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Products of completion regular measures

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Abstract. We investigate the products of topological measure spaces, discussing conditions under which all open sets will be measurable for the simple completed product measure, and under which the product of completion regular measures will be completion regular. In passing, we describe a new class of spaces on which all completion regular Borel probability measures are τ-additive, and which have other interesting properties.

1. Introduction. Suppose that $(X,\mathfrak{T},\Sigma,\mu)$ and $(Y,\mathfrak{S},T,\nu)$ are topological probability spaces, that is, probability spaces with topologies such that every open set is measurable. We can form product measures on $X\times Y$ in various ways. First, we have the ordinary completed product measure λ derived by Carathéodory's method from the outer measure λ^*, where

$$\lambda^*C=\inf\Big\{\sum_{i\in\mathbb{N}}\mu E_i\nu F_i : E_i\in\Sigma,\ F_i\in T\ \forall i\in\mathbb{N},\ C\subseteq\bigcup_{i\in\mathbb{N}}E_i\times F_i\Big\}.$$

It can happen that λ is again a topological measure in that every open set in $X\times Y$, for the product topology, is λ-measurable, but even for apparently well-behaved spaces (e.g., completion regular compact Radon probability spaces) this is not necessarily true (see [4]); the conditions under which it occurs are not well understood.

For a wide variety of important cases, it is known that λ does at least have an extension to a topological measure. Recall that a topological probability space $(X,\mathfrak{T},\Sigma,\mu)$ is *τ-additive* if $\mu(\bigcup\mathcal{G})=\sup_{G\in\mathcal{G}}\mu G$ for every non-empty upwards-directed family $\mathcal{G}$ of open sets. If $(X,\mathfrak{T},\Sigma,\mu)$ is a topological probability space, and $(Y,\mathfrak{S},T,\nu)$ is a τ-additive topological probability space, then we have a topological measure λ_R on $X\times Y$, given by setting

$$\lambda_R C=\int\nu C_x\,\mu(dx)$$

for Borel sets $C\subseteq X\times Y$, writing $C_x=\{y:(x,y)\in C\}$ for $C\subseteq X\times Y$, $x\in X$; for consistency with other constructions, we will take it that λ_R is

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to be the completion of its restriction to the Borel sets of $X \times Y$. If now μ is also τ-additive then we have a similar measure λ'_R given by

$$\lambda'_R C = \int \mu C^y \, \nu(dy)$$

for Borel $C \subseteq X \times Y$, writing $C^y = \{x : (x,y) \in C\}$, and under these circumstances λ_R and λ'_R are both τ-additive, therefore equal (since they agree on open rectangles). (See [18]. We remark that if μ and ν are both Radon measures, then so is $\lambda_R = \lambda'_R$.) On Borel sets in the domain of λ, λ agrees with λ_R and λ'_R if these are defined (using Fubini's theorem on λ); so if μ and ν are inner regular for the Borel sets—for instance, if they are completions of Borel measures—then λ_R, if it is defined, will extend λ, and λ will be a topological measure iff it is equal to λ_R.

Yet another aspect of the problem concerns the algebra $\Sigma \mathbin{\widehat{\otimes}_\sigma} T$, the σ-algebra of subsets of $X \times Y$ generated by $\{E \times F : E \in \Sigma,\ F \in T\}$. λ is the completion of its restriction to $\Sigma \mathbin{\widehat{\otimes}_\sigma} T$—that is, it is inner regular for $\Sigma \mathbin{\widehat{\otimes}_\sigma} T$—so (if ν is τ-additive and μ, ν are inner regular for the Borel sets) $\lambda = \lambda_R$ iff λ_R is inner regular for $\Sigma \mathbin{\widehat{\otimes}_\sigma} T$.

While the original impulse to study such questions arose from a simple desire to understand the nature of product Radon measures, there are important problems in functional analysis which depend on the analysis here; see, for instance, the distinction between "stable" and "R-stable" set which is necessary in [20], §9.

An allied question refers to "completion regular" spaces. Let us say that a topological measure space $(X, \mathfrak{T}, \Sigma, \mu)$ is *completion regular* if μ is inner regular for the zero sets (that is, sets of the form $f^{-1}[\{0\}]$ for some continuous $f : X \to \mathbb{R}$). Note that if $\mathcal{B}_0 = \mathcal{B}_0(X)$ is the Baire σ-algebra of X, that is, the σ-algebra generated by the zero sets in X, then $\mu \restriction \mathcal{B}_0$ is necessarily inner regular for the zero sets ([6], Theorem 4.2), so $(X, \mathfrak{T}, \Sigma, \mu)$ is completion regular iff μ is inner regular for $\mathcal{B}_0$. Many important spaces are completion regular; in particular, if X is a locally compact topological group and μ is (left or right) Haar measure on X, then μ is completion regular ([13], or [10], §64, Theorem I), and if X is a product of compact metric spaces, and μ is a product of strictly positive Radon probabilities on the factors, then μ is completion regular ([12], Theorem 3, or [1], Theorem 3). Now the question is, when is the product of completion regular measures again completion regular? Because $\mathcal{B}_0(X \times Y)$ always includes $\mathcal{B}_0(X) \mathbin{\widehat{\otimes}_\sigma} \mathcal{B}_0(Y)$ (and these are equal if X and Y are compact), the "product" in this question should be taken to be a topological product measure; normally, of course, the product measure λ_R discussed above. If $(X, \mathfrak{T}, \Sigma, \mu)$ and $(Y, \mathfrak{S}, T, \nu)$ are compact completion regular topological probability spaces, with ν τ-additive, then we see that λ_R is completion regular precisely when all open sets are λ-measurable.

A variety of conditions have been found which are sufficient to ensure that open sets are λ-measurable, or that λ_R is completion regular. Most of those with which we are acquainted amount to special cases of the main result of the present paper, which is based on a particular topological condition on one of the factors. Recall that a Hausdorff space is *dyadic* if it is a continuous image of $\{0,1\}^I$ for some set I. For the elementary properties of dyadic spaces see [3], 3.12.12 and 4.5.9–11; the essential ones relevant to us here are that

(i) compact metric spaces are dyadic,
(ii) finite unions of dyadic spaces are dyadic,
(iii) zero sets in dyadic spaces are dyadic,
(iv) all products of dyadic spaces are dyadic,
(v) compact Hausdorff topological groups are dyadic

([16]; see also [21] and [2], pp. 93–94). We generalise this concept by saying that a topological space is *quasi-dyadic* if it is the continuous image of a product of separable metrizable spaces. Now our first result is that if $(X, \mathfrak{T}, \Sigma, \mu)$ and $(Y, \mathfrak{S}, T, \nu)$ are τ-additive topological probability spaces, and one of them is quasi-dyadic and completion regular, then every open set in $X \times Y$ is λ-measurable (Theorem 5). It follows that if both factors are completion regular, so is λ_R. The argument of Theorem 5 depends on both factors having τ-additive measures. As it happens, however, this assumption can be omitted in the case of the quasi-dyadic factor, because any completion regular topological probability on a quasi-dyadic space is τ-additive (Theorem 4).

Of course, these results have consequences for the product of many factors. If we have finitely many completion regular quasi-dyadic topological probability spaces $(X_\alpha, \mathfrak{T}_\alpha, \Sigma_\alpha, \mu_\alpha)$, then every open set in $Z = \prod_\alpha X_\alpha$ is measurable for the simple product measure on Z; if we have infinitely many such spaces, and all but countably many of the μ_α are strictly positive (that is, give non-zero measure to every non-empty open set), the same will be true on the infinite product (Corollary 6).

We should mention the following known special cases of these results.

(i) Kakutani's theorem on the products of measures on compact metric spaces, already discussed, is of course the fundamental version of Corollary 6.

(ii) [1], Theorem 3, allows one factor to be non-metrizable, and introduces the asymmetry to be found in the hypotheses of our Theorem 5.

(iii) [9], Theorem 3.1, covers the case in which the quasi-dyadic factor is actually a product of separable metric spaces; it strengthens Theorem 2 of [7] and the main theorem of [8].

(iv) Concerning our Theorem 4, the case in which X is itself a product of separable metric spaces is given in [9], Theorem 3.1.

For simplicity, we express our results in terms of probability measures. Of course, everything we say about finite products applies at least to all totally finite measures, and much of it can be extended to non-σ-finite measures if care is taken over the definitions of the product measures.

2. Lemma. *If $\langle S_\alpha\rangle_{\alpha\in I}$ is a family of separable metric spaces, then every member E of the Baire σ-algebra of $S=\prod_{\alpha\in I}S_\alpha$ depends on a countable subset J of I, that is, $E=\pi_J^{-1}[F]$ for some $F\subseteq\prod_{\alpha\in J}S_\alpha$, writing $\pi_J:\prod_{\alpha\in I}S_\alpha\to\prod_{\alpha\in J}S_\alpha$ for the canonical map.*

Proof. See [19], or [3], 2.7.12.

3. To help give meaning to the concept of "quasi-dyadic" space, we give some elementary properties of these spaces.

Proposition. (a) *A continuous image of a quasi-dyadic space is quasi-dyadic. Any product of quasi-dyadic spaces is quasi-dyadic.*

(b) *A space with countable network is quasi-dyadic.*

(c) *If X is a quasi-dyadic space and Y belongs to the Baire σ-algebra of X, then Y is quasi-dyadic.*

(d) *A countable union of quasi-dyadic subspaces of a given topological space is quasi-dyadic.*

Proof. (a) Immediate from the definition.

(b) (See [3], 3.1.J and elsewhere, for basic facts concerning countable networks.) Let $\mathcal{D}$ be a countable network for the topology of X. On X let $\sim$ be the equivalence relation in which $x\sim y$ if they belong to just the same members of $\mathcal{D}$; let Y be the space $X/{\sim}$ of equivalence classes, and $\theta:X\to Y$ the canonical map. Y has a separable metrizable topology with base $\{\theta[D]:D\in\mathcal{D}\}\cup\{\theta[X\setminus D]:D\in\mathcal{D}\}$. Let I be any set such that $\#(\{0,1\}^I)\ge\#(X)$, and for each $y\in Y$ let $f_y:\{0,1\}^I\to y$ be a surjection. Then we have a continuous surjection $f:Y\times\{0,1\}^I\to X$ given by saying that $f(y,z)=f_y(z)$ for $y\in Y$, $z\in\{0,1\}^I$.

(c) Let $\langle S_\alpha\rangle_{\alpha\in I}$ be a family of separable metric spaces with product S and $f:S\to X$ a continuous surjection. Let $\mathcal{A}$ be the family of subsets of S which factor through countable sub-products, and $\mathcal{E}$ the set $\{E:E\subseteq X,\ f^{-1}[E]\in\mathcal{A}\}$. Then $\mathcal{A}$ and $\mathcal{E}$ are σ-algebras. By Lemma 2, every zero set in S belongs to $\mathcal{A}$; consequently, every zero set in X belongs to $\mathcal{E}$ and $\mathcal{E}$ includes the Baire σ-algebra of X. But if $E\in\mathcal{E}$, there is a countable $J\subseteq I$ such that $E=f[F\times\prod_{\alpha\in I\setminus J}S_\alpha]$ for some $F\subseteq\prod_{\alpha\in J}S_\alpha$; and F, being a subset of a countable product of separable metric spaces, is separable and metrizable, so E is quasi-dyadic.

(d) If $E_n\subseteq X$ is quasi-dyadic for each $n\in\mathbb{N}$, then $Z=\mathbb{N}\times\prod_{n\in\mathbb{N}}E_n$ is quasi-dyadic, and $f:Z\to\bigcup_{n\in\mathbb{N}}E_n$ is a continuous surjection, where $f(n,\langle x_i\rangle_{i\in\mathbb{N}})=x_n$.

R e m a r k. We include (b) to emphasize that the ideas here may be applied to non-Hausdorff spaces. Any continuous image of a product of spaces with countable networks will be quasi-dyadic.

4. THEOREM. *Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a quasi-dyadic completion regular topological probability space. Then μ is τ-additive.*

P r o o f. Suppose, if possible, otherwise.

(a) The first step is the standard reduction to the case in which X is covered by open sets of zero measure. In detail: suppose that X_0 is a quasi-dyadic space and μ_0 is a completion regular topological probability measure on X_0 which is not τ-additive. Let $\mathcal{G}$ be an upwards-directed family of open sets in X_0 such that $\mu_0(\bigcup \mathcal{G}) > \sup_{G\in\mathcal{G}} \mu_0 G$. Set $\mu_1 E = \mu_0 E - \sup_{G\in\mathcal{G}} \mu_0(E \cap G)$ for every Borel set $E \subseteq X_0$; then μ_1 is a completion regular Borel measure on X_0, $\mu_1 G = 0$ for every $G \in \mathcal{G}$, and $\mu_1(\bigcup \mathcal{G}) > 0$. Let $X \subseteq \bigcup \mathcal{G}$ be a Baire set such that $\mu_1 X > 0$. Then X is quasi-dyadic (3c). For Borel sets $E \subseteq X$, set $\mu E = \mu_1 E/\mu_1 X$; then μ is a completion regular Borel probability measure on X, and $\{X \cap G : G \in \mathcal{G}\}$ is a cover of X by open negligible sets.

(b) Now let $\langle S_\alpha\rangle_{\alpha\in I}$ be a family of separable metric spaces such that there is a continuous surjection $f : S \to X$, where $S = \prod_{\alpha\in I} S_\alpha$. For each $\alpha \in I$ let $\mathcal{B}_\alpha$ be a countable base for the topology of S_α. For $J \subseteq I$ let $\mathcal{C}(J)$ be the family of all subsets of S expressible in the form

$$\{s : s(\alpha) \in B_\alpha \ \forall \alpha \in K\},$$

where K is a finite subset of J and $B_\alpha \in \mathcal{B}_\alpha$ for each $\alpha \in K$; thus $\mathcal{C}(I)$ is a base for the topology of S. Set $\mathcal{C}_0(J) = \{U : U \in \mathcal{C}(J),\ \mu^* f[U] = 0\}$ for each $J \subseteq I$.

For each negligible set $E \subseteq X$, let $\langle F_n(E)\rangle_{n\in\mathbb{N}}$ be a family of zero subsets of $X \setminus E$ such that $\sup_{n\in\mathbb{N}} \mu F_n = 1$. Then each $f^{-1}[F_n(E)]$ is a zero subset of S, so there is a countable set $M(E) \subseteq I$ such that all the sets $f^{-1}[F_n(E)]$ depend on $M(E)$ (Lemma 2). Let $\mathcal{J}$ be the family of countable subsets J of I such that $M(f[U]) \subseteq J$ for every $U \in \mathcal{C}_0(J)$; then $\mathcal{J}$ is cofinal with $[I]^{\le\omega}$, that is, every countable subset of I is included in some member of $\mathcal{J}$. (If we start from any countable subset J_0 of I and set

$$J_{n+1} = J_0 \cup \bigcup \{M(f[U]) : U \in \mathcal{C}_0(J_n)\}$$

for each $n \in \mathbb{N}$, then $\bigcup_{n\in\mathbb{N}} J_n \in \mathcal{J}$.)

(c) For each $J \in \mathcal{J}$, set

$$Q_J = \bigcap \Big\{ \bigcup_{n\in\mathbb{N}} F_n(f[U]) : U \in \mathcal{C}_0(J) \Big\}.$$

Then $\mu Q_J = 1$ and $f^{-1}[Q_J]$ depends on J.

If $G \subseteq X$ is a negligible open set, then $G \cap Q_J = \emptyset$ whenever $J \in \mathcal{J}$ and there is a negligible Baire set $Q \supseteq G$ such that $f^{-1}[Q]$ depends on J. For set $H = \pi_J^{-1}[\pi_J[f^{-1}[G]]]$, where $\pi_J : S \to \prod_{\alpha \in J} S_\alpha$ is the canonical map; then H is a union of members of $\mathcal{C}(J)$, because $f^{-1}[G]$ is open. Also, because $f^{-1}[Q]$ depends on J, $H \subseteq f^{-1}[Q]$, so $f[H] \subseteq Q$ and $\mu^* f[H] = 0$; thus all the members of $\mathcal{C}(J)$ included in H actually belong to $\mathcal{C}_0(J)$, and $H \cap f^{-1}[Q_J] = \emptyset$. But this means that $f^{-1}[G] \cap f^{-1}[Q_J] = \emptyset$ and $G \cap Q_J = \emptyset$, as claimed.

In particular, if G is a negligible open set in X, then $G \cap Q_J = \emptyset$ whenever $J \in \mathcal{J}$ and $J \supseteq M(G)$.

(d) If $J \in \mathcal{J}$, there are $s, s' \in f^{-1}[Q_J]$ such that $s{\restriction}J = s'{\restriction}J$ and $f(s)$, $f(s')$ can be separated by open sets in X. To see this, start from any $x \in Q_J$ and take a negligible open set G including x (recall that our hypothesis is that X is covered by negligible open sets). For each $n \in \mathbb{N}$ let $h_n : X \to \mathbb{R}$ be a continuous function such that $F_n(G) = h_n^{-1}[\{0\}]$. We know that $G \cap Q_J \neq \emptyset$, while $G \subseteq X \setminus (\bigcup_{n\in\mathbb{N}} F_n(G) \cap Q_J)$, which is a negligible Baire set; by (c), $f^{-1}[X \setminus (\bigcup_{n\in\mathbb{N}} F_n(G) \cap Q_J)]$ does not depend on J, and there must be some n such that $f^{-1}[F_n(G) \cap Q_J]$ does not depend on J. Accordingly, there must be $s, s' \in S$ such that $s{\restriction}J = s'{\restriction}J$, $s \in f^{-1}[F_n(G) \cap Q_J]$ and $s' \notin f^{-1}[F_n(G) \cap Q_J]$. Now s and s' both belong to $f^{-1}[Q_J]$, because $f^{-1}[Q_J]$ depends on J; while $f(s) \in F_n(G)$ and $f(s') \notin F_n(G)$, so $h_n(f(s)) \neq h_n(f(s'))$ and $f(s)$, $f(s')$ can be separated by open sets.

(e) We are now ready to embark on the central construction of the argument. We may choose inductively, for ordinals $\xi < \omega_1$, sets $J_\xi \in \mathcal{J}$, negligible open sets $G_\xi, G'_\xi \subseteq X$, points $s_\xi, s'_\xi \in S$, sets $U_\xi, V_\xi, V'_\xi \in \mathcal{C}(I)$ such that

- $J_\eta \subseteq J_\xi$, U_η, V_η, V'_η all belong to $\mathcal{C}(J_\xi)$, $G_\eta \cap Q_{J_\xi} = \emptyset$ whenever $\eta < \xi < \omega_1$ (using the results of (b) and (c) to choose J_ξ);
- $s_\xi{\restriction}J_\xi = s'_\xi{\restriction}J_\xi$, $s_\xi \in f^{-1}[Q_{J_\xi}]$, $f(s_\xi)$ and $f(s'_\xi)$ can be separated by open sets in X (using (d) to choose s_ξ, s'_ξ);
- G_ξ, G'_ξ are disjoint negligible open sets containing $f(s_\xi)$, $f(s'_\xi)$ respectively (choosing G_ξ, G'_ξ);
- $U_\xi \in \mathcal{C}(J_\xi)$, $V_\xi, V'_\xi \in \mathcal{C}(I \setminus J_\xi)$, $s_\xi \in U_\xi \cap V_\xi \subseteq f^{-1}[G_\xi]$, $s'_\xi \in U_\xi \cap V'_\xi \subseteq f^{-1}[G'_\xi]$ (choosing U_ξ, V_ξ, V'_ξ, using the fact that $s_\xi{\restriction}J_\xi = s'_\xi{\restriction}J_\xi$).

On completing this construction, take for each $\xi < \omega_1$ a finite set $K_\xi \subseteq J_{\xi+1}$ such that U_ξ, V_ξ and V'_ξ all belong to $\mathcal{C}(K_\xi)$. By the Δ-system lemma ([14], II.1.5), there is an uncountable $A \subseteq \omega_1$ such that $\langle K_\xi \rangle_{\xi\in A}$ is a Δ-system with root K say, that is, $K_\xi \cap K_\eta = K$ for all distinct ξ, $\eta \in A$. For $\xi \in A$, express U_ξ as $\widetilde{U}_\xi \cap U'_\xi$ where $\widetilde{U}_\xi \in \mathcal{C}(K)$, $U'_\xi \in \mathcal{C}(K_\xi \setminus K)$. Then there are only countably many possibilities for $\widetilde{U}_\xi$, so there is an uncountable $B \subseteq A$ such that $\widetilde{U}_\xi$ is constant for $\xi \in B$; write $\widetilde{U}$ for the constant value.

Let $C \subseteq B$ be an uncountable set, not containing $\min A$, such that $K_\xi \setminus K$ does not meet J_η whenever $\xi, \eta \in C$ and $\eta < \xi$. Let $D \subseteq C$ be such that D and $C \setminus D$ are both uncountable.

Note that $K \subseteq K_\eta \subseteq J_\xi$ whenever η, $\xi \in A$ and $\eta < \xi$, so that $K \subseteq J_\xi$ for every $\xi \in C$. Consequently, U'_ξ, V_ξ and V'_ξ all belong to $\mathcal{C}(K_\xi \setminus K)$ for every $\xi \in C$.

(f) Consider the open set

$$G = \bigcup_{\xi \in D} G_\xi \subseteq X.$$

At this point the argument divides.

Case 1. Suppose $\mu^*(G \cap f[\widetilde{U}]) > 0$. Then there is a Baire set $Q \subseteq G$ such that $\mu^*(Q \cap f[\widetilde{U}]) > 0$. Let $J \subseteq I$ be a countable set such that $f^{-1}[Q]$ depends on J. Let $\gamma \in C \setminus D$ be so large that $K_\xi \setminus K$ does not meet J for any $\xi \in A$ with $\xi \geq \gamma$. Then $Q \cap Q_{J_\gamma} \cap f[\widetilde{U}]$ is not empty; take $s \in \widetilde{U} \cap f^{-1}[Q \cap Q_{J_\gamma}]$. Because the $K_\xi \setminus K$ are disjoint from each other and from $J \cup J_\gamma$ for $\xi \in D$, $\xi > \gamma$, we may modify s to form s' such that $s' \restriction J \cup J_\gamma = s \restriction J \cup J_\gamma$ and $s' \in U'_\xi \cap V'_\xi$ for every $\xi \in D$, $\xi > \gamma$; now $s' \in \widetilde{U}$ (because $K \subseteq J_\gamma$), so $s' \in \widetilde{U} \cap U'_\xi \cap V'_\xi \subseteq f^{-1}[G'_\xi]$ and $f(s') \notin G_\xi$ whenever $\xi \in D$, $\xi > \gamma$. On the other hand, if $\xi \in D$ and $\xi < \gamma$, $G_\xi \cap Q_{J_\gamma} = \emptyset$, while $s' \in f^{-1}[Q_{J_\gamma}]$ (because $f^{-1}[Q_{J_\gamma}]$ depends on J_γ), so again $f(s') \notin G_\xi$.

Thus $f(s') \notin G$. But $s' \restriction J = s \restriction J$ so $f(s) \in Q \subseteq G$, which is impossible.

This contradiction disposes of the possibility that $\mu^*(G \cap f[\widetilde{U}]) > 0$.

Case 2. Suppose that $\mu^*(G \cap f[\widetilde{U}]) = 0$. In this case there is a negligible Baire set $Q \supseteq G \cap f[\widetilde{U}]$. Let $J \subseteq I$ be a countable set such that $f^{-1}[Q]$ depends on J. Let $\xi \in D$ be such that $K_\xi \setminus K$ does not meet J. Then

$$\widetilde{U} \cap U'_\xi \cap V_\xi \subseteq f^{-1}[G_\xi] \cap \widetilde{U} \subseteq f^{-1}[G \cap f[\widetilde{U}]] \subseteq f^{-1}[Q],$$

so $\widetilde{U} \subseteq f^{-1}[Q]$, because $U'_\xi \cap V_\xi$ is a non-empty member of $\mathcal{C}(I \setminus J)$. But this means that $\mu^* f[\widetilde{U}] = 0$ and $\mu^* f[U_\xi] = 0$. On the other hand, we have $s_\xi \in U_\xi \cap f^{-1}[Q_{J_\xi}]$, so $U_\xi \notin \mathcal{C}_0(J_\xi)$ and $\mu^* f[U_\xi] > 0$.

Thus this route is also blocked and we must abandon the original hypothesis that there is a quasi-dyadic space with a completion regular topological probability measure which is not τ-additive.

5. Theorem. *Let $(X, \mathfrak{T}, \Sigma, \mu)$ and $(Y, \mathfrak{S}, T, \nu)$ be topological probability spaces; suppose that $(X, \mathfrak{T}, \Sigma, \mu)$ is completion regular and quasi-dyadic, and that $(Y, \mathfrak{S}, T, \nu)$ is τ-additive. Then every open set in $X \times Y$ is measurable for the ordinary completed product measure λ on $X \times Y$.*

Proof. Suppose, if possible, otherwise. By Theorem 4, we know that μ is τ-additive. We may suppose that the domain T of ν is precisely the algebra of Borel subsets of Y, since restricting the domain of ν will tend to decrease the domain of λ.

(a) There must be a closed set $C \subseteq X \times Y$ which is not λ-measurable. Because λ_R is τ-additive, $\lambda_R((X \times Y) \setminus C) \le \lambda_*((X \times Y) \setminus C)$; but also $\lambda_R C \le \lambda^* C$, so in fact we must have $\lambda_R C = \lambda^* C$.

We are supposing that X is quasi-dyadic; take a family $\langle S_\alpha \rangle_{\alpha \in I}$ of separable metric spaces and a continuous surjection $f : S \to X$, where $S = \prod_{\alpha \in I} S_\alpha$. Choose $\mathcal{B}_\alpha$ and define $\mathcal{C}(J)$, for $J \subseteq I$, as in part (b) of the proof of Theorem 4.

(b) If $J \subseteq I$ is countable, there are G, U, V such that $G \subseteq Y$ is open, $U \in \mathcal{C}(J)$, $V \in \mathcal{C}(I \setminus J)$, $V \ne \emptyset$, $C \cap (f[U \cap V] \times G) = \emptyset$ and $\lambda^*(C \cap (f[U] \times G)) > 0$. To see this, write

$$\begin{aligned} \mathcal{G}_U = \{G : G \subseteq Y \text{ is open}, \\ &\exists V \in \mathcal{C}(I \setminus J),\ V \ne \emptyset,\ C \cap (f[U \cap V] \times G)) = \emptyset\}, \\ G_U = \bigcup \mathcal{G}_U, \end{aligned}$$

and choose $F_U \supseteq f[U]$ such that $F_U \in \Sigma$ and $\mu F_U = \mu^*(f[U])$, for each $U \in \mathcal{C}(J)$. We have $G_U \in T$ for every U, and $\mathcal{C}(J)$ is countable, so

$$C_1 = (X \times Y) \setminus \bigcup_{U \in \mathcal{C}(J)} F_U \times G_U \in \Sigma \mathbin{\widehat{\otimes}_\sigma} T;$$

also $C_1 \subseteq C$ because C is closed and

$$\{f[U \cap V] \times G : G \subseteq Y \text{ is open},\ U \in \mathcal{C}(J),\ V \in \mathcal{C}(I \setminus J)\}$$

is a network for the topology of $X \times Y$. So $\lambda_R(C \setminus C_1) = \lambda_R C - \lambda_R C_1 = \lambda^*(C \setminus C_1) > 0$, because C is not λ-measurable. Accordingly, there is a $U \in \mathcal{C}(J)$ such that $\lambda_R(C \cap (F_U \times G_U)) > 0$. Next, because ν is τ-additive, there is a countable $\mathcal{G} \subseteq \mathcal{G}_U$ such that $\nu(G_U \setminus \bigcup \mathcal{G}) = 0$, and now $\lambda_R(C \cap (F_U \times \bigcup \mathcal{G})) = \lambda_R(C \cap (F_U \times G_U)) > 0$, so there is a $G \in \mathcal{G}$ such that $\lambda_R(C \cap (F_U \times G)) > 0$, that is,

$$\int_{F_U} \nu(C_x \cap G)\, \mu(dx) > 0.$$

But this means that $\mu\{x : x \in F_U,\ \nu(C_x \cap G) > 0\} > 0$, so that $\mu^*\{x : x \in f[U],\ \nu(C_x \cap G) > 0\} > 0$, and $\lambda^*(C \cap (f[U] \times G)) > 0$. Finally, because $G \in \mathcal{G}_U$, there is a $V \in \mathcal{C}(I \setminus J)$ such that $C \cap (f[U \cap V] \times G) = \emptyset$.

(c) We may therefore choose inductively families $\langle J_\xi \rangle_{\xi < \omega_1}$, $\langle G_\xi \rangle_{\xi < \omega_1}$, $\langle U_\xi \rangle_{\xi < \omega_1}$, $\langle V_\xi \rangle_{\xi < \omega_1}$ in such a way that, for every $\xi < \omega_1$,

- J_ξ is a countable subset of I,
- G_ξ is an open subset of Y,
- $U_\xi \in \mathcal{C}(J_\xi)$, $V_\xi \in \mathcal{C}(I \setminus J_\xi)$, $V_\xi \neq \emptyset$,
- $C \cap (f[U_\xi \cap V_\xi] \times G_\xi) = \emptyset$,
- $\lambda^*(C \cap (f[U_\xi] \times G_\xi)) > 0$,
- $\bigcup_{\eta<\xi} J_\eta \subseteq J_\xi$,
- $V_\xi \in \mathcal{C}(J_{\xi+1})$.

For each $\xi < \omega_1$, let K_ξ be a finite subset of I such that U_ξ, V_ξ both belong to $\mathcal{C}(K_\xi)$. By the Δ-system lemma, there is an uncountable set $A \subseteq \omega_1$ such that $\langle K_\xi \rangle_{\xi\in A}$ is a Δ-system with root K say. Express each U_ξ as $\widetilde{U}_\xi \cap U'_\xi$ where $\widetilde{U}_\xi \in \mathcal{C}(K)$ and $U'_\xi \in \mathcal{C}(K_\xi \setminus K)$; because $\mathcal{C}(K)$ is countable, there is a $\widetilde{U}$ such that $B = \{\xi : \xi \in A,\ \xi \neq \min A,\ \widetilde{U}_\xi = \widetilde{U}\}$ is uncountable. Note that $\mu^*(f[\widetilde{U}]) \geq \mu^*(f[U_\xi]) > 0$ for any $\xi \in B$. Also, $K \subseteq K_{\min A} \subseteq J_\xi$ for each $\xi \in B$, so $V_\xi \in \mathcal{C}(K_\xi \setminus J_\xi) \subseteq \mathcal{C}(K_\xi \setminus K)$ for each $\xi \in B$.

(e) Set $H_\zeta = \bigcup_{\xi\in B,\,\xi\geq\zeta} G_\xi$ for each $\zeta < \omega_1$. The family $\langle \nu H_\zeta \rangle_{\zeta<\omega_1}$ is non-increasing in $\mathbb{R}$, so there must be a $\delta < \omega_1$ such that $\nu H_\zeta = \nu H_\delta$ whenever $\delta \leq \zeta < \omega_1$. Now consider

$$D = C \cap (X \times H_\delta).$$

Because $D \supseteq C \cap (f[U_\xi] \times G_\xi)$ for any $\xi \in B \setminus \delta$, $\lambda_R D > 0$. Set

$$F = \{x : \nu(C_x \cap H_\delta) > 0\} \subseteq X;$$

because Fubini's theorem applies to λ_R, $\mu F > 0$. At this point we use the hypothesis that μ is completion regular to see that there is a zero set $F_0 \subseteq F$ with $\mu F_0 > \mu F - \mu^*(f[\widetilde{U}])$. Now $W = \widetilde{U} \cap f^{-1}[F_0]$ is non-empty because $F_0 \cap f[\widetilde{U}]$ is non-empty. Let $J \subseteq I$ be a countable set such that $K \subseteq J$ and W depends on J (see §§2–3 above). Let $\zeta < \omega_1$ be such that $\zeta \geq \delta$ and $K_\xi \setminus K = \emptyset$ for every $\xi \in B \setminus \zeta$; this exists because $\langle K_\xi \setminus K \rangle_{\xi\in B}$ is disjoint.

Take any $w \in W$, and modify it to obtain $w' \in S$ such that

$$w' \restriction J = w \restriction J \quad \text{and} \quad w' \in U'_\xi \cap V_\xi \quad \text{for every } \xi \in B \setminus \zeta;$$

this is possible because $U'_\xi \cap V_\xi \in \mathcal{C}(K_\xi \setminus K)$ for each $\xi \in B \setminus \zeta$, and $\langle K_\xi \setminus K \rangle_{\xi\in B}$ is disjoint. Set $x = f(w')$; then $x \in F_0 \subseteq F$, so $\nu D_x > 0$.

If $\xi \in B \setminus \zeta$, then $x \in f[\widetilde{U} \cap U'_\xi \cap V_\xi] = f[U_\xi \cap V_\xi]$, so $(x, y) \notin C$ for $y \in G_\xi$ and $C_x \cap G_\xi = \emptyset$. Thus

$$D_x \subseteq H_\delta \setminus H_\zeta,$$

and $\mu H_\zeta < \mu H_\delta$, contrary to the choice of δ.

This contradiction proves the theorem.

6. Corollary. *Let* $\langle (X_\alpha, \mathfrak{T}_\alpha, \Sigma_\alpha, \mu_\alpha) \rangle_{\alpha\in I}$ *be a family of quasi-dyadic completion regular topological probability spaces, and suppose that all but* *countably many of the μ_α are strictly positive. Then the simple* (*completed*) *product measure on $Z = \prod_{\alpha\in I} X_\alpha$ is also a completion regular topological measure.*

Proof. For finite I, this is a simple induction on $\#(I)$, using Theorems 4 and 5. For infinite I, use Theorem 2.9 of [9], which says that if all the τ-additive topological product measures on finite subproducts are completion regular, and all but countably many of the factor measures μ_α are strictly positive, the τ-additive topological product measure on Z will also be completion regular, and therefore will coincide with the simple completed product measure.

7. Remarks. In Theorems 4 and 5 it is not of course necessary that X itself should be quasi-dyadic. If $(X, \mathfrak{T}, \Sigma, \mu)$ is a completion regular τ-additive topological probability space in which μ is inner regular for the quasi-dyadic subsets of X, this will do just as well. This will be so, for instance, if X is an open subset of a dyadic space and μ is a completion regular Radon measure on X.

In the same way, it will be enough, in Corollary 6, if every μ_α is inner regular for the quasi-dyadic subsets of X_α.

Theorem 4 can be generalized as follows: Let X be a quasi-dyadic space and $\mathcal{I}$ a proper σ-ideal of subsets of X such that for every open $G \subseteq X$ there are Baire subsets E, F of X such that $E \subseteq G \subseteq F$ and $F \setminus E \in \mathcal{I}$. Then X cannot be covered by the open sets belonging to $\mathcal{I}$; indeed, $\bigcup\{G : G \in \mathcal{I},\ G$ is open$\} \in \mathcal{I}$.

We do not know of a ZFC example of a completion regular measure on a completely regular Hausdorff space which is not τ-additive. (We note that there is a probability measure on the Baire σ-algebra of $\mathbb{R}^{\mathfrak{c}}$ which is not τ-additive, in the strong sense that $\mathbb{R}^{\mathfrak{c}}$ is covered by the negligible cozero sets; see [17], or [6], 15.4; but this measure is not completion regular.)

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