, Corollary 1.11], but stated here for self maps of the p-completion of BG. The main point is that any self map of $BG\hat{_p}$ lifts to some essentially unique self map of $BT\hat{_p}$. This applies, in fact, to arbitrary maps between classifying spaces of distinct compact connected Lie groups (see [AM, Theorem 1.1]), but the following (simpler) case suffices for our purposes here.

PROPOSITION 1.2. *Fix a compact connected Lie group G, let $T \subseteq G$ be a maximal torus, and let $W = N(T)/T$ be the Weyl group. Then for any $\mathbb{Q}$-equivalence $f : BG\hat{_p} \to BG\hat{_p}$, there exists an epimorphism $\phi : T_\infty \twoheadrightarrow T_\infty$, inducing a $\mathbb{Q}$-equivalence $f_T = B\phi : BT\hat{_p} \to BT\hat{_p}$, and such that the following square commutes up to homotopy*:

$$\text{(1)}\qquad \begin{array}{ccc} BT\hat{_p} & \xrightarrow[f_T]{\simeq} & BT\hat{_p} \\ \Big\downarrow{\scriptstyle \text{incl}} & & \Big\downarrow{\scriptstyle \text{incl}} \\ BG\hat{_p} & \xrightarrow[f]{\simeq} & BG\hat{_p} \end{array}$$

Furthermore, the following hold.

(i) *For any other $f'_T = B\phi' : BT\hat{_p} \to BT\hat{_p}$ for which* (1) *commutes up to homotopy, there exists $w \in W$ such that $\phi' = \mathrm{conj}(w) \circ \phi$.*

(ii) *ϕ is admissible: there is an automorphism $\beta \in \mathrm{Aut}(W)$ which permutes the reflections in W and such that $\phi \circ \mathrm{conj}(w) \simeq \mathrm{conj}(\beta(w)) \circ \phi$ for all $w \in W$.*

(iii) *f_T is a homotopy equivalence if and only if f is.*

Proof. Let T_∞ be the subgroup in T of elements of p-power order. By Theorem 1.1 above, there is a homomorphism $\phi : T_\infty \to G$ such that $f|BT \simeq B\phi : BT\hat{_p} \to BG\hat{_p}$. After composing ϕ with an inner automorphism of G, if necessary, we may assume that $\mathrm{Im}(\phi) \subseteq T$. So from now on, we regard ϕ as a homomorphism $T_\infty \to T_\infty$. Set

$$f_T = B\phi : BT\hat{_p} \simeq (BT_\infty)\hat{_p} \to BT\hat{_p};$$

then square (1) commutes up to homotopy.

Let $\iota : T \to G$ be the inclusion. For any $w \in N(T)$, the commutative diagram

$$\begin{array}{ccccc} BT\hat{_p} & \xrightarrow{w_*} & BT\hat{_p} & \xrightarrow[=f_T]{B\phi} & BT\hat{_p} \\ \Big\downarrow{\scriptstyle B\iota} & & \Big\downarrow{\scriptstyle B\iota} & & \Big\downarrow{\scriptstyle B\iota} \\ BG\hat{_p} & \xrightarrow[\simeq \mathrm{Id}]{w_*} & BG\hat{_p} & \xrightarrow{f} & BG\hat{_p} \end{array}$$

shows that $B(\iota \circ \phi \circ \mathrm{conj}(w)) \simeq B(\iota \circ \phi)$. By Theorem 1.1 again, this means that

$$\text{(2)}\qquad \phi \circ \mathrm{conj}(w) = \mathrm{conj}(\beta(w)) \circ \phi \quad \text{for some } \beta(w) \in N(\mathrm{Im}(\phi)).$$

Set $n=\dim(T)$. Consider the group $H_2(BT\hat{_p};\mathbb{Q})\cong(\hat{\mathbb{Q}}_p)^n$, regarded as a $\hat{\mathbb{Q}}_p[W]$-representation. By (2), $\mathrm{Ker}(H_2(f_T;\mathbb{Q}))$ is W-invariant. Also, if G_i is a simple factor in G with maximal torus $T_i=T\cap G_i$, then $H_2(BT_i\hat{_p};\mathbb{Q})$ is W-irreducible (cf. [Bb1, p. 82, Proposition 5(v)]). Thus, if $H_2(f_T;\mathbb{Q})$ is not an isomorphism, then either it vanishes on some such summand $H_2(BT_i\hat{_p};\mathbb{Q})$, or it fails to be injective on $H_2(BT\hat{_p};\mathbb{Q})^W=H_2(BZ(G)\hat{_p};\mathbb{Q})$. And in either case, this contradicts the assumption that f is a $\mathbb{Q}$-equivalence.

This shows that $H_2(f_T;\mathbb{Q})$ is an isomorphism. In particular, since $T_\infty\cong(\mathbb{Z}[1/p]/\mathbb{Z})^n$, ϕ sends T_∞ onto itself with finite kernel, and hence $f_T=B\phi$ is a $\mathbb{Q}$-equivalence. Also, since $\phi: T_\infty\to T_\infty$ is onto, and since W acts effectively on T_∞, the element $\beta(w)\in W$ which satisfies (2) is unique (for any $w\in W$); and $\beta: W\to W$ is an automorphism. Then ϕ is β-equivariant by construction, and this finishes the proof of (ii).

Assume now that $f'_T: BG\hat{_p}\to BG\hat{_p}$ is such that $B\iota\circ f_T\simeq B\iota\circ f'_T$. Write $f'_T=B\phi'$, where $\phi'(T_\infty)=T_\infty$. Then by Theorem 1.1, ϕ and ϕ' are conjugate via some element $g\in G$. In particular, $gT_\infty g^{-1}=T_\infty$, and so $g\in N(T)$. In other words, if we set $w=gT\in W$, then $\phi=\mathrm{conj}(w)\circ\phi'$, and hence $f_T\simeq f'_T\circ w_*$. This proves point (i).

It remains to prove point (iii): that f_T is a homotopy equivalence if and only if f is. If f is a homotopy equivalence, then let $\bar f_T$ be a lifting of f^{-1} to $BT\hat{_p}$. The composites $f_T\circ\bar f_T$ and $\bar f_T\circ f_T$ are homotopy equivalences by (i) (the uniqueness of the lifting), and so f_T is also a homotopy equivalence.

Now assume that f_T is a homotopy equivalence. Each term in the fibration $G/T\xrightarrow{i}BT\to BG$ is simply connected, so the completed sequence $G/T\hat{_p}\to BT\hat{_p}\to BG\hat{_p}$ is again a fibration [BK, VI.6.5]. Since square (1) commutes up to homotopy, f_T restricts to a map $f_{G/T}: G/T\hat{_p}\to G/T\hat{_p}$. Also, $i_*: H_*(G/T;\mathbb{Q})\to H_*(BT;\mathbb{Q})$ is injective by [Br, Theorem 20.3(b)], and $H_*(G/T;\mathbb{Z})$ is torsion free by [Bt]. So for each n, $H_n(f_T;\mathbb{Z})$ is an automorphism of the lattice $H_n(BT;\mathbb{Z})$ which sends the sublattice $H_n(G/T;\mathbb{Z})$ into itself, and hence it restricts to an automorphism of $H_n(G/T;\mathbb{Z})$. Thus, $H_*(f_{G/T};\mathbb{Z})$ is an automorphism, and $f_{G/T}$ is a homotopy equivalence. Since f_T is a homotopy equivalence by assumption, f must also be a homotopy equivalence. ∎

Proposition 1.2 says that there is a well defined map from the monoid of $\mathbb{Q}$-equivalences $BG\hat{_p}\to BG\hat{_p}$ to the monoid of admissible epimorphisms $T_\infty\twoheadrightarrow T_\infty$ modulo the action of the Weyl group. This will be stated more precisely in Proposition 1.4 below, and the map will be shown to be injective in Section 2.

In [JMO, Theorem 3.4], we saw that when G is simple and $p\,|\,|W|$, then any $\mathbb{Q}$-equivalence $BG\hat{_p}\to BG\hat{_p}$ is a homotopy equivalence. This was in turn a generalization of Ishiguro's theorem [Is], that unstable Adams operations on BG are defined only for degrees prime to $|W|$. The next proposition describes what happens for general connected G.

PROPOSITION 1.3. *Let G be a compact connected semisimple Lie group such that p divides the order of the Weyl group of each simple component of G. Then any $\mathbb{Q}$-equivalence $f : BG\hat{_p} \to BG\hat{_p}$ is a homotopy equivalence.*

Proof. Fix a $\mathbb{Q}$-equivalence $f : BG\hat{_p} \to BG\hat{_p}$. Let $f_T = B\phi : BT\hat{_p} \to BT\hat{_p}$ be the map of Proposition 1.2, where $\phi : T_\infty \to T_\infty$ is β-equivariant for some $\beta \in \mathrm{Aut}(W)$. Since W is finite, there is some $r > 0$ such that ϕ^r is W-equivariant. Upon replacing f by f^r, we can assume that ϕ itself is W-equivariant. We will show that ϕ is an isomorphism: then f_T is a homotopy equivalence, and f is a homotopy equivalence by Proposition 1.2(iii).

Set $n = \mathrm{rk}(G) = \dim(T)$. Then $H_2(B\phi;\mathbb{Q}) = H_2(f_T;\mathbb{Q})$ is a W-equivariant automorphism of $H_2(BT\hat{_p};\mathbb{Q}) \cong (\hat{\mathbb{Q}}_p)^n$, and sends $H_2(BT\hat{_p};\mathbb{Z}) \cong (\hat{\mathbb{Z}}_p)^n$ to itself. As a W-representation, $H_2(BT\hat{_p};\mathbb{Q}) \cong \hat{\mathbb{Q}}_p \otimes_{\mathbb{Q}} H_2(BT;\mathbb{Q})$ splits as a sum of distinct irreducible summands, one for each simple factor in G (cf. [Bb1, p. 82, Proposition 5(v)]). Hence, for each simple factor $G_i \lhd G$ with maximal torus $T_i = T \cap G_i$, $H_2(B\phi;\mathbb{Q})$ restricted to $H_2(BT_{i}\hat{_p};\mathbb{Q})$ is multiplication by some constant $k_i \in \hat{\mathbb{Z}}_p$.

We must show that $p \nmid k_i$ for all i. Assume otherwise: fix a simple factor $G_i \lhd G$ such that $p \mid k_i$, set $T_i = T \cap G_i$, and let $W_i = N(T_i)/T_i$ be the Weyl group of G_i. Choose any $w \in N_{G_i}(T_i) \smallsetminus T_i$ of p-power order ($p \mid |N(T_i)/T_i|$ by assumption), and let $t \in T_i$ be any element conjugate in G_i to w. Set $P = \langle T_i, w\rangle$, a p-toral subgroup. By Theorem 1.1, $f|BP_\infty \simeq B\bar{\phi}$ for some $\bar{\phi} : P_\infty = \langle T_{i\infty}, w\rangle \to G$. Then $\bar{\phi}(P_\infty) = P_\infty$, since the restriction of ϕ sends $T_{i\infty}$ onto itself and is W_i-equivariant (and W_i acts effectively). Also, since w and t are conjugate in G, $\bar{\phi}^i(w)$ and $\bar{\phi}^i(t)$ are conjugate in G for all i (by Theorem 1.1 again). Since $\phi(t) = t^{k_i}$ and $p \mid k_i$, we have $\phi^m(t) = 1$ for some m, and hence $\phi^m(w) = 1$. On the other hand, $\phi : T_{i\infty} \to T_{i\infty}$ is W_i-equivariant and onto, and W_i acts effectively on $T_{i\infty}$; so $\bar{\phi}(wT_{i\infty}) = wT_{i\infty}$. And this implies that $\bar{\phi}^i(w) \neq 1$ for all i; which is a contradiction. ∎

So far, we have associated, with each $\mathbb{Q}$-equivalence $f \in [BG\hat{_p}, BG\hat{_p}]_{\mathbb{Q}}$, an admissible map $\phi : T_\infty \twoheadrightarrow T_\infty$. When formulating later results, it will be convenient to work with the integral lattice in T instead of with T itself.

Write $T = L(T)/\Lambda$, where $L(T) \cong \mathbb{R}^n$ is the Lie algebra (or universal covering group) for T, and where

$$\Lambda = \mathrm{Ker}[\exp : L(T) \twoheadrightarrow T]$$

is the integral lattice. Set $\Lambda\hat{_p} = \hat{\mathbb{Z}}_p \otimes \Lambda = \varprojlim (\Lambda/p^n\Lambda)$.

Consider the standard isomorphism $\mathrm{Hom}(T,T) \cong \mathrm{Hom}(\Lambda,\Lambda)$, which sends $\phi : T \to T$ to $L(\phi)|\Lambda$. We want to define the corresponding isomorphism in the p-adic situation. One way to do this is to set

$$L(T_\infty) = \varprojlim(\cdots \to T_\infty \xrightarrow{p\cdot} T_\infty \xrightarrow{p\cdot} T_\infty) \cong (\hat{\mathbb{Q}}_p)^n$$

(where $n = \dim(T)$); so that $\Lambda_p^\wedge = \mathrm{Ker}[L(T_\infty) \twoheadrightarrow T_\infty]$. Any $\phi \in \mathrm{End}(T_\infty)$ lifts to the map $L(\phi) = \varprojlim(\phi) \in \mathrm{End}(L(T_\infty))$. The isomorphism

$$\mathrm{Hom}(T_\infty, T_\infty) \xrightarrow{\cong} \mathrm{Hom}(\Lambda_p^\wedge, \Lambda_p^\wedge) \cong M_n(\hat{\mathbb{Z}}_p)$$

is now defined by sending $\phi \in \mathrm{End}(T_\infty)$ to $L(\phi)|\Lambda_p^\wedge$. Equivalently, this identification can be obtained by identifying $H_2(BT_p^\wedge; \mathbb{Z})$ with $\Lambda_p^\wedge$.

Under this identification, the monoid of epimorphisms from T_∞ to itself is identified with $\mathrm{Aut}(L(T_\infty)) \cap \mathrm{End}(\Lambda_p^\wedge)$. An admissible epimorphism is by definition one which is equivariant with respect to some endomorphism of the Weyl group W; and if we regard W as a subgroup of $\mathrm{Aut}(T_\infty)$ this allows us to identify:

$$\mathrm{AdmEpi}(T_\infty, T_\infty) \cong N_{\mathrm{Aut}(L(T_\infty))}(W) \cap \mathrm{End}(\Lambda_p^\wedge).$$

Also, if we let $\mathrm{AdmIso}(T_\infty, T_\infty)$ denote the group of admissible *iso*morphisms, then

$$\mathrm{AdmIso}(T_\infty, T_\infty) \cong N_{\mathrm{Aut}(T_\infty)}(W).$$

As before, $[BG_p^\wedge, BG_p^\wedge]_{\mathbb{Q}}$ denotes the monoid of homotopy classes of $\mathbb{Q}$-equivalences of $BG_p^\wedge$ to itself, and $[BG_p^\wedge, BG_p^\wedge]_{\mathrm{h}}$ denotes the subgroup of homotopy equivalences. The following reformulation of Propositions 1.2 and 1.3 was suggested to us by Bill Dwyer (see also [DW, Proposition 5.5]).

Proposition 1.4. *There is a well defined homomorphism of monoids*

$$\Theta : [BG_p^\wedge, BG_p^\wedge]_{\mathbb{Q}} \to \mathrm{AdmEpi}(T_\infty, T_\infty)/W \cong [N_{\mathrm{Aut}(L(T_\infty))}(W) \cap \mathrm{End}(\Lambda_p^\wedge)]/W,$$

where $\Theta(f) = H_2(f_T; \mathbb{Q}) = L(\phi)$ *for any* $\mathbb{Q}$*-equivalence* $f : BG_p^\wedge \to BG_p^\wedge$ *and any lifting of* f *to* $f_T = B\phi : BT_p^\wedge \to BT_p^\wedge$. *Furthermore*, Θ *restricts to a group homomorphism*

$$\Theta_{\mathrm{h}} : [BG_p^\wedge, BG_p^\wedge]_{\mathrm{h}} \to \mathrm{AdmIso}(T_\infty, T_\infty)/W \cong N_{\mathrm{Aut}(T_\infty)}(W)/W \cong N_{\mathrm{Aut}(\Lambda_p^\wedge)}(W)/W;$$

and

$$[BG_p^\wedge, BG_p^\wedge]_{\mathrm{h}} = \Theta^{-1}(N_{\mathrm{Aut}(T_\infty)}(W)/W) \cong \Theta^{-1}(N_{\mathrm{Aut}(\Lambda_p^\wedge)}(W)/W).$$

Proof. By definition, an epimorphism $\phi : T_\infty \twoheadrightarrow T_\infty$ is admissible if and only if $H_2(B\phi; \mathbb{Q}) = L(\phi) \in \mathrm{Aut}(L(T_\infty))$ lies in the normalizer of W. Hence Θ and Θ_{h} are well defined by Proposition 1.2. And by Proposition 1.2(iii), a $\mathbb{Q}$-equivalence $f : BG_p^\wedge \to BG_p^\wedge$ is a homotopy equivalence if and only if $\Theta(f)$ is an automorphism of $\Lambda_p^\wedge$. ∎

2. Throughout this section, we concentrate on showing that Θ and Θ_{h} are injective. This requires the machinery developed in [JMO].

Recall that a compact Lie group P is called *p-toral* if the identity component P_0 is a torus, and if P/P_0 is a finite p-group. If $W_p = N_p(T)/T$ is a Sylow p-subgroup of W, then $N_p(T)$ is a *maximal* p-toral subgroup of G in the strong sense that any p-toral subgroup is conjugate to a subgroup of $N_p(T)$ (cf. [JMO, Lemma A.1]). We now fix (for the rest of this section) such a maximal p-toral subgroup $N_p(T)$.

Proving the injectivity of the map Θ above means showing that two $\mathbb{Q}$-equivalences $BG^{\wedge}_p \to BG^{\wedge}_p$ are homotopic if they agree on BT. This will be done using an approximation of BG by a limit of classifying spaces of p-toral subgroups. So we must first show that maps $f, f' : BG^{\wedge}_p \to BG^{\wedge}_p$ which are homotopic on BT are homotopic on $BN_p(T)$; and hence (by maximality) on BP for all p-toral $P \subseteq G$. This was shown in [JMO] in the special case where f, f' are homotopy equivalences; and the next proposition extends this to the case of $\mathbb{Q}$-equivalences.

PROPOSITION 2.1. *Let G be any compact connected Lie group. Assume that $f, f' : BG^{\wedge}_p \to BG^{\wedge}_p$ are $\mathbb{Q}$-equivalences, and that $f|BT \simeq f'|BT$. Then for any p-toral subgroup $P \subseteq G$, $f|BP \simeq f'|BP$.*

Proof. Since $N_p(T)$ is a maximal p-toral subgroup, it suffices to show that $f|BN_p(T) \simeq f'|BN_p(T)$.

By Theorem 1.1, there are quasihomomorphisms $\varrho, \varrho' : N_p(T) \to G$ such that $f|BN_p(T) \simeq B\varrho$ and $f'|BN_p(T) \simeq B\varrho'$. Since $f|BT \simeq f'|BT$, we may assume that $\varrho|T = \varrho'|T$.

Let $G' \lhd G$ be the product of those simple factors for which p divides the order of the Weyl group. Then it suffices to show that ϱ and ϱ' are conjugate after restriction to $N_p(T) \cap G'$. In other words, we are reduced to the case where $G = G'$. But then f and f' are homotopy equivalences by Proposition 1.3, and so $f|BN_p(T) \simeq f'|BN_p(T)$ by [JMO, Proposition 3.5]. ∎

We now recall some definitions from [JMO]. A p-toral subgroup $P \subseteq G$ is called *p-stubborn* if $N(P)/P$ is finite and contains no nontrivial normal p-subgroups. For example, if $G = SO(3)$ and $p = 2$, then the subgroup $P \cong \mathbb{Z}/2 \times \mathbb{Z}/2$ is p-stubborn since the only normal 2-subgroup of $N(P)/P \cong \Sigma_3$ (the symmetric group of order 6) is the trivial group. Note that a maximal p-toral subgroup is always p-stubborn.

We let $\mathcal{R}_p(G)$ denote the category whose objects are orbits G/P for p-stubborn $P \subseteq G$, and where $\mathrm{Mor}(G/P, G/P')$ is the set of all G-maps between the orbits. By [JMO, Proposition 1.5], $\mathcal{R}_p(G)$ is a finite category, in the sense that it contains finitely many isomorphism classes of objects, and has finite morphism sets.

One of the main results in [JMO] is that the map

$$\underset{G/P\in\mathcal{R}_p(G)}{\underrightarrow{\mathrm{hocolim}}}\big(EG/P\big) \cong EG\times_G\Big(\underset{G/P\in\mathcal{R}_p(G)}{\underrightarrow{\mathrm{hocolim}}}\big(G/P\big)\Big)\to BG$$

(induced by projection) is an $\mathbb{F}_p$-homology equivalence. In particular,

$$[BG\hat{_p}, BG\hat{_p}]\cong[BG, BG\hat{_p}]\cong[\underset{G/P\in\mathcal{R}_p(G)}{\underrightarrow{\mathrm{hocolim}}}\big(EG/P\big), BG\hat{_p}].$$

This makes it natural to study the restriction map

$$\begin{aligned} R:[BG\hat{_p}, BG\hat{_p}]\to \underset{G/P\in\mathcal{R}_p(G)}{\underleftarrow{\lim}}[EG/P, BG\hat{_p}] &\cong \underset{G/P\in\mathcal{R}_p(G)}{\underleftarrow{\lim}}[BP, BG\hat{_p}]\\ &\cong \underset{G/P\in\mathcal{R}_p(G)}{\underleftarrow{\lim}}\mathrm{QRep}(P,G)\end{aligned}$$

(where the last step follows from Theorem 1.1).

An $\mathcal{R}_p$-*invariant* quasirepresentation of G is defined here to be a homomorphism $\varrho : N_p(T)_\infty \to G'$ into some compact connected Lie group G' which extends (via restriction and conjugation) to an element in the inverse limit $\underleftarrow{\lim}\,\mathrm{QRep}(P,G')$. Thus, for any $f : BG\to BG'\hat{_p}$, $f|BN_p(T)\simeq B\varrho$ for some unique $\mathcal{R}_p$-invariant quasirepresentation ϱ.

For any $\mathcal{R}_p$-invariant quasirepresentation $\varrho : N_p(T)_\infty\to G'$ of G,

$$\mathrm{map}(BG, BG'\hat{_p})_{[\varrho]}$$

will denote the space of "maps of type ϱ"; i.e., the space of maps $f : BG \to BG'\hat{_p}$ such that $f|BN_p(T)\simeq B\varrho$. The brackets are added to emphasize that this is a (possibly empty) *union* of connected components in $\mathrm{map}(BG, BG'\hat{_p})$. We next need to describe the relationship between the homotopy groups of $\mathrm{map}(BG, BG'\hat{_p})_{[\varrho]}$, and those of the spaces $\mathrm{map}(BP, BG'\hat{_p})_{B\varrho|P}$ for p-stubborn $P\subseteq G$. When doing this, it is convenient to use the following functors.

DEFINITION 2.2. For any $\mathcal{R}_p$-invariant quasirepresentation $\varrho : N_p(T)_\infty \to G'$ of G, where G' is connected, define a contravariant functor $\Pi^\varrho_* : \mathcal{R}_p(G)\to\mathbb{Z}_{(p)}$-mod by setting

$$\Pi^\varrho_*(G/P) := \pi_*(\mathrm{map}(EG/P, BG'\hat{_p})_{B\varrho|P})\cong \pi_*(BC_{G'}(\varrho(P_\infty)))\hat{_p}$$

for each p-stubborn $P\subseteq G$.

When making Π^ϱ_* into a well defined functor to abelian groups, there are, of course problems with choosing base points, and with possibly nonabelian fundamental groups. This is discussed in detail in Wojtkowiak [Wo]. In all of the cases which occur in this paper, the centralizer $C_{G'}(\varrho(P_\infty))$ is abelian, and so these difficulties do not arise.

It is convenient to think of the higher derived functors of inverse limits as "cohomology groups" of a category. For this reason, and to simplify notation, we write $H^i(\mathcal{C}; F) = \varprojlim^i(F)$ for any contravariant functor $F : \mathcal{C} \to \mathcal{A}b$ defined on a small category $\mathcal{C}$. As one might expect, the obstructions to the restriction map R displayed above being a bijection are the higher limits of the functors Π_*^{ϱ}.

THEOREM 2.3. *Fix a compact connected Lie group* G', *and an* $\mathcal{R}_p$*-invariant quasirepresentation* $\varrho : N_p(T) \to G'$. *Then* $\mathrm{map}(BG, BG'^{\wedge}_p)_{[\varrho]}$ *is nonempty if* $H^{i+1}(\mathcal{R}_p(G); \Pi_i^{\varrho}) = 0$ *for all* $i \geq 1$, *and is connected if* $H^i(\mathcal{R}_p(G); \Pi_i^{\varrho}) = 0$ *for all* $i \geq 1$.

Proof. See Wojtkowiak [Wo], who deals more generally with spaces of the form $\mathrm{map}(\underrightarrow{\mathrm{hocolim}}(X_\alpha), Y)$. ■

Theorem 2.3 is a special case of a second quadrant spectral sequence, which converges to the homotopy of $\mathrm{map}(\underrightarrow{\mathrm{hocolim}}(X_\alpha), Y)_{[\hat{f}]}$ for $\hat{f} \in \varprojlim[X_\alpha, Y]$. See Bousfield & Kan [BK, Propositions XII.4.1 and XI.7.1] and Bousfield [Bf] for details.

We want to show, for a pair $f, f' : BG^{\wedge}_p \to BG^{\wedge}_p$ of $\mathbb{Q}$-equivalences, that $f \simeq f'$ if $f|BT \simeq f'|BT$. We have already seen that $f, f' \in [BG, BG^{\wedge}_p]_{[\varrho]}$ for the same $\mathcal{R}_p$-invariant quasirepresentation ϱ; and so they are homotopic by Theorem 2.3 if the higher limits $H^i(\mathcal{R}_p(G); \Pi_i^{\varrho})$ all vanish.

In fact, it will suffice to consider the case where ϱ is the inclusion. In this case, we write Π_* for Π_*^{ϱ}. Thus, for any G/P in $\mathcal{R}_p(G)$,

$$\Pi_i(G/P) = \pi_i(\mathrm{map}(BP, BG^{\wedge}_p)_{\mathrm{incl}}) \cong \pi_i(BC_G(P))^{\wedge}_p \cong \pi_{i-1}(C_G(P))^{\wedge}_p$$

by Theorem 1.1. Also, for any p-stubborn $P \subseteq G$, $C_G(P) = Z(P)$ [JMO, Lemma 1.5(ii)], and is in particular abelian. The identity component of $C_G(P)$ is thus a torus, and hence $\pi_i(BC_G(P)) = 0$ for $i \geq 3$. In other words, $\Pi_i = 0$ for all $i \geq 3$. And for $i = 1$ or 2,

$$\Pi_i(G/P) \cong \pi_{i-1}(Z(P))$$

for each G/P in $\mathcal{R}_p(G)$.

We thus need the following computation of the higher limits of the functors Π_i.

THEOREM 2.4. *Fix a compact connected Lie group* G. *Then if* $\pi_1(G)$ *is torsion free and* $i \geq 1$, *or if* G *is arbitrary and* $i \geq 2$, *then*

$$H^*(\mathcal{R}_p(G); \Pi_i) \cong \begin{cases} \pi_{i-1}(Z(G))^{\wedge}_p & \text{if } * = 0, \\ 0 & \text{if } * > 0. \end{cases} \tag{1}$$

Proof. See [JMO, Theorem 4.1 and Sections 5–6]. When $i = 1$, this is shown in [JMO] only when G is simply connected. But if $G' \lhd G$ is the maximal semisimple component, then G' is simply connected by assumption,

$\mathcal{R}_p(G) \cong \mathcal{R}_p(G')$ [JMO, Proposition 1.6(i,ii)]; and Π_1 is the same on both categories. ■

In fact, Theorem 2.4 holds for any compact connected Lie group G. For $i \geq 2$, this is proven in [JMO]. For $i = 1$, a proof of this was given in the first version of [JMO], before Notbohm showed us how to get by with the result in the simply connected case.

We are now ready to apply these techniques. The following theorem really consists of two separate results. But we combine them here, since there is quite a bit of overlap in the proofs.

THEOREM 2.5. *The map*

$$\Theta : [BG\hat{_p}, BG\hat{_p}]_{\mathbb{Q}} \to N_{\mathrm{Aut}(L(T_\infty))}(W)/W$$

defined in Proposition 1.4 *is an injection of monoids*, *and its image is closed in the p-adic topology.*

Proof. The theorem will be shown in three steps. The injectivity of Θ is shown in Step 2, and the fact that $\mathrm{Im}(\Theta)$ is closed in Step 3. The first step is somewhat more technical, and is needed (in part) to get around the problem that the higher limits $H^j(\mathcal{R}_p(G); \Pi_i)$ have been completely computed only when $\pi_1(G)$ is torsion free.

Step 1. Let n be the exponent of $\pi_1(G/Z(G)_0)$, where $Z(G)_0$ is the identity component of the center. Let $\widetilde{G} \xrightarrow{\gamma} G$ be the finite covering such that $\pi_1(\widetilde{G}) = n \cdot \pi_1(G)$ (i.e., the subgroup of all nth powers in $\pi_1(G)$). Then $\widetilde{G}$ splits as a product $\widetilde{G} = S \times G'$, where S is a torus and G' is simply connected. For any $f : BG\hat{_p} \to BG\hat{_p}$, $\pi_2(f)$ sends $\pi_2(B\widetilde{G}\hat{_p})$ ($\subseteq \pi_2(BG\hat{_p})$) to itself, so that the composite

$$B\widetilde{G}\hat{_p} \xrightarrow{B\gamma} BG\hat{_p} \xrightarrow{f} BG\hat{_p} \longrightarrow K([\pi_1(G)/\pi_1(\widetilde{G})]\hat{_p}, 2)$$

is nullhomotopic. Since the fibration $B\widetilde{G} \to BG \to K(\pi_1(G)/\pi_1(\widetilde{G}), 2)$ is still a fibration after completion (cf. [BK, VI.6.5]), this shows that f lifts to a map $\widetilde{f} : B\widetilde{G}\hat{_p} \to B\widetilde{G}\hat{_p}$.

Set $\widetilde{T} = \gamma^{-1}(T)$, a maximal torus in $\widetilde{G}$. A homomorphism $\phi : T_\infty \to T_\infty$ lifts to at most one homomorphism $\widetilde{\phi} : \widetilde{T}_\infty \to \widetilde{T}_\infty$, since two liftings of ϕ differ by an element of $\mathrm{Hom}(T_\infty, \mathrm{Ker}(\gamma)) = 0$.

Write $G' = G_1 \times G_2$, where G_2 is the product of those simple factors for which p divides the order of the Weyl group. Then $\widetilde{T} = S \times T_1 \times T_2$, where T_i is a maximal torus in G_i; and $N_p(\widetilde{T}) = S \times T_1 \times N_p(T_2)$ for some maximal p-toral subgroup $N_p(T_2) \subseteq G_2$. If $P \subseteq N_p(\widetilde{T})$ is a p-stubborn subgroup of $\widetilde{G}$, then since $N(P)/P$ is finite (by definition of p-stubborn), P must have the form $P = S \times T_1 \times P'$ for some p-stubborn $P' \subseteq G_2$.

Now let $\varrho : N_p(T) \to G$ be any $\mathcal{R}_p$-invariant quasirepresentation, and set $f_P = B(\varrho|P) : BP \to BG^\wedge_p$ for each p-toral $P \subseteq N_p(T)$. In other words, the f_P define an element $\widehat{\varrho} \in \varprojlim [BP, BG^\wedge_p]$. By [BK, XII.4.1], $\mathrm{map}(BG^\wedge_p, BG^\wedge_p)_{[\varrho]}$ is the homotopy inverse limit (over $\mathcal{R}_p(G)$) of the spaces $\mathrm{map}(BP, BG^\wedge_p)_{f_P}$.

For each p-stubborn $P \subseteq G$ we write $\widetilde{P} = \gamma^{-1}(P)$. By [JMO, Proposition 1.6(i)], the correspondence $G/P \leftarrow \widetilde{G}/\widetilde{P}$ induces an isomorphism of categories $\mathcal{R}_p(G) \cong \mathcal{R}_p(\widetilde{G})$. Also, Theorem 1.1 applies to show that $\mathrm{map}(BP, BG^\wedge_p)_{f_P} \simeq \mathrm{map}(B\widetilde{P}, BG^\wedge_p)_{f_P \circ B\gamma}$ for each P. Upon taking homotopy inverse limits [BK, XII.4.1], we now get a homotopy equivalence

$$\mathrm{map}(BG^\wedge_p, BG^\wedge_p)_{[\varrho]} \xrightarrow[\simeq]{-\circ B\gamma} \mathrm{map}(B\widetilde{G}^\wedge_p, BG^\wedge_p)_{[\varrho\circ\gamma|\widetilde{P}]}. \tag{1}$$

In particular, $B\varrho$ extends to a map $BG^\wedge_p \to BG^\wedge_p$ if and only if it extends to a map $B\widetilde{G}^\wedge_p \to BG^\wedge_p$; and the map defined on $BG^\wedge_p$ is unique if and only if it is unique when defined on $B\widetilde{G}^\wedge_p$.

Step 2. Fix a pair $f, f' : BG^\wedge_p \to BG^\wedge_p$ of $\mathbb{Q}$-equivalences such that $f|BT \simeq f'|BT$. We want to show that $f \simeq f'$. Choose liftings of f and f' to $\widetilde{f}, \widetilde{f}' : B\widetilde{G}^\wedge_p \to B\widetilde{G}^\wedge_p$. Then $\widetilde{f}|B\widetilde{T} \simeq \widetilde{f}'|B\widetilde{T}$ (by the uniqueness of the lifting). By Proposition 2.1, $f|BN_p(T) \simeq f'|BN_p(T) \simeq B\varrho$ for some $\varrho \in \mathrm{QRep}(N_p(T), G)$. In other words,

$$f, f' \in \mathrm{map}(BG^\wedge_p, BG^\wedge_p)_{[\varrho]}.$$

If we can show that $\widetilde{f} \simeq \widetilde{f}'$, then $f \circ B\gamma \simeq B\gamma \circ \widetilde{f} \simeq B\gamma \circ \widetilde{f}' \simeq f' \circ B\gamma$, and (1) applies to show that $f \simeq f'$. So to simplify the notation, we can just assume that $G = \widetilde{G}$.

Let $\varrho \in \mathrm{QRep}(N_p(T), G)$ be an $\mathcal{R}_p$-invariant quasirepresentation such that $f|BN_p(T) \simeq B\varrho$ (Theorem 1.1). We may assume (after conjugating, if necessary) that $\varrho(N_p(T)_\infty) \subseteq N_p(T)_\infty$.

Recall (Step 1) that G factors as a product $G = S \times G_1 \times G_2$, where S is a torus, where G_1 and G_2 are simply connected, and where p divides the orders of the Weyl groups of all simple summands of G_2 but not of any of the simple summands of G_1. Let f_2 denote the composite

$$f_2 : (BG_2)^\wedge_p \hookrightarrow BG^\wedge_p \xrightarrow{f} BG^\wedge_p \xrightarrow{\mathrm{proj}} (BG_2)^\wedge_p.$$

Then f_2 is a $\mathbb{Q}$-equivalence, and is a homotopy equivalence by Proposition 1.3. Upon composing with $1 \times f_2^{-1}$, we can thus assume that $f|BN_p(T_2)$ is homotopic to the inclusion; or equivalently that $\varrho|N_p(T_2)$ is the inclusion of $N_p(T_2)$ into G.

Now fix a p-stubborn subgroup $P \subseteq G$. We may assume that $P \subseteq N_p(T)$, and hence (by Step 1) that $P = S \times T_1 \times P'$ for some $P' \subseteq N_p(T_2)$. In particular, $\varrho(P) = P$; and so upon referring to Definition 2.2, we see that $\Pi^{\varrho}_* = \Pi_*$. Theorem 2.4 now applies to show that $H^i(\mathcal{R}_p(G); \Pi^{\varrho}_j) = 0$ for all $i, j > 0$. And so $f \simeq f'$ by Theorem 2.3.

Step 3. It remains to show that $\mathrm{Im}(\Theta)$ is closed in $N_{\mathrm{Aut}(L(T_\infty))}(W)/W$ in the p-adic topology. So consider an element $\omega = \lim_{i\to\infty}(\omega_i)$, where $\omega_i \cdot W \in \mathrm{Im}(\Theta)$ for each i. Let $\phi, \phi_i : T_\infty \twoheadrightarrow T_\infty$ be the corresponding admissible maps. We may assume that $\phi = \lim(\phi_i)$ in $\mathrm{Hom}(T_\infty, T_\infty)$ (i.e., not only mod W).

For each i, choose an extension $f_i : BG^\wedge_p \to BG^\wedge_p$ of $B\phi_i$ (i.e., $\omega_i = \Theta(f_i)$); and let $\widetilde{f}_i : B\widetilde{G}^\wedge_p \to B\widetilde{G}^\wedge_p$ be a lifting. Then $\widetilde{f}_i|B\widetilde{T} \simeq B\widetilde{\phi}_i$, where $\widetilde{\phi}_i \in \mathrm{End}(\widetilde{T}_\infty)$ is a (unique) lifting of ϕ_i. Let p^m be the exponent of $\mathrm{Ker}(\gamma)$. If ϕ_i and ϕ_j agree on all p^{k+m}-torsion in T, for any $k > 0$, then $\widetilde{\phi}_i$ and $\widetilde{\phi}_j$ agree on all p^k-torsion in $\widetilde{T}$. Hence, since the ϕ_i converge to ϕ, the $\widetilde{\phi}_i$ converge to some $\widetilde{\phi} \in \mathrm{End}(\widetilde{T}_\infty)$, and $\widetilde{\phi}$ is a lifting of ϕ.

Using Theorem 1.1, choose elements $\varrho_i \in \mathrm{QRep}(N_p(\widetilde{T}), \widetilde{G})$ such that $\widetilde{f}_i|BN_p(\widetilde{T}) \simeq B\varrho_i$. The ϱ_i are $\mathcal{R}_p$-invariant (by Theorem 1.1 again). We may assume (since the ϱ_i are well-defined only up to conjugation) that $\varrho_i|\widetilde{T}_\infty = \widetilde{\phi}_i$ and $\mathrm{Im}(\varrho_i) \subseteq N_p(\widetilde{T})_\infty$ for each i. And since $\widetilde{\phi}_i(\widetilde{T}_\infty) = \widetilde{T}_\infty$, a counting argument shows that $\mathrm{Im}(\varrho_i) = N_p(\widetilde{T})_\infty$.

Now write $N_p(\widetilde{T})_\infty = \bigcup_{n=1}^\infty P_n$, where $P_1 \subseteq P_2 \subseteq \ldots$ are finite p-groups. For each n, the set

$$\{\varrho_i|P_n : i \ge 1\} \subseteq \mathrm{Rep}(P_n, N_p(\widetilde{T}))$$

is finite (since $\mathrm{Rep}(P_n, N_p(\widetilde{T}))$ is finite: this follows easily from [MZ, Theorems 1.10.5 & 5.3]). Hence, we can successively choose elements ϱ_{ni}, where $\{\varrho_{n1}, \varrho_{n2}, \ldots\}$ is a subsequence of $\{\varrho_{n-1,1}, \varrho_{n-1,2}, \ldots\}$ for each i (and $\varrho_{0i} = \varrho_i$); and where $\varrho_{ni}|P_n$ and $\varrho_{nj}|P_n$ are conjugate for each n and each i, j.

Now set $\sigma_n = \varrho_{nn}$ for each n. Upon replacing the σ_n by conjugate homomorphisms, if necessary, we may assume that $\sigma_n|P_m = \sigma_n|P_n$ for each $m \ge n$. The σ_n thus converge to a homomorphism $\sigma : N_p(\widetilde{T})_\infty \to \widetilde{G}$; and $\sigma|\widetilde{T}_\infty$ is conjugate to $\widetilde{\phi}$ by construction. For any p-toral subgroup $P \subseteq N_p(\widetilde{T})$ and any $g \in \widetilde{G}$ for which $gPg^{-1} \subseteq N_p(\widetilde{T})$, $(\sigma_n|gP_\infty g^{-1}) \circ \mathrm{conj}(g)$ is (for each n) conjugate in $\widetilde{G}$ to $\sigma_n|P_\infty$; and so the same holds for σ. In other words, σ is an $\mathcal{R}_p$-invariant quasirepresentation of $\widetilde{G}$.

By Step 1 again, $N_p(\widetilde{T}) = S \times T_1 \times N_p(T_2)$, and each p-stubborn subgroup $P \subseteq N_p(\widetilde{T})$ of G has the form $P = S \times T_1 \times P'$ for some p-stubborn $P' \subseteq G_2$. Also, $\sigma = \varrho' \times \varrho''$, where $\varrho' \in \mathrm{End}((S \times T_1)_\infty)$ and $\varrho'' \in \mathrm{Aut}(N_p(T_2)_\infty)$. Set $\varrho = 1 \times \varrho''$, also an $\mathcal{R}_p$-invariant quasirepresentation of $\widetilde{G}$. For any

p-stubborn $P \subseteq \widetilde{G}$ contained in $N_p(\widetilde{T})$, $\varrho(P) = \sigma(P)$ by construction, and hence $\Pi_*^\sigma \cong \Pi_*^\varrho$ as functors defined on $\mathcal{R}_p(\widetilde{G})$.

We next want to show that $\Pi_*^\varrho \cong \Pi_*$ as functors on $\mathcal{R}_p(\widetilde{G})$. Fix a p-stubborn subgroup $P \subseteq N_p(\widetilde{T})$. By [JMO, Lemma 1.5], $C_G(P) = Z(P) \subseteq P$. We claim that $C_G(\varrho P) = Z(\varrho P)$. By [JMO, Proposition A.4], $\pi_0(C_G(\varrho P))$ is a p-group, and so $C_G(\varrho P)$ is a union of p-toral subgroups. Hence, if $C_G(\varrho P) \not\subseteq \varrho P$, we can choose $g \in C_G(\varrho P) \smallsetminus \varrho P$ of p-power order, and thus such that $\langle g, \varrho P\rangle$ is still p-toral. So after conjugating (in G), we may assume that $\langle g, \varrho P_\infty\rangle \subseteq N_p(\widetilde{T})_\infty$. But then $\varrho^{-1}(g) \in C_G(P) = Z(P)$, and this is a contradiction. Thus, ϱ induces isomorphisms

$$\varrho_P : \Pi_*(G/P) \cong \pi_*(BZ(P)) \xrightarrow{\cong} \pi_*(BZ(\varrho P)) \cong \Pi_*^\varrho(G/P)$$

for each G/P in $\mathcal{R}_p(G)$; and this induces an isomorphism $\Pi_*^\varrho \cong \Pi_*$ of functors on $\mathcal{R}_p(\widetilde{G})$.

Theorem 2.4 can now be applied to show that

$$H^j(\mathcal{R}_p(\widetilde{G}); \Pi_i^\sigma) \cong H^j(\mathcal{R}_p(\widetilde{G}); \Pi_i^\varrho) \cong H^j(\mathcal{R}_p(\widetilde{G}); \Pi_i) = 0$$

for all $i, j \geq 1$. Hence $\mathrm{map}(B\widetilde{G}\hat{_p}, B\widetilde{G}\hat{_p})_{[\sigma]}$ is nonempty (and connected) by Theorem 2.3. Choose some element $\widetilde{f} \in \mathrm{map}(B\widetilde{G}\hat{_p}, B\widetilde{G}\hat{_p})_{[\sigma]}$.

Recall that we are working with the finite covering $\widetilde{G} \xrightarrow{\gamma} G$ of Step 1. By construction, $\sigma|\widetilde{T}_\infty = \widetilde{\phi}$, and hence $\sigma(\mathrm{Ker}(\gamma)) = \mathrm{Ker}(\gamma)$. So σ factors through a unique map $\tau \in \mathrm{End}(N_p(T)_\infty)$, and τ is also an $\mathcal{R}_p$-invariant quasirepresentation. Since $B\gamma \circ \widetilde{f} \in \mathrm{map}(B\widetilde{G}\hat{_p}, BG\hat{_p})_{\hat{\tau}\circ\gamma} \neq \emptyset$, formula (1) above applies to show that $\mathrm{map}(BG\hat{_p}, BG\hat{_p})_{\hat{\tau}} \neq \emptyset$. And for any $f \in \mathrm{map}(BG\hat{_p}, BG\hat{_p})_{\hat{\tau}}$, $f|BT\hat{_p} \simeq B\tau|BT \simeq B\phi$, so $\Theta(f) = \omega$, and we are done. ■

The following corollary to Theorem 2.5 is immediate.

Corollary 2.6. *For any pair of $\mathbb{Q}$-equivalences $f, f' : BG\hat{_p} \to BG\hat{_p}$, $f \simeq f'$ if and only if $f|BT \simeq f'|BT$, if and only if $H^*(f;\mathbb{Q}) = H^*(f';\mathbb{Q})$.*

Proof. We have just seen that $f \simeq f'$ if and only if $f|BT \simeq f'|BT$. And $f|BT \simeq f'|BT$ if and only if $H^*(f;\mathbb{Q}) = H^*(f';\mathbb{Q})$ by a theorem of Notbohm [No1, Proposition 4.1]. ■

Using the same techniques, we get the following description of the individual connected components of the space of $\mathbb{Q}$-equivalences. This result has also been proven recently by Dwyer & Wilkerson [DW2, Theorem 1.3], using quite different methods.

Proposition 2.7. *For any compact connected Lie group G, and any $\mathbb{Q}$-equivalence $f : BG\hat{_p} \to BG\hat{_p}$, the natural homomorphism $Z(G) \times G \to G$*

induces homotopy equivalences

$$BZ(G)\hat{}_p \xrightarrow{\simeq} \mathrm{map}(BG\hat{}_p, BG\hat{}_p)_{\mathrm{Id}} \underset{\simeq}{\xrightarrow{f\circ -}} \mathrm{map}(BG\hat{}_p, BG\hat{}_p)_f.$$

Proof. We first show that the first map is a homotopy equivalence. By the obstruction theory of Wojtkowiak [Wo] (more precisely, by the version of his result given in [JMO, Theorem 3.9]), this will follow immediately if the formula for $H^j(\mathcal{R}_p(G); \varPi_i)$ in Theorem 2.4 (formula (1)) holds for all $i > j \ge 0$, and $i = j \ge 2$. So the only case which it remains to check is that

$$H^0(\mathcal{R}_p(G); \varPi_1) := \varprojlim_{\mathcal{R}_p(G)} (\varPi_1) \cong \pi_1(BZ(G)\hat{}_p). \tag{1}$$

Whether or not this holds, we have

$$\pi_1\big(\mathrm{map}(BG\hat{}_p, BG\hat{}_p)_{\mathrm{Id}}\big) \cong \varprojlim_{\mathcal{R}_p(G)} (\varPi_1) \cong \varprojlim_{G/P\in\mathcal{R}_p(G)} \pi_0(Z(P))_{(p)}.$$

So by [JMO, Theorem 4.2], (1) does hold if G is simple. If G is a product of a torus and simple groups G_i, then $\mathcal{R}_p(G)$ is the product of the $\mathcal{R}_p(G_i)$ [JMO, Proposition 1.6(ii)]; and so (1) again holds. Finally, if G is arbitrary, then there are monomorphisms

$$\pi_1(BZ(G)) \cong \pi_0(Z(G)) \rightarrowtail \varprojlim_{G/P\in\mathcal{R}_p(G)} \pi_0(Z(P))$$

and

$$\Big(\varprojlim_{G/P\in\mathcal{R}_p(G)} \pi_0(Z(P))\Big)/\pi_0(Z(G)) \rightarrowtail \varprojlim_{G/P\in\mathcal{R}_p(G)} \pi_0(Z(P/Z(G))) \cong \pi_0(Z(G/Z(G))) = 1$$

(since $\mathcal{R}_p(G/Z) \cong \mathcal{R}_p(G)$ by [JMO, Proposition 1.6(i)]). So (1) also holds in this case.

To see that the second map is a homotopy equivalence, let $G_{\mathrm{s}} \subseteq G$ be the maximal semisimple subgroup, and note that $\mathcal{R}_p(G_{\mathrm{s}}) \cong \mathcal{R}_p(G)$ (any p-stubborn subgroup of G contains the connected component of $Z(G)$, and its intersection with G_{s} is p-stubborn in G_{s}). Hence the obstructions to $(f \circ -)$ being a homotopy equivalence are the same for G and G_{s}, and we can assume that $G = G_{\mathrm{s}}$ is semisimple. In this case, we can write

$$BG\hat{}_p \simeq (BG_1)\hat{}_p \times (BG_2)\hat{}_p \quad \text{and} \quad f \simeq f_1 \times f_2,$$

where p divides the orders of all simple components of G_1 but of none of the simple components of G_2. (The simple components for which p does not divide the order of the Weyl groups all have center and fundamental group of order prime to p.) Then f_1 is a homotopy equivalence by Proposition 1.3, and we can assume $f_1 = \mathrm{Id}$. Write $f|BN_p(T) \simeq B\varrho$. The only p-stubborn subgroups of G_2 are the maximal tori, so $\Pi_*^{\varrho} \cong \Pi_*$, and $(f\circ -)$ is a homotopy equivalence by the Bousfield–Kan spectral sequence again. ■

3. The main goal of this section is to describe the images of the maps

$$\Theta : [BG_p^{\wedge}, BG_p^{\wedge}]_{\mathbb{Q}} \rightarrowtail N_{\mathrm{Aut}(L(T_\infty))}(W)/W$$

and

$$\Theta_{\mathrm{h}} : [BG_p^{\wedge}, BG_p^{\wedge}]_{\mathrm{h}} \rightarrowtail N_{\mathrm{Aut}(\Lambda_p^{\wedge})}(W)/W$$

of Proposition 1.4. The image of Θ is described explicitly in Theorem 3.4. In Corollary 3.5, we then show that Θ_{h} is an isomorphism if $p \neq 2$, or if G contains no factor of the form $Sp(n) \times SO(2n+1)$. The following example shows why this restriction is necessary.

EXAMPLE 3.1. Assume that $p = 2$, and that $G = Sp(n) \times SO(2n+1)$ for some $n \geq 1$. Let $T = T_1 \times T_2$ be the product of the standard maximal tori, where

$$T_1 = \mathrm{Im}[\alpha : \mathbb{R}^n \to Sp(n)] \quad \text{and} \quad T_2 = \mathrm{Im}[\beta : \mathbb{R}^n \to SO(2n+1)],$$

and

$$\alpha(\theta_1, \ldots, \theta_n) = \mathrm{diag}(e^{2\pi i\theta_1}, \ldots, e^{2\pi i\theta_n}) \in Sp(n),$$

$$\beta(\theta_1, \ldots, \theta_n) = \begin{pmatrix} \cos(2\pi\theta_1) & -\sin(2\pi\theta_1) \\ \sin(2\pi\theta_1) & \cos(2\pi\theta_1) \end{pmatrix} \oplus \ldots \oplus \begin{pmatrix} \cos(2\pi\theta_n) & -\sin(2\pi\theta_n) \\ \sin(2\pi\theta_n) & \cos(2\pi\theta_n) \end{pmatrix} \oplus (1).$$

Let $\Lambda = \Lambda_1 \times \Lambda_2 \subseteq \widetilde{T}$ be the integral lattice for T, and let $\phi \in \mathrm{Aut}(T) \cong \mathrm{Aut}(\Lambda)$ be the involution which switches the factors (sending $\alpha(\theta_1, \ldots, \theta_n)$ to $\beta(\theta_1, \ldots, \theta_n)$ and vice versa). Then ϕ is an admissible map for G, but $B\phi$ does not extend to any self map of $BG_2^{\wedge}$. In other words,

$$\Theta_{\mathrm{h}} : \mathrm{Aut}(BG_p^{\wedge}) \to N_{\mathrm{Aut}(\Lambda_p^{\wedge})}(W)/W$$

is not onto in this case.

Proof. Note first that ϕ is admissible, since $Sp(n)$ and $SO(2n+1)$ have the same Weyl groups under this identification of their maximal tori. If $B\phi$ could be extended to a self map of $BG_2^{\wedge}$, then it would be a homotopy equivalence by Proposition 1.2(iii), and hence would restrict to a homotopy equivalence between $BSp(n)_2^{\wedge}$ and $BSO(2n+1)_2^{\wedge}$. But $BSp(n)_2^{\wedge}$ is 3-connected, while $\pi_2(BSO(2n+1)_2^{\wedge}) \cong \mathbb{Z}/2$. ■

The roots in G play an important role in both the statements and the proofs of the results later in this section. In the next proposition, we collect some of the facts about roots which will be needed later. But we first set up some notation.

Let R denote the set of roots of G: regarded as a subset of $L(T)^* = \mathrm{Hom}(L(T),\mathbb{R})$. Then the Lie algebra $L(G)$, under the adjoint (conjugation) action of T, splits as a sum

$$L(G) = L(T) \oplus \sum_{\pm\theta\in R} V_\theta,$$

where each V_θ is a 2-dimensional irreducible T-representation with character $\chi_\theta : T \to S^1$, and where $\chi_\theta \circ \exp(v) = \exp(2\pi i \cdot \theta(v))$ for each $\theta \in R$ and each $v \in L(T)$. (Note that V_θ defines θ only up to sign.) In other words, the roots are defined to be the liftings to $\mathrm{Hom}(L(T),\mathbb{R})$ of the irreducible characters for the adjoint action of T on $L(G)$. In particular, since $\Lambda = \mathrm{Ker}[\exp : L(T) \to T]$, $\theta(\Lambda) \subseteq \mathbb{Z}$ for each $\theta \in R$; and $\theta(v) \in \mathbb{Z}$ (for $v \in L(T)$) if and only if $\exp(v)$ commutes with all elements of $\exp(V_\theta)$.

Proposition 3.2. *Let R be the set of roots of G, regarded as a subset of $\mathrm{Hom}(T, S^1) \cong \mathrm{Hom}(\Lambda,\mathbb{Z}) \subseteq L(T)^*$. Let $W = N(T)/T$ be the Weyl group of G, with the induced action on $L(T)$. Let $R_{\mathrm{s}} \subseteq R$ be the set of simple roots with respect to some fixed Weyl chamber. Assume that $L(T)$ has been given a fixed W-invariant inner product. Then the following hold.*

(i) *W is the group generated by the reflections in the hyperplanes $\mathrm{Ker}(\theta)$ for $\theta \in R_{\mathrm{s}}$. Conversely, for any $w \in W$, if w is a reflection (if $\widetilde{T}^w$ has codimension one), then $\widetilde{T}^w$ is the kernel of some root $\theta \in R$.*

(ii) *$R = W \cdot R_{\mathrm{s}}$: each root is in the W-orbit of some simple root.*

(iii) *The inclusion $T \subseteq G$ induces a surjection $\Lambda \cong \pi_1(T) \twoheadrightarrow \pi_1(G)$.*

(iv) *If $Z(G) = 1$, then $\Lambda = \{x \in L(T) : R(x) \subseteq \mathbb{Z}\}$.*

(v) *For each $\theta \in \mathbb{R}$, there exists a unique element $v_\theta \in \Lambda$ such that $v_\theta \perp \mathrm{Ker}(\theta)$ and $\theta(v_\theta) = 2$ (v_θ is the nodal vector of θ in the notation of Bourbaki). And $\pi_1(G) \cong \Lambda/\langle v_\theta : \theta \in R\rangle$.*

(vi) *For any $\theta \in \mathbb{R}$, $\Lambda \cap \mathrm{Ker}(\theta)^\perp = \mathbb{Z} \cdot v_\theta$ or $\mathbb{Z} \cdot \frac{1}{2} v_\theta$, where the second possibility occurs if and only if G contains a direct factor $SO(2n+1)$ for some $n \geq 1$, and θ is a short root of such a factor.*

Proof. The first statement in (i)—that W is generated by reflections by simple roots—is shown in [Ad, 5.13(iv) & 5.34]. To see the second statement, note that if $wT \in N(T)/T$ is a reflection in $\widetilde{T}$, then $w \in C_G(S)$ (the centralizer) for some codimension one subtorus $S \subseteq T$. By [Bb2, p. 31, Lemma 2], $C_G(S) \supsetneqq \langle T, w\rangle$ is connected (and nonabelian). So it must have at least one root $\theta \in R$, and $\mathrm{Ker}(\theta) = S$.

Point (ii) is shown in [Bb1, p. 154, Prop. 15]. Point (iii) is shown in [Ad, 5.47] or [Bb2, p. 34, Prop. 11]. Point (iv) is shown in [Ad, Proposition 5.3], and is equivalent to [Bb2, Proposition 8(b)].

Fix some $\theta \in R$, let $v_\theta \in L(T)$ be the unique element such that $v_\theta \perp \mathrm{Ker}(\theta)$ and $\theta(v_\theta) = 2$, and let $w_\theta \in N(T)$ be such that $w_\theta T \in W$

is the reflection in $\mathrm{Ker}(\theta)$. Then $w_\theta \in H_\theta := C_G(\exp(\mathrm{Ker}(\theta)))$, and H_θ is a connected subgroup by [Bb2, p. 31] again. Also, $L(H_\theta) = L(T) \oplus V_\theta$ in the notation above; and so $Z(H_\theta) = \mathrm{Ker}(\chi_\theta) = \exp(\theta^{-1}(\mathbb{Z}))$. In particular, if $v \in L(T)$ and $\theta(v) = 1$, then $\exp(v) \in Z(H_\theta)$, $v_\theta = v - w_\theta(v)$, and so $\exp(v_\theta) = [\exp(v), w_\theta] = 1$.

This shows that $v_\theta \in \Lambda = \mathrm{Ker}(\exp)$ for each $\theta \in R$. The formula $\pi_1(G) \cong \Lambda/\langle v_\theta : \theta \in R\rangle$ is shown in [Ad, 5.47 & 5.48] and [Bb2, p. 34, Prop. 11].

It remains to check point (vi). Fix a root $\theta \in R$. Since $\theta(\Lambda) \subseteq \mathbb{Z}$ and $\theta(v_\theta) = 2$, we must have

$$\Lambda \cap \mathrm{Ker}(\theta)^\perp = \mathbb{Z} \cdot v_\theta \quad \text{or} \quad \mathbb{Z} \cdot \tfrac{1}{2} v_\theta.$$

Assume that the second possibility occurs; i.e., that $\frac{1}{2}v_\theta \in \Lambda$. Fix any other root $\eta \in R \setminus \{\pm\theta\}$. Since the projection of η to the line $\mathbb{R} \cdot \theta$ is $\frac{\langle \eta,\theta\rangle}{\langle \theta,\theta\rangle} \cdot \theta$, and since $(\mathbb{R} \cdot \theta)^\perp(v_\theta) = 0$, we have

$$\frac{\langle \eta,\theta\rangle}{\langle \theta,\theta\rangle} = \left(\frac{\langle \eta,\theta\rangle}{\langle \theta,\theta\rangle} \cdot \theta\right)\left(\tfrac{1}{2}v_\theta\right) = \eta\left(\tfrac{1}{2}v_\theta\right) \in \mathbb{Z}.$$

In other words, the projection of η to $\mathbb{R} \cdot \theta$ is an integral multiple of θ. A quick check of the different possibilities (cf. [Ad, Proposition 5.25]) shows that either $\eta \perp \theta$, or the angle between η and θ is $45°$ or $135°$ and θ is the shorter root.

By point (ii), we can assume that $\theta \in R_{\mathrm{s}}$. The Dynkin diagram of G thus contains a node which is not connected to any other node by single or triple lines; and which, if connected to another node by a double line, represents the shorter of those two roots. The classification of simple Lie groups now shows that G contains a normal simple subgroup of type B_n (i.e., $H \cong SO(2n+1)$ or $Spin(2n+1)$) for some $n \geq 1$.

Assume $H \lhd G$, where $H \cong SO(2n+1)$ or $Spin(2n+1)$. Identify $L(T \cap H)$ with $\mathbb{R}^n$ via the map β of Example 3.1. Let $x_i \in \mathrm{Hom}(\mathbb{R}^n, \mathbb{R})$ denote projection to the ith coordinate, and let $\varepsilon_i \in \mathbb{R}^n$ denote the ith element of the standard basis. Then the integral lattice of H is $\mathbb{Z}^n$ if $H \cong SO(2n+1)$, and $\langle \varepsilon_i \pm \varepsilon_j\rangle$ if $H \cong Spin(2n+1)$. The roots of H are the elements $\pm x_i$ and $\pm x_i \pm x_j$ (for $i \neq j$). Then $v_\theta = \pm 2\varepsilon_i$ if $\theta = \pm x_i$, and $v_\theta = \pm\varepsilon_i \pm \varepsilon_j$ if $\theta = \pm x_i \pm x_j$. Hence $\frac{1}{2}v_\theta \in \Lambda = \mathbb{Z}^n$ if and only if $H \cong SO(2n+1)$ and $\theta = \pm x_i$.

Thus, $\frac{1}{2}v_\theta \in \Lambda$ for some $\theta \in R$ if and only if θ is a short root of some simple component $H \lhd G$, where $H \cong SO(2n+1)$ for some $n \geq 1$. Since $Z(SO(2n+1)) = 1$, this subgroup is in fact a direct factor of G. ∎

We now regard a root $\theta \in R \subseteq \mathrm{Hom}(\Lambda, \mathbb{Z})$ also as an element of $\mathrm{Hom}(\Lambda_p^\wedge, \hat{\mathbb{Z}}_p)$ or $\mathrm{Hom}(L(T_\infty), \hat{\mathbb{Q}}_p)$ (recall that $L(T_\infty) \cong \mathbb{Q} \otimes_{\mathbb{Z}} \Lambda_p^\wedge$). We can assume that the W-invariant inner product on $L(T)$ takes integer values on Λ (take any integer valued inner product on Λ and sum over its W-orbit).

Hence the inner product extends to a $\hat{\mathbb{Q}}_p$-valued inner product on $L(T_\infty)$. Note that the cosine of angles between elements is no longer defined in this setting (since there is no unique choice of square roots); but the square of the cosine is defined. It will be convenient to say that two angles α_1, α_2 are "the same up to sign" if $\cos^2(\alpha_1) = \cos^2(\alpha_2)$; i.e., if $\alpha_1 = \alpha_2$ or $\alpha_1 = \pi - \alpha_2$.

LEMMA 3.3. *Let $\omega \in \mathrm{Aut}(L(T_\infty))$ be any admissible map*; *i.e.*, *ω lies in the normalizer of the Weyl group $W \subseteq \mathrm{Aut}(\Lambda) \subseteq \mathrm{Aut}(L(T_\infty))$. Then ω permutes the components of $L(T_\infty)$ corresponding to simple factors of G. The dual map ω^* sends each root in G to some $\hat{\mathbb{Q}}_p$-multiple of another root*; *and preserves the angles between roots up to sign. Furthermore*, *after replacing ω by $w \circ \omega$ for some $w \in W$ if necessary*, *we may assume that ω permutes any given set of simple roots of G* (*again up to scalar multiples*); *and hence induces an automorphism of the Dynkin diagram* (*though possibly reversing arrows*).

Proof. By Proposition 3.2(i), the kernels of the roots of G are precisely those hyperplanes in $L(T)$ (hence in $L(T_\infty)$) for which the corresponding reflection lies in W. Since ω is equivariant with respect to some automorphism of W, it permutes the reflections, hence the kernels of roots, and hence the roots themselves up to scalar multiple.

Write $\mathbb{Q} \otimes \Lambda = V_0 \times V_1 \times \ldots \times V_m$, where V_0 comes from the torus factor in G, and the $V_1, \ldots, V_m$ from the simple factors. Then $V_0 = (\mathbb{Q} \otimes \Lambda)^W$, and the spaces $\mathbb{C} \otimes_{\mathbb{Q}} V_1, \ldots, \mathbb{C} \otimes_{\mathbb{Q}} V_m$ are distinct irreducible representations of W (cf. [Bb1, p. 82, Proposition 5(v)]). Hence the $\hat{\mathbb{Q}}_p[W]$-representations $V_{i\hat{p}} = \hat{\mathbb{Q}}_p \otimes_{\mathbb{Q}} V_i$ ($1 \le i \le m$) are also irreducible. Since ω is equivariant with respect to some automorphism of W, it must permute the $V_{i\hat{p}}$. Furthermore, since the W-invariant inner product on each irreducible summand is unique up to scalar multiple, ω preserves the inner product on each of the $V_{1\hat{p}}, \ldots, V_{m\hat{p}}$ up to scalar multiple (and $V_{i\hat{p}} \perp V_{j\hat{p}}$ for $i \ne j$). Since each root $\theta \in R$ lies in one of the $(V_{i\hat{p}})^*$ for $1 \le i \le m$, this shows that ω preserves angles between roots up to sign.

Now choose a permutation $\sigma \in \Sigma(R)$ and elements $0 \ne a_\theta \in \hat{\mathbb{Q}}_p$ such that $\omega^*(\theta) = a_\theta \cdot \sigma\theta$ for each root $\theta \in R$. Fix a Weyl chamber C, and let R_{s} be the corresponding set of simple roots. Since the Dynkin diagram is a union of trees, we can arrange (by switching pairs of roots $\pm\theta$) that for any pair $\theta_1 \ne \theta_2$ in R_{s}, the angle between $\sigma\theta_1$ and $\sigma\theta_2$ is between $\pi/2$ and π. Then for each $\theta_1, \theta_2 \in R_{\mathrm{s}}$, θ_1 and θ_2 form the same angle as $\sigma\theta_1$ and $\sigma\theta_2$.

Let T_{s} be the intersection of T with the maximal semisimple subgroup of G. Then $L(T_{\mathrm{s}}) = \sum_{i=1}^m \mathbb{R} \otimes_{\mathbb{Q}} V_i$; and R_{s} and $\sigma(R_{\mathrm{s}})$ are bases of $L(T_{\mathrm{s}})$ (cf. [Ad, Prop. 5.33]). There is thus a unique $\mu : L(T_{\mathrm{s}}) \xrightarrow{\cong} L(T_{\mathrm{s}})$ such that $\mu^*(\theta/\|\theta\|) = \sigma\theta/\|\sigma\theta\|$ for each $\theta \in R_{\mathrm{s}}$; and μ is orthogonal since σ preserves angles between the simple roots. Also,

(1) $$\mu^{-1}(C)=\{x\in L(T_{\rm s}):\sigma\theta(x)\ge 0 \text{ for all } \theta\in R_{\rm s}\};$$

and hence $\mu^{-1}(C)$ is a union of Weyl chambers. Since the Weyl chambers are permuted transitively by the orthogonal action of W (cf. [Ad, Theorem 5.13]), they and $\mu^{-1}(C)$ all have the same volume after intersection with the unit ball; and hence $\mu^{-1}(C)$ is itself a Weyl chamber. We can thus assume (after composing ω and μ by some element of W) that $\mu(C)=C$. But then (1) implies that

$$\{\mathrm{Ker}(\theta):\theta\in R_{\rm s}\}=\{\mathrm{Ker}(\sigma\theta):\theta\in R_{\rm s}\}$$

(the walls of $C=\mu(C)$); and hence that $\sigma(R_{\rm s})=R_{\rm s}$.

In particular, since the nodes in the Dynkin diagram correspond to simple roots of G, and the connectors correspond to the angles between roots, this shows that σ induces an automorphism (possibly reversing arrows) of the Dynkin diagram of G. ■

We are now ready to show which admissible maps extend to $\mathbb{Q}$-equivalences of $BG^{\wedge}_p$ to itself. In other words, we will describe the image of the homomorphism

$$\Theta:[BG^{\wedge}_p,BG^{\wedge}_p]_{\mathbb{Q}}\rightarrowtail \mathrm{AdmEpi}(T_\infty,T_\infty)/W\cong[N_{\mathrm{Aut}(L(T_\infty))}(W)\cap\mathrm{End}(\Lambda^{\wedge}_p)]/W$$

of Proposition 1.4. Roughly, the following theorem says that the only necessary conditions for an admissible epimorphism ϕ to lift to a self map of $BG^{\wedge}_p$ are the ones imposed by Proposition 1.3 and Example 3.1.

THEOREM 3.4. *Let $G_{\langle p\rangle}\lhd G$ be the product of all simple summands whose Weyl group has order a multiple of p. Let $R\subseteq\mathrm{Hom}(\Lambda,\mathbb{Z})$ be the set of roots of G, and let $R_{\langle p\rangle}\subseteq R$ be the subset of roots in $G_{\langle p\rangle}$. For each $n\ge 1$, let $G_n\lhd G$ be the product of all normal subgroups isomorphic to $SO(2n+1)$. Then, for any admissible epimorphism $\phi:T_\infty\twoheadrightarrow T_\infty$, $\phi W\in\mathrm{Im}(\Theta)$ if and only if the following two conditions are satisfied*:

(a) $\mathrm{Ker}(\phi)\cap G_{\langle p\rangle}=1$ *and*
(b) (*if* $p=2$) $\phi(G_n\cap T_\infty)\subseteq G_n\cap T_\infty\ \forall n\ge 1$;

if and only if $\omega:=L(\phi)|\Lambda^{\wedge}_p\in\mathrm{End}(\Lambda^{\wedge}_p)$ satisfies the two conditions

(i) $\omega^*(R)\subseteq\hat{\mathbb{Z}}_p\cdot R$ *and*
(ii) $\omega^*(R_{\langle p\rangle})\subseteq(\hat{\mathbb{Z}}_p)^*\cdot R_{\langle p\rangle}$.

Proof. We will prove the implications

$$(\phi W\in\mathrm{Im}(\Theta))\xRightarrow{\text{Step1}}(\text{a,b})\xRightarrow{\text{Step2}}(\text{i,ii})\xRightarrow{\text{Step3}}(\omega W\in\mathrm{Im}(\Theta)).$$

Step 1. Assume that $\phi W = \Theta(f)$; i.e., $f : BG\hat{_p} \to BG\hat{_p}$ is such that $f|BT\hat{_p} \simeq B\phi$. We will show that ϕ satisfies conditions (a) and (b) above.

Write $G' = G_{\langle p\rangle}$ for short, set $G'' = G/G'$, and let $\alpha : G' \to G$ and $\beta : G \to G''$ denote the induced maps. Let $T' = T \cap G'$ and $T'' = T/T'$ be their maximal tori, and set $\phi' = \phi|T'_\infty : T'_\infty \to T'_\infty$. The composite

$$BG'\hat{_p} \xrightarrow{B\alpha} BG\hat{_p} \xrightarrow{f} BG\hat{_p} \xrightarrow{B\beta} BG''\hat{_p}$$

is nullhomotopic, since it is trivial in rational cohomology (cf. [JMO, Theorem 3.11]). Hence $f \circ B\alpha$ pulls back along the fibration $BG'\hat{_p} \to BG\hat{_p} \to BG''\hat{_p}$ (cf. [BK, VI.6.5]) to a map $f' : BG'\hat{_p} \to BG'\hat{_p}$. By Proposition 1.3, f' is a homotopy equivalence, and so $\mathrm{Ker}(\phi') = \mathrm{Ker}(\phi) \cap G_{\langle p\rangle} = 1$ by Proposition 1.2(iii). This proves point (a).

Now assume $p = 2$; we must prove condition (b). We have just seen that f restricts to a self map (and homotopy equivalence) of $G_{\langle 2\rangle}$. So we can assume that $G = G_{\langle 2\rangle}$; i.e., that G is semisimple and $\phi \in \mathrm{Aut}(T_\infty)$.

By Lemma 3.3, ϕ permutes the simple summands of G. Fix some $SO(2n+1) \cong H \lhd G$, and let $H' \lhd G$ be the simple summand such that $\phi(T_\infty \cap H) = T_\infty \cap H'$. Since $Z(H) = 1$, H must be a direct factor of G (i.e., not just up to a finite covering). Hence $T_\infty \cap H'$ is a direct factor of T_∞, and so $BH'\hat{_2}$ is a direct factor of $BG\hat{_2}$ (i.e., H' is a direct factor of G up to odd degree covering). The composite

$$BH\hat{_2} \xrightarrow{\text{incl}} BG\hat{_2} \xrightarrow{f} BG\hat{_2} \xrightarrow{\text{proj}} BH'\hat{_2}$$

is a homotopy equivalence; so $\pi_2(BH'\hat{_2}) \cong \pi_2(BSO(2n+1)\hat{_2}) \cong \mathbb{Z}/2$. Also, by Lemma 3.3, H' has the same Dynkin diagram as H, except possibly for the direction of the arrows; and hence (since $\pi_1(H') \neq 1$) must be isomorphic to one of the groups $SO(2n+1)$ or $PSp(n)$. These two are isomorphic if $n \le 2$; while if $n \ge 3$ then

$$\pi_5(BSO(2n+1)) \cong \pi_5(BO) \cong 0 \quad \text{and} \quad \pi_5(BPSp(n)) \cong \pi_5(BSp) \cong \mathbb{Z}/2$$

(cf. [Ml, p. 142]). Thus, $H' \cong H \cong SO(2n+1)$. Alternatively, this last point can be proven by showing that any admissible homomorphism between the (2-adic) integral lattices of $SO(2n+1)$ and $PSp(n)$ (for $n \ge 3$) must (after composing with a Weyl group element) be a scalar multiple of the map used in Example 3.1—and hence is not an isomorphism.

We have now shown that (for any $n \ge 1$), ϕ permutes those simple factors isomorphic to $SO(2n+1)$ among themselves. This proves condition (b).

Step 2. Let $R_{\mathrm{s}} \subseteq R$ be the simple roots with respect to some Weyl chamber. Then $R_{\langle p\rangle} \cap R_{\mathrm{s}}$ is a $\hat{\mathbb{Q}}_p$-basis for $L(\Lambda\hat{_p}_{\langle p\rangle})$ (cf. [Ad, Prop. 5.33]).

By Lemma 3.3, there is a permutation $\sigma \in \Sigma(R)$ such that for each $\theta \in R$, $\omega^*(\theta) = a_\theta \cdot \sigma\theta$ for some $a_\theta \in (\hat{\mathbb{Q}}_p)^*$. We may assume (by Lemma 3.3

again) that $\sigma(R_{\mathrm{s}}) = R_{\mathrm{s}}$. For each $\theta \in R$, let $v_\theta \in \Lambda$ be the element defined in Proposition 3.2(v,vi): $v_\theta \perp \mathrm{Ker}(\theta)$ and $\theta(v_\theta) = 2$. Since ω preserves angles (and in particular orthogonality), we see that $\omega(v_{\sigma\theta}) = a_\theta \cdot v_\theta$ for each θ.

Assume now that conditions (a) and (b) hold. If p is odd, then by Proposition 3.2(vi), $(\hat{\mathbb{Q}}_p \cdot v_\theta) \cap \Lambda\hat{_p} = \hat{\mathbb{Z}}_p \cdot v_\theta$ for each $\theta \in R$. Hence, since $\omega(v_{\sigma\theta}) = a_\theta \cdot v_\theta$ and $\omega(\Lambda) \subseteq \Lambda$, we get $a_\theta \in \hat{\mathbb{Z}}_p$ for each θ, and so $\omega^*(R) \subseteq \hat{\mathbb{Z}}_p \cdot R$.

If $p = 2$, then let $R_0 \subseteq R$ be the set of those roots θ such that $\frac{1}{2}v_\theta \in \Lambda$. By 3.2(vi) again, the elements $\theta \in R_0$ are precisely the short roots of summands $SO(2n+1) \lhd G$; i.e., the short roots in the G_n (for $n \geq 1$). Also, for each i, $\phi(G_n \cap T_\infty) = G_n \cap T_\infty$ by condition (b), and $\phi|G_n \cap T_\infty$ is injective by (a). Thus, ω restricts to an admissible automorphism of the 2-adic integral lattice of G_n, which permutes the simple factors by Lemma 3.3. Also, the only admissible automorphisms of the 2-adic integral lattice of $SO(2n+1)$ are given by multiplication by scalars $a \in (\hat{\mathbb{Z}}_2)^*$, and so we can conclude that $\sigma(R_0) = R_0$. The same argument as for odd p now shows that $a_\theta \in \hat{\mathbb{Z}}_2$ for all θ; and so condition (i) also holds in this case.

Finally, as noted above, the elements of $R_{\mathrm{s}} \cap R_{\langle p\rangle}$ form a $\hat{\mathbb{Q}}_p$-basis for $L(\Lambda\hat{_p}_{\langle p\rangle})$. Hence by (a),

$$p \nmid \det(\omega|L(G_{\langle p\rangle} \cap T_\infty)) = \pm \prod_{\theta \in R_{\mathrm{s}} \cap R_{\langle p\rangle}} a_\theta; \tag{2}$$

and so $p \nmid a_\theta$ for $\theta \in R_{\langle p\rangle} = W \cdot (R_{\mathrm{s}} \cap R_{\langle p\rangle})$. And this proves condition (ii).

Step 3a. Assume that $G = S \times H$, where S is a torus and H is a semisimple Lie group with trivial center. We show here that for such G, conditions (i) and (ii) suffice to imply that $\omega \cdot W \in \mathrm{Im}(\Theta)$.

Write $G = S \times H_1 \times \ldots \times H_m$, where the H_m are simple. Let $\Lambda = \Lambda_0 \times \Lambda_1 \times \ldots \times \Lambda_m$ and $W = W_1 \times \ldots \times W_m$ be the corresponding decompositions of Λ and W. By Lemma 3.3, there is some $\tau \in \Sigma_m$ such that $\omega(\Lambda_i\hat{_p}) = (\Lambda_{\tau i})\hat{_p}$ for all $1 \leq i \leq m$ (and $\omega(\Lambda_0) = \Lambda_0$); and H_i and $H_{\tau i}$ have the same Dynkin diagram (up to arrow reversal) for each i. Also, since $\omega(R_{\langle p\rangle}) \subseteq (\hat{\mathbb{Z}}_p)^* \cdot R_{\langle p\rangle}$ (by condition (ii)), the arrow on a double connector can be reversed only if $p \neq 2$, and the arrow on a triple connector can be reversed only if $p \neq 3$.

Recall that $Z(H_i) = 1$ for all $i \geq 1$. Thus, for each i, either H_i and $H_{\tau i}$ are isomorphic, or one of them is isomorphic to $SO(2n+1)$ and the other to $PSp(n)$ for some $n \geq 3$. And by the remark on reversing arrows in the Dynkin diagram, this last case can occur only if $p \neq 2$. By a result of Friedlander [Fr], $BSO(2n+1)\hat{_p} \simeq BSp(n)\hat{_p}$ for any n and any odd p. So we can compose ω with $\Theta_{\mathrm{h}}(B\alpha)$ for some appropriate $\alpha \in \mathrm{Aut}(G)$, to arrange that ω sends each simple factor to itself.

We can now write $\omega = \prod_{i=0}^{m} \omega_i$, where $\omega_i : \Lambda_{ip}\hat{} \rightarrowtail \Lambda_{ip}\hat{}$ for each i, and where $\phi_0 \in \Theta([BS_p\hat{}, BS_p\hat{}]_{\mathbb{Q}})$ by Theorem 1.1. We will be done upon showing that $\omega_i \in \Theta([BH_{ip}\hat{}, BH_{ip}\hat{}]_{\mathbb{Q}})$ for each i. In particular, we can simplify the notation, and assume that $G = H_i$ is simple.

We have seen that ω^* permutes the roots and simple roots of G up to scalar multiple, and hence induces an automorphism of the Dynkin diagram of G, possibly reversing arrows. The only simple groups whose Dynkin diagrams have arrow reversing automorphisms are B_2 ($= SO(5) \cong PSp(2)$), G_2, and F_4. Also, as noted above, such arrow reversing can occur only if $p \neq 2$ and $G \cong B_2$ or F_4; or if $p \neq 3$ and $H_i \cong G_2$. In all of these cases, self maps $BG_p\hat{} \to BG_p\hat{}$ have been constructed by Friedlander (in [Fr] again), to realize the arrow reversing automorphisms. So if necessary we can compose with one of these maps, to arrange that ω acts on the Dynkin diagram preserving arrows.

Since $Z(G) = 1$, any arrow preserving automorphism of the Dynkin diagram can be realized by some automorphism $\alpha \in \mathrm{Aut}(G)$ (cf. [Bb2, p. 42, Corollaire]). So upon replacing ω by $L(\alpha|T) \circ \omega$ for some α, we are reduced to the case where ω^* acts on the Dynkin diagrams via the identity, and sends each root to some scalar multiple of itself. In particular, since the Weyl group W is generated by reflections in the kernels of the roots (3.2(i)), $\omega \in \mathrm{End}(\Lambda)$ is W-equivariant; and is multiplication by some $k \in \hat{\mathbb{Z}}_p$ since $L(T_\infty) = \hat{\mathbb{Q}}_p \otimes \Lambda$ is irreducible as a W-representation (cf. [Bb1, p. 82]). Also, by (ii), $k \in (\hat{\mathbb{Z}}_p)^*$ if $p \mid |W|$.

For such k, unstable Adams operations $\psi^k : BG_p\hat{} \to BG_p\hat{}$ have been constructed by Sullivan [Su] (when $G = SU(n)$) or Wilkerson [Wi] (in general). And since the restriction of ψ^k to $BT_p\hat{}$ is induced by the kth power map on T, we see that $\Theta(\psi^k) = \omega$.

Step 3b. Now let G be arbitrary, and fix some admissible map $\omega \in \mathrm{End}(\Lambda_p\hat{})$ such that $\omega^*(R) \subseteq (\hat{\mathbb{Z}}_p \cdot R)$ and $\omega^*(R_{\langle p \rangle}) \subseteq ((\hat{\mathbb{Z}}_p)^* \cdot R_{\langle p \rangle})$. We will show that ω extends to a $\mathbb{Q}$-equivalence $BG_p\hat{} \to BG_p\hat{}$.

Let n be the exponent of the center of the semisimple part of G, and let $\pi = \{z \in Z(G) : z^n = 1\}$. Set $\overline{G} = G/\pi$: a quotient group which satisfies the condition in Step 3a. Let $\overline{\Lambda} \supseteq \Lambda$ be the integral lattice in $\overline{G}$; then

$$\overline{\Lambda} = \{x \in \mathbb{Q} \otimes \Lambda : R(x) \subseteq \mathbb{Z},\ nx \in \Lambda\}$$

by Proposition 3.2(iv). For any $x \in \overline{\Lambda}_p\hat{}$, $n \cdot \omega x = \omega(nx) \in \Lambda_p\hat{}$ and

$$R(\omega x) = (\omega^* R)(x) \subseteq (\hat{\mathbb{Z}}_p \cdot R)(x) \subseteq \hat{\mathbb{Z}}_p$$

(using (i)). So $\omega(\overline{\Lambda}_p\hat{}) \subseteq \overline{\Lambda}_p\hat{}$; and ω extends to a map $\bar{f} : B\overline{G}_p\hat{} \to B\overline{G}_p\hat{}$ by Step 3a.

Identify $\overline{\Lambda}^{\wedge}_p/\Lambda^{\wedge}_p \cong \pi_p$ (the Sylow p-subgroup of π), and let $\omega' \in \mathrm{Aut}(\pi_p)$ be the map induced by ω. The composite

$$BG^{\wedge}_p \longrightarrow B\overline{G}^{\wedge}_p \xrightarrow{\bar f} B\overline{G}^{\wedge}_p \longrightarrow K(\pi_p, 2)$$

is nullhomotopic, since

$$\begin{aligned} H^2(BG;\pi_p) \cong \mathrm{Hom}(\pi_1(G),\pi_p) &\subseteq \mathrm{Hom}(\pi_1(T),\pi_p) \qquad \text{(Prop. 3.2(iii))} \\ &\cong \mathrm{Hom}(\Lambda,\pi_p) \\ &\cong \mathrm{Hom}(\Lambda^{\wedge}_p,\pi_p). \end{aligned}$$

So $\bar f$ pulls back along the fibration $BG^{\wedge}_p \to B\overline{G}^{\wedge}_p \to K(\pi_p,2)$ to a map $f : BG^{\wedge}_p \to BG^{\wedge}_p$; and f extends the original admissible map $B\phi$. ∎

An inspection of the proof of Theorem 3.4 shows, at least when G is semisimple with trivial center, that $[BG^{\wedge}_p, BG^{\wedge}_p]_{\mathbb{Q}}$ is generated by products of unstable Adams operations on the separate simple factors of G, by automorphisms of G, and by the "exceptional isogenies" of Friedlander. This is the generalization to connected groups of the theorem of Hubbuck [Hu], which says that for simple G, $[BG, BG]$ is generated by automorphisms and unstable Adams operations.

The following description of the self homotopy equivalences of $BG^{\wedge}_p$ is now easy.

COROLLARY 3.5. *If p is odd, then for any compact connected Lie group G, any admissible map $\omega \in \mathrm{Aut}(\Lambda^{\wedge}_p)$ extends to a homotopy equivalence $f : BG^{\wedge}_p \to BG^{\wedge}_p$. In other words,*

$$\Theta_{\mathrm{h}} : [BG^{\wedge}_p, BG^{\wedge}_p]_{\mathrm{h}} \to N_{\mathrm{Aut}(T_\infty)}(W)/W \cong N_{\mathrm{Aut}(\Lambda^{\wedge}_p)}(W)/W$$

is an isomorphism of groups in this case. If $p = 2$, then Θ_{h} is onto if and only if G contains no direct factor of the form $Sp(n) \times SO(2n+1)$ (for some $n \ge 1$). And if G does contain such a factor, then $\mathrm{Im}(\Theta_{\mathrm{h}})$ is the subgroup of all elements which send factors $SO(2n+1)$ to factors of the same type.

Proof. Recall that $[BG^{\wedge}_p, BG^{\wedge}_p]_{\mathrm{h}} = \Theta^{-1}(N_{\mathrm{Aut}(T_\infty)}(W)/W)$ (Proposition 1.4). Using this, Theorem 3.4 implies that Θ_{h} is onto (and hence an isomorphism) if p is odd, or if $p = 2$ and G has no direct factor $SO(2n+1)$.

If $p = 2$ and G does contain a factor $SO(2n+1)$, then it can only be sent to another direct factor which is either isomorphic to $SO(2n+1)$, or which has the same integral lattice (2-adically) and root system $\overline{R}$ (restricted to this summand). And a check of the root systems shows that the only other possibility is for it to be sent to a direct factor $Sp(n)$. Thus, if Θ_{h} is not onto, then G must contain a direct factor $SO(2n+1) \times Sp(n)$; and $\mathrm{Im}(\Theta_{\mathrm{h}})$ is the group of all admissible maps which send factors $SO(2n+1)$ to factors of the same type.

Finally, Example 3.1 shows that Θ_{h} is never onto when G has a direct factor $SO(2n+1)\times Sp(n)$. ∎

Using Sullivan's arithmetic pullback square for completions and localizations of simply connected spaces, these results can now be converted to results about global self maps of BG.

THEOREM 3.6. *There is a monomorphism*

$$\Theta : [BG, BG]_{\mathbb{Q}} \rightarrowtail \mathrm{AdmEpi}(T,T) \cong [N_{\mathrm{Aut}(\mathbb{Q}\otimes\Lambda)}(W)\cap \mathrm{End}(\Lambda)]/W$$

such that for any $\mathbb{Q}$*-equivalence* $f : BG \to BG$, $\Theta(f) = \phi W$ *for some* $\phi : T \twoheadrightarrow T$ *with* $f|BT \simeq B\phi$.

For each prime p, *let* $G_{\langle p\rangle} \lhd G$ *be the product of all simple summands whose Weyl group has order a multiple of* p, *and set* $\Lambda_{\langle p\rangle} = \Lambda\cap L(G_{\langle p\rangle}\cap T)$. *For each* $n\ge 1$, *let* $H_n \lhd G$ *be the product of all normal subgroups isomorphic to* $SO(2n+1)$. *Then*, *for any* $\phi\in \mathrm{AdmEpi}(T,T)$, $\phi W\in \mathrm{Im}(\Theta)$ *if and only if*

(a) $\mathrm{Ker}(\phi)\cap G_{\langle p\rangle} = 1$ *for all* $p\,|\,|W|$, *and*
(b) $\phi(G_n\cap T) = G_n\cap T$ *for all* $n\ge 1$.

Proof. For any $f : BG\to BG$, $f|BT\simeq B\phi$ for some $\phi : T\to T$ by Notbohm's theorem [No1]; and $\omega = L(\phi)|\Lambda$ satisfies conditions (a,b) since it satisfies them after p-completion for each p (Theorem 3.4). Thus, there is a well defined homomorphism Θ as above, and $\phi W\in \mathrm{Im}(\Theta)$ only if ϕ satisfies the given conditions.

If $\Theta(f) = \Theta(f')$, then $\Theta(f) = \Theta(f')$ for each p, and so $f^{\wedge}_p \simeq f'^{\wedge}_p$ for each p by the injectivity of Θ (Theorem 2.5). And by [JMO, Theorem 3.1], this implies that $f\simeq f'$.

Now fix some admissible map $\phi : T\twoheadrightarrow T$ which satisfies conditions (a) and (b). For each prime p, Theorem 3.4 applies to show that $B\phi$ extends to a $\mathbb{Q}$-equivalence $f_p : BG^{\wedge}_p \to BG^{\wedge}_p$. And then by [JMO, Theorem 3.1] (applied with $f_T = B(\mathrm{incl}\circ\phi)$), there exists $f : BG\to BG$ such that $f^{\wedge}_p\simeq f_p$ for each p, and hence such that $f|BT\simeq B\phi$.

Theorem 3.1 in [JMO] was used here to show both uniqueness and existence of maps $f : BG\to BG$. Its proof is based on the homotopy pullback square of mapping spaces

$$\begin{array}{ccc}
\mathrm{map}(BG,BG) & \longrightarrow & \prod_p \mathrm{map}(BG,BG^{\wedge}_p) \\
\downarrow & & \downarrow \\
\mathrm{map}(BG,BG_{\mathbb{Q}}) & \longrightarrow & \mathrm{map}(BG,(\prod_p BG^{\wedge}_p)_{\mathbb{Q}}),
\end{array}$$

which is induced by Sullivan's arithmetic pullback square for BG [BK, VI.8.1]. It also uses the fact that $BG_{\mathbb{Q}}$ and $(\prod_p BG^{\wedge}_p)_{\mathbb{Q}}$ are both products of Eilenberg–MacLane spaces. ∎

As a final application of these results, we get the following (disappointing) result about the global self homotopy equivalences of BG.

COROLLARY 3.7. *For any compact connected Lie group G, any homotopy equivalence $f : BG \to BG$ is homotopic to $B\alpha$ for some $\alpha \in \mathrm{Aut}(G)$.*

Proof. Assume $f|BT \simeq B\phi$, where $\phi \in \mathrm{AdmEpi}(T,T)$. Then $\phi \in \mathrm{Aut}(T)$, since f is a homotopy equivalence. Set $\omega = L(\phi)|\Lambda \in \mathrm{Aut}(\Lambda)$. By Theorem 3.4, ω^* sends each root of G to an integral multiple of some other root; and since the simple roots are linearly independent those integers must be ± 1. In other words, ω^* permutes the roots; and so ω is an automorphism of the root system with integral lattice. It follows that $\phi = \alpha|T$ for some $\alpha \in \mathrm{Aut}(G)$ (cf. [Bb2, p. 41, Prop. 17]); and $f \simeq B\alpha$ since Θ is injective (Theorem 3.6). ∎

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