FUNDAMENTA
MATHEMATICAE
147 (1995)

The minimum uniform compactification of a metric space

by

R. Grant Woods (Winnipeg, Man.)

Abstract. It is shown that associated with each metric space (X,d) there is a compactification u_dX of X that can be characterized as the smallest compactification of X to which each bounded uniformly continuous real-valued continuous function with domain X can be extended. Other characterizations of u_dX are presented, and a detailed study of the structure of u_dX is undertaken. This culminates in a topological characterization of the outgrowth $u_d\mathbb{R}^n \setminus \mathbb{R}^n$, where $(\mathbb{R}^n, d)$ is Euclidean n-space with its usual metric.

1. Introduction. Let X be a completely regular Hausdorff (i.e. Tikhonov) topological space. As usual, a *compactification* of X is a compact Hausdorff space αX that contains X as a dense subspace. Two compactifications αX and γX are called *equivalent* if there is a homeomorphism h from αX onto γX such that $h(x) = x$ for each $x \in X$. (We denote this by writing $\alpha X \cong \gamma X$.) Equivalent compactifications of X are "the same" (except for notation); if we identify equivalent compactifications of X then the class $\mathcal{K}(X)$ of compactifications of X can be regarded as a set, and partially ordered as follows: $\alpha X \le \gamma X$ if there is a continuous surjection $f : \gamma X \to \alpha X$ such that $f(x) = x$ for each $x \in X$. Thus ordered, $(\mathcal{K}(X), \le)$ is a complete upper semilattice whose largest member is the Stone–Čech compactification βX. (It is a complete lattice iff X is locally compact.) Let $C^*(X)$ denote the set of all bounded real-valued continuous functions with domain X; then βX can be characterized (up to equivalence) as the compactification X to which each member of $C^*(X)$ can be continuously extended.

[Everything in the preceding paragraph is well known; the reader is referred to [GJ], [Wa], [PW], and [Ma] for more details.]

Now let (X, d) be a fixed metric space. We also regard it as a topological space with the metric topology τ_d induced by d.

1991 *Mathematics Subject Classification*: 54D35, 54E35.

This research was supported by NSERC Grant No. OGP00D7592.

Define a binary relation δ_d on the power set $\mathcal{P}(X)$ of X as follows:

$$A\delta_d B \quad \text{if } d(A,B)=0.$$

[Here, as usual, $d(A,B)=\inf\{d(a,b): a\in A \text{ and } b\in B\}$.] It is well known that δ_d is a separated Efremovich proximity on X (see Chapter 1 of [NW]). The theory of proximities then tells us that there exists a compactification u_dX of X, called the *Smirnov* or *Samuel compactification* of X. We formalize its known properties in the following theorem.

THEOREM 1.1. *Let (X,d) be a metric space. Then the topological space (X,τ_d) has a compactification u_dX with these properties*:

(a) *If $A,B\in\mathcal{P}(X)$ then $\mathrm{cl}_{u_dX} A\cap \mathrm{cl}_{u_dX} B\neq\emptyset$ iff $d(A,B)=0$.*

(b) *If (X,d) and (Y,ϱ) are metric spaces and $f:X\to Y$ is uniformly continuous*, *then there is a continuous function $f^u: u_dX\to u_\varrho Y$ such that $f^u|X=f$.*

(c) *Let $U_d^*(X)$ denote the ring of all bounded real-valued uniformly continuous functions with domain (X,d). If $f\in U_d^*(X)$ then there is a* (*necessarily unique*) *continuous function $f^*: u_dX\to\mathbb{R}$ such that $f^*|X=f$.*

The above results are essentially straightforward applications of the theory of proximity spaces as expounded in [Wi], [PW], and especially [NW]. (I have also benefited from consulting the unpublished monograph [R] by my colleague Dr. M. C. Rayburn.) Specifically, a proof of (a) above appears in 7.7 of [NW], and (b) can be proved by combining 4.8 and 7.10 of [NW]. Clearly (c) is a special case of (b) with $Y=\mathrm{cl}_{\mathbb{R}} f[X]$.

We shall call u_dX the *minimum uniform compactification* of the metric space (X,d) (see Theorem 2.3(a) for the rationale behind this terminology). We will sometimes write "uX" instead of "u_dX" when it is clear what metric d is under consideration. Similarly we shall write $U^*(X)$ rather than $U_d^*(X)$.

The purpose of this paper is to investigate the properties of this compactification, particularly in the case where (X,d) is a locally compact separable metric space. Although partial results have been known for some time (e.g. see [M]), to our knowledge no systematic extensive study of the compactification has been undertaken. Perhaps the most interesting result in this paper is the structure theorem (4.9) which tells us that $u\mathbb{R}^n\setminus\mathbb{R}^n$ can be written as a union of 2^n copies of $[0,1]^n\times(\beta\omega\setminus\omega)$ (where ω is the countably infinite discrete space) "glued together" in a nontrivial fashion. (Here, and throughout this paper, $\mathbb{N}$ will denote the set of positive integers and $\mathbb{R}^n$ will denote Euclidean n-space with the usual metric.) We also investigate the structure of $u\mathbb{R}\setminus\mathbb{R}$ in more detail, and discuss its relation to $\beta\mathbb{R}\setminus\mathbb{R}$.

I would like to thank the referee and Prof. Jorge Martinez for a number of useful suggestions.

2. Characterizations of $u_d X$. In this section we develop several characterizations of the compactification $u_d X$ of the metric space (X,d).

DEFINITION 2.1. Let (X,d) be a metric space. If $A \subseteq X$, define the function $g_A : X \to \mathbb{R}$ by $g_A(x) = \min\{d(x,A),1\}$.

Clearly if $A \subseteq X$ then $g_A \in U_d^*(X)$. In what follows we shall frequently make use of Taĭmanov's theorem (see, for example, 4.2(h) of [PW]), as follows.

THEOREM 2.2. *Let X be a Tikhonov space and let αX, $\gamma X \in \mathcal{K}(X)$. The following are equivalent*:

(a) $\alpha X \geq \gamma X$.

(b) *If A and B are disjoint closed subsets of X and if* $\mathrm{cl}_{\gamma X} A \cap \mathrm{cl}_{\gamma X} B = \emptyset$, *then* $\mathrm{cl}_{\alpha X} A \cap \mathrm{cl}_{\alpha X} B = \emptyset$.

We now characterize uX (up to equivalence).

THEOREM 2.3. *Let (X,d) be a metric space. Then*

(a) *uX is the smallest compactification of X* (*in the poset* $(\mathcal{K}(X),\leq)$) *to which each member of $U^*(X)$ can be continuously extended.*

(b) $uX = \max\{\alpha X \in \mathcal{K}(X)$: *if A and B are subsets of X and* $d(A,B) = 0$ *then* $\mathrm{cl}_{\alpha X} A \cap \mathrm{cl}_{\alpha X} B \neq \emptyset\}$.

P r o o f. (a) As noted in Theorem 1.1(b), each $f \in U^*(X)$ can be continuously extended to $uf^* \in C^*(uX)$. Suppose that αX is another compactification of X to which each member of $U^*(X)$ can be continuously extended. We will prove that $\alpha X \geq uX$. To do this it suffices by Theorem 2.2 to show that if A and B are disjoint closed subsets of X for which $\mathrm{cl}_{uX} A \cap \mathrm{cl}_{uX} B = \emptyset$ then $\mathrm{cl}_{\alpha X} A \cap \mathrm{cl}_{\alpha X} B = \emptyset$. If $\mathrm{cl}_{uX} A \cap \mathrm{cl}_{uX} B = \emptyset$ then by 1.1(a) there exists $r > 0$ such that $d(A,B) > r$. As $g_A \in U^*(X)$, by hypothesis it can be continuously extended to $f : \alpha X \to \mathbb{R}$. Suppose that $p \in \mathrm{cl}_{\alpha X} A \cap \mathrm{cl}_{\alpha X} B$ and let $f(p) = s$. Then there exist $a \in A \cap f^{\leftarrow}[(s - r/8, s + r/8)]$ and $b \in B \cap f^{\leftarrow}[(s-r/8, s+r/8)]$, and so $|f(a)-f(b)| < r/4$. Thus $d(b,A) < r/4$, contradicting the definition of r. Consequently, $\mathrm{cl}_{\alpha X} A \cap \mathrm{cl}_{\alpha X} B = \emptyset$ as required.

(b) By Theorem 1.1(a), uX belongs to the set whose maximum we are taking. Suppose that αX is any other member of the set. Let A and B be subsets of X for which $\mathrm{cl}_{\alpha X} A \cap \mathrm{cl}_{\alpha X} B = \emptyset$. By hypothesis $d(A,B) > 0$, so by Theorem 1.1(a), $\mathrm{cl}_{uX} A \cap \mathrm{cl}_{uX} B = \emptyset$. Hence by Theorem 2.2, it follows that $\alpha X \leq uX$. ∎

COROLLARY 2.4. *The compactification uX is characterized uniquely* (*up to equivalence*) *by the fact that it has the following two properties*:

(i) *If $f \in U^*(X)$ then f extends continuously to $uf \in C^*(uX)$.*

(ii) *If A and B are subsets of X and $d(A,B)=0$ then* $\mathrm{cl}_{uX} A \cap \mathrm{cl}_{uX} B \neq \emptyset$.

P r o o f. If $\alpha X \in \mathcal{K}(X)$ and αX has both (i) and (ii), by Theorem 2.3(a), $\alpha X \geq uX$ and by Theorem 2.3(b), $\alpha X \leq uX$. Hence αX is equivalent to uX. ∎

We can also characterize uX in the following ways.

THEOREM 2.5. *Let (X, d) be a metric space and let $\alpha X \in \mathcal{K}(X)$. The following are equivalent*:

(a) $\alpha X \cong uX$ (*as compactifications of* X).
(b) *If* $A, B \subseteq X$ *then* $\mathrm{cl}_{\alpha X} A \cap \mathrm{cl}_{\alpha X} B \neq \emptyset$ *iff* $d(A, B) = 0$.
(c) $\{f \in C^*(X) : f$ *can be continuously extended to* $\alpha X\} = U_d^*(X)$.

P r o o f. We know, from Theorem 1.1(a), that if $A, B \subseteq X$ then $\mathrm{cl}_{uX} A \cap \mathrm{cl}_{uX} B = \emptyset$ iff $d(A, B) > 0$. But as noted on p. 42 of [NW], Smirnov [S] has proved that uX is equivalent to αX iff $\{(A, B) \in \mathcal{P}(X) \times \mathcal{P}(X) : \mathrm{cl}_{\alpha X} A \cap \mathrm{cl}_{\alpha X} B = \emptyset\} = \{(A, B) \in \mathcal{P}(X) \times \mathcal{P}(X) : \mathrm{cl}_{uX} A \cap \mathrm{cl}_{uX} B = \emptyset\}$. The equivalence of (a) and (b) now follows.

To prove that (a) and (c) are equivalent, first note that by Theorem 1.1(c), $U_d^*(X) \subseteq \{f \in C^*(X) : f$ can be continuously extended to $uX\}$. Conversely, suppose $g \in C(uX)$ and let $K = g[uX]$. As uX and K are compact, they have unique compatible proximities (see 3.7 of [NW]) and the subspace proximity inherited from uX by X is just δ_d (see 7.9 of [NW]). Hence by 7.7 of [NW], $g|X : X \to K$ is a proximity map from (X, δ_d) to (K, δ_ϱ), where ϱ is the subspace metric induced on K by the Euclidean metric on $\mathbb{R}$. Hence by 4.8 of [NW], g is uniformly continuous. Thus $g|X \in U_d^*(X)$ and so $\{f \in C^*(X) : f$ can be continuously extended to $uX\} = U_d^*(X)$. But as a compactification αX of a Tikhonov space X is determined (up to equivalence of compactifications of X) by $\{f|X : f \in C(\alpha X)\}$ (see 4.5(q) of [PW], for example), it follows that (a) and (c) are equivalent. ∎

One useful consequence of Theorem 2.5 is the following.

COROLLARY 2.6. *Let (X, d) be a metric space. If $A \subseteq X$ and $x \in uX \setminus \mathrm{cl}_{uX} A$ then there exists a closed subset B of X such that $x \in \mathrm{cl}_{uX} B$ and $d(A, B) > 0$.*

P r o o f. There exist disjoint open sets U and V of uX such that $x \in U$ and $\mathrm{cl}_{uX} A \subseteq V$. One quickly verifies that $x \in \mathrm{cl}_{uX}(\mathrm{cl}_X(U \cap X))$ and that $\mathrm{cl}_{uX}(\mathrm{cl}_X(U \cap X)) \cap \mathrm{cl}_{uX} A = \emptyset$. Let $B = \mathrm{cl}_X(U \cap X)$; it follows from Theorem 2.5 that $d(A, B) > 0$. ∎

A *zero-set* of a space Y is a subset of the form $Z(f) = f^{\leftarrow}(0)$, where $f \in C^*(Y)$. As in [GJ], [Wa] and [PW] we denote the set of zero-sets of Y by $\mathbf{Z}(Y)$. If (X, d) is a metric space and $f \in U_d^*(X)$, we will denote the (unique) continuous extension of f to uX by uf.

We now analyze the zero-sets of uX, and use them to provide an alternate characterization of uX.

THEOREM 2.7. *Let* (X,d) *be a metric space and let* $A \subseteq X$. *Then*:

(a) $Z(ug_A) = \mathrm{cl}_{uX} A$ (see Definition 2.1 and Corollary 2.4(i) for notation).

(b) *Let* $B(n) = \{x \in X : d(x,A) \geq 1/n\}$. *Then*

$$\mathrm{cl}_{uX} A = \bigcap \{uX \setminus \mathrm{cl}_{uX} B(n) : n \in \mathbb{N}\}.$$

(c) *Let* $\mathcal{S} = \{S \subseteq uX : S$ *is the intersection of countably many sets of the form* $\mathrm{cl}_{uX} E$, *where* E *is a subset of* $X\}$. *Then* $\mathbf{Z}(uX) = \mathcal{S}$.

Proof. (a) Suppose that $x \in \mathrm{cl}_{uX} A$. Then $ug_A(x) \in ug_A[\mathrm{cl}_{uX} A] = \mathrm{cl}_{[0,1]}\, g_A[A] = \{0\}$ so $x \in Z(ug_A)$. Conversely, suppose that $x \notin \mathrm{cl}_{uX} A$. By Corollary 2.6 there exists a closed subset B of X such that $x \in \mathrm{cl}_{uX} B$ and $d(A,B) = r > 0$. Thus $B \subseteq g_A^{\leftarrow}[[r,1]]$. Consequently, $ug_A(x) \in \mathrm{cl}_{uX}\, g_A[B] \subseteq [r,1]$ and so $x \notin Z(ug_A)$. The result follows.

(b) Clearly $d(B(n),A) \geq 1/n$ so $\mathrm{cl}_{uX} A \cap \mathrm{cl}_{uX} B = \emptyset$. Hence $\mathrm{cl}_{uX} A \subseteq \bigcap\{uX \setminus \mathrm{cl}_{uX} B(n) : n \in \mathbb{N}\}$. Conversely, suppose $x \notin \mathrm{cl}_{uX} A$. By Corollary 2.6 there exists a closed subset F of X and $k \in \mathbb{N}$ such that $x \in \mathrm{cl}_{uX} F$ and $d(A,F) > 1/k$. Thus $F \subseteq B(k)$ and so $x \in \mathrm{cl}_{uX} B(k)$. Thus $x \notin \bigcap\{uX \setminus \mathrm{cl}_{uX} B(n) : n \in \mathbb{N}\}$ and the result follows.

(c) By (a) we see that $\mathbf{Z}(uX) \supseteq \mathcal{S}$. If αX is any compactification of X, and if $f \in C(\alpha X)$, then $Z(f) = \bigcap\{\mathrm{cl}_{\alpha X}(X \cap f^{\leftarrow}[(-1/n,1/n)]) : n \in \mathbb{N}\}$; in particular $\mathbf{Z}(uX) \subseteq \mathcal{S}$. The result follows. ∎

THEOREM 2.8. *Let* (X,d) *be a metric space. Then the compactification* uX *is characterized uniquely* (*up to equivalence*) *by the fact that for each closed subset* A *of* X, *the function* g_A *extends continuously to* $ug_A \in C^*(uX)$ *and* $\mathrm{cl}_{uX} A = Z(ug_A)$.

Proof. The proof of Theorem 2.3 shows that uX is the smallest compactification of X to which each g_A can be extended. Now suppose that γX were a compactification of X for which g_A could be extended continuously to $\gamma g_A \in C(\gamma X)$ and for which $\mathrm{cl}_{\gamma X} A = Z(\gamma g_A)$. By the above, $uX \leq \gamma X$. Now suppose that A and B were closed subsets of X such that $d(A,B) = 0$. For each $n \in \mathbb{N}$ choose $a_n \in A$ and $b_n \in B$ for which $d(a_n,b_n) \leq 1/n$. As γX is compact there exists $p \in \mathrm{cl}_{\gamma X}\{b_n : n \in \mathbb{N}\}$. Thus $p \in \mathrm{cl}_{\gamma X} B$. If $\gamma g_A(p) = \varepsilon > 0$, find $j \in \mathbb{N}$ such that $1/j < \varepsilon/4$ and $b_j \in (\gamma g_A)^{\leftarrow}(\varepsilon/2, 3\varepsilon/2)$. Thus $d(b_j,A) > \varepsilon/2$ while $d(b_j,a_j) < \varepsilon/4$, which is a contradiction. Thus $p \in Z(\gamma g_A)$ and so by hypothesis $p \in \mathrm{cl}_{\gamma X} B \cap \mathrm{cl}_{\gamma X} A$. Thus γX belongs to a set of compactifications of which uX was shown in Theorem 2.3(b) to be the maximum. Thus $\gamma X \leq uX$ and so uX and γX are equivalent as claimed. ∎

If $Z \in \mathbf{Z}(X)$ it is not in general true that $\mathrm{cl}_{\beta X} Z \in \mathbf{Z}(\beta X)$, so Theorem 2.8 gives a way in which βX and uX behave differently. Also, if A and B are disjoint noncompact closed subsets of X for which $d(A, B) = 0$, then $\mathrm{cl}_{uX} A \cap \mathrm{cl}_{uX} B$ is a nonempty zero-set of uX that is disjoint from X, so not every zero-set of uX need be of the form $\mathrm{cl}_{uX} A$ in general, where A is a closed subset of X.

THEOREM 2.9. *If* (X, d) *is a metric space, and if* $S \subseteq X$, *then* $\mathrm{cl}_{u_d X} S \cong uS$ (*up to equivalence*), *where* uS *is the minimum uniform compactification of the metric space* $(S, d|S)$.

Proof. By a theorem in [K] (as quoted in 2.3 of [LR]) if $f \in U^*(S)$ then f can be extended to $f^{\#} \in U^*(X)$. But $f^{\#}$ extends to $f^* \in C^*(uX)$ and $f^*|\mathrm{cl}_{uX} S$ extends f to $C^*(\mathrm{cl}_{uX} S)$. Observe that $\delta|S$, the subspace proximity on S induced by δ, is given by $A(\delta|S)B$ iff $(d|S)(A, B) = 0$ (where A and B are subsets of S). Hence we see that if $(d|S)(A, B) = 0$ then $A\delta B$ and so $\mathrm{cl}_{uX} A \cap \mathrm{cl}_{uX} B = \emptyset$ whence $\mathrm{cl}_{\mathrm{cl}_{uX} S} A \cap \mathrm{cl}_{\mathrm{cl}_{uX} S} B = \emptyset$. Thus by Corollary 2.4 our result follows. ∎

Finally, recall that if (X, d) and (Y, s) are metric spaces then a bijection $f : X \to Y$ is called a *uniform isomorphism* if both f and f^{-1} are uniformly continuous (see, for example, 35.10 of [Wi]). In this case we say that the metric spaces (X, d) and (Y, s) are *uniformly equivalent*. If $X = Y$, then the metrics d and s on the common underlying set X are said to be uniformly equivalent. Clearly this happens iff the identity function $\mathrm{id} : (X, d) \to (X, s)$ is a uniform isomorphism; it is well known that this is true iff there are positive constants m and M such that for all $x, y \in X$, $md(x, y) \le s(x, y) \le Md(x, y)$. This implies that $\tau_d = \tau_s$, but the converse implication fails.

The following is an immediate consequence of Theorem 1.1(b).

THEOREM 2.10. *If* (X, d) *and* (Y, s) *are metric spaces and if* $f : X \to Y$ *is a uniform isomorphism*, *then* f *extends to a homeomorphism* $F : u_d X \to u_s Y$; *in particular*, $u_d X \setminus X$ *is homeomorphic to* $u_s Y \setminus Y$.

Proof. By Theorem 1.1(b), f continuously extends to $F : u_d X \to u_s Y$ and f^{-1} continuously extends to $G : u_s Y \to u_d X$. Then $G \circ F|_X$ is the identity on X, so $G \circ F$ is the identity on $u_d X$. The theorem follows. ∎

Let (X, τ) be a metrizable topological space. Define $\mathcal{M}(X)$ to be $\{u_d X : d$ is a metric for X such that $\tau_d = \tau\}$. We conclude this section by investigating the order-theoretic properties of $\mathcal{M}(X)$ when viewed as a subset of the poset $\mathcal{K}(X)$ of compactifications of X. As usual, if (X, d) is a metric space and $x \in X$ we denote by $S_d(x, \varepsilon)$ the open sphere with centre x and radius ε; we write "$S(x, \varepsilon)$" if there is only one metric d under discussion.

THEOREM 2.11. *Let (X,τ) be a metrizable topological space. Then*:

(a) $\bigvee \mathcal{M}(X) = \beta X$ *(where the supremum is taken in $\mathcal{K}(X)$).*

(b) *If (X,τ) is locally compact and noncompact then the one-point compactification of X belongs to $\mathcal{M}(X)$ iff X is second countable.*

Proof. (a) Let d be a metric on X for which $\tau_d = \tau$. Let A and B be disjoint nonempty closed sets of X. Then there exists $f \in C^*(X)$ for which $f[A] = \{0\}$, $f[B] = \{1\}$, and $\mathbf{0} \le f \le \mathbf{1}$. Define $d_{A,B} : X \times X \to \mathbb{R}$ by

$$d_{A,B}(x,y) = \max\{|f(x) - f(y)|, d(x,y)\}.$$

It is straightforward to verify that $d_{A,B}$ is a metric on X. Clearly $d_{A,B}(x,y) \ge d(x,y)$ if $x,y \in X$, and consequently $\tau_d \subseteq \tau_{d_{A,B}}$. Conversely, let $p \in X$ and let $\varepsilon > 0$ be given. As $f \in C^*(X)$ there exists $\delta(p,\varepsilon) > 0$ such that $d(p,x) < \delta(p,\varepsilon)$ implies $|f(p) - f(x)| < \varepsilon$. Let $\alpha(p,\varepsilon) = \min\{\varepsilon, \delta(p,\varepsilon)\}$. It is easy to verify that

$$S_d(p,\alpha(p,\varepsilon)) \subseteq S_{d_{A,B}}(p,\varepsilon), \quad \text{and so} \quad \tau_{d_{A,B}} \subseteq \tau_d.$$

Hence $u_{d_{A,B}}X \in \mathcal{M}(X)$ and as $d_{A,B}(A,B) = 1$, by Theorem 2.5(b) it follows that $\mathrm{cl}_{u_{d_{A,B}}X} A \cap \mathrm{cl}_{u_{d_{A,B}}X} B = \emptyset$. Hence by Taĭmanov's theorem (see Theorem 2.2) we see that if A and B are any pair of disjoint closed sets of (X,τ), then $\mathrm{cl}_{\bigvee \mathcal{M}(X)} A \cap \mathrm{cl}_{\bigvee \mathcal{M}(X)} B = \emptyset$. But this is the characterizing property of the Stone–Čech compactification of a normal space (see 6.5 of [GJ]), and so $\bigvee \mathcal{M}(X) \cong \beta X$.

(b) Let μX denote the one-point compactification of X. If there is a metric s on X such that $\tau_s = \tau$ and $\mu X \cong u_s X$ then $|u_s X \setminus X| = 1$ and so, by Theorem 3.3 below, $u_s X$ is the metric completion of (X,s). Hence μX is compact and metrizable and hence second countable; consequently, X is second countable.

Conversely, if X is second countable then so is μX, and hence μX is metrizable. If s is a compatible metric on μX then it is easily seen that $(\mu X, s)$ is a complete metric space in which $(X, s|X)$ is densely and isometrically embedded. Hence $(\mu X, s)$ is the metric completion of $(X, s|X)$ and by Theorem 3.3 it follows that $\mu X \cong u_{s|X} X$. Clearly $\tau_{s|X} = \tau$ and so $\mu X \in \mathcal{M}(X)$. ∎

3. Elementary properties of uX. We begin by noting that the minimum uniform compactification of a metric space is "the same as" the minimum uniform compactification of its metric completion. This means that we can confine our attention to studying minimum uniform compactifications of complete metric spaces.

DEFINITION 3.1. Let (X, d) be a metric space.

(a) If $\varepsilon > 0$ the subset D of X is said to be *ε-discrete* if $S(x, \varepsilon) \cap D = \{x\}$ for each $x \in D$.

(b) The metric completion of (X, d) will be denoted by (X^*, d^*).

THEOREM 3.2. *Let (X, d) be a metric space. Then $u_{d^*}X^*$ is equivalent* (*as a compactification of X*) *to u_dX.*

Proof. This follows immediately from Theorem 2.9. ∎

Part (b) of Theorem 3.3 below is a special case of 2.4 of [C], and essentially also of 3.1 of [M].

THEOREM 3.3. *Let (X, d) be a metric space.*

(a) *If (X, d) is totally bounded then u_dX is the metric completion of X.*

(b) *If (X, d) is not totally bounded then $u_dX \setminus X$ contains a copy of $\beta\omega \setminus \omega$* (*and hence u_dX is not metrizable*).

Proof. (a) As (X, d) is totally bounded, it follows that (X^*, d^*) is compact. Consequently, $u_{d^*}X^* = X^*$, so by Theorem 3.2, $u_dX = X^*$.

(b) As (X, d) is not totally bounded, it contains, for some $\varepsilon > 0$, a countably infinite ε-discrete subset D. If A and B are disjoint subsets of D then $d(A, B) \geq \varepsilon$ and by Theorem 2.5(b) it follows that $\mathrm{cl}_{u_dX} A \cap \mathrm{cl}_{u_dX} B = \emptyset$. Hence (by 6.5 of [GJ], for example) D is C^*-embedded in $\mathrm{cl}_{u_dX} D$ and so $\mathrm{cl}_{u_dX} D \cong \beta D \cong \beta\omega$. As D is closed in X, $\mathrm{cl}_{u_dX} D \setminus D$ is a subset of $u_dX \setminus X$ and is homeomorphic to $\beta\omega \setminus \omega$. As $\beta\omega \setminus \omega$ is not metrizable, $u_dX \setminus X$ cannot be. ∎

An obvious task is to characterize those metric spaces (X, d) for which $uX \cong \beta X$. This is easily accomplished using results in [A] and [Ra].

THEOREM 3.4. *The following are equivalent for a metric space (X, d)*:

(a) $uX \cong \beta X$.

(b) $C^*(X) = U^*(X)$.

(c) $C(X) = U(X)$.

(d) *There is a compact subset K of X such that $X \setminus K$ consists of isolated points of X, and for each $r > 0$ there exists $\varepsilon_r > 0$ such that $\{x \in X : d(x, K) > r\}$ is ε_r-discrete.*

Proof. As βX is characterized up to equivalence as the compactification of X in which X is C^*-embedded (see 6.5 of [GJ]), the equivalence of (a) and (b) follows from Theorem 2.5(c). The equivalence of (c) and (d) is proved in [A]. The equivalence of (b) and (c) is proved in [Ra]. ∎

Recall (see [BSw]) that a Tikhonov space X is called *Oz* if every open subset of X is z-embedded in X (a subset S of X is *z-embedded* in X if each zero-set of S is the intersection with S of a zero-set of X). Considerable attention has been devoted to characterizing those X for which βX is Oz. Corollary 3.5 below, which characterizes when βX is Oz if X is metrizable, is due to the referee.

COROLLARY 3.5. *The following conditions are equivalent for a metrizable space X*:

(a) *There exists a compatible metric d on X such that $\beta X = u_d X$.*
(b) *βX is Oz.*
(c) *The set of nonisolated points of X is compact.*

Proof. (a) implies (b): By Theorem 2.7(a) and problem 3B of [PW], any regular closed subset of $u_d X$ is a zero-set of $u_d X$. By Theorem 5.1 of [Bl], a space Y is Oz iff each regular closed subset of Y is a zero-set of Y. It immediately follows that (a) implies (b).

(b) implies (c): We refer the reader to [BSw] for undefined terminology used herein. By Theorem 6.4 of [BSw], X is extremally pseudocompact. Hence by Theorem 5.2 of [BSw], $X = E \cup F$, where X is an open extremally disconnected subset of X and $f|F$ is bounded for each $f \in C(X)$. But as extremally disconnected metrizable spaces are discrete (see 14N(2) of [GJ], for example), the points of E are isolated in X, so the set K of nonisolated points of X is a subset of F. Consequently, $f|K$ is bounded for each $f \in C(X)$, and as K is C-embedded in X (as K is closed and X is normal), K is pseudocompact. But pseudocompact metrizable spaces are compact; the implication follows.

(c) implies (a): Let σ be a compatible metric on X that is bounded by 1. For each $n \in \mathbb{N}$ let $K_n = \{x \in X : \sigma(x, K) < 1/n\}$. Define $d_n : X \times X \to \mathbb{R}$ as follows:

$$d_n(x,y) = \begin{cases} 1 & \text{if } x \neq y \text{ and } \{x,y\} \setminus K_n \neq \emptyset, \\ 0 & \text{if } x = y, \\ \sigma(x,y) & \text{if } \{x,y\} \subseteq K_n. \end{cases}$$

Let $d(x,y) = \sum_{n=1}^{\infty} 2^{-n} d_n(x,y)$. It is straightforward to verify that d is a compatible metric on X satisfying condition (d) of Theorem 3.4. Consequently, (a) holds by Theorem 3.4. ∎

Let (X,d) and (Y,s) be two metric spaces. One defines two "standard" metrics σ and t on $X \times Y$ by $\sigma((x_1,y_1),(x_2,y_2)) = \sqrt{d(x_1,x_2)^2 + s(y_1,y_2)^2}$ and $t((x_1,y_1)(x_2,y_2)) = d(x_1,x_2) + s(y_1,y_2)$. Since $(a+b)/\sqrt{2} \leq \sqrt{a^2+b^2} \leq a+b$ if $a \geq 0$ and $b \geq 0$, it follows that σ and t are uniformly equivalent metrics on $X \times Y$ and so $u_\sigma(X \times Y) \cong u_t(X \times Y)$ (see the discussion preceding Theorem 2.10, and also Theorem 2.10 itself). If $\mathbf{C}$ is the category of metric spaces and uniformly continuous mappings, then uniform isomorphisms are isomorphisms in the category-theoretic sense and $(X \times Y, t)$ is easily checked to be the category-theoretic product (in $\mathbf{C}$) of (X,d) and (Y,s) (see, for example, Chapter 10 of [Wa] or Chapter 9 of [PW]). So t (or equivalently σ) is the "correct" metric to put on $X \times Y$ to form the product of the spaces (X, d) and (Y, s); in what follows we will use t for the ease of computation that it affords.

If (X, d) and (Y, s) are two metric spaces then $u_d X \times u_s Y$ and $u_t(X \times Y)$ are both compactifications of the space $X \times Y$ (observe that the product topology induced on $X \times Y$ by τ_d and τ_s is just τ_t, so there is no ambiguity about what topology $X \times Y$ is to carry). An obvious question is to determine under what conditions these compactifications of $X \times Y$ are equivalent. Recall (see 4AG of [PW] or 8.12 of [Wa]) that "Glicksberg's theorem" answers the corresponding question for the Stone–Čech compactification: $\beta(X \times Y) \cong (\beta X) \times (\beta Y)$ iff $X \times Y$ is pseudocompact. Also note that the completion $((X \times Y)^*, t^*)$ of $(X \times Y, t)$ is uniformly equivalent to the product $(X^* \times Y^*, \widehat{t}\,)$ of the completions (X^*, d^*) and (Y^*, s^*), where $\widehat{t}$ is defined from d^* and s^* in the same way in which t was defined from d and s; the map f that takes the equivalence class $[(x_n, y_n)_{n \in \mathbb{N}}]$ (where $(x_n)_{n \in \mathbb{N}}$ and $(y_n)_{n \in \mathbb{N}}$ are Cauchy in (X, d) and (Y, s) respectively) to $([(x_n)_{n \in \mathbb{N}}], [(y_n)_{n \in \mathbb{N}}])$, and is the identity on $X \times Y$, is a uniform isomorphism from $((X \times Y)^*, t^*)$ onto $(X^* \times Y^*, \widehat{t}\,)$.

I have been informed by Professor M. Hušek that a version of the following theorem (couched in the language of uniformities) may be found in [Če]. I have been unable to locate it, and hence include the proof below for completeness.

Theorem 3.6. *The following are equivalent for two metric spaces* (X, d) *and* (Y, s):

(a) $u_t(X \times Y) \cong u_d X \times u_s Y$ (*where t is as described above*).
(b) *At least one of* (X, d) *and* (Y, s) *is totally bounded.*

Proof. Suppose that (b) fails. Then neither (X, d) nor (Y, s) is totally bounded so there exist positive numbers δ_1 and δ_2 such that (X, d) has an infinite δ_1-discrete set D_1 and (Y, s) has an infinite δ_2-discrete subset D_2. Let $\delta = \min\{\delta_1, \delta_2\}$; then $D_1 \times D_2$ is an infinite δ-discrete subset of $(X \times Y, t)$. By the proof of Theorem 3.3(b) it follows that $\mathrm{cl}_{u_d X} D_1 \cong \beta D_1$ and $\mathrm{cl}_{u_s Y} D_2 \cong \beta D_2$; thus $\mathrm{cl}_{u_d X \times u_s Y} D = \mathrm{cl}_{u_d X} D_1 \times \mathrm{cl}_{u_s Y} D_2 \cong \beta D_1 \times \beta D_2$. However, $\mathrm{cl}_{u_t(X \times Y)} D \cong \beta D = \beta(D_1 \times D_2)$ (also by the proof of 3.3(b)). Thus if $u_d X \times u_s Y \cong u_t(X \times Y)$ it would follow that $\beta D_1 \times \beta D_2 \cong \beta(D_1 \times D_2)$ (as compactifications of $D_1 \times D_2$), which would contradict Glicksberg's theorem as $D_1 \times D_2$ is not pseudocompact. Hence (a) fails; thus (a) implies (b).

To show that (b) implies (a) let us first assume that (a) holds whenever the totally bounded factor (say (Y, s)) is compact. Then if (b) holds and (Y, s) is totally bounded, we see that $u_t(X \times Y) = u_{t^*}((X \times Y)^*)$ (by Theorem 3.2). But $((X \times Y)^*, t^*)$ is uniformly equivalent to $(X^* \times Y^*, \widehat{t}\,)$

as noted in the remarks preceding this theorem, and so $u_{t^*}((X^* \times Y)^*)$ and $u_{\hat{t}}(X^* \times Y^*)$ are equivalent compactifications of $X \times Y$ (here, of course, X^* and Y^* carry the metrics d^* and s^* respectively.) But (Y, s) is totally bounded by hypothesis, so its completion (Y^*, s^*) is compact; since we are assuming that (a) holds when dealing with a product whose one factor is compact, we know that $u_{\hat{t}}(X^* \times Y^*) \cong u_{d^*}X^* \times u_{s^*}Y^*$. Thus $u_t(X \times Y) = u_{t^*}((X \times Y)^*) \cong u_{\hat{t}}(X^* \times Y^*) \cong u_{d^*}X^* \times u_{s^*}Y^*$. But $u_{d^*}X^* \cong u_d X$ and $u_{s^*}Y^* \cong u_s Y$ by Theorem 3.2, so we conclude that $u_t(X \times Y) \cong u_d X \times u_s Y$. So it suffices to prove that (b) implies (a) in the special case where (Y, s) is compact.

So assume that (Y, s) is compact, and let A and B be subsets of $X \times Y$ such that $t(A, B) = 0$. We will show that $\mathrm{cl}_{uX \times Y} A \cap \mathrm{cl}_{uX \times Y} B \neq \emptyset$ (of course $u_d X \times u_s Y = u_d X \times Y$ since (Y, s) is compact). For each $n \in \mathbb{N}$ there exist $a_n \in A$ and $b_n \in B$ such that $t(a_n, b_n) < 1/n$. Consider $\{p_Y(a_n) : n \in \mathbb{N}\}$, where p_Y is the projection map from $X \times Y$ onto Y. Since Y is compact, there is an infinite subset I of N and a point of $q \in Y$ such that $\{p_Y(a_n) : n \in I\}$ converges to q. Now consider $\{p_X(a_n) : n \in I\}$. If this is finite there exists an infinite subset J of I and there exists $p \in X$ such that $p_X(a_n) = p$ for each $n \in J$. Then for each $n \in J$ we have

$$\begin{aligned} t(a_n, (p, q)) &= t((p_X(a_n), p_Y(a_n)), (p, q)) \\ &= d(p_X(a_n), p) + s(p_Y(a_n), q) = s(p_Y(a_n), q); \end{aligned}$$

as $n \in J$ becomes large, this approaches zero. Hence $(p, q) \in \mathrm{cl}_{uX \times Y} A$. As $t(a_n, b_n) < 1/n$, it follows that $t(b_n, (p, q)) < s(p_Y(a_n), q) + 1/n$ for each $n \in J$ and similarly we conclude that $(p, q) \in \mathrm{cl}_{uX \times Y} A \cap \mathrm{cl}_{uX \times Y} B$.

If $\{p_X(a_n) : n \in I\}$ (henceforth denoted by S) is infinite, denote $\{p_X(b_n) : n \in I\}$ by T. As $d(p_X(a_n), p_X(b_n)) \leq t(a_n, b_n) < 1/n$ for each $n \in I$ it follows that $d(S, T) = 0$ and so there exists $p \in \mathrm{cl}_{uX} S \cap \mathrm{cl}_{uX} T$. We now claim that $(p, q) \in \mathrm{cl}_{uX \times Y} A \cap \mathrm{cl}_{uX \times Y} B$. Let $(p, q) \in V \times W$ where V is open in uX and W is open in Y. As $\lim\{p_Y(a_n) : n \in I\} = q$ and $s(p_Y(a_n), p_Y(b_n)) \leq t(a_n, b_n) < 1/n$, we see that $p_Y(a_n) \in W$ and $p_Y(b_n) \in W$ for all but finitely many $n \in I$. As $p \in V \cap \mathrm{cl}_{uX} S$ we see that $p_X(a_n) \in V$ for infinitely many $n \in I$. Hence there exists $n \in I$ such that $a_n \in V \times W$ and so $(p, q) \in \mathrm{cl}_{uX \times Y} A$. As $p \in V \cap \mathrm{cl}_{uX} T$ a similar argument shows that $(p, q) \in \mathrm{cl}_{uX \times Y} B$. Thus we have shown that if $t(A, B) = 0$ then $\mathrm{cl}_{uX \times Y} A \cap \mathrm{cl}_{uX \times Y} B \neq \emptyset$.

Next we claim that if $f \in U_t^*(X \times Y)$ then f extends continuously to $uX \times Y$. For each $q \in Y$ we define $f_q : X \to \mathbb{R}$ by $f_q(x) = f(x, q)$. Then $f_q \in U_d^*(X)$ and so f_q extends to $uf_q \in C(u_d X)$. Define $f^* : uX \times Y \to \mathbb{R}$ by $f^*(x, q) = uf_q(x)$ for each $(x, q) \in uX \times Y$. Clearly $f^*|X \times Y = f$ so it remains to show that f^* is continuous. To do this it suffices to show that if $a \in uX \setminus X$ and $q \in Y$ then $f^*|(X \times Y) \cup \{(a, q)\}$ is continuous (see 6H of [GJ], for example). Let $\varepsilon > 0$ be given. We must find V open in uX and W open in Y for which $(a,q) \in V \times W$ and $f^*[(V \times W) \cap (X \times Y)] \subseteq (f^*(a,q) - \varepsilon, f^*(a,q) + \varepsilon)$. As uf_q is continuous there exists an open subset V of uX such that $a \in V$ and $uf_q[V] \subseteq (uf_q(a) - \varepsilon/4, uf_q(a) + \varepsilon/4)$. Thus, if $x \in V$ then

$$(*) \qquad f^*(x,q) \in \left(f^*(a,q) - \frac{\varepsilon}{4}, f^*(a,q) + \frac{\varepsilon}{4}\right).$$

As f is uniformly continuous there exists $\delta > 0$ such that if (x,y) and (s,w) are in $X \times Y$ and $t((x,y),(v,w)) < \delta$ then $|f(x,y) - f(v,w)| < \varepsilon/4$. So, let $W = S_s(q,\delta)$. Then if $(x,y) \in (V \cap X) \times W$ then $|f(x,y) - f(x,q)| < \varepsilon/4$. Combine this with $(*)$ and conclude that $f^*[(V \times W) \cap (X \times Y)] \subseteq (f^*(a,q) - \varepsilon, f^*(a,q) + \varepsilon)$. Thus f^* is continuous as claimed. Hence by Corollary 2.4 it follows that $u(X \times Y) \cong uX \times uY$ when Y is compact; as noted above, the theorem follows from this. ■

4. The minimum uniform compactification of a locally compact σ-compact metric space. In this section we investigate the structure of $u_d X \setminus X$ in the case where (X,d) is a locally compact σ-compact noncompact metric space. It is well known (see 11.7.2 of [D], for example) that if X is a locally compact σ-compact noncompact Hausdorff space then there exists a sequence $\{K(n) : n \in \mathbb{N}\}$ of nonempty compact subsets of X such that $K(n)$ is a proper subset of $\operatorname{int} K(n+1)$, $K(n) = \operatorname{cl} \operatorname{int} K(n)$, and $X = \bigcup\{K(n) : n \in \mathbb{N}\}$. Observe that this means that a closed subset of X is compact iff it is a subset of some $K(n)$. We will use the following notation.

NOTATION 4.1. If A is a subspace of the metric space (X,d) we will denote $\operatorname{cl}_{\beta X} A \setminus X$ (resp. $\operatorname{cl}_{u_d X} A \setminus X$) by A^* (resp. A^u). Clearly $(\operatorname{cl}_X A)^* = A^*$ and $(\operatorname{cl}_X A)^u = A^u$ if $A \subseteq X$.

THEOREM 4.2. *Let (X,d) be a locally compact σ-compact metric space, and let A and B be two closed noncompact subsets of X. The following are equivalent*:

(a) $\lim_{n\to\infty} \sup\{d(x,A) : x \in B \setminus K(n)\} = 0$,
(b) $B^u \subseteq A^u$.

Proof. (a) implies (b): Suppose (b) fails, and let $p \in B^u \setminus A^u$. By Theorem 2.5 and Corollary 2.6 there exists a closed subset F of X such that $p \in \operatorname{cl}_{uX} F$ and $\operatorname{cl}_{uX} A \cap \operatorname{cl}_{uX} F = \emptyset$. Hence there exists $r > 0$ such that $d(A,F) > r$ by our choice of F (see Theorem 2.5). Clearly $p \in (B \setminus K(n))^u \cap (F \setminus K(n))^u$ for each $n \in \mathbb{N}$, so it follows by Theorem 1.1(a) that $d(B \setminus K(n), F \setminus K(n)) = 0$ for each $n \in \mathbb{N}$. Thus for each $n \in \mathbb{N}$ there exist $x(n) \in B \setminus K(n)$ and $y(n) \in F \setminus K(n)$ for which $d(x(n),y(n)) < r/2$. Hence $d(x(n),A) \geq r/2$ by our choice of r. Consequently, (a) fails.

(b) implies (a): Suppose (a) fails. Then there exists some $r > 0$ such that for each $n \in \mathbb{N}$ there exists an $x(n) \in B \setminus K(n)$ for which $d(x(n), A) \geq r$. Let $L = \{x(n) : n \in \mathbb{N}\}$. It is not hard to see that L is a closed discrete noncompact subset of X (see the comments preceding the statement of this theorem) and that $d(L, A) \geq r$. Thus $L^u \cap A^u = \emptyset$ by Theorem 1.1, and $\emptyset \neq L^u \subseteq B^u$. Thus (b) fails. ∎

Recall (see 6.5(a) of [PW], for example) that a continuous closed surjection $f : Y \to Z$ is called *irreducible* if proper closed subsets of Y are taken to proper subsets of Z by f. If there is an irreducible continuous surjection from one compact space onto another then those spaces share many topological properties (see 6.5(d) and 6B of [PW]); consequently, it is of interest to know under what conditions a continuous surjection with compact domain will be irreducible.

COROLLARY 4.3. *Let (X, d) be locally compact and σ-compact. Let $f : \beta X \to uX$ extend the identity map and let $g = f|\beta X \setminus X$. The following are equivalent*:

(a) *g is not an irreducible map from $\beta X \setminus X$ onto $uX \setminus X$.*

(b) *There is an open subset V of X whose X-closure is noncompact and for which* $\lim_{n\to\infty} \sup\{d(x, X \setminus V) : x \in X \setminus K(n)\} = 0$.

Proof. (a) implies (b): There is a proper closed subset H of $\beta X \setminus X$ for which $g[H] = uX \setminus X$. By 6.5(b) of [GJ] there is a closed subset A of X for which $A^* \neq \beta X \setminus X$ but $g[A^*] = uX \setminus X$. Let $V = X \setminus A$. As $A^* \neq \beta X \setminus X$ there is a noncompact closed subset of X disjoint from A, and so the X-closure of V is noncompact. Clearly $g[A^*] = A^u$ and so $X^u \subseteq A^u$. The result now follows from Theorem 4.2.

(b) implies (a): Let $A = X \setminus V$. By hypothesis $\mathrm{cl}_X(X \setminus A)$ is not compact and hence not pseudocompact. It follows from 2.5 of [Wo2] that $A^* \neq X^*$. As above, $g[A^*] = A^u$, and by (b) and Theorem 4.2 it follows that $A^u = X^u = uX \setminus X$. ∎

EXAMPLE 4.4. (a) If $X = \mathbb{R}^n$ (Euclidean n-space) then $g : \beta\mathbb{R}^n \setminus \mathbb{R}^n \to u\mathbb{R}^n \setminus \mathbb{R}^n$ is not irreducible as we can let $K(j) = \{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x}\| \leq j\}$ and $V = \bigcup\{\{\mathbf{x} \in \mathbb{R}^n : k < \|\mathbf{x}\| < k + 1/k\} : k \in \mathbb{N}\}$.

(b) If X is ω with the discrete metric (distinct points are a distance 1 apart) then $uX = \beta X$ and so g is the identity and hence irreducible.

THEOREM 4.5. *Let (X, d) be a locally compact σ-compact complete space. Then every nonempty G_δ-subset of $uX \setminus X$ contains a copy of $\beta\omega \setminus \omega$.*

Proof. Let G be a nonempty G_δ-set of $uX \setminus X$. As X is σ-compact, G is a G_s-set of uX. Hence there exists $f \in C(uX)$ such that $\emptyset \neq Z(f) \subseteq G$. Consequently, one can inductively choose a sequence $(x_n)_{n\in\mathbb{N}}$ of points of

X, and a subsequence $\{m_n : n \in \mathbb{N}\}$ of $\mathbb{N}$ such that $n < j$ implies $m_n < m_j$, such that $x_n \in f^{\leftarrow}[(1/m_{n+1}, 1/m_n)] \setminus K(n)$. Then $\mathrm{cl}_X\{x_n : n \in \mathbb{N}\} = L$ is not compact (see the remarks preceding Theorem 4.2), but is complete (as X is), and hence is not totally bounded. Hence there exists $\varepsilon > 0$ and an infinite ε-discrete subset D of L. Clearly $\mathrm{cl}_{uX} D \setminus X \subseteq \mathrm{cl}_{uX} L \setminus X \subseteq Z(f) \subseteq G$. It follows from the proof of Theorem 3.3(b) that $\mathrm{cl}_{uX} D \setminus X$ is homeomorphic to $\beta\omega \setminus \omega$. The result follows. ■

Next we show that if (X, d) is a locally compact σ-compact noncompact metric space, we can find a discrete metric space (X, σ) for which $u_\sigma Y \setminus Y$ is "the same" as $u_d X \setminus X$. Specifically:

THEOREM 4.6. *Let (X, d) be a locally compact σ-compact noncompact metric space. Then there is a countable set D and a metric σ on D such that τ_σ is the discrete topology and $u_d X \setminus X$ is homeomorphic to $u_\sigma D \setminus D$.*

Proof. Clearly $\{K(n+1) \setminus \mathrm{int}\, K(n), d|K(n+1) \setminus \mathrm{int}\, K(n)\}$ is a compact metric space for each $n \in \mathbb{N}$ (here $K(n)$ is as defined in the paragraph preceding Notation 4.1). Consequently, it has a finite $(1/n)$-net $D(n)$ (since it is totally bounded). Now let $D = \bigcup\{D(n) : n \in \mathbb{N}\}$ and let $\sigma = d|D$. By Theorem 2.9, $u_\sigma D = \mathrm{cl}_{u_d X} D$. Let $\varepsilon > 0$ and choose $n_\varepsilon \in \mathbb{N}$ so that $n_\varepsilon \geq 2$ and $1/n_\varepsilon < \varepsilon$. Observe that $X \setminus K(n_\varepsilon) \subseteq \bigcup\{K(n+1) \setminus \mathrm{int}\, K(n) : n > n_\varepsilon\}$; hence if $x \in X \setminus K(n_\varepsilon)$ there exists $k > n_\varepsilon$ such that $x \in K(k+1) \setminus \mathrm{int}\, K(k)$. There exists $y \in D(k)$ such that $d(x, y) < 1/k$; hence $d(x, D) < \varepsilon$. It follows that $\lim_{n\to\infty} \sup\{d(x, D) : x \in X \setminus K(n)\} = 0$ and hence by Theorem 4.2 that $u_\sigma D \setminus D = \mathrm{cl}_{u_d X} D \setminus D \supseteq u_d X \setminus X$. If $z \in X$ find $n_z \in X$ such that $z \in \mathrm{int}\, K(n_z)$; then $\bigcup\{D(n) : n \leq n_z\} = F$ is a finite set and so $(\mathrm{int}\, K(n_z)) \setminus (F \setminus \{z\})$ is a neighborhood of z disjoint from D. Consequently, D is a closed discrete subset of (X, τ_d); it follows that τ_σ is the discrete topology and that $\mathrm{cl}_{u_d X} D \setminus X \subseteq u_d X \setminus X$. Hence $u_\sigma D \setminus D = \mathrm{cl}_{u_d X} D \setminus D = u_d X \setminus X$ and the theorem follows. ■

Observe that this means that if (X, d) is a locally compact σ-compact noncompact metric space without isolated points then $u_d X$ has no "remote points"; in other words, each point of $u_d X \setminus X$ is in the $u_d X$-closure of a closed nowhere dense subset of (X, d). (By contrast, it is known (see [vD]) that $\beta X \setminus X$ has a dense subset of 2^c remote points).

Now we investigate the structure of $u_d \mathbb{R}^n \setminus \mathbb{R}^n$, where $(\mathbb{R}^n, d)$ is the usual Euclidean space. We begin with a lemma.

LEMMA 4.7. *Let (K, d) be a compact metric space, let $\mathbb{Z}$ denote the set of all integers and let s be the metric on $\mathbb{Z}^n$ given by*

$$s((i_1, \ldots, i_n), (j_1, \ldots, j_n)) = \sum_{k=1}^{n} |i_k - j_k|.$$

If t is the metric on $K \times \mathbb{Z}^n$ defined by

$$t((k, i_1, \ldots, i_n), (x, j_1, \ldots, j_n)) = d(k, x) + \sum_{k=1}^{n} |i_k - j_k|$$

then $u_t(K \times \mathbb{Z}^n) \cong K \times \beta(\mathbb{Z}^n)$, which is homeomorphic to $K \times \beta\omega$.

Proof. By Theorem 3.4, $u_s(\mathbb{Z}^n) \cong \beta(\mathbb{Z}^n)$ which is homeomorphic to $\beta\omega$. Hence by Theorem 3.5, $u_t(K \times \mathbb{Z}^n) \cong K \times u_s\mathbb{Z}^n$, which is homeomorphic to $K \times \beta\omega$. ∎

Theorem 4.9 below is one of the principal results of this paper. Its proof will be by induction, so we begin by proving the special case of Theorem 4.9 in which $n = 1$ (see Theorem 4.8 below). As the proof of Theorem 4.9 is motivated by geometric considerations, our ability to visualize $\mathbb{R}$ and its subsets will assist in an understanding of the general situation. Throughout what follows, I will denote the closed unit interval and d will denote both the Euclidean metric on $\mathbb{R}$ and its restriction to I.

THEOREM 4.8. *The space $u_d\mathbb{R} \setminus \mathbb{R}$ can be written as a union of two copies of $I \times (\beta\omega \setminus \omega)$; each is a regular closed subset of $u_d\mathbb{R} \setminus \mathbb{R}$, and their intersection is a nowhere dense copy of $\beta\omega \setminus \omega$.*

Proof. Let (K, d) be (I, d) and define t as in Lemma 4.7 (with $n = 1$); thus $t((k, i), (x, j)) = |k - x| + |i - j|$. Define a function $f : I \times \mathbb{Z} \to \mathbb{R}$ by

$$f(r, i) = r + i.$$

Clearly f is a well-defined surjection. Let $\varepsilon > 0$ and set $\delta = \min\{1/2, \varepsilon\}$. If $t((k, i), (x, j)) < \delta$ then $i = j$ and $d(f(k, i), f(x, j)) = |(k + i) - (x + i)| = |k - x| < \varepsilon$. Hence f is uniformly continuous and hence by Theorem 2.10 it has a continuous extension $F : u_t(I \times \mathbb{Z}) \to u_d\mathbb{R}$.

Let $A = I \times \{2j : j \in \mathbb{Z}\}$ and $B = I \times \{2j + 1 : j \in \mathbb{Z}\}$. Clearly $I \times \mathbb{Z} = A \cup B$. Observe that

$$f[A] = \{r + 2j : j \in \mathbb{Z} \text{ and } r \in [0, 1]\} = \bigcup\{[2j, 2j + 1] : j \in \mathbb{Z}\}.$$

Evidently $f|A : A \to f[A]$ is a bijection, and is uniformly continuous as f is. Now $(f|A)^{-1} : (f[A], d|f[A]) \to (A, t|A)$ is also uniformly continuous; for if $\varepsilon > 0$ is given, let $\delta = \min\{\varepsilon, 1/2\}$. If $i, j \in \mathbb{Z}$ and $r, x \in I$, suppose $d(r + 2i, x + 2j) < \delta$; then $i = j$ and $t((f|A)^{\leftarrow}(r + 2i), (f|A)^{\leftarrow}(s + 2j)) = t((r, 2i), (x, 2i)) = |r - x| = d(r + 2i, x + 2i) < \varepsilon$. Thus $(f|A)^{\leftarrow}$ is uniformly continuous and hence $f|A$ is a uniform isomorphism from $(A, t|A)$ onto $(f[A], d|f[A])$. A similar proof shows that $f|B$ is a uniform isomorphism from B onto $\bigcup\{[2j - 1, 2j] : j \in \mathbb{Z}\}$ (equipped with the subspace metric inherited from d).

It now follows from Theorems 2.9 and 2.10 that the restrictions $F|\mathrm{cl}_{u_t(I\times\mathbb{Z})} A$ and $F|\mathrm{cl}_{u_t(I\times\mathbb{Z})} B$ are respectively homeomorphisms from

$\mathrm{cl}_{u_t(I\times\mathbb{Z})} A$ and $\mathrm{cl}_{u_t(I\times\mathbb{Z})} B$ onto $\mathrm{cl}_{u_d\mathbb{R}} f[A]$ and $\mathrm{cl}_{u_d\mathbb{R}} f[B]$. But since $\mathbb{Z}$ is 1-discrete, it follows from Theorems 3.4, 3.6 and 2.9 that $\mathrm{cl}_{u_t(I\times\mathbb{Z})}(I \times \{s \in \mathbb{Z} : s \text{ is even}\}) = \mathrm{cl}_{u_t(I\times\mathbb{Z})} A$ and $\mathrm{cl}_{u_t(I\times\mathbb{Z})}(I \times \{s \in \mathbb{Z} : s \text{ is odd}\}) = \mathrm{cl}_{u_t(I\times\mathbb{Z})} B$ are both homeomorphic to $I \times \beta\omega$. From this it readily follows that $(\mathrm{cl}_{u_t(I\times\mathbb{Z})} A) \setminus A$ and $(\mathrm{cl}_{u_t(I\times\mathbb{Z})} B) \setminus B$ are each homeomorphic to $I \times (\beta\omega \setminus \omega)$. Consequently, $f[A]^u$ and $f[B]^u$ are both homeomorphic to $I \times (\beta\omega \setminus \omega)$. Hence $u_d\mathbb{R} \setminus \mathbb{R}$ can be written as the union of these two copies of $I \times (\beta\omega \setminus \omega)$, since $\mathbb{R} = f[A] \cup f[B]$.

Next we show that $f[A]^u$ is a regular closed subset of $u\mathbb{R} \setminus \mathbb{R}$. To do this, it suffices to show that if $L \subseteq \mathbb{R}$ and $p \in f[A]^u \setminus L^u$ then $((u\mathbb{R} \setminus \mathbb{R}) \setminus L^u) \cap ((u\mathbb{R} \setminus \mathbb{R}) \setminus f[B]^u) \neq \emptyset$, since clearly $(u\mathbb{R} \setminus \mathbb{R}) \setminus f[B]^u \subseteq \mathrm{int}_{u\mathbb{R}\setminus\mathbb{R}} f[A]^u$. In other words, by Theorem 4.2 it suffices to show that

$$(*) \qquad \text{if } p \in f[A]^u \setminus L^u \text{ then } \lim_{n\to\infty} \sup\{d(x, L \cup f[B]) : |x| > n\} \neq 0.$$

Suppose that $(*)$ fails; we will derive a contradiction. Since $p \notin L^u$, by Corollary 2.6 there exist $G \subseteq \mathbb{R}$ and $\delta \in (0, 1/4)$ such that $p \in G^u$ and $d(G, L) \geq \delta$. Since $(*)$ fails, there exists $n(\delta) \in \mathbb{N}$ such that if $|x| > n(\delta)$ then $d(x, L \cup f[B]) < \delta/4$. Consequently, we would have

$$(**) \qquad \bigcup\left\{\left[2n + \frac{\delta}{4}, 2n + 1 - \frac{\delta}{4}\right] : n \geq n(\delta)\right\} \subseteq \left\{x : d(x, L) < \frac{\delta}{4}\right\}.$$

Since $d(G, L) \geq \delta$ it would follow that

$$(***) \qquad G \setminus [-n(\delta), n(\delta)] \subseteq \bigcup\left\{\left[2n - 1 + \frac{3\delta}{4}, 2n - \frac{3\delta}{4}\right] : n \in \mathbb{Z}\right\}.$$

However, as $p \in u\mathbb{R} \setminus \mathbb{R}$ and $p \in G^u \cap f[A]^u$, it is clear that

$$p \in [G \setminus (-n(\delta), n(\delta))]^u \cap [f[A] \setminus (-n(\delta), n(\delta))]^u,$$

and so

$$d(G \setminus (-n(\delta), n(\delta)),\ f[A] \setminus (-n(\delta), n(\delta))) = 0.$$

But it follows from $(***)$ that $d(G \setminus (-n(\delta), n(\delta)), f[A] \setminus (-n(\delta), n(\delta))) \geq 3\delta/4$, which is a contradiction. This shows that $(*)$ holds, and so $f[A]^u$ is a regular closed subset of $u\mathbb{R} \setminus \mathbb{R}$. Clearly $f[B]^u$ is also a regular closed subset of $u\mathbb{R} \setminus \mathbb{R}$.

Finally, note that as $\mathbb{Z} = f[A] \cap f[B]$, it immediately follows that $\mathbb{Z}^u \subseteq f[A]^u \cap f[B]^u$. Conversely, if $p \in (u\mathbb{R} \setminus \mathbb{R}) \setminus \mathbb{Z}^u$, by Corollary 2.6 there exists $D \subseteq \mathbb{R}$ such that $p \in D^u$ and $d(D, \mathbb{Z}) = r > 0$.

Let $C = \bigcup\{[n - r/2, n + r/2] : n \in \mathbb{Z}\}$. Clearly $d(D, C) = r/2$ so by Theorem 2.5, $p \notin C^u$. Thus $p \in f[B]^u \cap f[A]^u$ would imply that $p \in (f[B]\setminus C)^u \cap (f[A]\setminus C)^u$. But $d(f[B]\setminus C, f[A]\setminus C) = r$, which by Theorem 2.5 is a contradiction. It follows that $\mathbb{Z}^u = f[B]^u \cap f[A]^u$. By Theorems 2.9 and 3.4 we see that $\mathbb{Z}^u \cong \beta\omega \setminus \omega$, and by Theorem 4.2, $\mathrm{int}_{u\mathbb{R}\setminus\mathbb{R}} \mathbb{Z}^u = \emptyset$. Thus

$f[B]^u = \mathrm{cl}_{u\mathbb{R}\setminus\mathbb{R}}((u\mathbb{R} \setminus \mathbb{R}) \setminus f[A]^u)$ and the common boundary of $f[B]$ and $f[A]$ is homeomorphic to $\beta\omega \setminus \omega$. ∎

Observe that not every subset of $u\mathbb{R} \setminus \mathbb{R}$ of the form A^u, where A is a regular closed subset of $\mathbb{R}$, is a regular closed subset of $u\mathbb{R} \setminus \mathbb{R}$. For example, let $A = \bigcup\{[n, n+1/(2n)] : n \in \mathbb{N}\}$. Clearly $\mathbb{N}^u \subseteq A^u$ and it follows quickly from Theorem 4.2 that $A^u \subseteq \mathbb{N}^u$; consequently, $A^u = \mathbb{N}^u$ (and $\mathbb{N}^u \cong \beta\omega \setminus \omega$ by Theorems 2.9 and 3.4). But $(\mathbb{R} \setminus \mathbb{N})^u = u\mathbb{R} \setminus \mathbb{R}$ by Theorem 4.2, and so $\mathrm{int}_{u\mathbb{R}\setminus\mathbb{R}} A^u = \mathrm{int}_{u\mathbb{R}\setminus\mathbb{R}} \mathbb{N}^u = \emptyset$. Thus A^u is nowhere dense in $u\mathbb{R} \setminus \mathbb{R}$, and hence not regular closed.

We now prove the "general case" (Theorem 4.9) below. The reader is advised to "draw pictures" for the case $n = 2$ to aid intuitive understanding.

THEOREM 4.9. *Let $n \in \mathbb{N}$, and let d denote both the Euclidean metric on $\mathbb{R}^n$ and its restriction to I^n. Then*:

(a) *$\mathbb{R}^n$ can be written as a union of 2^n regular closed subsets, each of which* (*with the subspace metric induced by d*) *is uniformly isomorphic to* $(I^n \times \mathbb{Z}^n, t)$ (*where t is the metric described in Lemma* 4.7) *and any two of which intersect in a nowhere dense subset of $\mathbb{R}^n$.*

(b) *$u_d\mathbb{R}^n \setminus \mathbb{R}^n$ can be written as the union of 2^n copies of $I^n \times (\beta\omega \setminus \omega)$; each copy is a regular closed subset of $u_d\mathbb{R}^n \setminus \mathbb{R}^n$, and the intersection of any two copies is a nowhere dense subset of $u_d\mathbb{R}^n \setminus \mathbb{R}^n$.*

Proof. By Theorem 4.8 the result holds when $n = 1$. Assume inductively that it holds when $n = k$; we will prove that it holds when $n = k+1$. So, let $\mathbb{R}^k = \bigcup\{A_i : 1 \le i \le 2^k\}$, where each A_i is a regular closed subset of $\mathbb{R}^k$, $i \ne j$ implies $\mathrm{int}_{\mathbb{R}^k}(A_i \cap A_j) = \emptyset$, and $(A_i, d|A_i)$ is uniformly isomorphic to $(I^k \times \mathbb{Z}^k, t)$ (as described in Lemma 4.7). Let $C = \bigcup\{[2j, 2j+1] : j \in \mathbb{Z}\}$ and $E = \bigcup\{[2j-1, 2j] : j \in \mathbb{Z}\}$. Then $\mathbb{R} = C \cup E$, and C and E are regular closed subsets of $\mathbb{R}$ with $\mathrm{int}_{\mathbb{R}}(C \cap E) = \emptyset$. Also, $(C, \sigma|C)$ and $(E, \sigma|E)$ are uniformly isomorphic to $I \times \mathbb{Z}$ (by Theorem 4.8); here σ denotes the Euclidean metric on $\mathbb{R}$.

Then $\mathbb{R}^{k+1} = \bigcup\{G_j : 1 \le j \le 2^{k+1}\}$, where $G_j = C \times A_j$ and $G_{2^k+j} = E \times A_j$ if $1 \le j \le 2^k$. It is routine to verify that each G_j is a regular closed subset of $\mathbb{R}^{k+1}$ and that $\mathrm{int}_{\mathbb{R}^{k+1}}(G_i \cap G_j) = \emptyset$ if $i \ne j$. As products of uniform isomorphisms are uniform isomorphisms and as the subspace metric m induced on G_j from $\mathbb{R}^{k+1}$ is uniformly equivalent to the product metric induced on G_j by $\sigma|C$ and $d|A_j$ or $\sigma|E$ and $d|A_j$ (as the case may be), it is easily seen that (G_j, m) is uniformly equivalent to $(C, \sigma|C) \times (A_j, d|A_j)$ and hence to $(I \times \mathbb{Z}) \times (I^k \times \mathbb{Z}^k) = (I^{k+1} \times \mathbb{Z}^{k+1}, t)$ where t is the metric described in Lemma 4.7. Hence by Theorems 2.9, 3.4 and 3.6, and the fact that $\mathbb{Z}^{k+1}$ is 1-discrete (see 4.7 for details), it follows that $\mathrm{cl}_{u_d\mathbb{R}^{k+1}} G_j$ is homeomorphic to $I^{k+1}\times\beta\omega$ so $(\mathrm{cl}_{u_d\mathbb{R}^{k+1}} G_j)\setminus G_j$ is homeomorphic to $I^{k+1}\times(\beta\omega\setminus\omega)$. Thus $\mathrm{cl}_{u_d\mathbb{R}^{k+1}} G_j\setminus\mathbb{R}^{k+1}$ is also homeomorphic to $I^{k+1}\times(\beta\omega\setminus\omega)$.

As $\mathbb{R}^{k+1}=\bigcup\{G_j:1\le j\le 2^{k+1}\}$ it follows that $u_d\mathbb{R}^{k+1}\setminus\mathbb{R}^{k+1}$ can be written as the union of 2^{k+1} copies of $I^{k+1}\times(\beta\omega\setminus\omega)$. A proof similar to that used towards the end of the proof of Theorem 4.8 can be applied to show that each of these copies is a regular closed subset of $u\mathbb{R}^{k+1}\setminus\mathbb{R}^{k+1}$, and that the intersection of two distinct copies is a nowhere dense subset of $u\mathbb{R}^{k+1}\setminus\mathbb{R}^{k+1}$. We omit the tedious details; the proof of Theorem 4.8 will serve as a guide to those who wish to construct them. The inductive step of the proof is now completed; it follows that the theorem holds for all $n\in\mathbb{N}$. ∎

We now turn to a more detailed examination of the structure of $u\mathbb{R}\setminus\mathbb{R}$. In what follows $\mathbb{R}$ is given the Euclidean metric and its subspaces are given the restriction of that metric. It is known that $\beta\mathbb{R}\setminus\mathbb{R}$ has two connected components, namely $\mathrm{cl}_{\beta\mathbb{R}}[0,\infty)\setminus\mathbb{R}$ and $\mathrm{cl}_{\beta\mathbb{R}}(-\infty,0]\setminus\mathbb{R}$ (see 6.10 of [GJ]). Since $\mathbb{R}=(-\infty,-1]\cup[-1,1]\cup[1,\infty)$ and $d((-\infty,-1],[1,\infty))=2$, it follows that $u\mathbb{R}\setminus\mathbb{R}=(-\infty,-1]^u\cup[1,\infty)^u$ and $(-\infty,-1]^u\cap[1,\infty)^u=\emptyset$. Since $[1,\infty)^u$ is a continuous image of $\mathrm{cl}_{\beta\mathbb{R}}[1,\infty)\setminus[0,\infty)=\mathrm{cl}_{\beta\mathbb{R}}[0,\infty)\setminus\mathbb{R}$, it is a continuum. Thus $u\mathbb{R}\setminus\mathbb{R}$ also has two connected components. Clearly these are homeomorphic.

Let $[0,\infty)=H$. Then $H^u=[0,\infty)^u$ is a continuum and one of the connected components of $u\mathbb{R}\setminus\mathbb{R}$. By Theorem 2.9 it is clear that $H^u=uH\setminus H$, so $u\mathbb{R}\setminus\mathbb{R}$ is the topological sum of two copies of the continuum $uH\setminus H$. Hence to study $u\mathbb{R}\setminus\mathbb{R}$ it suffices to study $uH\setminus H$.

Let αX be a compactification of the Tikhonov space X and let $f:\beta X\to\alpha X$ be the Čech map fixing X pointwise. Recall that αX is called a *perfect compactification* of X if $f^{\leftarrow}(p)$ is a connected subspace of βX for each $p\in\alpha X$ (equivalently, for each $p\in\alpha X\setminus X$).

THEOREM 4.10. (a) *uH is a perfect compactification of H.*
(b) *$u\mathbb{R}$ is a perfect compactification of $\mathbb{R}$.*

Proof (sketch). From the remarks above relating uH and $u\mathbb{R}$ it is clear that (a) implies (b). We indicate the main outline of the proof of (a), but omit verification of some tedious but routine details.

Let $f:\beta H\to uH$ be the Čech map. Let $p\in uH\setminus H$ and suppose $f^{\leftarrow}(p)$ were not connected. As $f^{\leftarrow}(p)$ is compact, a routine compactness argument (using the fact that $\{\mathrm{cl}_{\beta H} Z: Z\in\mathbf{Z}(X)\}$ is a base for the closed sets of βH; see 6.5 of [GJ]) shows that there exist disjoint zero-sets Z and S of H such that

$$f^{\leftarrow}(p)\cap\mathrm{cl}_{\beta H} Z\ne\emptyset\ne f^{\leftarrow}(p)\cap\mathrm{cl}_{\beta H} S \tag{1}$$

and

$$f^{\leftarrow}(p)\subseteq\mathrm{cl}_{\beta H}(Z\cup S). \tag{2}$$

Now $p \in f[\mathrm{cl}_{\beta H} Z] \cap f[\mathrm{cl}_{\beta H} S] = \mathrm{cl}_{uH} Z \cap \mathrm{cl}_{uH} S$, so by Theorem 2.5(b),

$$d(Z,S) = 0. \tag{3}$$

(Here d denotes the usual Euclidean metric on H.)

We can write $H \setminus (S \cup Z \cup \{0\})$ as the union of pairwise disjoint open intervals, i.e.

$$H \setminus (S \cup Z \cup \{0\}) = \bigcup \{(a_k, b_k) : k \in \mathbb{N}\}.$$

Let $S^* = S \cup \{\bigcup (a_k, b_k) : a_k \in S,\ b_k \in S,\ k \in \mathbb{N}\}$ and $Z^* = Z \cup \{\bigcup (a_k, b_k) : a_k \in Z,\ b_k \in Z,\ k \in \mathbb{N}\}$. A slightly involved but straightforward calculation shows that S^* and Z^* are disjoint closed subsets of H containing S and Z respectively. Hence we can replace S and Z by S^* and Z^* respectively in (1), (2), and (3); in other words, we can assume that (1), (2), (3) hold and also

$$H \setminus (S \cup Z) = \bigcup \{(a_k, b_k) : k \in \mathbb{N}\} \tag{4}$$

where $k \neq i$ implies $(a_k, b_k) \cap (a_i, b_i) = \emptyset$ and for each $k \in \mathbb{N}$, either $a_k \in S$ and $b_k \in Z$ or else $a_k \in Z$ and $b_k \in S$.

Let $\varepsilon_k = b_k - a_k$. We claim that if $M > 0$, and if we define J to be $\{k \in \mathbb{N} : (a_k, b_k) \subseteq [0, M]\}$, then J is finite. To verify this, suppose not; then $\lim_{k\to\infty}\{\varepsilon_k : k \in J\} = 0$ (as is easily verified). Let q be an accumulation point of $\{a_k : k \in J\}$. It now follows from the fact that $\varepsilon_k \to 0$, and (4), that $q \in \mathrm{cl}_H S \cap \mathrm{cl}_H Z = S \cap Z$, which contradicts the fact that $S \cap Z = \emptyset$. Hence our claim holds.

Define α_1 to be $\min\{\varepsilon_1/4, 1\}$ and if $k \in \mathbb{N}$ and $k > 1$ define α_k to be $\min\{\varepsilon_k/4, 1/k, \alpha_{k-1}\}$. Then let $A_k = [a_k + \alpha_k, b_k - \alpha_k]$ (where a_k and b_k are as in (4)). By the previous claim $\{A_k : k \in \mathbb{N}\}$ is a locally finite family of closed subsets of H, so if we define A to be $\bigcup\{A_k : k \in \mathbb{N}\}$ then A is closed, and clearly $A \cap (Z \cup S) = \emptyset$.

By (2) above, $f^{\leftarrow}(p) \subseteq \mathrm{cl}_{\beta H}(Z \cup S)$, so $p \notin \mathrm{cl}_{uH} A$. Hence by Corollary 2.6 there exists a closed subset B of H and $r > 0$ such that $p \in \mathrm{cl}_{uH} B$ and $d(A, B) = r$. As $p \notin H$, clearly B is unbounded.

Choose $k_0 \in \mathbb{N}$ such that $\alpha_{k_0} < r/4$. By a previous claim, there is $M \in H$ such that $k \le k_0$ implies $(a_k, b_k) \subseteq M$ and $k > k_0$ implies $(a_k, b_k) \subseteq [M, \infty)$.

We now claim that if $y \in B \cap [M, \infty)$ then $d(y, H \setminus (S \cup Z)) \ge r/4$; this follows from the definition of the "α_k"s and from the triangle inequality. We omit the details.

Next we claim that $d(B \cap S \cap [M, \infty), Z) \ge r/4$. To see this, suppose that $x \in B \cap S \cap (M, \infty)$ and $y \in Z$. If $(x, y) \subseteq S \cup Z$ then as $S \cap Z = \emptyset$ and (x, y) is connected, either $(x, y) \subseteq S$ (and so $y \in S$) or else $(x, y) \in Z$ (and so $x \in Z$). Either possibility yields $S \cap Z \neq \emptyset$, which is a contradiction; hence there exists $q \in (x, y) \setminus (Z \cup S)$. As $x \in B$, by the previous paragraph

$q - x \geq r/4$. Thus $y - x \geq r/4$ as $x < q < y$. Our claim therefore holds. Similarly we can prove that $d(B \cap Z \cap [M, \infty), S) \geq r/4$.

By the paragraph before last, $B \cap [M, \infty) \subseteq S \cup Z$. But $p \in \mathrm{cl}_{uH}(B \cap [M, \infty))$ so either $p \in \mathrm{cl}_{uH}(B \cap [M, \infty) \cap S)$ or $p \in \mathrm{cl}_{uH}(B \cap [M, \infty) \cap Z)$. But as $p \in \mathrm{cl}_{uH} S \cap \mathrm{cl}_{uH} Z$, by Theorem 2.5 this contradicts the claim verified in the previous paragraph. This ultimate contradiction yields the theorem. ∎

In view of Theorem 4.10 it is natural to ask whether $u\mathbb{R}^n$ is a perfect compactification of $\mathbb{R}^n$ if $n > 1$. The methods of proof of Theorem 4.10 do not generalize readily, and we leave this as an open question.

The space $\mathrm{cl}_{\beta\mathbb{R}}[0, \infty) \setminus \mathbb{R} = \beta H \setminus H$ is known to be an indecomposable continuum—i.e. a compact connected space which cannot be written as the union of two proper compact connected subspaces (see [Wo1] or [B] or 6AA of [PW] for a proof of this). It follows from the next result that $uH \setminus H$ shares this property. We thank the referee for drawing the following argument to our attention; our original proof of this was considerably more involved.

THEOREM 4.11. *If αX is a perfect compactification of the Tikhonov space X, and if $\beta X \setminus X$ is an indecomposable continuum, then so is $\alpha X \setminus X$. Consequently, $uH \setminus H$ is an indecomposable continuum.*

Proof. Let $f : \beta X \to \alpha X$ be the Čech map. As $f^{\leftarrow}(p)$ is a connected subspace of βX for each $p \in \alpha X$ and f is a closed continuous surjection, it follows from Theorem 6.1.29 of [E] that for every connected subset C of αX, the inverse image $f^{\leftarrow}[C]$ is a connected subspace of βX. Hence if $\alpha X \setminus X$ is the union of proper subcontinua K and L, then $\beta X \setminus X$ is the union of proper subcontinua $f^{\leftarrow}[K]$ and $f^{\leftarrow}[L]$, in contradiction to hypothesis. Consequently, $\alpha X \setminus X$ is indecomposable; in particular, $uH \setminus H$ is. ∎

References

[A] M. Atsuji, *Uniform continuity of continuous functions of metric spaces*, Pacific J. Math. 8 (1958), 1–16.

[B] D. Bellamy, *A non-metric indecomposable continuum*, Duke Math. J. 38 (1971), 15–20.

[Bl] R. L. Blair, *Spaces in which special sets are Z-embedded*, Canad. J. Math. 28 (1976), 673–690.

[BSw] R. L. Blair and M. A. Swardson, *Spaces with an Oz Stone–Čech compactification*, Topology Appl. 36 (1990), 73–92.

[C] S. Carlson, *Completely uniformizable proximity spaces*, Topology Proc. 10 (1985), 237–250.

[Če] E. Čech, *Topological Spaces*, Wiley Interscience, Prague, 1966.

[vD] E. K. van Douwen, *Remote points*, Dissertationes Math. 188 (1981).

[D] J. Dugundji, *Topology*, Allyn and Bacon, 1965.

[E] R. Engelking, *General Topology*, Polish Scientific Publishers, Warszawa, 1977.

[GJ] L. Gillman and M. Jerison, *Rings of Continuous Functions*, Van Nostrand, Princeton, 1960.

[HY] J. Hocking and G. Young, *Topology*, Addison-Wesley, Reading, 1961.

[K] M. Katětov, *On real-valued functions in topological spaces*, Fund. Math. 38 (1951), 85–91.

[LR] R. Levy and M. D. Rice, *Techniques and examples in U-embedding*, Topology Appl. 22 (1986), 157–174.

[Ma] K. D. Magill, Jr., *The lattice of compactifications of a locally compact space*, Proc. London Math. Soc. (3) 18 (1968), 231–244.

[M] D. Mattson, *Discrete subsets of proximity spaces*, Canad. J. Math. 31 (1979), 225–230.

[NW] S. Naimpally and B. Warrack, *Proximity Spaces*, Cambridge Univ. Press, 1970.

[PW] J. R. Porter and R. G. Woods, *Extensions and Absolutes of Hausdorff Spaces*, Springer, 1987.

[R] M. C. Rayburn, *Proximities*, unpublished manuscript.

[Ra] J. Rainwater, *Spaces whose finest uniformity is metric*, Pacific J. Math. 9 (1959), 567–570.

[S] Yu. M. Smirnov, *Mappings of systems of open sets*, Mat. Sb. (73) 31 (1952), 152–166 (in Russian).

[Wa] R. C. Walker, *The Stone–Čech Compactification*, Springer, New York, 1974.

[Wi] S. Willard, *General Topology*, Addison-Wesley, Reading, 1970.

[Wo1] R. G. Woods, *Certain properties of $\beta X - X$ for σ-compact X*, doctoral dissertation, McGill Univ., Montreal, 1969.

[Wo2] —, *Coabsolutes of remainders of Stone–Čech compactifications*, Pacific J. Math. 37 (1971), 545–560.

DEPARTMENT OF MATHEMATICS AND ASTRONOMY
UNIVERSITY OF MANITOBA
WINNIPEG, MANITOBA
CANADA R3T 2N2

Received 15 February 1994;
in revised form 31 August 1994 and 2 December 1994