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Inessentiality with respect to subspaces

by

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Abstract. Let X be a compactum and let $\mathcal{A} = \{(A_i, B_i) : i = 1, 2, \ldots\}$ be a countable family of pairs of disjoint subsets of X. Then $\mathcal{A}$ is said to be essential on $Y \subset X$ if for every closed F_i separating A_i and B_i the intersection $(\bigcap F_i) \cap Y$ is not empty. So $\mathcal{A}$ is inessential on Y if there exist closed F_i separating A_i and B_i such that $\bigcap F_i$ does not intersect Y.

Properties of inessentiality are studied and applied to prove:

THEOREM. *For every countable family $\mathcal{A}$ of pairs of disjoint open subsets of a compactum X there exists an open set $G \subset X$ on which $\mathcal{A}$ is inessential and for every positive-dimensional $Y \subset X \setminus G$ there exists an infinite subfamily $\mathcal{B} \subset \mathcal{A}$ which is essential on Y.*

This theorem and its generalization provide a new approach for constructing hereditarily infinite-dimensional compacta not containing subspaces of positive dimension which are weakly infinite-dimensional or C-spaces.

1. Introduction

DEFINITION 1.1. A family $\mathcal{A} = \{(A_i, B_i) : i = 1, 2, \ldots\}$ of pairs of disjoint subsets of a separable metric space X is said to be *essential on* $Y \subset X$ if for every closed F_i separating A_i and B_i the intersection $(\bigcap F_i) \cap Y$ is not empty. So $\mathcal{A}$ is inessential on Y if there exist closed F_i separating A_i and B_i such that $(\bigcap F_i) \cap Y = \emptyset$.

In Section 2 we study basic properties of inessentiality which are applied in Section 3 (together with some ideas presented in [3]) to prove our main result:

THEOREM 1.2. *For every countable family $\mathcal{A}$ of pairs of disjoint open subsets of a compactum X there exists an open set $G \subset X$ on which $\mathcal{A}$ is inessential and for every $Y \subset X \setminus G$ of positive dimension there exists an infinite subfamily $\mathcal{B} \subset \mathcal{A}$ which is essential on Y.*

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This theorem provides a new approach for constructing hereditarily infinite-dimensional compacta. The first example of such compacta was given by Walsh in 1979 [7] (see also [4]).

Definition 1.3. An infinite-dimensional compactum X is called *hereditarily infinite-dimensional* if every subspace of X is either zero-dimensional or infinite-dimensional.

Definition 1.4. A separable metric space X is said to be *strongly infinite-dimensional* if there exists a countable family of pairs of disjoint closed subsets of X which is essential on X.

The Hilbert cube $[0,1]\times[0,1]\times\ldots$ is an example of a strongly infinite-dimensional compactum (as the family of pairs of opposite faces $A_i = \{(x_1,x_2,\ldots) : x_i = 0\}$ and $B_i = \{(x_1,x_2,\ldots) : x_i = 1\}$ is essential on the Hilbert cube (see [2])).

It is easy to see that X is strongly infinite-dimensional if and only if there exists a countable family of pairs of open sets with disjoint closures which is essential on X. So let X be a strongly infinite-dimensional compactum and let $\mathcal{D} = \{(V_i,U_i) : i = 0,1,2,\ldots\}$ be a family of pairs of open subsets of X with disjoint closures which is essential on X. Define $\mathcal{A} = \{(V_i,U_i) : i = 1,2,\ldots\}$ and let G be as in the conclusion of Theorem 1.2. Then every subspace of $Z = X\setminus G$ is either zero-dimensional or strongly infinite-dimensional.

Indeed, if $Y\subset Z$ is of positive dimension then there is an infinite $\mathcal{B}\subset\mathcal{A}$ which is essential on Y. It is easy to check that $\mathcal{C} = \{(Y\cap V, Y\cap U) : (V,U)\in\mathcal{B}\}$ is a family of pairs of open subsets of Y with disjoint closures which is essential on Y and therefore Y is strongly infinite-dimensional.

Hence Z is hereditarily strongly infinite-dimensional provided $\dim Z > 0$. Aiming at a contradiction suppose $\dim Z \le 0$. Then V_0 and U_0 can be separated by some closed F_0 not intersecting Z. As $\mathcal{A}$ is inessential on G take closed sets F_i, $i\ge 1$, separating the pairs of $\mathcal{A}$ such that $\bigcap_{i\ge1} F_i$ does not intersect G. Then $\bigcap_{i\ge0} F_i = \emptyset$. This contradicts the assumption of essentiality of $\mathcal{D}$ and shows that $\dim Z > 0$. So we have proved

Theorem 1.5 (Rubin [6]). *Every strongly infinite-dimensional compactum contains a hereditarily strongly infinite-dimensional compactum.*

In the end of Section 3 we will point out a connection of our approach with C-spaces.

Finally, I wish to thank Prof. R. Pol for encouraging me to write this note.

2. Basic properties of inessentiality. Throughout this section X is assumed to be a compactum and $\mathcal{A},\mathcal{B}$ are two countable families of pairs of disjoint open subsets of X.

We say that $Y_1, Y_2, \ldots \subset X$ are *discrete* if Y_i can be enlarged to disjoint neighbourhoods $Y_i \subset G_i$.

For $Y \subset X$ define

$$H_c(\mathcal{A}, Y) = \text{the union of all components of } Y \text{ on which } \mathcal{A} \text{ is inessential},$$

$$H_q(\mathcal{A}, Y) = \text{the union of all quasi-components of } Y \text{ on which } \mathcal{A} \text{ is inessential.}$$

PROPOSITION 2.1. *Let $\mathcal{A}$ be inessential on $Y \subset X$. Then there is a neighbourhood V of Y on which $\mathcal{A}$ is also inessential.*

Proof. Take closed partitions F_i for the pairs of $\mathcal{A} = \{(V_i, U_i) : i = 1, 2, \ldots\}$ such that $Y \cap (\bigcap F_i) = \emptyset$ and define $V = X \setminus \bigcap F_i$. ■

PROPOSITION 2.2. *Let $\mathcal{A}$ be inessential on each set of a discrete family $Y_1, Y_2, \ldots \subset X$. Then $\mathcal{A}$ is also inessential on $\bigcup Y_i$.*

Proof. Enlarge Y_i to disjoint neighbourhoods $Y_i \subset G_i$; by Proposition 2.1 we can assume without loss of generality that $\mathcal{A} = \{(V_j, U_j) : j = 1, 2, \ldots\}$ is inessential on each G_i. For every i take a partition F_{ij} between V_j and U_j such that $(\bigcap_j F_{ij}) \cap G_i = \emptyset$. Let V_{ij} and U_{ij} be disjoint neighbourhoods of V_j and U_j respectively such that $X \setminus F_{ij} = V_{ij} \cup U_{ij}$. Define

$$V_j^* = V_j \cup \bigcup_i (V_{ij} \cap G_i) \quad \text{and} \quad U_j^* = U_j \cup \bigcup_i (U_{ij} \cap G_i).$$

It is not difficult to show that $F_j = X \setminus (V_j^* \cup U_j^*)$ is a partition between V_j and U_j such that $\bigcap F_j$ does not meet $\bigcup G_i$ which contains $\bigcup Y_i$. ■

PROPOSITION 2.3. *Let Y be a closed subset of X. Then $\mathcal{A}$ is inessential on $H_c(\mathcal{A}, Y)$.*

Proof. Let A be a component of Y contained in $H_c(\mathcal{A}, Y)$. Then by Proposition 2.1 one can find a clopen subset V_A of Y with $A \subset V_A$ on which $\mathcal{A}$ is inessential. $H_c(\mathcal{A}, Y)$ is covered by $\{V_A : A \subset H_c(\mathcal{A}, Y)\}$ and hence we can choose a sequence $V_{A_1}, V_{A_2}, \ldots$ which also covers $H_c(\mathcal{A}, Y)$. Define $Y_1 = V_{A_1}, Y_2 = V_{A_2} \setminus Y_1, \ldots, Y_{i+1} = V_{A_{i+1}} \setminus (Y_1 \cup \ldots \cup Y_i), \ldots$ The sequence $Y_1, Y_2, \ldots$ is discrete and by Proposition 2.2, $\mathcal{A}$ is inessential on $\bigcup Y_i$ which contains $H_c(\mathcal{A}, Y)$. ■

PROPOSITION 2.4. *Let $Y \subset X$ be open in* $\operatorname{cl} Y$. *Then $\mathcal{A} \cup \mathcal{B}$ is inessential on $H_q(\mathcal{A}, Y) \cap H_q(\mathcal{B}, Y)$.*

Proof. Let $f : \operatorname{cl} Y \to [0, 1]$ be a continuous map with $Y = f^{-1}(0, 1]$ and consider the compact "rings" $K_n = f^{-1}[1/(n+1), 1/n]$, $n = 1, 2, \ldots$ Take a point $x \in K_n$ and let A be the quasi-component of Y containing x. Then A is the intersection of all clopen subsets of Y which contain x. Hence the component of K_n containing x is contained in A. So we have

$$(*)\qquad H_q(\mathcal{A},Y)\cap K_n\subset H_c(\mathcal{A},K_n)\quad\text{and}\quad H_q(\mathcal{B},Y)\cap K_n\subset H_c(\mathcal{B},K_n).$$

The collections of "odd rings" $K_1,K_3,K_5,\ldots$ and of "even rings" $K_2,K_4,K_6,\ldots$ are discrete and by Proposition 2.3, $\mathcal{A}$ and $\mathcal{B}$ are inessential on $H_c(\mathcal{A},K_n)$ and $H_c(\mathcal{B},K_n)$ respectively. So by Proposition 2.2, $\mathcal{A}$ is inessential on $Y_1=\bigcup_{i\ge1}H_c(\mathcal{A},K_{2i-1})$ and $\mathcal{B}$ is inessential on $Y_2=\bigcup_{i\ge1}H_c(\mathcal{B},K_{2i})$. From $(*)$ it follows that $H_q(\mathcal{A},Y)\cap H_q(\mathcal{B},Y)\subset Y_1\cup Y_2$ and since $\mathcal{A}\cup\mathcal{B}$ is inessential on $Y_1\cup Y_2$ we get the required result. ∎

3. Proof of Theorem 1.2. Take any countable decomposition of $\mathcal{A}$ into disjoint infinite subfamilies $\mathcal{A}=\mathcal{A}_1\cup\mathcal{B}_1\cup\mathcal{A}_2\cup\mathcal{B}_2\cup\ldots$

Let $\mathcal{F}$ be a countable family of continuous functions from X to $[0,1]$ which distinguish the points of X and let $\mathcal{C}$ be a countable family of Cantor sets in $[0,1]$ such that every non-degenerate interval in $[0,1]$ contains some Cantor set from $\mathcal{C}$. Arrange $\mathcal{F}\times\mathcal{C}$ into a sequence $(f_1,C_1),(f_2,C_2),\ldots,$ where $f_i\in\mathcal{F}$ and $C_i\in\mathcal{C}$.

Let $g_i:C_i\to 2^X$ be continuous and onto.

Define

$$Y_i=\bigcup\{f_i^{-1}(c)\setminus g_i(c):c\in C_i\}=f_i^{-1}(C_i)\setminus\bigcup\{f_i^{-1}(c)\cap g_i(c):c\in C_i\}.$$

It is not difficult to see that $\bigcup\{f_i^{-1}(c)\cap g_i(c):c\in C_i\}$ is closed in X and therefore Y_i is open in $\operatorname{cl}Y_i$. Hence by Proposition 2.4, $\mathcal{A}_i\cup\mathcal{B}_i$ is inessential on $Z_i=H_q(\mathcal{A}_i,Y_i)\cap H_q(\mathcal{B}_i,Y_i)$. So $\mathcal{A}$ is inessential on $Z=Z_1\cup Z_2\cup\ldots$ Thus by Proposition 2.1 there is a neighbourhood G of Z on which $\mathcal{A}$ is inessential and we claim that G is the desired set.

Indeed, let $Y\cap G=\emptyset$ and $\dim Y>0$.

Since $\mathcal{F}$ distinguishes the points of X, the map $x\to(f(x))_{f\in\mathcal{F}}$ embeds X in $\prod_{f\in\mathcal{F}}f(X)$ and so Y is embedded in $\prod_{f\in\mathcal{F}}f(Y)$. Hence $\dim Y>0$ implies that there is some $f\in\mathcal{F}$ such that $\dim f(Y)>0$, so $f(Y)$ contains a non-degenerate interval and we can choose some $C\in\mathcal{C}$ such that $C\subset f(Y)$. Take i such that $(f_i,C_i)=(f,C)$; we are going to show that either $\mathcal{A}_i$ or $\mathcal{B}_i$ is essential on Y.

Assume the contrary. Then by Proposition 2.1 there is a neighbourhood $Y\subset V$ on which both $\mathcal{A}_i$ and $\mathcal{B}_i$ are inessential. Let c in C_i be such that $g_i(c)=X\setminus V$. Set $D_i=f_i^{-1}(c)\setminus g_i(c)$. It is obvious that $D_i=\bigcap\{f_i^{-1}(U)\cap Y_i:$ U contains c and U is clopen in $C_i\}$ and since for U clopen in C_i, $f_i^{-1}(U)\cap Y_i$ is clopen in Y_i, it follows that D_i equals the intersection of all clopen subsets of Y_i containing D_i. Therefore the quasi-components of Y_i intersecting D_i are contained in D_i. Both $\mathcal{A}_i$ and $\mathcal{B}_i$ are inessential on D_i as $D_i\subset V$. Hence $D_i\subset H_q(\mathcal{A}_i,Y_i)$ and $D_i\subset H_q(\mathcal{B}_i,Y_i)$, that is, $D_i\subset Z_i$. Clearly, D_i intersects

Y and so Y intersects Z_i. This contradicts the assumption $Y \cap G = \emptyset$ and proves the theorem. ■

As a matter of fact we have proved the following version of Theorem 1.2:

THEOREM 3.1. *Let $\mathcal{A}$ be a countable family of pairs of disjoint open subsets of a compactum X. Then for every countable decomposition of $\mathcal{A}$ into disjoint subfamilies $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2 \cup \ldots$ there exists an open set $G \subset X$ such that $\mathcal{A}$ is inessential on G and for every Y with* $\dim Y > 0$ *not meeting G there is some $\mathcal{A}_i$ which is essential on Y.*

A remark concerning C-spaces. R. Pol noticed that the approach of this note admits a more general setting.

DEFINITION 3.2. We will call a family $\mathcal{P}$ of subsets of a separable metric space X *admissible for* X if

(i) $A \in \mathcal{P}$ and $B \subset A$ imply that $B \in \mathcal{P}$,

(ii) for every $A \in \mathcal{P}$ there is a neighbourhood $A \subset V$ which belongs to $\mathcal{P}$,

(iii) for disjoint open sets $V_i \in \mathcal{P}$, $i = 1, 2, \ldots$, the union $\bigcup V_i$ belongs to $\mathcal{P}$.

Let $\mathcal{A}$ be a family of pairs of disjoint open subsets of X. Denote by $\mathcal{P}_{\mathcal{A}}$ the family of all subsets of X on which $\mathcal{A}$ is inessential. We have shown that $\mathcal{P}_{\mathcal{A}}$ is admissible for X.

The following example of admissible families is based on the notion of C-spaces (for more information see [1]).

DEFINITION 3.3. Let $\mathcal{A} = \{\mathcal{U}_1, \mathcal{U}_2, \ldots\}$ be a countable family of open covers of a separable metric space X. We will say that $\mathcal{A}$ is *inessential on* $Y \subset X$ if there exist families $\mathcal{V}_1, \mathcal{V}_2, \ldots$ of disjoint open sets such that $\mathcal{V}_i$ refines $\mathcal{U}_i$ and $\mathcal{V}_1, \mathcal{V}_2, \ldots$ cover Y. Otherwise $\mathcal{A}$ is *essential on* Y. Moreover, X is called a *C-space* if every countable family of open covers of X is inessential on X.

For a family $\mathcal{A}$ of open covers of X we will use the same notation $\mathcal{P}_{\mathcal{A}}$ that we have used for a family of pairs to denote the family of all subsets of X on which $\mathcal{A}$ is inessential. It is easy to verify that in this case $\mathcal{P}_{\mathcal{A}}$ is also admissible.

For families $\mathcal{P}_1, \mathcal{P}_2, \ldots$ of subsets of X define $\mathcal{P}_1 \vee \mathcal{P}_2 = \{A_1 \cup A_2 : A_i \in \mathcal{P}_i\}$ and $\bigvee \mathcal{P}_i = \bigvee_{i=1}^{\infty} \mathcal{P}_i = \{\bigcup A_i : A_i \in \mathcal{P}_i\}$. It is not difficult to check that for a family of pairs or covers $\mathcal{A}$ and every decomposition of $\mathcal{A}$ into disjoint subfamilies $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2 \cup \ldots$ we have $\bigvee \mathcal{P}_{\mathcal{A}_i} \subset \mathcal{P}_{\mathcal{A}}$. Now Theorem 3.1 can be reformulated as follows.

THEOREM 3.4. *Let $\mathcal{P}_1, \mathcal{P}_2, \ldots$ be admissible families for a compactum X. Then there is an open set $G \in \bigvee \mathcal{P}_i$ such that for every $Y \subset X \setminus G$ of positive dimension there is some $\mathcal{P}_i$ which does not contain Y.*

THEOREM 3.5. *Let $\mathcal{A}$ be a countable family of open covers of a compactum X. Then for every countable decomposition of $\mathcal{A}$ into disjoint subfamilies $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2 \cup \ldots$ there exists an open set $G \subset X$ such that $\mathcal{A}$ is inessential on G and for every Y with $\dim Y > 0$ not meeting G there is some $\mathcal{A}_i$ which is essential on Y.*

And Theorem 1.5 can be stated as

THEOREM 3.6. *Suppose a compactum X is not a C-space. Then X contains a compactum $Z \subset X$ which is not a C-space such that no $Y \subset Z$ of positive dimension is a C-space.*

This theorem generalizes the analogous result of [5] where Y is assumed to be a closed subset of Z.

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