FUNDAMENTA
MATHEMATICAE
147 (1995)

Dense orderings, partitions and weak forms of choice

by

Carlos G. González (Campinas)

Abstract. We investigate the relative consistency and independence of statements which imply the existence of various kinds of dense orders, including dense linear orders. We study as well the relationship between these statements and others involving partition properties. Since we work in ZF (i.e. without the Axiom of Choice), we also analyze the role that some weaker forms of AC play in this context.

Since 1922, when Fraenkel presented his proof of the independence of the Axiom of Choice, a considerable amount of research has been done on the consistency and independence of several principles concerning the existence of orderings, e.g., the assertion that every set can be linearly ordered, the assertion that maximal antichains in a partially ordered set exist, the Kinna–Wagner Principle, etc. However, as far as we know, there does not seem to exist in the literature any study about principles affording the existence of dense (partial) orderings, in the sense that there is an element between two different ones. These principles are intimately related to statements about the existence of certain partitions, which are of independent interest. All such principles, as well as other statements analyzed here, constitute weak versions of the Axiom of Choice (AC), in the sense that AC implies each of them in ZF, though the converse is not true. This paper contains an initial investigation in this field of research.

In order to simplify the hypotheses of the various set-theoretic interrelationships we prove all set-theoretic statements in ZF and assume the consistency of ZF in all independence proofs. The main results of this paper can be expressed by the following theorem:

THEOREM 1. AC $\Rightarrow$ DO $\Rightarrow$ O $\Rightarrow$ DPO; *moreover*, *none of the implications is reversible in* ZF *and* DPO *is independent of* ZF. (See Section 1 and the Glossary at the end of this paper.)

1991 *Mathematics Subject Classification*: Primary 03E20.

The various components of Theorem 1 are the following: AC $\Rightarrow$ DO is Lemma 1; DO $\Rightarrow$ O is immediate; O $\Rightarrow$ DPO is Corollary 1; DO does not imply AC is Corollary 13; O does not imply DO is Corollary 11; DPO does not imply O is Corollary 10, and the independence of DPO is Corollary 5.

The first section presents the definitions of various kinds of dense orders and gives preliminary results. In the second section we introduce some statements involving partitions and prove the first independence results. Finally, in the last section we offer some independence results about principles which imply the existence of various kinds of orderings. Theorem 8 in this section shows that the Mostowski model has an infinite set that cannot be linearly densely ordered.

Acknowledgements. The author wishes to express his gratitude to Prof. Carlos A. Di Prisco for several remarks and suggestions during the writing of the first part of this paper. The results presented here are a part of the author's Doctoral Thesis, written under the supervision of Prof. Luiz Paulo de Alcantara. The author is indebted to Prof. de Alcantara, as well as to Prof. Walter A. Carnielli, Prof. Carlos A. Lungarzo, Prof. Max Dickmann and Prof. D. Pincus for valuable comments on previous versions. This work has been supported by a thesis grant at the State University of Campinas from CNPq (Brazil) and by CONICIT during my stay at IVIC, Caracas, Venezuela. Thanks are also due to the referee whose valuable comments helped to improve substantially earlier versions of the present paper.

1. Preliminaries

Definition 1. A *partial order* is an irreflexive and transitive relation $<$. We always use the symbol $<$ in this sense. The formula $x < y \vee x = y$ is denoted by $x \leq y$. A *dense partial order* on a set A is a relation $<$ such that

$$\forall x \in A \, \forall y \in A \, (x < y \Rightarrow \exists z \in A \, (x < z \wedge z < y)).$$

This concept of dense partial order is not enough for representing the intuitive idea of a dense order, since $\emptyset$ is a trivial dense partial order on any set. For this reason, we make the following definition:

Definition 2. A *non-trivial dense order* on a set A is a dense partial order $<$ satisfying the additional condition

$$\exists x \in A \, \exists y \in A \, (x < y).$$

There are two other concepts closely related to the above:

Definition 3. An *anywhere dense order* on A is a dense partial order $<$ satisfying the additional condition

$$\forall x \in A \, \exists y \in A \, (x < y \vee y < x).$$

Definition 4. A partial order $<$ is called *somewhere dense* on A if

$$\exists a \in A\, \exists b \in A\, (a < b \wedge \forall c \in A\, \forall d \in A\, (a \le c < d \le b \Rightarrow \\ \Rightarrow \exists h \in A\, (c < h < d))).$$

Definition 5. A (*linear*) *order* is a partial order which satisfies

$$\forall x\, \forall y\, (x = y \vee x < y \vee y < x).$$

A *dense* (*linear*) *order* has an obvious definition.

The existence of a non-trivial dense partial order on a set implies that the set is infinite. By this fact, our principles must have the following form:

Definition 6. DPO (O, DO, resp.) are the statements "every infinite set can be non-trivially densely ordered" ("every set can be linearly ordered", "every infinite set can be linearly and densely ordered", resp.). (Note that DO implies O.)

Definition 7. Given a partial order $\langle A, < \rangle$, an *interval* is a set $I \subseteq A$ such that $x \in I \wedge y \in I \wedge x < z < y \Rightarrow z \in I$ (cf. [5], p. 10). An interval is *non-trivial* if $|I| > 1$.

Lemma 1. AC $\Rightarrow$ DO.

Proof. (The countable case is trivial. The intuitive idea for the non-trivial case is to divide an uncountable cardinal κ into intervals, each of them isomorphic to ω, and then to define a dense linear order on each one via a bijection with $\mathbb{Q}^+$.)

Let x be a set and let κ be its cardinal number. If $\kappa = \omega$, we use the order induced by a bijection from ω to $\mathbb{Q}^+$ (i.e., the set of positive rationals). Now, assuming that $\kappa > \omega$, we will define a linear dense order on κ which induces a linear dense order on x. For this, let $\alpha, \delta < \kappa$ be ordinals.

Let

$$A = \{\gamma \le \alpha : \gamma \text{ is a limit ordinal}\}, \qquad \beta = \bigcup A.$$

Then β is a limit ordinal (possibly 0) and α has the form $\beta + n$ with $n \in \omega$ (possibly $n = 0$). Let f be a bijection between ω and $\mathbb{Q}^+$ and let $<_{\mathbb{Q}^+}$ be the natural order of $\mathbb{Q}^+$. In order to define $<_D$, a linear dense order on κ, we distinguish three cases.

If $\delta < \beta$ then $\delta <_D \alpha$.

If $\beta + \omega \le \delta$ then $\alpha <_D \delta$.

If $\beta \le \delta < \beta + \omega$ then we first observe that δ can be written as $\beta + m$, with $m \in \omega$ (possibly $m = 0$). Then we define: $\delta <_D \alpha \Leftrightarrow \beta + n <_D \beta + m \Leftrightarrow f(n) <_{\mathbb{Q}^+} f(m)$.

We want to show that $<_D$ is a dense linear order. It is clearly a linear order. To see that it is dense, we fix $\alpha, \beta < \kappa$, $\alpha <_D \beta$. Then there exists γ, which is either a limit ordinal or $\gamma = 0$, such that $\alpha = \gamma + n$ with

$n \in \omega$. We now proceed by cases. First, if $\beta = \gamma + m$ with $m \in \omega$, by definition of $<_D$, we have $f(n) <_{\mathbb{Q}^+} f(m)$. Hence, by the density of $<_{\mathbb{Q}^+}$, also $\exists q \in \mathbb{Q}^+ \, (f(n) <_{\mathbb{Q}^+} q <_{\mathbb{Q}^+} f(m))$. As f is a bijection, there is an s such that $f(s) = q$, and thus $\alpha = \gamma + n <_D \gamma + s <_D \gamma + m = \beta$. In the second case, note that $\gamma + \omega \leq \beta$. But then, since $\mathbb{Q}^+$ does not have a last element, we conclude that there is a q such that $f(n) <_{\mathbb{Q}^+} q = f(s)$ for some s, i.e., $\alpha <_D \gamma + s$. Finally, by the definition of $<_D$, we have $\gamma + s <_D \beta$. ■

There exists a simpler argument for the above proof based on model-theoretic ideas, namely using the theorem of Löwenheim–Skolem (though one must be careful in interpreting the symbol =). Our proof, however, defines explicitly the dense linear order and can also be adapted to the use of weaker forms of choice (for example, "every set can be ordered as the linear sum of canonically countable intervals", where "canonically" means that there is a single function enumerating simultaneously the elements of all intervals). In this paper "countable" always means "infinite countable". The definition of sum of orders is the usual one (see [5], p. 19).

In the sequel we analyze some equivalences between principles entailing the existence of dense orders.

Lemma 2. *Let $<$ be a non-trivial dense order on a set A. Then $<$ can be extended to an anywhere dense order on A.*

Proof. Let $\langle A, < \rangle$ be a non-trivial dense order. Then there exist a and b in A such that $a < b$. Let $(a, b) = \{x : a < x < b\}$. Furthermore, let B be the set of all elements of A which are not comparable with any other one: $B = \{x : \neg\exists y \, (x < y \vee y < x)\}$. We define then $<' = (< \cup ((a, b) \times B))$. Obviously $<'$ is an anywhere dense order on A. ■

Lemma 3. *Let A be a set such that there exists a somewhere dense order $<$ on it. Then there exists an anywhere dense order on A.*

Proof. Let $\langle A, < \rangle$ be a somewhere dense order. Then there are a and b in A such that $a < b$ and $[a, b] = \{x : a \leq x \leq b\}$ is a dense interval (i.e. a non-trivial interval which is anywhere dense). Now we define $<'$, the restriction of $<$ to $[a, b]$, as: $<' = < \restriction [a, b] = < \cap \{\langle x, y \rangle : a \leq x, y \leq b\}$. Furthermore, let $B = A - [a, b] = \{x : a \not\leq x \vee x \not\leq b\}$. Then we define $<'' = <' \cup ((a, b) \times B)$. It is easy to see that $<''$ is an anywhere dense order on A. ■

The existence of a linear order on an infinite set implies the existence of a non-trivial dense order on this set. The construction employed here (namely, the finite collapsing of a partial order) is closely related to the condensation method (see [5], Ch. 4).

LEMMA 4. *Let $\langle A,<\rangle$ be a linear order and A an infinite set. Then there exists a non-trivial dense order on A.*

Proof. If A contains an infinite well-ordered, or anti-well-ordered interval, we define a linear dense interval as in Lemma 1. Hence we can suppose with no loss of generality that there are no intervals of these forms.

For $x \in A$ we define I_x, the *finite collapsing interval* of x, as follows:

$$I_x = \{y \in A : \text{both } [x,y] \text{ and } [y,x] \text{ are finite}\},$$

where $[x,y]$ is the closed interval with respect to $<$. We will see that I_x is finite for $x \in A$. Otherwise, there exists an x such that either $I_{<x} = \{y \in I_x : y < x\}$ or $I_{>x} = \{y \in I_x : x < y\}$ is infinite. Then, by the definition of I_x, every non-empty subset of $I_{<x}$ has a last element and every non-empty subset of $I_{>x}$ has a first element, contradicting our assumption.

Now, for $x,y \in A$ we define

$$x <' y \Leftrightarrow x < y \wedge I_x \neq I_y.$$

We claim that $<'$ is a partial order. The irreflexivity is obvious. For the transitivity, assume $x <' y$ and $y <' z$. Clearly $x < z$. It remains to be proved that $I_x \neq I_z$. Otherwise, the interval $[x,z]$ is finite and thus $I_r = I_x$ for every r such that $x < r < z$. In particular, $I_x = I_y$, a contradiction. To see that it is dense, let $x,y \in A$ be such that $x <' y$; then $[x,y]$ is infinite. Since I_x, I_y are finite, there exists z such that $x < z < y$ and $z \notin I_x, I_y$, i.e., $I_z \neq I_x$ and $I_z \neq I_y$. Then we have $x <' z <' y$.

Our case assumption implies that every finite collapsing interval is finite. Since A is infinite, it contains at least two such intervals, and hence $<'$ is non-trivial. ∎

COROLLARY 1. O $\Rightarrow$ DPO. ∎

From the proof of the last lemma we obtain immediately the following:

COROLLARY 2. *Let $\langle A,<\rangle$ be an infinite linear order. Then either A has a countable subset or there exists a dense suborder of $\langle A,<\rangle$.* ∎

2. Dense orderings and partitions. In this section we analyze the relationship between the principles of dense partial order and the existence of certain partitions of infinite sets.

DEFINITION 8. A set is called *partible* if it is the union of two pairwise disjoint infinite sets. PP is the statement: "every infinite set is partible". An infinite set is called *amorphous* if it is not partible (see [2], p. 52).

PP is independent of ZF (see [3], p. 12, and [2], pp. 52 and 96). Moreover, PP is a very weak consequence of AC, since it is implied by "every infinite set has a countable subset". Let $\mathrm{Inf}(x)$ be the predicate "x is infinite". Then the following holds:

LEMMA 5. *Let x be a set and let $p \subseteq \mathcal{P}(x)$ be such that* $\mathrm{Inf}(p)$ *and* $\forall z \in p$ $(\neg \mathrm{Inf}(z) \Rightarrow \exists t \in p\,(\mathrm{Inf}(t) \wedge z \subseteq t))$. *Then the conditions*:

(i) *for every infinite set h in p there exists a partition of h into two disjoint infinite sets h_1, h_2 belonging to p*;

(ii) $\forall h_1 \forall h_2\,(h_1, h_2 \in p \Rightarrow (h_1 \cup h_2) \in p \wedge (h_1 - h_2) \in p)$;

imply that there exists an anywhere dense order on p.

Proof. We define in p: $a < b \Leftrightarrow (a \subseteq b \wedge \mathrm{Inf}(b-a))$. We see immediately that it is a partial order. To see that it is dense, let $a, b \in p$. First, we divide $b-a$: $b-a = c \cup d$, with c and d infinite and disjoint, and then take $r = c \cup a$, so that $a < r < b$. Finally, let $a \in p$. If a is finite, there is an infinite b such that $a \subseteq b$ and thus $a < b$. If a is infinite, we divide it into c and d, both infinite and disjoint, and obtain $c < a$, so that $<$ is anywhere dense. ■

COROLLARY 3. *Let x be a set and let h be such that $h \subseteq \mathcal{P}(x)$ and $\forall z \in h\, \mathrm{Inf}(z)$. Then the conditions* (i) *and* (ii) *of the preceding lemma imply that there is an anywhere dense order on h.* ■

COROLLARY 4. *Let x be an infinite set. Then* PP *implies that there exists an anywhere dense order on $\mathcal{P}(x)$.* ■

The last corollary and, more notably, the next lemma show the close relationship existing between PP and dense orders, since the existence of non-trivial dense orders on every infinite set implies that every infinite set is partible.

LEMMA 6. DPO $\Rightarrow$ PP.

Proof. Let x be an infinite set. By DPO there exists a dense order $<$ on x and a and b in x such that $a < b$. Let c be such that $a < c < b$ and let $s = \{z \in x : z < c \vee z = c\}$ and $t = \{z \in x : z \not< c \wedge z \neq c\}$. Then s and t satisfy the condition required by PP. ■

COROLLARY 5. DPO *is independent of* ZF. ■

We now introduce a statement stronger than PP.

DEFINITION 9. A set x is called $\aleph_0$-*partible* if there exists a partition of x of cardinality $\aleph_0$. P-$\aleph_0$ is the statement "every infinite set is $\aleph_0$-partible".

If a set x is $\aleph_0$-partible then there exists a countable partition of x formed by infinite sets (and *a fortiori* x is partible). To see this let $x_0, x_1, \ldots, x_i, \ldots$ be a countable partition of x. Then for each prime number p we define $y_p = \bigcup\{x_n : n = p^m \text{ for some } m \in \omega,\ m \neq 0\}$ and $y_0 = \bigcup\{x_n : n \text{ has at least two prime factors or } n = 0 \text{ or } n = 1\}$.

LEMMA 7. P-$\aleph_0 \Rightarrow$ DPO.

Proof. Let x be an infinite set and let $B = \{A_n : n \in \omega\}$ be a countable partition of x. Furthermore, let $f : \omega \to \mathbb{Q}$ be a bijection and let $<_{\mathbb{Q}}$ be the natural order of $\mathbb{Q}$. Then we define for $a, a' \in A$: $a < a' \Leftrightarrow f(n) <_{\mathbb{Q}} f(m) \wedge a \in A_n \wedge a' \in A_m$. It can be easily shown that $<$ is a non-trivial dense order, using the facts that (1) A_i are disjoint and non-empty, (2) f is a bijection, and (3) the properties of $<_{\mathbb{Q}}$. ∎

If a set has a countable subset then it is $\aleph_0$-partible, and consequently, there exists a dense partial order on it. Alternatively, we can use the existence of a countable subset of this set to define a somewhere dense order on it and then proceed as in Lemma 3. Hence, the statement "every infinite set has a countable subset" implies DPO in ZF.

Since in ZF if a set x has a countable subset then x is $\aleph_0$-partible, the question of whether or not the converse holds poses itself naturally. The following counterexample shows that the answer is negative.

COUNTEREXAMPLE 1. *There exists a model of* ZF *containing a set x which is $\aleph_0$-partible but has no countable subset.*

Proof. We assume that there is a set of reals without a countable subset (this property holds in various models found in the literature). Fixing such an x, we want to show that this set is $\aleph_0$-partible. We proceed by cases:

Case 1. If x has no upper bound or if it has no lower bound, we use the set of integers $\mathbb{Z}$ to define a countable partition in the obvious way.

Case 2. If x has both lower and upper bounds, then there exists an infinite $y \subseteq x$ without a first element, since x is an infinite set without a countable subset, and hence the natural order of the reals $<$ restricted to x is not a well-order. Let r be the greatest lower bound of y in $\mathcal{R}$, and let $\{q_n : n \in \omega\}$ be a strictly decreasing sequence of rationals converging to r (it is easily proved in ZF that such a sequence exists for every real r). Then $\{[q_{n+1}, q_n) : n \in \omega\}$ is an $\aleph_0$-partition of a subset of x. ∎

DEFINITION 10 (Dedekind's Axiom). D is the statement "every infinite set has a countable subset".

The question whether P-$\aleph_0$ implies D in ZF arises very naturally. There is a well-known symmetric model (as we shall see below) in which P-$\aleph_0$ is true, whilst both AC and D fail. Hence the statement "P-$\aleph_0$ implies D" is not deducible in ZF. In order to deal with this question we need a preliminary result.

THEOREM 2. *Let T be a family of sets such that for a fixed $n \in \omega$, we have $t \in T \Rightarrow |t| \leq n$. Let $\{A_i : i \in \omega\}$ be a countable partition of $\bigcup T$. Then there exists a countable partition of T.*

P r o o f. Let $B_i = \{t \in T : t \cap A_i \neq \emptyset\}$. Inductively we define a sequence $\{D_i : i \in \omega\}$ which will turn out to be a countable partition of some subset of T. Suppose D_r is already defined for $r < i$. Let C_i be the condition "there are infinitely many A_j's such that $A_j \cap \bigcup(T - (\bigcup_{r<i} D_r \cup B_i)) \neq \emptyset$", and F_i the condition $B_i - (\bigcup_{r<i} D_r) \neq \emptyset$. Then we define

$$D_i = \begin{cases} B_i - (\bigcup_{r<i} D_r) & \text{if } C_i \text{ and } F_i, \\ \emptyset & \text{otherwise.} \end{cases}$$

We prove successively:

(1) *C_i holds for at least one index i.*

Assuming otherwise we show that $\bigcup T$ is covered by finitely many A_j's, a contradiction. Firstly, the fact that the elements of T have cardinality $\leq n$ entails that the set $\bigcup(\bigcap_{i<n} B_i)$ is covered by the $A_0, A_1, \ldots, A_{n-1}$. The complement of this set is

$$(*)\qquad \bigcup\Big(T - \bigcap_{i<n} B_i\Big) = \bigcup\Big(\bigcup_{i<n}(T - B_i)\Big) = \bigcup_{i<n}\Big(\bigcup(T - B_i)\Big);$$

it suffices, then, to show that $\bigcup(T - B_i)$ is covered by finitely many A_j's, for $i = 0, \ldots, n-1$.

The failure of C_i entails, by the definition above, that $D_i = \emptyset$ for all $i \in \omega$; hence $\neg C_i$ is equivalent to

$$\text{“}A_j \cap \bigcup(T - B_i) \neq \emptyset \text{ for finitely many } j\text{'s”}.$$

Note that, for $X \subseteq \bigcup T$, X is covered by finitely many A_j's iff $X \cap A_j \neq \emptyset$ for finitely many A_j's. From the last two sentences it follows that $\bigcup(T - B_i)$ is covered by finitely many A_j's, as claimed.

(2) *$C_i \wedge F_i$ holds for infinitely many indices i.*

We note first that $C_i \wedge F_i$ holds for at least one index i, e.g., for the first i such that C_i holds (since $\bigcup_{r<i} D_r = \emptyset$ for such an i).

Assuming (2) fails, let k be the largest index i for which $C_i \wedge F_i$ holds. Under this assumption $D_i = \emptyset$ for $i > k$. Hence, for $i > k$,

$$(**)\qquad \bigcup_{r<i} D_r = \bigcup_{r\leq k} D_r = \bigcup_{r<k} D_r \cup B_k$$

(since $C_k \wedge F_k$ holds). Let

$$(***)\qquad T_k = T - \Big(\bigcup_{r<k} D_r \cup B_k\Big) = T - \Big(\bigcup_{r\leq k} D_r\Big),$$

and

$$J = \Big\{A_j : j > k \text{ and } A_j \cap \bigcup T_k \neq \emptyset\Big\}.$$

Since C_k holds, $\bigcup T_k$ is not covered by finitely many A_j's and J is infinite.

Next we prove:

(2.a) $A_j \in J \Rightarrow \neg C_j$.

From $A_j \in J$, we have $A_j \cap \bigcup T_k \neq \emptyset$ and hence $B_j \cap T_k \neq \emptyset$. By $(*{*}*)$, there is $t \in B_j - \bigcup_{r \le k} D_r$, and using $(**)$, $B_j - \bigcup_{r<j} D_r \neq \emptyset$. This shows F_j holds; since $j > k$, C_j must fail.

(2.b) $\bigcup T_k$ *is covered by finitely many* A_j*'s.*

Let $A_{j_0}, \ldots, A_{j_{n-1}}$ be n distinct elements of J. Note that $j_i > k$, for each j_i, since each $A_{j_i} \in J$. By the argument used in (1), $\bigcup(\bigcap_{i<n} B_{j_i})$ is covered by $A_{j_0}, \ldots, A_{j_{n-1}}$. Since C_{j_i} fails (see (2.a)), $\bigcup(T - (\bigcup_{r<j_i} D_r \cup B_{j_i})) = \bigcup(T_k - B_{j_i})$ (see $(**)$ and $(*{*}*)$) is covered by finitely many A_j's, for $0 \le i \le n-1$. As in $(1.(*))$, it follows that $\bigcup(T_k - \bigcap_{i<n} B_{j_i})$ is also covered by finitely many A_j's, and hence so is $\bigcup T_k$.

This contradiction proves (2) and hence Theorem 2. ∎

In [2], p. 77, a symmetric model $\mathcal{N}$ was defined in which there exists an infinite set of reals A without a countable subset (in $\mathcal{N}$) and also possessing an injective function $\mathcal{F} : \mathcal{N} \to I \times \mathrm{On}$, where I is the set of all finite subsets of A. We claim that in this model every set is $\aleph_0$-partible.

THEOREM 3. *For all* $x \in \mathcal{N}$ *there exists a countable partition of* x *in* $\mathcal{N}$.

Proof. Let $x \in \mathcal{N}$ be such that $(x \text{ is infinite})^{\mathcal{N}}$. If $(\mathcal{F}[x])^{\mathcal{N}}$ has only finitely many $a \subseteq A$ (a finite) such that $\langle a, \alpha\rangle \in \mathcal{F}[x]$ for some α, we see that $R = \{\alpha : \langle a, \alpha\rangle \in \mathcal{F}[x] \text{ for some } a \subseteq A\}$ is infinite. Then, using the well-ordering of the ordinals we single out countably many elements of R and, since $\mathcal{F}$ is injective, we can thus define a countable subset of x.

On the other hand, suppose that there are infinitely many $a \subseteq A$ such that $\langle a, \alpha\rangle \in \mathcal{F}[x]$ for some α, and let S be the set of such a. If the cardinality of the elements of S is unbounded, in the sense that for $n \in \omega$ there is an $a \in S$ such that $|a| > n$, then S is partitioned into classes each of which contains the elements of S which have the same cardinality, and in this way we obtain a countable partition of $\mathcal{F}[x]$. If this is not the case, then there exists $n \in \omega$ such that $|x| \le n$ for all $x \in S$. But then, as we have seen, an infinite set of reals without a countable subset has a countable partition. Hence we can use Theorem 2 to get a countable partition of S. ∎

COROLLARY 6. P-$\aleph_0$ *does not imply* D *in* ZF. ∎

We shall prove in the sequel a generalization of Theorem 2, which was first proved by Kuratowski. (See [7], pp. 94–95 for Kuratowski's proof.) However, Kuratowski's argument to prove Theorem 4 is different from the argument presented here.

Theorem 4. *Let x be a set such that there exists $Y \subseteq \mathcal{P}(x)$ such that $|Y| = \omega$. Then x is $\aleph_0$-partible.*

Proof. Let $Y = \{y_n : n \in \omega\} \subseteq \mathcal{P}(x)$. We can suppose without loss of generality that $y_n \neq \emptyset$ and $y_n \neq x$ for all $n \in \omega$. We will define a sequence $\{z_n : n \in \omega\}$, and we claim that it includes an $\aleph_0$-partition of a subset of x. Suppose we have defined the sequence up to z_n.

We shall say that y_r is *n-equivalent* to y_m, $y_r \sim_n y_m$, iff $y_r - \bigcup_{s<n} z_s = y_m - \bigcup_{s<n} z_s$. Furthermore, let $S_n = \bigcup_{r<n} z_r \cup y_n$. Next we introduce another equivalence relation, $\approx_n$. Given $n \in \omega$, we define: $y_m \approx_n y_r \Leftrightarrow y_m - S_n = y_r - S_n$. Let $[y_m]_{\sim_n}$ and $[y_m]_{\approx_n}$ be the equivalence classes corresponding to these relations. Furthermore, we define

$$H_n = \{[y_m]_{\sim_n} : m \in \omega\} \quad \text{and} \quad K_n = \{[y - m]_{\approx_n} : m \in \omega\}.$$

Finally, C_n is the condition "K_n is infinite". Now we can define z_n:

$$z_n = \begin{cases} y_n - \bigcup_{m<n} z_m & \text{if } C_n, \\ x - (\bigcup_{m<n} z_m \cup y_n) & \text{otherwise.} \end{cases}$$

We claim that there are infinitely many $n \in \omega$ such that $z_n \neq \emptyset$. First, we note that $z_0 \neq \emptyset$. Now to get a contradiction, suppose that there exist only finitely many $i \in \omega$ such that $z_i \neq \emptyset$, and let n be the last such i. According to the next lemma, H_{n+1} is infinite, and hence there exist infinitely many y_m such that $y_m - \bigcup_{r \le n} z_r \neq \emptyset$. We fix some y_m with this condition such that $m > n$. Since for every $r > m$, $z_r = \emptyset$, we have $\bigcup_{r \le n} z_r = \bigcup_{r<m} z_r$, and thus $y_m - \bigcup_{r<m} z_r \neq \emptyset$. From this and $z_m = \emptyset$ we obtain $\neg C_m$, and hence $z_m = x - (\bigcup_{r<m} z_r \cup y_m)$, from which $\bigcup_{r<m} z_r \cup y_m = x$. As a consequence we obtain $x - \bigcup_{r<m} z_r \subseteq y_m$, and since $z_r = \emptyset$ for $r > n$, $x - \bigcup_{r \le n} z_r \subseteq y_m$. If we repeat the same argument for $[y_{m'}]_{\sim_n} \neq [y_m]_{\sim_n}$ with $m' > n$, we obtain $x - \bigcup_{r \le n} z_r \subseteq y_{m'}$. We claim $y_m \sim_n y_{m'}$. For this, let $h \in y_{m'} - \bigcup_{r \le n} z_r$. By the inclusion above we have $h \in y_m$, and $h \in (y_m - \bigcup_{r \le n} z_r)$. Exchanging m' for m, we obtain finally $y_m \sim_n y_{m'}$, a contradiction. ■

Lemma 8. *For all $n \in \omega$, H_n is infinite.*

Proof. We proceed by induction on n. The assertion is true for $n = 0$, since Y is infinite and $x \in H_0$ iff $x = \{y_r\}$ for some r (note that $y_m \sim_0 y_r$ iff $y_m = y_r$). We now suppose that it is true for $n \in \omega$. We proceed by analyzing two cases.

(i) C_n holds: This means that K_n is infinite and since $S_n = \bigcup_{r<n+1} z_r$, we have $[y_r]_{\approx_r} = [y_r]_{\sim_{n+1}}$, and $K_n = H_{n+1}$.

(ii) $\neg C_n$ holds: By the induction hypothesis H_n is infinite. We show that there are infinitely many $[y_m]_{\sim_{n+1}}$. Note that $[y_j]_{\sim_n} \subseteq [y_j]_{\approx_n}$ for every j. Since K_n is finite and H_n infinite, there exists an m such that $\{[y_j]_{\sim_n} : [y_j]_{\sim_n} \subseteq [y_m]_{\approx_n}\}$ is infinite. We fix one such m. Let m', m'' be such that

$[y_{m'}]_{\sim_n} \neq [y_{m''}]_{\sim_n}$ and $[y_{m'}]_{\sim_n}, [y_{m''}]_{\sim_n} \subseteq [y_m]_{\not\sim_n}$. It suffices to show that $[y_{m'}]_{\sim_{n+1}} \neq [y_{m''}]_{\sim_{n+1}}$. Assuming otherwise, we have

$$y_{m'} - \Big(\bigcup_{s<n} z_s \cup z_n\Big) = y_{m''} - \Big(\bigcup_{s<n} z_s \cup z_n\Big), \tag{1}$$

$$y_{m'} - \Big(\bigcup_{s<n} z_s \cup y_n\Big) = y_{m''} - \Big(\bigcup_{s<n} z_s \cup y_n\Big), \tag{2}$$

$$y_{m'} - \Big(\bigcup_{s<n} z_s\Big) \neq y_{m''} - \Big(\bigcup_{s<n} z_s\Big). \tag{3}$$

Beginning with an element realizing (3) and using (1) leads to a contradiction with (2). ∎

This last Theorem gives an alternative way of showing that the statement "if x is infinite then $\mathcal{P}(x)$ is Dedekind infinite" does not imply in ZF the statement "if x is infinite then x is Dedekind infinite" (for the standard proof of this, see [2], p. 81).

3. Further independence results. An important question in this context is whether DPO implies O in ZF. Within the theory ZFA, a suitable modification of ZF where the existence of atoms or *Urelemente* is admitted, the answer is negative, since in the permutation model presented below DPO is true, while O fails.

Let $\mathcal{N}$ be the second Fraenkel model, which is defined from the permutations of a countable set of pairs of *Urelemente*. (See, e.g., [2], p. 48.) This model is constructed from a countable set of atoms A (but this set is *not* countable in the model). The set A is partitioned in countably many disjoint pairs. The set of all these pairs is called B and is in the model. We use the following notation for B and its elements: $B = \{\{a_0, b_0\}, \{a_1, b_1\}, \ldots \ldots, \{a_n, b_n\}, \ldots\}$. Now we consider the group $\mathcal{G}$ of all those permutations of A which preserve pairs, i.e., $\pi(\{a, b\}) = \{a, b\}$, and the filter generated by the ideal of finite subsets of A (see [2], p. 47). We define $\mathrm{sym}_{\mathcal{G}}(x) = \{\pi \in \mathcal{G} : \pi(x) = x\}$ and $\mathrm{fix}_{\mathcal{G}}(y) = \{\pi \in \mathcal{G} : \pi(x) = x$ for every $x \in y\}$. For $x \in A$, we say that an element E of the ideal is a *support* of x iff $\mathrm{fix}(E) \subseteq \mathrm{sym}(x)$. (For the details of this construction, see [2], pp. 45–48.)

COUNTEREXAMPLE 2. *In the model $\mathcal{N}$ there exist sets without a least support.*

We consider the set $\{a_0\}$ with $a_0 \in A$. If $\pi \in \mathcal{G}$, $\pi(a_0) = a_0$, we have $\pi(b_0) = b_0$, because $\pi(\{a_0, b_0\}) = \{a_0, b_0\}$. We note that both $\{a_0\}$ and $\{b_0\}$ are supports of $\{a_0\}$, but $\emptyset = \{a_0\} \cap \{b_0\}$ is not.

DEFINITION 11. Let S be a support. Then S is called *normal* if it is of the form $\{a_{i_0}, b_{i_0}, a_{i_1}, b_{i_1}, \ldots, a_{i_n}, b_{i_n}\}$.

We remark that if S is a normal support and $\pi \in \mathcal{G}$, then $\pi[S] = S$ and $\pi[A - S] = A - S$. If $\pi \in \mathcal{G}$, then $\pi(\pi(a)) = a$ for all $a \in A$.

Lemma 9. *Let $x \in \mathcal{N}$. Then the intersection of two normal supports of x is also a normal support of x.*

Proof. Let S_1 and S_2 be two normal supports of x. We claim that $S = S_1 \cap S_2$ is a normal support of x. Normality is obvious. To show that it is a support, let $\pi \in \mathrm{fix}(S)$. We prove $\pi \in \mathrm{sym}(x)$.

We now define two permutations π_1, π_2 such that $\pi_1 \in \mathrm{fix}(S_1), \pi_2 \in \mathrm{fix}(S_2)$ by setting for $a \in A$,

$$\pi_1(a) = \begin{cases} \pi(a) & \text{if } a \notin S_1, \\ a & \text{otherwise,} \end{cases} \qquad \pi_2(a) = \begin{cases} \pi(a) & \text{if } a \in S_1 - S_2, \\ a & \text{otherwise.} \end{cases}$$

Since $S_1, A - S_1$, etc., are normal supports and complements of normal supports, by the above remark we see that π_1 and π_2 are permutations. Furthermore, they preserve pairs, and thus they belong to $\mathcal{G}$.

Straightforward verification using the fact that $\pi(a) = a$ for $a \in S$ shows that $\pi(a) = \pi_1(\pi_2(a))$ for all $a \in A$.

Since a permutation of A yields an automorphism of the universe, we have $\pi = \pi_1 \circ \pi_2$. We observe that $\pi_2(x) = x$ since $\pi_2 \in \mathrm{fix}(S_2)$ and S_2 is a support of x; similarly, $\pi_1(x) = x$. Thus $\pi(x) = \pi_1(\pi_2(x)) = x$, and $\pi \in \mathrm{sym}(x)$, showing that S is a support of x. ∎

Corollary 7. *Let $x \in \mathcal{N}$. If K is the set of normal supports of x, there exists a unique $S_0 \in K$ such that $|S_0| \leq |S|$ for all $S \in K$. S_0 is the least normal support of x.*

Proof. Standard, using Lemma 9 (pick S_0 to be a member of K of minimal cardinality). ∎

We now consider the (proper class) function $\mathcal{H} : \mathcal{N} \to \mathcal{P}^{<\omega}(A)$ ($\mathcal{P}^{<\omega}(A)$ is the set of finite subsets of A), where $\mathcal{H}(x)$ is the least normal support of x. Note that $\mathcal{H}$ is symmetric in the sense of [2], p. 49.

Lemma 10. *Let $x \in \mathcal{N}$, and let S be a support such that $\mathcal{H}(y) = S$ for all $y \in x$. Then x can be well-ordered.*

Proof. Let $\mathcal{F}$ be the filter that is used to define the model $\mathcal{N}$. Recall that $\mathrm{fix}(x) \in \mathcal{F}$ implies x well-orderable (see [2], p. 47). We will show that $\mathrm{fix}(x) \in \mathcal{F}$. For this we claim that $\mathrm{fix}(S) \subseteq \mathrm{fix}(x)$. Let $\pi \in \mathrm{fix}(S)$ and let $y \in x$. Since S is a support of y, we have $\mathrm{fix}(S) \subseteq \mathrm{sym}(y)$, i.e., $\pi(y) = y$. Hence $\pi \in \mathrm{fix}(x)$. ∎

Lemma 11. *There are countably many normal supports.*

Proof. Let R be the set of normal supports. Then we define $f : R \to \mathcal{P}^{<\omega}(\omega)$ in the following way: for $S \in R$, $S = \{a_{i_0}, b_{i_0}, a_{i_1}, b_{i_1}, \ldots, a_{i_k}, b_{i_k}\}$, let $f(S) = \{i_0, i_1, \ldots, i_k\}$. Note that $f \in \mathcal{N}$ because the enumeration of B is in $\mathcal{N}$. It is easy to see that f is a bijection between R and $\mathcal{P}^{<\omega}(\omega)$, hence R is countable. ∎

THEOREM 5. *If x is infinite and $x \in \mathcal{N}$, then x is $\aleph_0$-partible in $\mathcal{N}$.*

Proof. Let $\mathcal{H}$ be as above. If the cardinal number of the elements of $\mathcal{H}[x]$ is unbounded, we define a partition of x by putting in the same class the elements of x whose least normal supports have the same cardinality. Otherwise, we analyze two cases. If $\mathcal{H}[x]$ is infinite, then it is countable, as follows immediately from the preceding lemma. Then an enumeration of $\mathcal{H}[x]$ induces a countable partition of x. If $\mathcal{H}[x]$ is finite, there exists an infinite subset of x such that $\mathcal{H}$ assigns the same least normal support to all of them. But then this subset is well-ordered by Lemma 10, and it yields a countable partition of x. ∎

COROLLARY 8. DPO *does not imply* O *in* ZFA. ∎

DEFINITION 12. Let AC_2^ω be the Axiom of Choice restricted to countable families of pairs.

COROLLARY 9. P-$\aleph_0$ *does not imply* AC_2^ω *in* ZFA. ∎

Now we briefly sketch the transfer of the above result into ZF. This transfer was found independently by D. Pincus and the author. The technique by Jech and Sochor seems too weak to transfer P-$\aleph_0$. Indeed, on the one hand, P-$\aleph_0$ is not immediately equivalent to the existential closure of a formula in which every quantifier is bounded to some rank of the permutation model (see Problem 1 in [2], p. 94). On the other hand, there exist sets in $\mathcal{N}$ without least support and thus we cannot use Theorem 6.6 of [2], p. 90. For this transfer, we use the method developed in [4] from which the definitions of *boundable*, *injectively boundable* and *surjectively boundable* formulas and statements are taken (see [4], pp. 721–722). A statement Φ is called *transferable* if there is a metatheorem: "If Φ is true in a permutation model, then Φ is consistent with ZF". Pincus [4] proves:

THEOREM 6. *An injectively boundable statement is transferable. Hence, so is any surjectively boundable statement.* ∎

We also need the following:

DEFINITION 13. For a set x, the *surjective cardinal* $|x|^-$ is

$$|x|^- = \sup\{\alpha : \text{there is a surjection from } x \text{ onto } \alpha\}.$$

Let $\theta(x)$ denote the property "if x is infinite, then x is $\aleph_0$-partible". Since $|x|^- > \omega$ implies $\theta(x)$, the following is provable in set theory with atoms:

$$\forall x\,(|x|^- \le \omega \Rightarrow \theta(x)) \Leftrightarrow \forall x\,\theta(x).$$

Thus, P-$\aleph_0$ is equivalent to a surjectively boundable statement in set theory with atoms. Furthermore, the statement "there exists a set of pairs without a choice set" is boundable and hence surjectively boundable. From this we deduce that the statement "P-$\aleph_0$ and there exists a set of pairs without a choice set" is transferable. Thus, we have:

THEOREM 7. P-$\aleph_0$ *does not imply* AC_2^ω *in* ZF. ∎

COROLLARY 10. DPO *does not imply* O *in* ZF. ∎

In the sequel we will show that O does not imply DO in ZF. Let $\mathcal{N}$ be the ordered Mostowski model given by a set A of *Urelemente* isomorphic to the rationals (see [2], p. 49). Let a, b, c be in A such that $a < b$ and $c \notin [a, b]$, and let $B = [a, b] \cup \{c\}$ in A. We note that B is in $\mathcal{N}$ with support $\{a, b, c\}$. Then we have the following:

THEOREM 8. *B cannot be linearly and densely ordered in $\mathcal{N}$.*

Proof. To get a contradiction suppose that $<_D$ is a linear dense order on B in $\mathcal{N}$. Since $<_D \in \mathcal{N}$, it has a finite support S: $\mathrm{fix}(S) \subseteq \mathrm{sym}(<_D)$. Let $S' = (S \cap B) - \{a, b, c\}$. If $S' = \emptyset$, let $I_0 = (a, b)$. (Intervals in this theorem are always taken with respect to $<$, the order on A.) If not, let $S' = \{s_0, \ldots, s_{n-1}\}$ with $|S'| = n$ and $s_i < s_{i+1}$. Then $I_0 = (a, s_0)$, $I_i = (s_{i-1}, s_i)$ for $0 < i < n$, and $I_n = (s_{n-1}, b)$. We observe that there is no $s \in S$ between two successive elements of S'. We claim that there is no r such that $r \notin I_i$ for $i \le n$, and $x <_D r <_D y$, with $x <_D y$ and $x, y \in I_i$. Otherwise, we fix such x, y and r. Then we note that there exists a $\pi \in \mathrm{fix}(S')$ such that $\pi(z) = z$ and $\pi(x) = y$ for all $z \notin I_i$. In this case, since $\pi \in \mathrm{fix}(S')$, π preserves $<_D$, and from $x <_D r$ we obtain $\pi(x) <_D \pi(r)$, i.e. $y <_D r$, contradicting $r <_D y$. Thus $B - (S \cup \{a, b, c\})$ is partitioned into sets $I_{i_0}, \ldots, I_{i_n}$ so that $\forall x \,\forall y\, (x \in I_{i_j} \wedge y \in I_{i_{j+1}} \Rightarrow x <_D y)$. Let $T = S' \cup \{a, b, c\}$. Since S is finite, and $<_D$ is a linear dense order, the elements of T must be either extremes of $<_D$, or we must have each of them separating two intervals $I_{i_j}, I_{i_{j+1}}$. Since we have $n + 1$ intervals, there can be at most $n + 2$ such elements. But T has $n + 3$ elements, a contradiction. ∎

The result for ZFA proved by Theorem 8 is easily transfered into ZF by using a technique due to Jech and Sochor (see [2], pp. 85, 90, and also Problem 1 on p. 94). This method makes it possible to construct, from the ordered Mostowski model, a symmetric model in which the Boolean Prime Ideal Theorem (PI) and O hold (see op. cit., p. 113). In this way we obtain the following:

COROLLARY 11. O *does not imply* DO *in* ZF. ∎

COROLLARY 12. PI *does not imply* DO *in* ZF. ∎

A final remark: Sageev has constructed a model where the statements "for every infinite cardinal m, $m = \aleph_0 \cdot m$" and O hold but AC fails (see [6], p. 148). Actually, Sageev shows that "for every infinite cardinal m, $m = 2{\cdot}m$" holds, but Halpern and Howard have proved the equivalence with the former (see [1], p. 489). It is easy to see that these statements imply "every set can be ordered as the linear sum of canonically countable intervals" (i.e. the same function enumerates each interval). Using the argument of Lemma 1 it can be shown that DO holds in this model. In this way we have the following:

COROLLARY 13. DO *does not imply* AC *in* ZF. ∎

Glossary

DPO is "every infinite set can be non-trivially densely ordered".
O is "every set can be linearly ordered".
DO is "every infinite set can be linearly and densely ordered".
D is "every infinite set has a countable subset".
PP is "every infinite set is partible".
P-$\aleph_0$ is "every infinite set is $\aleph_0$-partible".
PI is the Boolean Prime Ideal Theorem.
AC_2^ω is the Axiom of Choice restricted to countable families of pairs.

References

[1] J. D. Halpern and P. E. Howard, *Cardinals m such that* $2m = m$, Proc. Amer. Math. Soc. 26 (1970), 487–490.
[2] T. Jech, *The Axiom of Choice*, North-Holland, Amsterdam, 1973.
[3] A. Levy, *The independence of various definitions of finiteness*, Fund. Math. 46 (1958), 1–13.
[4] D. Pincus, *Zermelo–Fraenkel consistency results by Fraenkel–Mostowski methods*, J. Symbolic Logic 37 (1972), 721–743.
[5] J. G. Rosenstein, *Linear Orderings*, Academic Press, New York, 1982.
[6] G. Sageev, *An independence result concerning the axiom of choice*, Ann. Math. Logic 8 (1975), 1–184.
[7] A. Tarski, *Sur les ensembles finis*, Fund. Math. 6 (1924), 45–95.

CENTRO DE LÓGICA, EPISTEMOLOGIA E HISTÓRIA DA CIÊNCIA
UNIVERSIDADE ESTADUAL DE CAMPINAS (UNICAMP)
P.O. BOX 6133
13081-000 CAMPINAS SP, BRAZIL
E-mail: GONZAL@CESAR.UNICAMP.BR

Received 24 August 1993;
in revised form 5 October 1994