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The hyperspace of finite subsets of a stratifiable space

by

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Abstract. It is shown that the hyperspace of non-empty finite subsets of a space X is an ANR (an AR) for stratifiable spaces if and only if X is a 2-hyper-locally-connected (and connected) stratifiable space.

0. Introduction. For a space X, let $\mathfrak{F}(X)$ denote the hyperspace of non-empty finite subsets of X with the Vietoris topology, i.e., the topology generated by the sets

$$\langle U_1, \ldots, U_n \rangle = \{A \in \mathfrak{F}(X) \mid A \subset U_1 \cup \ldots \cup U_n,\ U_i \cap A \neq \emptyset\ (\forall i = 1, \ldots, n)\},$$

where $n \in \mathbb{N}$ and $U_1, \ldots, U_n$ are open in X. We denote by $\mathcal{S}$ the class of stratifiable spaces [Bo_1] and by $\mathcal{M}$ the class of metrizable spaces. Note that $\mathfrak{F}(X) \in \mathcal{S}$ if $X \in \mathcal{S}$ (cf. [MK, Theorem 3.6]). In [CN], it is shown that $\mathfrak{F}(X)$ is an ANR($\mathcal{M}$) (an AR($\mathcal{M}$)) if and only if $X \in \mathcal{M}$ is locally path-connected (and connected). In this paper, we consider the condition for non-metrizable $X \in \mathcal{S}$ under which $\mathfrak{F}(X)$ is an ANR($\mathcal{S}$) (or an AR($\mathcal{S}$)).

A T_1-space X is 2-*hyper-locally-connected* (2-HLC) [$\mathrm{Bo}_{2,3}$] if there exist a neighborhood U of the diagonal ΔX in X^2 and a function $\lambda : U \times \mathbf{I} \to X$ satisfying the following conditions:

(a) $\lambda(x, y, 0) = x$ and $\lambda(x, y, 1) = y$ for each $(x, y) \in U$;
(b) the function $t \mapsto \lambda(x, y, t)$ is continuous for each $(x, y) \in U$;
(c) for each $x \in X$ and each neighborhood V of x, there is a neighborhood W of x such that $W^2 \subset U$ and $\lambda(W^2 \times \mathbf{I}) \subset V$.

The condition (c) means that $\lambda(x, x, t) = x$ for any $x \in X$ and $t \in \mathbf{I}$ and that λ is continuous at each point of $\Delta X \times \mathbf{I}$. In case $U = X^2$, X is said to be 2-*hyper-connected* (2-HC). In the above definition of 2-HLC or 2-HC,

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if λ is continuous then X is *locally equi-connected* (LEC) or *equi-connected* (EC) [Du]. Obviously if X is 2-HLC (2-HC) then X is locally path-connected (and path-connected). Conversely, it will be shown that every locally path-connected *metrizable* space is 2-HLC (Theorem 1.4). The following is our result:

MAIN THEOREM. *For a space* X, $\mathfrak{F}(X)$ *is an* ANR($\mathcal{S}$) (*an* AR($\mathcal{S}$)) *if and only if* $X \in \mathcal{S}$ *is* 2-HLC (*and connected*).

Since an ANR($\mathcal{S}$) is LEC (see [Ca$_3$]), it is 2-HLC. Thus we have the following:

COROLLARY. *For a connected* ANR($\mathcal{S}$) X, $\mathfrak{F}(X)$ *is an* AR($\mathcal{S}$).

1. Cauty's test space and the 2-HLC-ness. Let K be a simplicial complex. The nth skeleton of K is denoted by $K^{(n)}$. Let $|K|$ denote the polyhedron of K, i.e., $|K| = \bigcup K$ with the weak topology. For each $\sigma \in K$, the barycenter and the boundary of σ are denoted by $\widehat{\sigma}$ and $\partial\sigma$, respectively. And for any $0 < t \le 1$, let

$$\sigma(t) = \{x \in \sigma \mid 0 \le x(\widehat{\sigma}) < t\} \quad \text{and} \quad \sigma[t] = \{x \in \sigma \mid 0 \le x(\widehat{\sigma}) \le t\},$$

where $(x(\widehat{\sigma}))_{\sigma\in K}$ are the barycentric coordinates of x with respect to the barycentric subdivision of K. Each $x \in \sigma(1) = \sigma \setminus \widehat{\sigma}$ can be uniquely written as follows:

$$x = (1 - x(\widehat{\sigma}))\pi_\sigma(x) + x(\widehat{\sigma})\widehat{\sigma}, \qquad \pi_\sigma(x) \in \partial\sigma.$$

Then the map $\pi_\sigma : \sigma(1) \to \partial\sigma$ is called the *radial projection*. The simplex σ with vertices $v_0, \ldots, v_n$ is denoted by $\langle v_0, \ldots, v_n\rangle$ and a point $x \in \sigma$ is represented by $x = \sum_{i=0}^n x(v_i)v_i$, where $(x(v))_{v\in K^{(0)}}$ are the barycentric coordinates of x. Here we abuse the notation "$\langle \ldots \rangle$". But it can be recognized from the context to represent a simplex or a basic open set.

In [Ca$_3$], the first author constructed a space $Z(X)$ for every space X and proved that a stratifiable space X is an AR($\mathcal{S}$) (resp. an ANR($\mathcal{S}$)) if and only if X is a retract (resp. a neighborhood retract) of $Z(X)$. Let $F(X)$ denote the full simplicial complex with X the set of vertices (i.e., $X = F(X)^{(0)}$). Then $Z(X)$ is defined as $|F(X)|$ with the topology generated by open sets W in $|F(X)|$ such that

$$W \cap X \text{ is open in } X \quad \text{and} \quad |F(W \cap X)| \subset W.$$

The second condition above means that each $\tau \in F(X)$ is contained in W if all vertices of τ are contained in $W \cap X$. For each $A \subset X$, $F(A)$ is a subcomplex of $F(X)$ and $Z(A)$ is a subspace of $Z(X)$. If A is closed in X, then $Z(A)$ is closed in $Z(X)$. For each $n \in \mathbb{Z}_+ = \mathbb{N} \cup \{0\}$, let $Z_n(X) = |F(X)^{(n)}|$ viewed as a subspace of $Z(X)$. Then $Z_0(X) = X$ and $Z(X) =$

$\bigcup_{n\in\mathbb{Z}_+} Z_n(X)$. We use the following notations (see [GS]):

$$T(A) = \{\sigma \in F(X) \setminus F(A) \mid \sigma \cap A \neq \emptyset\},$$
$$M(A) = \{x \in Z(X) \mid \exists\sigma \in F(A) \text{ such that } x(\widehat{\sigma}) > 0\},$$
$$T_n(A) = T(A) \cap (F(X)^{(n)} \setminus F(X)^{(n-1)}) \quad \text{and}$$
$$M_n(A) = Z(A) \cup (M(A) \cap Z_n(X)).$$

For each $\varepsilon \in (0,1)^{T(A)}$, we define

$$M(A,\varepsilon) = \bigcup_{n\in\mathbb{Z}_+} M_n(A,\varepsilon),$$

where $M_0(A,\varepsilon) = Z(A) = |F(A)|$ and

$$M_n(A,\varepsilon) = Z(A) \cup \bigcup\{\sigma(\varepsilon(\sigma)) \cap \pi^{-1}(M_{n-1}(A,\varepsilon)) \mid \sigma \in T_n(A)\}$$

for each $n \in \mathbb{N}$. Then $M(A,\varepsilon) \cap X = A$. For any open set U in X, $M(U,\varepsilon)$ is an open set in $Z(X)$. Note that $M_n(A,\varepsilon)$ can also be defined for $\varepsilon \in (0,1)^{T_1(A)\cup\ldots\cup T_n(A)}$. The following is the same as [GS, Lemma 4.1].

1.1. Lemma. *The family*

$$\{M(U,\varepsilon) \mid U \text{ is open in } X,\ \varepsilon \in (0,1)^{T(U)}\}$$
$$(\text{resp. } \{M_1(U,\varepsilon) \mid U \text{ is open in } X,\ \varepsilon \in (0,1)^{T_1(U)}\})$$

is an open base for $Z(X)$ (*resp.* $Z_1(X)$). ■

The 2-HLC-ness is characterized as follows:

1.2. Theorem. *A space X is* 2-HLC (*resp.* 2-HC) *if and only if X is a neighborhood retract* (*resp. retract*) *of* $Z_1(X)$.

Proof. We only show the 2-HLC case since the 2-HC case is the same and easy.

To prove the "only if" part, give X a total order "$\leq$". Then each $z \in Z_1(X) \setminus X$ can be uniquely represented as follows:

$$z = (1-t_z)x_z + t_z y_z, \qquad x_z < y_z \in X,\ 0 < t_z < 1.$$

Let b_z be the barycenter of $\langle x_z, y_z\rangle$. Then observe $z = (1-2t_z)x_z + 2t_z b_z$ if $t_z \leq 1/2$ and $z = (1-2(1-t_z))y_z + 2(1-t_z)b_z$ if $t_z \geq 1/2$. Let $\lambda : U \times \mathbf{I} \to X$ be a function in the definition of 2-HLC-ness. Then X has an open cover $\mathcal{V}$ such that $W = \bigcup_{V\in\mathcal{V}} V^2 \subset U$. Then $N = \bigcup_{V\in\mathcal{V}} M_1(V,1/2)$ is an open neighborhood of X in $Z_1(X)$. Observe that $z \in N \setminus X$ and $(x_z, y_z) \notin W$ imply $t_z < 1/4$ or $t_z > 3/4$. Now we define a retraction $r : N \to X$ by $r|X = \mathrm{id}$ and for each $z \in N \setminus X$,

$$r(z) = \begin{cases} \lambda(x_z, y_z, t_z) & \text{if } (x_z, y_z) \in W, \\ x_z & \text{if } (x_z, y_z) \notin W \text{ and } t_z < 1/4, \\ y_z & \text{if } (x_z, y_z) \notin W \text{ and } t_z > 3/4. \end{cases}$$

This is well defined by the condition (a). We show that r is continuous. Since $N \setminus X$ is a subspace of $|F(X)^{(1)}|$, $r|N \setminus X$ is continuous by the condition (b). Hence r is continuous at each point of $N \setminus X$. To see the continuity of r at any point $x \in X$, let U' be a neighborhood of $r(x) = x$ in X. By the condition of (c), x has a neighborhood V' in X such that $V' \subset V$ for some $V \in \mathcal{V}$ and $\lambda(V'^2 \times \mathbf{I}) \subset U'$. By Lemma 1.1, $M_1(V', 1/2)$ is an open neighborhood of x in $Z_1(X)$. By the definition, $r(M(V', 1/2)) \subset U'$, that is, r is continuous at any $x \in X$.

To prove the "if" part, let N be a neighborhood of X in $Z_1(X)$ and $r : N \to X$ a retraction. By Lemma 1.1, X has an open cover $\mathcal{V}$ and $\varepsilon_V \in (0,1)^{T_1(V)}$, $V \in \mathcal{V}$, such that $\bigcup_{V\in\mathcal{V}} M_1(V, \varepsilon_V) \subset N$. Then $U = \bigcup_{V\in\mathcal{V}} V^2$ is an open neighborhood of ΔX in X^2. We can define $\lambda : U \times \mathbf{I} \to X$ by

$$\lambda(x, y, t) = r((1-t)x + ty).$$

The condition (a) is obvious. The condition (b) follows from the continuity of $r|\langle x, y\rangle$. To see the condition (c), let U' be a neighborhood of x in X. By the continuity of r and Lemma 1.1, we can choose a neighborhood V' of x in U' and $\varepsilon'_{V'} \in (0,1)^{T_1(V')}$ so that V' is contained in some $V \in \mathcal{V}$ and $r(M_1(V', \varepsilon'_{V'})) \subset U'$, whence $\lambda(V'^2 \times \mathbf{I}) \subset r(M_1(V', \varepsilon'_{V'})) \subset U'$. ∎

The following is easily proved.

1.3. Theorem. *A connected* 2-HLC *space* X *is* 2-HC.

Proof. Let $\lambda : U \times \mathbf{I} \to X$ be a function in the definition of 2-HLC-ness. Since X is connected and locally path-connected, X is path-connected. For each $(x, y) \in X^2 \setminus U$, we have a path $\lambda_{(x,y)} : \mathbf{I} \to X$ such that $\lambda_{(x,y)}(0) = x$ and $\lambda_{(x,y)}(1) = y$. Then λ can be extended to $\widehat{\lambda} : X^2 \times \mathbf{I} \to X$ by $\widehat{\lambda}(x, y, t) = \lambda_{(x,y)}(t)$ for each $(x, y, t) \in (X^2 \setminus U) \times \mathbf{I}$. Obviously $\widehat{\lambda}$ satisfies the conditions (a), (b) and (c). Hence X is 2-HC. ∎

In the class $\mathcal{M}$, the 2-HLC-ness is identical with the local path-connectedness.

1.4. Theorem. *Every locally path-connected metrizable space is* 2-HLC. *Hence every connected and locally path-connected metrizable space is* 2-HC.

Proof. Let $X = (X, d)$ be a locally path-connected metric space. By $C(\mathbf{I}, X)$, we denote the set of all paths in X. We define

$$U = \{(x, y) \in X^2 \mid \exists f \in C(\mathbf{I}, X) \text{ such that } f(0) = x \text{ and } f(1) = y\}.$$

By the local path-connectedness, it is easy to see that U is a neighborhood of the diagonal ΔX in X^2. For each $(x, y) \in U \setminus \Delta X$, choose $\lambda_{(x,y)} \in C(\mathbf{I}, X)$

so that $\lambda_{(x,y)}(0) = x$, $\lambda_{(x,y)}(1) = y$ and

$$\operatorname{diam} \lambda_{(x,y)}(\mathbf{I}) < 2 \inf\{\operatorname{diam} f(\mathbf{I}) \mid f \in C(\mathbf{I}, X) \text{ such that } f(0) = x \text{ and } f(1) = y\}.$$

We define $\lambda : U \times \mathbf{I} \to X$ by $\lambda(x, x, t) = x$ and $\lambda(x, y, t) = \lambda_{(x,y)}(t)$ if $x \neq y$. Then λ satisfies the conditions (a) and (b). To see the condition (c), let $x \in X$ and V be a neighborhood of x. Choose $\delta > 0$ so that the δ-neighborhood of x is contained in V. Since X is locally path-connected, x has a neighborhood W such that for each $y, z \in W$, there is $f \in C(\mathbf{I}, X)$ such that $f(0) = y$, $f(1) = z$ and $f(\mathbf{I})$ is contained in the $\frac{1}{5}\delta$-neighborhood of x, whence $\operatorname{diam} f(\mathbf{I}) < \frac{2}{5}\delta$. For each $(y, z, t) \in W^2 \times \mathbf{I}$,

$$\begin{aligned} d(x, \lambda(y, z, t)) &\le d(x, y) + d(y, \lambda(y, z, t)) \\ &< \tfrac{1}{5}\delta + \operatorname{diam} \lambda_{(y,z)}(\mathbf{I}) < \tfrac{1}{5}\delta + \tfrac{4}{5}\delta = \delta. \end{aligned}$$

This means that $\lambda(W^2 \times \mathbf{I}) \subset V$. ■

2. Proof of the Main Theorem. The "only if" part of the Main Theorem follows from the following theorem:

2.1. THEOREM. *For a space* X, *if* $\mathfrak{F}(X)$ *is an* ANR($\mathcal{S}$) (*resp. an* AR($\mathcal{S}$)) *then* $X \in \mathcal{S}$ *is* 2-HLC (*resp.* 2-HC).

Proof. First note that X is homeomorphic to $\mathfrak{F}_1(X) \subset \mathfrak{F}(X) \in \mathcal{S}$, whence $X \in \mathcal{S}$ [Ce, Theorem 2.3].

In case $\mathfrak{F}(X)$ is an ANR($\mathcal{S}$), there exist an open neighborhood U of the diagonal ΔX in X^2 and a map $\gamma : U \times \mathbf{I} \to \mathfrak{F}(X)$ such that $\gamma(x, x, t) = \{x\}$ for any $x \in X$ and $t \in \mathbf{I}$, and $\gamma(x, y, 0) = \{x\}$ and $\gamma(x, y, 1) = \{y\}$ for any $(x, y) \in U$. For each $(x, y) \in U$, let

$$\Gamma(x, y) = \bigcup \gamma(\{(x, y)\} \times \mathbf{I}) = \bigcup_{t \in \mathbf{I}} \gamma(x, y, t) \subset X.$$

Then $\Gamma(x, y)$ is compact (cf. [Mi, 2.5.2]), whence it is metrizable [Ce, Corollary 5.7]. And as is easily observed, $\Gamma(x, y)$ is connected. Note that $\gamma(\{(x, y)\} \times \mathbf{I}) \subset \mathfrak{F}(\Gamma(x, y))$. By [CN, Lemma 2.2], $\Gamma(x, y)$ is locally connected. Thus each $\Gamma(x, y)$ is a Peano continuum, which is path-connected. For each $(x, y) \in U$, choose a path $\lambda_{(x,y)} : \mathbf{I} \to \Gamma(x, y)$ such that $\lambda_{(x,y)}(0) = x$ and $\lambda_{(x,y)}(1) = y$. We define $\lambda : U \times \mathbf{I} \to X$ by $\lambda(x, y, t) = \lambda_{(x,y)}(t)$. Then λ satisfies the conditions (a) and (b). To see the condition (c), let $x \in X$ and V be a neighborhood of x. Then $\mathfrak{F}(V)$ is a neighborhood of $\gamma(x, x, t) = \{x\}$ for each $t \in \mathbf{I}$. From the continuity of γ, there is a neighborhood W of x such that $\gamma(W^2 \times \mathbf{I}) \subset \mathfrak{F}(V)$, which implies that $\Gamma(y, z) \subset V$ for each $(y, z) \in W^2$. Thus $\lambda(W^2 \times \mathbf{I}) \subset V$.

In case $\mathfrak{F}(X)$ is an AR($\mathcal{S}$), $U = X^2$ in the above, whence X is 2-HC. ■

Before the proof of the "if" part of the Main Theorem, we note that the connected case implies the general case. In fact, if X is 2-HLC then X is locally connected, hence each component of X is open and closed. As is easily observed,

$$\{\langle X_1,\ldots,X_n\rangle \mid n\in\mathbb{N},\ \text{each } X_i \text{ is a component of } X\}$$

is a discrete open cover of $\mathfrak{F}(X)$ and each $\langle X_1,\ldots,X_n\rangle$ is homeomorphic to the product space $\mathfrak{F}(X_1)\times\ldots\times\mathfrak{F}(X_n)$. Thus $\mathfrak{F}(X)$ is an ANR($\mathcal{S}$) if $\mathfrak{F}(X_0)$ is an AR($\mathcal{S}$) for each component X_0 of X.

To prove the connected case, it suffices to construct a retraction $r: Z(\mathfrak{F}(X))\to\mathfrak{F}(X)$ by [Ca_3, Theorem 1.3]. By Theorems 1.2 and 1.3, we have a retraction of $Z_1(X)$ onto X, which induces a retraction $r^*:\mathfrak{F}(Z_1(X))\to\mathfrak{F}(X)$. In the following, we first construct a map $\theta: |F(\mathfrak{F}(X))^{(1)}|\to\mathfrak{F}(Z_1(X))$ such that $\theta|\mathfrak{F}(X)=\mathrm{id}$ and define a retraction $r_1=r^*\circ\theta: |F(\mathfrak{F}(X))^{(1)}|\to\mathfrak{F}(X)$. And then we extend r_1 to a retraction $r:|F(\mathfrak{F}(X))|\to\mathfrak{F}(X)$ by applying the following:

LEMMA 2.2 ([CN, Lemma 3.3]). *For any $(n+1)$-simplex σ $(n\ge 1)$, there exists a map $\varphi_\sigma:\sigma\to\mathfrak{F}_3(\partial\sigma)$ such that $\varphi_\sigma(x)=\{x\}$ for every $x\in\partial\sigma$, where $\mathfrak{F}_3(X)=\{A\in\mathfrak{F}(X)\mid \mathrm{card}\,A\le 3\}$.* ∎

Finally, we show the continuity of $r: Z(\mathfrak{F}(X))\to\mathfrak{F}(X)$ by using the following:

LEMMA 2.3. *Let $f:|F(X)|\to Y$ be continuous. Suppose that for each $x\in X$ and each neighborhood V of $f(x)$ in Y, there exists a neighborhood U of x in X such that $f(|F(U)|)\subset V$. Then $f: Z(X)\to Y$ is continuous.*

Proof. It suffices to verify the continuity of $f: Z(X)\to Y$ at each point $x\in X$. Let V be an open neighborhood of $f(x)$ in Y. By the assumption, we have an open neighborhood U of x in X such that $f(|F(U)|)\subset V$. Let $W=f^{-1}(V)\setminus(X\setminus U)$. Then W is open in $|F(X)|$ and $|F(W\cap X)|=|F(U)|\subset W$, whence W is open in $Z(X)$. Thus we have a neighborhood W of x in $Z(X)$. Since $f(W)\subset V$, f is continuous at x. ∎

Proof of the "if" part. As observed above, we only have to prove the connected case. In this case, we have a retraction $r^*:\mathfrak{F}(Z_1(X))\to\mathfrak{F}(X)$ which is induced by a retraction of $Z_1(X)$ onto X.

First we construct a map $\theta:|F(\mathfrak{F}(X))^{(1)}|\to\mathfrak{F}(Z_1(X))$ such that $\theta|\mathfrak{F}(X)=\mathrm{id}$. By [Ce, Theorem 2.2] and [Bo_1, Lemma 8.2], X has a continuous metric d. Let $\tau=\langle A,B\rangle\in F(\mathfrak{F}(X))^{(1)}$. For each $x\in A$, choose $y_x\in B$ such that $d(x,y_x)=\mathrm{dist}_d(x,B)$. Similarly, for each $y\in B$, choose $x_y\in A$ such that $d(y,x_y)=\mathrm{dist}_d(y,A)$. We define a map $\theta_\tau:\langle A,B\rangle\to\mathfrak{F}(Z_1(X))$

by

$$\theta_\tau((1-t)A+tB) = \{(1-t)x+ty_x \mid x \in A\} \cup \{(1-t)x_y+ty \mid y \in B\} \subset \bigcup_{x\in A}\langle x, y_x\rangle \cup \bigcup_{y\in B}\langle y, x_y\rangle \subset Z_1(X).$$

Then $\theta_\tau(A) = A$ and $\theta_\tau(B) = B$. In case $\tau = A \in F(\mathfrak{F}(X))^{(0)} = \mathfrak{F}(X)$, we have $\theta_A(A) = A$. The desired map $\theta : |F(\mathfrak{F}(X)^{(1)}| \to \mathfrak{F}(Z_1(X))$ is defined by $\theta|\tau = \theta_\tau$ for each $\tau \in F(\mathfrak{F}(X))^{(1)}$. Then clearly $\theta|\mathfrak{F}(X) = \mathrm{id}$.

Next we inductively define retractions $r_n : |F(\mathfrak{F}(X))^{(n)}| \to \mathfrak{F}(X)$ ($n \in \mathbb{N}$) so that $r_{n+1}||F(\mathfrak{F}(X))^{(n)}| = r_n$. Let $r_1 = r^*\theta$ and assume r_n has been defined. For each $(n+1)$-simplex $\sigma \in F(\mathfrak{F}(X))$, $r_n|\partial\sigma : \partial\sigma \to \mathfrak{F}(X)$ induces the map $\gamma_\sigma : \mathfrak{F}_3(\partial\sigma) \to \mathfrak{F}_3(\mathfrak{F}(X))$. Let $\varsigma : \mathfrak{F}_3(\mathfrak{F}(X)) \to \mathfrak{F}(X)$ be the map defined by union, i.e., $\varsigma(\{A, B, C\}) = A \cup B \cup C$ (cf. [Ke]) and let $\varphi_\sigma : \sigma \to \mathfrak{F}_3(\partial\sigma)$ be the map of Lemma 2.2. Then the map $r_\sigma = \varsigma \circ \gamma_\sigma \circ \varphi_\sigma : \sigma \to \mathfrak{F}(X)$ extends $r_n|\partial\sigma$. In fact, for each $x \in \partial\sigma$,

$$r_\sigma(x) = \varsigma \circ \gamma_\sigma \circ \varphi_\sigma(x) = \varsigma \circ \gamma_\sigma(\{x\}) = r_1(x).$$

We can define $r_{n+1} : |F(\mathfrak{F}(X))^{(n+1)}| \to \mathfrak{F}(X)$ by $r_{n+1}|\sigma = r_\sigma$ for each $(n+1)$-simplex $\sigma \in F(\mathfrak{F}(X))$.

Finally, let $r : |F(\mathfrak{F}(X))| \to \mathfrak{F}(X)$ be the retraction defined by $r||F(\mathfrak{F}(X))^{(n)}| = r_n$ for each $n \in \mathbb{N}$. For each $A_0 \in \mathfrak{F}(X)$ and each neighborhood $\mathcal{V}$ of A_0 in $\mathfrak{F}(X)$, we will construct a neighborhood $\mathcal{U}$ of A_0 in $\mathfrak{F}(X)$ so that $r(|F(\mathcal{U})|) \subset \mathcal{V}$. Then by Lemma 2.3, $r : Z(\mathfrak{F}(X)) \to \mathfrak{F}(X)$ is continuous. Let $A_0 = \{x_1, \ldots, x_n\}$ ($x_i \neq x_j$ if $i \neq j$) and

$$\delta = \min\{d(x_i, x_j) \mid i \neq j\} > 0.$$

We may assume that $\mathcal{V} = \langle V_1, \ldots, V_n\rangle$, where each V_i is an open neighborhood of x_i in X. Then one should observe that $\varsigma(\mathfrak{F}_3(\mathcal{V})) \subset \mathcal{V}$. Since r^* is continuous, $(r^*)^{-1}(\mathcal{V})$ is a neighborhood of A_0 in $\mathfrak{F}(Z_1(X))$. By Lemma 1.1, each x_i has an open neighborhood U_i in X with $\eta_i \in (0,1)^{T_1(U_i)}$ such that $\mathrm{diam}_d\, U_i \leq \frac{1}{4}\delta$ and

$$\langle M(U_1, \eta_1), \ldots, M(U_n, \eta_n)\rangle \subset (r^*)^{-1}(\mathcal{V}).$$

Then $\mathcal{U} = \langle U_1, \ldots, U_n\rangle$ is a neighborhood of A_0 in $\mathfrak{F}(X)$. To see that $r(|F(\mathcal{U})|) \subset \mathcal{V}$, it suffices to prove that $r(|F(\mathcal{U})^{(n)}|) \subset \mathcal{V}$ for each $n \in \mathbb{N}$. Let $\langle A, B\rangle \in F(\mathcal{U})^{(1)}$. For each $x \in A \cap U_i$, $\mathrm{dist}_d(x, B) = \mathrm{dist}_d(x, B \cap U_i)$, whence $y_x \in B \cap U_i$, so $\langle x, y_x\rangle \subset |F(U_i)^{(1)}| \subset M(U_i, \eta_i)$. Similarly, $\langle y, x_y\rangle \subset |F(U_i)^{(1)}| \subset M(U_i, \eta_i)$ for each $y \in B \cap U_i$. Then

$$\theta(\langle A, B\rangle) \subset \langle M(U_1, \eta_1), \ldots, M(U_n, \eta_n)\rangle \subset (r^*)^{-1}(\mathcal{V}),$$

whence $r(\langle A, B\rangle) = r^*\theta(\langle A, B\rangle) \subset \mathcal{V}$. Thus we have $r(|F(\mathcal{U})^{(1)}|) \subset \mathcal{V}$. Assume that $r(|F(\mathcal{U})^{(n)}|) \subset \mathcal{V}$ and let $\sigma \in F(\mathcal{U})^{(n+1)}$. Since γ_σ is induced by $r|\partial\sigma$ and $r(\partial\sigma) \subset \mathcal{V}$, we have $\gamma_\sigma(\mathfrak{F}_3(\partial\sigma)) \subset \mathfrak{F}_3(\mathcal{V})$. Then

$$r(\sigma) = r_\sigma(\sigma) = \varsigma \circ \gamma_\sigma \circ \varphi_\sigma(\sigma) \subset \varsigma \circ \gamma_\sigma(\mathfrak{F}_3(\partial\sigma)) \subset \varsigma(\mathfrak{F}_3(\mathcal{V})) \subset \mathcal{V}.$$

Therefore $r(|F(\mathcal{U})^{(n+1)}|) \subset \mathcal{V}$. By induction, $r(|F(\mathcal{U})^{(n)}|) \subset \mathcal{V}$ for each $n \in \mathbb{N}$. The proof is complete. ∎

Remark. For each $n \in \mathbb{N}$, let $\mathfrak{F}_n(X) = \{A \in \mathfrak{F}(X) \mid \operatorname{card} A \le n\}$. In [Ca$_2$], it was asserted that each $\mathfrak{F}_n(X)$ is an ANR($\mathcal{S}$) (resp. AR($\mathcal{S}$)) for any ANR($\mathcal{S}$) (resp. AR($\mathcal{S}$)) X. However, one should note that the proof in [Ca$_2$] is based on some false results in [Ja] and [Ca$_1$] (cf. examples in [Ng] and [Sa]). Afterward Nguyen To Nhu [Ng] gave a proof for the metrizable case together with $\mathfrak{F}(X)$. The stratifiable case is still open, that is,

2.4. Problem. *For any* ANR($\mathcal{S}$) X, *is each* $\mathfrak{F}_n(X)$ *an* ANR($\mathcal{S}$)*?*

Concerning our result, the following problem is posed:

2.5. Problem. *Is a locally path-connected stratifiable space* X 2-HLC*?*

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