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Sierpiński's hierarchy and locally Lipschitz functions

by

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Abstract. Let Z be an uncountable Polish space. It is a classical result that if $I \subseteq \mathbb{R}$ is any interval (proper or not), $f : I \to \mathbb{R}$ and $\alpha < \omega_1$ then $f \circ g \in \mathcal{B}_\alpha(Z)$ for every $g \in \mathcal{B}_\alpha(Z) \cap {}^Z I$ if and only if f is continuous on I, where $\mathcal{B}_\alpha(Z)$ stands for the αth class in Baire's classification of Borel measurable functions. We shall prove that for the classes $\mathcal{S}_\alpha(Z)$ ($\alpha > 0$) in Sierpiński's classification of Borel measurable functions the analogous result holds where the condition that f is continuous is replaced by the condition that f is locally Lipschitz on I (thus it holds for the class of differences of semicontinuous functions, which is the class $\mathcal{S}_1(Z)$). This theorem solves the problem raised by the work of Lindenbaum ([L] and [L, Corr.]) concerning the class of functions not leading outside $\mathcal{S}_\alpha(Z)$ by outer superpositions.

1. Introduction. The classical Baire classification of Borel measurable real functions defined on a metric space X is built as follows: $\mathcal{B}_0(X)$ consists of all continuous real functions on X and then, inductively, for $0 < \alpha < \omega_1$ we define

$$\mathcal{B}_\alpha(X) = \{\lim f_n : \text{the sequence } (f_n(x))_n \text{ is convergent for every } x \in X, \text{ with each } f_n \in \mathcal{B}_{\alpha_n}(X) \text{ for some } \alpha_n < \alpha,\}.$$

The second classical classification, the one of Sierpiński, is built with the use of absolutely convergent series of functions. We define $\mathcal{S}_0(X)$ to consist of all continuous real functions on X (thus $\mathcal{S}_0(X) = \mathcal{B}_0(X) = C(X)$) and then, inductively, for $0 < \alpha < \omega_1$ we set

$$\mathcal{S}_\alpha(X) = \Big\{\sum_{n=1}^{\infty} f_n : \sum_{n=1}^{\infty} |f_n(x)| < \infty \text{ for every } x \in X, \text{ with each } f_n \in \mathcal{S}_{\alpha_n}(X) \text{ for some } \alpha_n < \alpha\Big\}.$$

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Note that $\mathcal{S}_1(X)$ is the class of differences of upper (or, equivalently, lower) semicontinuous functions on X.

It is obvious that $\mathcal{S}_\alpha(X) \subseteq \mathcal{B}_\alpha(X)$ for every $\alpha < \omega_1$. In [Ke] Kempisty posed the problem whether $\mathcal{S}_\alpha([0,1]) \neq \mathcal{B}_\alpha([0,1])$ for every $1 < \alpha < \omega_1$ (previously it was shown independently by Sierpiński (in [S$_1$]) and Mazurkiewicz (in [Maz]) that $\mathcal{S}_1([0,1]) \neq \mathcal{B}_1([0,1])$). Theorem 3.13 of [Mor] settled this question: $\mathcal{S}_\alpha(X) \neq \mathcal{B}_\alpha(X)$ for every $0 < \alpha < \omega_1$.

Let Z be an uncountable Polish space. It is a classical result that if $I \subseteq \mathbb{R}$ is any interval, $f : I \to \mathbb{R}$ and $\alpha < \omega_1$, then $f \circ g \in \mathcal{B}_\alpha(Z)$ for every $g \in {}^Z I \cap \mathcal{B}_\alpha(Z)$ if and only if f is continuous on I. We shall prove (Theorem 3.4) that for the classes $\mathcal{S}_\alpha(Z)$, $0 < \alpha < \omega_1$, the analogous result holds where the condition that f is continuous is replaced by the condition that f is locally Lipschitz on I.

Finally, applying the above mentioned theorem on superpositions we shall give a characterization of the positive functions in $\mathcal{S}_\alpha(X)$. For example, a positive function f is in $\mathcal{S}_1([0,1])$ if and only if $f = g \cdot h$, where g is positive and upper semicontinuous and h is positive and lower semicontinuous. Again as an application of the theorem on superpositions we show that in this statement we cannot replace "positive" by "nonnegative".

2. Definitions and auxiliary facts. We shall use standard set-theoretical notation. $\mathbb{N}$ will stand for the set of all positive integers, $\mathbb{R}$ will denote the set of all reals. An *interval* in $\mathbb{R}$ will be any connected subset of $\mathbb{R}$. The term *closed proper interval* will only refer to intervals of the form $[a,b]$, where $a, b \in \mathbb{R}$, $a \leq b$. $P(A)$ stands for the family of all subsets of a set A. If $\mathscr{A} \subseteq P(A)$ and $X \subseteq A$ then let $\mathscr{A}|X = \{Y \cap X : Y \in \mathscr{A}\}$. Suppose that $\mathscr{A} \subseteq P(A)$. We say that $\mathscr{A}$ is a *lattice of subsets of* A if $\{\emptyset, A\} \subseteq \mathscr{A}$ and $\mathscr{A}$ is closed under finite unions and intersections. The range of a function f will be denoted by $\operatorname{Rg} f$. If A and B are sets then ${}^A B$ will denote the set of all functions with domain A and range contained in B. If $f \in {}^A B$ and $C \subseteq A$ then $f|C$ denotes the restriction of f to C. If $f : A \to \mathbb{R}$ then $\inf f = \inf\{f(x) : x \in A\}$, $\sup f = \sup\{f(x) : x \in A\}$. If $\mathscr{H} \subseteq {}^D A$ and $B \subseteq A$ then let $(\mathscr{H})_B = \{f \in \mathscr{H} : \operatorname{Rg} f \subseteq B\}$. For $\mathscr{G} \subseteq {}^A\mathbb{R}$ and $\mathscr{H} \subseteq {}^A\mathbb{R}$ let $\mathscr{G} + \mathscr{H} = \{g + h : g \in \mathscr{G} \text{ and } h \in \mathscr{H}\}$ and define $\mathscr{G} - \mathscr{H}$ and $\mathscr{G} \cdot \mathscr{H}$ analogously. The space ${}^{\mathbb{N}}\{0,1\}$ with the usual product topology will be denoted by $\mathscr{C}$ (it is homeomorphic to the classical Cantor set).

Let I be an interval. A function $f : I \to \mathbb{R}$ is *locally Lipschitz* on I if for each point x in I there are a neighbourhood $U(x)$ of x and a constant L_x such that f satisfies the Lipschitz condition with the constant L_x in $U(x) \cap I$.

Let $F \in {}^{X \times Y}\mathbb{R}$ and $(x,y) \in X \times Y$. We put $F_x(y) = F(x,y)$. A function $F \in {}^{X \times Y}\mathbb{R}$ is called a *universal function* for a class $\mathscr{H} \subseteq {}^Y\mathbb{R}$ if $\mathscr{H} = \{F_x : x \in X\}$.

Let $\mathscr{A} \subseteq P(A)$. By $\overline{M}\mathscr{A}$ we denote the family of all functions $f \in {}^{A}\mathbb{R}$ such that $f^{-1}((-\infty, c)) \in \mathscr{A}$ for every $c \in \mathbb{R}$. Similarly, $\underline{M}\mathscr{A}$ is the family of all functions $f \in {}^{A}\mathbb{R}$ such that $f^{-1}((c, \infty)) \in \mathscr{A}$ for every $c \in \mathbb{R}$. Note that $f \in \underline{M}\mathscr{A}$ if and only if $-f \in \overline{M}\mathscr{A}$. We put $M\mathscr{A} = \underline{M}\mathscr{A} \cap \overline{M}\mathscr{A}$.

We use standard notation from Descriptive Set Theory. For example, for X being a metric space $\Sigma^0_\alpha(X)$ ($\Pi^0_\alpha(X)$, resp.) denotes the αth additive (multiplicative, resp.) class in the hierarchy of Borel subsets of X.

For X a metric space and $\alpha < \omega_1$, let $\mathbf{B}_\alpha(X) = \{f \in {}^{X}\mathbb{R} : f^{-1}(G) \in \Sigma^0_{1+\alpha}(X)$ for each G open in $\mathbb{R}\}$. We have $\mathbf{B}_\alpha(X) = M\Sigma^0_{1+\alpha}(X)$. We shall also write $\mathbf{L}_\alpha(X)$ and $\mathbf{U}_\alpha(X)$ to denote $\underline{M}\Sigma^0_{1+\alpha}(X)$ and $\overline{M}\Sigma^0_{1+\alpha}(X)$, respectively. $\mathbf{L}_0(X)$ and $\mathbf{U}_0(X)$ are, obviously, the classes of lower and upper semicontinuous functions on X with values in $\mathbb{R}$. The class $\mathbf{L}_\alpha(X) + \mathbf{U}_\alpha(X)$ will be denoted by $\mathbf{S}_\alpha(X)$.

We have the diagram

$$\begin{array}{ccccc} & \nearrow & \mathbf{U}_\alpha(X) & \searrow & \\ \mathbf{B}_\alpha(X) & & & & \mathbf{S}_\alpha(X) \to \mathbf{B}_{\alpha+1}(X), \\ & \searrow & \mathbf{L}_\alpha(X) & \nearrow & \end{array}$$

where the arrows stand for inclusions (the properness of the first four inclusions in the case of X being an uncountable Polish space is classical; for the last one, see [Mor, 3.13] or Corollary 4.1 of this paper). It is worth mentioning that $\mathbf{B}_{\alpha+1}(X)$ is the closure of $\mathbf{S}_\alpha(X)$ in the uniform convergence topology for each $\alpha < \omega_1$ (the method of proof is given in [S_2], although this result is formulated there for a more restrictive case; see also [H, IX, XVI], [Mau, Th. 3.5] and [CM_2, Th. 1] for more general results).

Remark. The Lebesgue–Hausdorff theorem ([Ku, 31, IX]) says that $\mathcal{B}_\alpha(X) = \mathbf{B}_\alpha(X)$ for $\alpha < \omega$ and $\mathcal{B}_\alpha(X) = \mathbf{B}_{\alpha+1}(X)$ for $\alpha \geq \omega$. It also implies that $\mathcal{S}_{\alpha+1}(X) = \mathbf{S}_\alpha(X)$ for $\alpha < \omega$ and $\mathcal{S}_\alpha(X) = \mathbf{S}_\alpha(X)$ for $\alpha \geq \omega$.

Some of our considerations will have a more abstract setting where we use a more general notion (introduced in [H]) than that of Baire's classes.

A family $\mathscr{F}$ of real functions defined on a common domain D (we always assume $D \neq \emptyset$) will be called a *complete function system* on D if

(a) every real function which is constant on D is in $\mathscr{F}$;

(b) the maximum and minimum of two functions from $\mathscr{F}$ is in $\mathscr{F}$;

(c) the sum, difference, product, and quotient (with nowhere vanishing denominator) of two functions from $\mathscr{F}$ is in $\mathscr{F}$;

(d) the limit of a uniformly convergent sequence of functions from $\mathscr{F}$ is in $\mathscr{F}$.

For us, the most important example of a complete function system is the class $\mathbf{B}_\alpha(X)$ for a metric space X.

Let

$$\mathscr{M}_{\mathscr{F}} = \{f^{-1}((0,\infty)) : f \in \mathscr{F}\} \quad \text{and} \quad \mathscr{N}_{\mathscr{F}} = \{f^{-1}([0,\infty)) : f \in \mathscr{F}\}.$$

For a complete function system $\mathscr{F}$ we have $\mathscr{F} = M(\mathscr{M}_{\mathscr{F}})$ ([H, XII, p. 275]) and $\mathscr{M}_{\mathscr{F}}$ is a lattice of sets closed under countable unions ([H, VIII, p. 273]).

Let $\mathscr{L}(\mathscr{F}) = \underline{M}(\mathscr{M}_{\mathscr{F}})$ and $\mathscr{U}(\mathscr{F}) = \overline{M}(\mathscr{M}_{\mathscr{F}})$ and $\mathscr{S}(\mathscr{F}) = \mathscr{L}(\mathscr{F}) + \mathscr{U}(\mathscr{F})$ $(= \mathscr{L}(\mathscr{F}) - \mathscr{L}(\mathscr{F}) = \mathscr{U}(\mathscr{F}) - \mathscr{U}(\mathscr{F}))$.

In the case of Baire classes defined on a metric space X for every $\alpha < \omega_1$ we have $\mathscr{M}(\mathbf{B}_\alpha(X)) = \Sigma^0_{1+\alpha}(X)$, $\mathscr{L}(\mathbf{B}_\alpha(X)) = \mathbf{L}_\alpha(X)$, $\mathscr{U}(\mathbf{B}_\alpha(X)) = \mathbf{U}_\alpha(X)$ and $\mathscr{S}(\mathbf{B}_\alpha(X)) = \mathbf{S}_\alpha(X)$.

THEOREM 2.A ([CM$_1$, Prop. 1.1]). *Let $\mathscr{A} \subseteq P(A)$ be a lattice of subsets of A closed under countable unions and let $B \subseteq A$. Let I be a closed proper interval. Then any function in $(\underline{M}(\mathscr{A}|B))_I$ has an extension to a function in $(\underline{M}\mathscr{A})_I$.* ∎

Theorem 2.B is proved in [H, p. 278].

THEOREM 2.B. *Let $\mathscr{F}$ be a complete function system. A function g is in $\mathscr{S}(\mathscr{F})$ if and only if it can be represented as the sum of a pointwise absolutely convergent series $g = \sum_{n=1}^{\infty} g_n$, where all functions g_n belong to $\mathscr{F}$.* ∎

THEOREM 2.B′. *If in Theorem* 2.B *we assume that* $\operatorname{Rg} g \subseteq I$, *where I is an interval*, *then the functions g_n can be taken in such a way that* $\operatorname{Rg} s_m \subseteq I$, *where $s_m = \sum_{n=1}^{m} g_n$.*

Proof. Let g_n be the functions of Theorem 2.B. Write $s_m = \sum_{n=1}^{m} g_n$. Let a_n be a nonincreasing sequence converging to $\inf I$, b_n be a nondecreasing sequence converging to $\sup I$, and $a_n, b_n \in I$. Set

$$s_n^* = \min(\max(s_n, a_n), b_n).$$

Then obviously

$$g(x) = s_1^* + \sum_{n=1}^{\infty} (s_{n+1}^*(x) - s_n^*(x)).$$

We have

$$|s_{n+1}^*(x) - s_n^*(x)| \le \max(|g_{n+1}(x)|, a_n - a_{n+1}, b_{n+1} - b_n).$$

If I is bounded this finishes the proof. When, say, $\inf I = -\infty$ then $s_n(x) > a_n$ for $n > n(x)$, for some $n(x)$, and for $n > n(x)$ we have

$$|s_{n+1}^*(x) - s_n^*(x)| \le \max(|g_{n+1}(x)|,\ b_{n+1} - b_n).$$

An analogous argument for $\sup I = \infty$ finishes the proof. ∎

Theorem 2.C follows from [CM_1, Th. 2.1] (applied to $\mathscr{A} = \Sigma^0_{1+\alpha}$, in the notation of [$\mathrm{CM}_1$]).

THEOREM 2.C. *Let Z be an uncountable Polish space and let $I \subset \mathbb{R}$ be a closed proper interval. Then there exists a function $G \in (\mathbf{U}_\alpha(\mathscr{C} \times Z))_I$ universal for $(\mathbf{U}_\alpha(Z))_I$.* ∎

It follows from [Mor, Lemma 3.8] (applied to $\mathscr{A} = \Sigma^0_{1+\alpha}$, in the notation of [Mor]) that the following theorem holds.

THEOREM 2.D. *Let Z be an uncountable Polish space and let $I \subset \mathbb{R}$ be a closed proper interval. Then there exists a function $G \in (\mathbf{U}_\alpha(\mathscr{C} \times Z))_I - (\mathbf{U}_\alpha(\mathscr{C} \times Z))_I$ universal for $(\mathbf{U}_\alpha(Z))_I - (\mathbf{U}_\alpha(Z))_I$.* ∎

LEMMA 2.E. *Let Z be an uncountable Polish space and let $I \subset \mathbb{R}$ be a closed proper interval. Then there exist two functions $\Psi \in (\mathbf{U}_\alpha(Z^2))_I$ and $\Phi \in (\mathbf{U}_\alpha(Z^2))_I - (\mathbf{U}_\alpha(Z^2))_I$ such that for any $\psi \in (\mathbf{U}_\alpha(Z))_I$ and $\phi \in (\mathbf{U}_\alpha(Z))_I - (\mathbf{U}_\alpha(Z))_I$ there is $x \in Z$ such that $\psi(y) = \Psi(x,y)$ and $\phi(y) = \Phi(x,y)$ for every $y \in Z$.*

Proof. Let $C \subseteq Z$ be homeomorphic to the Cantor set $\mathscr{C}$, where C is considered with the topology inherited from Z. As $\mathscr{C}$ is homeomorphic to $\mathscr{C}^2$ we can choose a homeomorphism $\zeta : C \to \mathscr{C}^2$. By Theorem 2.C there exists a function $F \in (\mathbf{U}_\alpha(\mathscr{C} \times Z))_I$ universal for $(\mathbf{U}_\alpha(Z))_I$ and, by Theorem 2.D, there exists a function $G \in (\mathbf{U}_\alpha(\mathscr{C} \times Z))_I - (\mathbf{U}_\alpha(\mathscr{C} \times Z))_I$ universal for $(\mathbf{U}_\alpha(Z))_I - (\mathbf{U}_\alpha(Z))_I$. Let now $\Psi^* : C \times Z \to \mathbb{R}$ and $\Phi^* : C \times Z \to \mathbb{R}$ be defined as follows:

$$\Psi^*(c,z) = F(\zeta_1(c), z) \quad \text{and} \quad \Phi^*(c,z) = G(\zeta_2(c), z),$$

where $(\zeta_1(c), \zeta_2(c)) = \zeta(c) \in \mathscr{C}^2$.

Obviously $\Psi^* \in (\mathbf{U}_\alpha(C \times Z))_I$ and $\Phi^* \in (\mathbf{U}_\alpha(C \times Z))_I - (\mathbf{U}_\alpha(C \times Z))_I$. By Theorem 2.A the function Ψ^* has an extension to a function $\Psi \in (\mathbf{U}_\alpha(Z^2))_I$, and Φ^* has an extension to $\Phi \in (\mathbf{U}_\alpha(Z^2))_I - (\mathbf{U}_\alpha(Z^2))_I$. It is now very easy to see that Ψ and Φ have the properties desired. ∎

From the equality $\mathscr{F} = M(\mathscr{M}_{\mathscr{F}})$ which holds for any complete function system we can immediately derive the following fact:

FACT 2.F. *If $\mathscr{F}$ is a complete function system and $f \in \mathscr{F}$ then for any function h continuous on $\operatorname{Rg} f$ we have $h \circ f \in \mathscr{F}$.* ∎

3. Superpositions with locally Lipschitz functions. This section of the paper is divided into two parts. In the first one we prove a theorem (Theorem 3.1) on superpositions with locally Lipschitz functions for complete function systems. In the second one we prove its converse for the particular case of Sierpiński's classes of Borel measurable functions on uncountable Polish spaces.

A. *A theorem on complete function systems*

THEOREM 3.1. *Let $I \subseteq \mathbb{R}$ be an interval, and let $f : I \to \mathbb{R}$ be a locally Lipschitz function on I. Let $\mathscr{F}$ be a complete function system on D. Then $f \circ g \in \mathscr{S}(\mathscr{F})$ for every $g \in (\mathscr{S}(\mathscr{F}))_I$.*

Proof. By Theorem 2.B′ the function g can be represented as the sum of a pointwise absolutely convergent series $g = \sum_{n=1}^{\infty} g_n$, where $g_n \in \mathscr{F}$ and $\mathrm{Rg}\, s_n \in I$, for each n, where $s_n = \sum_{i=1}^{n} g_i$. By continuity of f we have, for every $x \in D$,

$$\begin{aligned}(f \circ g)(x) &= \lim_n (f \circ s_n)(x) \\ &= (f \circ s_1)(x) + \sum_{n=1}^{\infty} ((f \circ s_{n+1})(x) - (f \circ s_n)(x)).\end{aligned}$$

By Fact 2.F the functions $f \circ s_n$, $n \in \mathbb{N}$, belong to $\mathscr{F}$.

As $s_n(x)$ tends to $g(x) \in I$ and f satisfies locally the Lipschitz condition on I we have, for n large enough,

$$|(f \circ s_{n+1})(x) - (f \circ s_n)(x)| \le L|s_{n+1}(x) - s_n(x)|$$

with some constant $L = L_{g(x)}$, whence

$$|(f \circ s_1)(x)| + \sum_{n=1}^{\infty} |(f \circ s_{n+1})(x) - (f \circ s_n)(x)| < \infty.$$

Thus by Theorem 2.B, $f \circ g \in \mathscr{S}(\mathscr{F})$. ∎

B. *The converse of Theorem* 3.1 *for Borel measurable functions*

LEMMA 3.2. *Let $\mathscr{A}$ be a lattice of subsets of A. Let $u_1 \in \overline{M}\mathscr{A}$, $u_2 \in \overline{M}\mathscr{A}$, $|u_1| < C$, $|u_2| < C$ for some $C \in \mathbb{R}$. Then for each $\varepsilon > 0$ there exist functions $w_1 \in \overline{M}\mathscr{A}$ and $w_2 \in \overline{M}\mathscr{A}$ such that*

$$2\varepsilon \le w_1 \le 2C + 3\varepsilon, \quad \varepsilon \le w_2 \le 2C + 2\varepsilon, \quad \mathrm{Rg}(w_1 - w_2) \subseteq \{0, \varepsilon\}$$

and

$$w_1(x) - w_2(x) \ne u_1(x) - u_2(x),$$

for every $x \in A$ (for short we shall write $w_1 - w_2 \cap u_1 - u_2 = \emptyset$).

Proof. Let $K = [C/\varepsilon] + 1$. We put

$$A_{i,j} = \{x : i\varepsilon \le u_1(x) < (i+1)\varepsilon \text{ and } j\varepsilon \le u_2(x) < (j+1)\varepsilon\},$$

for $-K \le i, j \le K - 1$. For $x \in A_{i,j}$ we put

$$w_1(x) = \begin{cases} (K + 1 + \max(i,j))\varepsilon & \text{if } i \ne j, \\ (K + i + 2)\varepsilon & \text{if } i = j, \end{cases}$$

$$w_2(x) = (K + 1 + \max(i,j))\varepsilon.$$

It is easy to see that w_1 and w_2 satisfy the conditions desired. ∎

LEMMA 3.3. *Let $\mathscr{F}$ be a complete function system. Let $\xi \in \mathscr{S}(\mathscr{F})$ and suppose that $f : [0,1] \to \mathbb{R}$ does not satisfy the Lipschitz condition in any neighbourhood of zero. Then there are functions $h \in (\mathscr{U}(\mathscr{F}))_{[0,1]} - (\mathscr{U}(\mathscr{F}))_{[0,1]}$ and $\widetilde{h} \in (\mathscr{U}(\mathscr{F}))_{[0,1]}$ such that $h \ge 0$ and $(f \circ h - f \circ \widetilde{h}) \cap \xi = \emptyset$.*

Proof. By our assumption, for each $n \in \mathbb{N}$ there exist sequences x_n, y_n in $[0,1]$ such that

(i) $x_n < 1/n$, $x_n > y_n \ge 0$, $y_1 \ge y_2 \ge \ldots$,
(ii) $|f(x_n) - f(y_n)| > n(x_n - y_n)$.

Without loosing generality we can assume that

(iii) $f(x_n) > f(y_n)$.

Let $\xi = u_1 - u_2$, where $u_1 \in \mathscr{U}(\mathscr{F})$ and $u_2 \in \mathscr{U}(\mathscr{F})$.

Assume first that at least one of the functions u_1, u_2 is not bounded. For $n \in \mathbb{N}$, let

$$\Lambda_n = u_1^{-1}((-n,n)) \cap u_2^{-1}((-n,n))$$

and write $\Lambda_n = \bigcup_{m=1}^{\infty} \Lambda_{n,m}$, where $\Lambda_{n,m} \in \mathscr{N}_{\mathscr{F}}$. Order all the sets $\Lambda_{n,m}$ into a sequence $\Gamma_1, \Gamma_2, \ldots$, and put $A_k = \bigcup_{i=1}^{k} \Gamma_i$.

Let $i(1) = \min\{i : A_i \neq \emptyset\}$. Then inductively for $k > 1$, set $i(k) = \min\{i : i > i(k-1) \text{ and } A_i \setminus A_{i-1} \neq \emptyset\}$. Because at least one of the functions u_1, u_2 is unbounded the numbers $i(k)$ are defined for all $k \in \mathbb{N}$. Let $B_k = A_{i(k)} \setminus A_{i(k)-1}$.

The functions u_1 and u_2 are bounded on each set B_k, $k \in \mathbb{N}$. Let $|u_1|B_k| < C_k$ and $|u_2|B_k| < C_k$, for some $C_k \in \mathbb{R}$.

Let now $m_1 \in \mathbb{N}$ be chosen in such a way that (compare (i))

$$2C_1 m_1^{-1} + 4x_{m_1} < 1 \tag{1}$$

and, inductively, let $m_k > m_{k-1}$, $k > 1$, satisfy

$$2C_k m_k^{-1} + 4x_{m_k} < x_{m_{k-1}} - y_{m_{k-1}}. \tag{2}$$

By Lemma 3.2, taking there $\varepsilon = f(x_{m_k}) - f(y_{m_k})$ and $C = C_k$, there are functions $w_1^{(k)} \in \overline{M}(\mathscr{M}_{\mathscr{F}}|B_k)$ and $w_2^{(k)} \in \overline{M}(\mathscr{M}_{\mathscr{F}}|B_k)$ such that

$$(w_1^{(k)} - w_2^{(k)}) \cap (u_1 - u_2) = \emptyset, \tag{3}$$

$$\mathrm{Rg}(w_1^{(k)} - w_2^{(k)}) \subseteq \{f(x_{m_k}) - f(y_{m_k}), 0\}, \tag{4}$$

$$2(f(x_{m_k}) - f(y_{m_k})) \le w_1^{(k)} \le 2C_k + 3(f(x_{m_k}) - f(y_{m_k})) \tag{5}$$

and

$$f(x_{m_k}) - f(y_{m_k}) \le w_2^{(k)} \le 2C_k + 2(f(x_{m_k}) - f(y_{m_k})). \tag{6}$$

For $z \in B_k$, let

$$v_1^{(k)}(z) = w_1^{(k)}(z)\frac{x_{m_k} - y_{m_k}}{f(x_{m_k}) - f(y_{m_k})} + y_{m_k} \tag{7}$$

and

$$v_2^{(k)}(z) = w_2^{(k)}(z)\frac{x_{m_k} - y_{m_k}}{f(x_{m_k}) - f(y_{m_k})}. \tag{8}$$

Then, by (i)–(iii), (1), (2), (5) and (6) we get

$$1 > \sup v_1^{(1)} \ge \inf v_1^{(1)} > \sup v_1^{(2)} \ge \inf v_1^{(2)} > \ldots \ge 0, \tag{9}$$

$$1 > \sup v_2^{(1)} \ge \inf v_2^{(1)} > \sup v_2^{(2)} \ge \inf v_2^{(2)} > \ldots \ge 0 \tag{10}$$

and $v_1^{(k)} \ge v_2^{(k)}$.

Now, put

$$v_i(z) = v_i^{(k)}(z), \tag{11}$$

for $z \in B_k$, $i = 1, 2$, and

$$h(z) = v_1(z) - v_2(z), \tag{12}$$

and

$$\widetilde{h}(z) = \sum_{k=1}^{\infty} y_{m_k}\chi_{B_k}(z), \tag{13}$$

for $z \in Z$.

We now check that h and $\widetilde{h}$ satisfy the conclusion of our lemma.

Let $z \in B_k$. By (4), (7), (8) and (11)–(13) we get

$$\begin{aligned}
&f(h(z)) - f(\widetilde{h}(z))\\
&\quad = f\left((w_1^{(k)}(z) - w_2^{(k)}(z))\frac{x_{m_k} - y_{m_k}}{f(x_{m_k}) - f(y_{m_k})} + y_{m_k}\right) - f(y_{m_k})\\
&\quad = \begin{cases} f(y_{m_k}) - f(y_{m_k}) = 0 & \text{if } w_1^{(k)}(z) - w_2^{(k)}(z) = 0,\\ f(x_{m_k}) - f(y_{m_k}) & \text{if } w_1^{(k)}(z) - w_2^{(k)}(z) = f(x_{m_k}) - f(y_{m_k}), \end{cases}\\
&\quad = w_1^{(k)}(z) - w_2^{(k)}(z),
\end{aligned}$$

whence $(f(h) - f(\widetilde{h})) \cap \xi = \emptyset$.

It follows from (9) and (10) that $v_1 \in \mathscr{U}(\mathscr{F})_{[0,1]}$ and $v_2 \in \mathscr{U}(\mathscr{F})_{[0,1]}$. By the definition of the sets B_k and by (i) one can easily see that $\widetilde{h} \in (\mathscr{U}(\mathscr{F}))_{[0,1]}$.

If u_1 and u_2 are bounded, we put $B_1 = D$ in the above reasoning and the construction consists of only one stage. ■

THEOREM 3.4. *Let $I \subseteq \mathbb{R}$ be an interval, $f : I \to \mathbb{R}$ and $\alpha < \omega_1$. Let Z be any uncountable Polish space. Then $f \circ g \in \mathbf{S}_\alpha(Z)$ for every $g \in (\mathbf{S}_\alpha(Z))_I$ if and only if f is locally Lipschitz on I.*

Proof. The easier "if" implication is contained in Theorem 3.1.

We now prove the "only if" part. Assume for contradiction that $f \circ g \in \mathbf{S}_\alpha(Z)$ for every $g \in (\mathbf{S}_\alpha(Z))_I$ and that f is not locally Lipschitz. It is enough to consider the case where $I = [0,1]$ and f does not satisfy the Lipschitz condition in any neighbourhood of 0. By Lemma 2.E there are functions

$$\Phi \in (\mathbf{U}_\alpha(Z^2))_I - (\mathbf{U}_\alpha(Z^2))_I \quad \text{and} \quad \Psi \in (\mathbf{U}_\alpha(Z^2))_I$$

such that for any $\phi \in (\mathbf{U}_\alpha(Z))_I - (\mathbf{U}_\alpha(Z))_I$ and $\psi \in (\mathbf{U}_\alpha(Z))_I$ there is $x \in Z$ for which $\phi(y) = \Phi(x,y)$ and $\psi(y) = \Psi(x,y)$ for every $y \in Z$. Let now

$$\xi(x) = f(\max(\Phi(x,x),0)) - f(\Psi(x,x)).$$

By our assumption $\xi \in \mathbf{S}_\alpha(Z)$, but by Lemma 3.3 applied to $\mathscr{F} = \mathbf{B}_\alpha(Z)$ there are $h \in (\mathbf{U}_\alpha(Z))_I - (\mathbf{U}_\alpha(Z))_I$ and $\widetilde{h} \in (\mathbf{U}_\alpha(Z))_I$, $h \geq 0$, such that

$$(f \circ h - f \circ \widetilde{h}) \cap \xi = \emptyset.$$

But there is $x \in Z$ such that $h(y) = \Phi(x,y)$ and $\widetilde{h}(y) = \Psi(x,y)$ for each $y \in Z$ and then $f(h(x)) - f(\widetilde{h}(x)) = \xi(x)$, which is a contradiction. ∎

R e m a r k. Theorem 3.4 settles the question about the class of functions not leading outside $\mathbf{S}_\alpha$ by outer superpositions, raised by the work of Lindenbaum (see [L] and [L, Corr.]).

4. Corollaries to Theorems 3.1 and 3.4

A. *Baire's and Sierpiński's classifications of Borel measurable functions.* Theorems 3.4 and 3.1 imply the following corollary solving the problem of Kempisty (mentioned in the introduction) in a different way than it was done in [Mor].

Corollary 4.1. *Let Z be an uncountable Polish space. Then $\mathbf{B}_{\alpha+1}(Z) \setminus \mathbf{S}_\alpha(Z) \neq \emptyset$. Moreover, we can find $f \in \mathbf{B}_{\alpha+1}(Z) \setminus \mathbf{S}_\alpha(Z)$ such that $Z = X \cup Y$, $X \cap Y = \emptyset$ and $f|X \in \mathbf{S}_\alpha(X)$ and $f|Y \equiv 0$.*

P r o o f. By Theorem 3.4 there is a function $h : Z \to [0,\infty)$, $h \in \mathbf{S}_\alpha(Z)$, such that $f = \sqrt{h} \notin \mathbf{S}_\alpha(Z)$. Let $X = \{x : h(x) > 0\}$ and $Y = \{x : h(x) = 0\}$. By Theorem 3.1, $f|X \in \mathbf{S}_\alpha(X)$ and, obviously, $f|Y \equiv 0$. On the other hand, by [Ku, 31, III, Th. 2], $f \in \mathbf{B}_{\alpha+1}(Z)$. ∎

B. *The class $\mathscr{L}(\mathscr{F}) \cdot \mathscr{U}(\mathscr{F})$.* Let $\mathscr{F}$ be a complete function system. The class $\mathscr{S}(\mathscr{F})$ is defined as $\mathscr{L}(\mathscr{F}) + \mathscr{U}(\mathscr{F})$. It is of interest to look at the class $\mathscr{L}(\mathscr{F}) \cdot \mathscr{U}(\mathscr{F})$. Since $\mathscr{S}(\mathscr{F})$ is closed under multiplication, it contains $\mathscr{L}(\mathscr{F}) \cdot \mathscr{U}(\mathscr{F})$. We shall show, using Theorem 3.1, that a positive function belongs to $\mathscr{S}(\mathscr{F})$ if and only if it belongs to $\mathscr{L}(\mathscr{F}) \cdot \mathscr{U}(\mathscr{F})$ (Corollary 4.2). However, we shall apply Theorem 3.4 to show that this is not the case for nonnegative functions (Corollary 4.3).

Corollary 4.2. *Let $\mathscr{F}$ be a complete function system on D. Then*

$$(\mathscr{S}(\mathscr{F}))_{(0,\infty)} = (\mathscr{L}(\mathscr{F}))_{(0,\infty)} \cdot (\mathscr{U}(\mathscr{F}))_{(0,\infty)}.$$

Proof. Let $f \in (\mathscr{S}(\mathscr{F}))_{(0,\infty)}$. By Theorem 3.1, $\log f \in \mathscr{S}(\mathscr{F})$. Thus $\log f = g + h$ for some $g \in \mathscr{L}(\mathscr{F})$ and $h \in \mathscr{U}(\mathscr{F})$. Since e^x is continuous increasing, it follows that $e^g \in \mathscr{L}(\mathscr{F})_{(0,\infty)}$ and $e^h \in \mathscr{U}(\mathscr{F})_{(0,\infty)}$, and clearly $f = e^g e^h$. ∎

COROLLARY 4.3. *If Z is an uncountable Polish space then* $(\mathbf{S}_\alpha(Z))_{[0,\infty)} \setminus \mathbf{L}_\alpha(Z) \cdot \mathbf{U}_\alpha(Z) \neq \emptyset$.

Proof. By Theorem 3.4 there exists a function $f \in (\mathbf{S}_\alpha(Z))_{[0,\infty)}$ such that $f^{1/3} \notin \mathbf{S}_\alpha(Z)$. If $f = g \cdot h$ where $g \in \mathbf{L}_\alpha(Z)$ and $h \in \mathbf{U}_\alpha(Z)$, then $f^{1/3} = g^{1/3} \cdot h^{1/3}$. But $g^{1/3} \in \mathbf{L}_\alpha(Z)$ and $h^{1/3} \in \mathbf{U}_\alpha(Z)$, whence $f^{1/3} \in \mathbf{L}_\alpha(Z) \cdot \mathbf{U}_\alpha(Z) \subseteq \mathbf{S}_\alpha(Z)$ and this is a contradiction. ∎

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