FUNDAMENTA
MATHEMATICAE
146 (1995)

Quasivarieties of pseudocomplemented semilattices

by

M. E. Adams (New Paltz, N.Y.), **W. Dziobiak** (Toruń),
Matthew Gould (Nashville, Tenn.) and **Jürg Schmid** (Bern)

Abstract. Two properties of the lattice of quasivarieties of pseudocomplemented semilattices are established, namely, in the quasivariety generated by the 3-element chain, there is a sublattice freely generated by ω elements and there are 2^ω quasivarieties.

1. Introduction. For background material on quasivarieties, the reader is referred to Burris and Sankappanavar [5] or Mal'cev [10]. The following definition of a quasivariety is but one of several equivalent formulations.

Given a class $\mathbf{K}$ of similar algebras, $\mathbf{I}(\mathbf{K})$ and $\mathbf{H}(\mathbf{K})$ respectively denote the classes of all isomorphic and homomorphic images of algebras in $\mathbf{K}$, $\mathbf{S}(\mathbf{K})$ denotes the class of all subalgebras of algebras in $\mathbf{K}$, and the classes $\mathbf{P}(\mathbf{K})$ and $\mathbf{P_u}(\mathbf{K})$ consist respectively of all direct products and all ultraproducts of members of $\mathbf{K}$. (The direct product of an empty set of algebras, the trivial algebra, is included in $\mathbf{P}(\mathbf{K})$.) If $\mathbf{K} \supseteq \mathbf{I}(\mathbf{K}) \cup \mathbf{S}(\mathbf{K}) \cup \mathbf{P}(\mathbf{K}) \cup \mathbf{P_u}(\mathbf{K})$ then $\mathbf{K}$ is said to be a *quasivariety*, while if $\mathbf{K} \supseteq \mathbf{H}(\mathbf{K}) \cup \mathbf{S}(\mathbf{K}) \cup \mathbf{P}(\mathbf{K})$ then $\mathbf{K}$ is a *variety*. (Every variety is a quasivariety.) The quasivariety generated by $\mathbf{K}$ (the least quasivariety containing $\mathbf{K}$) is denoted by $Q(\mathbf{K})$, and the variety generated by $\mathbf{K}$ is denoted by $V(\mathbf{K})$. In general, $Q(\mathbf{K}) = \mathbf{ISPP_u}(\mathbf{K})$ (and $V(\mathbf{K}) = \mathbf{HSP}(\mathbf{K})$), but when $\mathbf{K}$ and its members are finite, $Q(\mathbf{K}) = \mathbf{ISP}(\mathbf{K})$. If $\mathbf{K}$ is finite, say $\mathbf{K} = \{A_1, \ldots, A_n\}$, then $Q(\mathbf{K})$ will be denoted by $Q(A_1, \ldots, A_n)$, and likewise for $V(\mathbf{K})$.

The quasivarieties contained in a given quasivariety constitute a lattice under the inclusion ordering: the meet of two such is their intersection, and the join is the quasivariety generated by their union. The cardinality of the lattice is at most 2^ω if the algebras are of finite type.

Interest in the investigation of lattices of quasivarieties is directly or indirectly devoted to a problem of Mal'cev [9], who asked for a description of all lattices that can be represented isomorphically as lattices of quasivarieties.

1991 *Mathematics Subject Classification*: Primary 06A12, 08C15.

The present paper provides an example of a lattice of quasivarieties with an intricate lattice structure demonstrating the complexity of Mal'cev's problem. Amongst those papers referenced here, other examples of this kind are given in [1], [3], [4], [6], [12], [16], and [17].

A *pseudocomplemented semilattice* is an algebra $(S; \wedge, *, 0, 1)$ comprised of a semilattice $(S; \wedge)$ with a least element 0, a greatest element 1, and a unary operation $*$ such that, for all $s, t \in S$, $s \wedge t = 0$ if and only if $t \leq s^*$. Background material on pseudocomplemented semilattices and related topics is to be found in Grätzer [7] and the papers cited below.

The pseudocomplemented semilattice $\widehat{B}$ obtained by adjoining a new greatest element to a Boolean algebra B is of interest because Jones [8] proved that those pseudocomplemented semilattices of the form $\widehat{B}$ are precisely the subdirectly irreducible ones. Letting B_m $(m < \omega)$ denote the finite Boolean algebra with m atoms, we obtain a strictly increasing sequence $(Q(\widehat{B}_m) : m < \omega)$ of quasivarieties investigated in [13]. The only non-trivial variety of pseudocomplemented semilattices is $Q(\widehat{B}_0)$ (proved by Jones [8]), which is precisely the class of all Boolean algebras. Thus, $m = 0$ is the only case in which $Q(\widehat{B}_m) = V(\widehat{B}_m)$. The gap between $Q(\widehat{B}_1)$ and $V(\widehat{B}_1)$ is particularly striking because $V(\widehat{B}_1)$ is the entire class of pseudocomplemented semilattices (proved in Jones [8]; this and the aforementioned results of Jones are also proved in Sankappanavar [11]).

Our goal is to prove the following result.

THEOREM 1.1. *The lattice of all subquasivarieties of the quasivariety* $Q(\widehat{B}_1)$ *generated by the* 3*-element pseudocomplemented semilattice* $\widehat{B}_1$ *has a sublattice freely generated by* ω *elements* (*hence*, *satisfies no non-trivial lattice identity*) *and is of cardinality* 2^ω.

To put this theorem in perspective, it should be noted that the lattice of quasivarieties of pseudocomplemented distributive lattices has a sublattice freely generated by ω elements (Tropin [16]; see also [6]) and has cardinality 2^ω (independently proved in [1] and Wroński [17]). Moreover, the existence of 2^ω quasivarieties of pseudocomplemented semilattices was established in [4]; by applying a criterion of [13] it can be shown that the quasivarieties exhibited in [4] lie in $Q(\widehat{B}_3)$. Additional related results can be found in [13] and [14]. We remark that, from an abstract algebra point of view, it is of interest to observe that $\widehat{B}_1$ is a 3-element algebra: Shafaat [15] showed that, for any 2-element algebra A, $Q(A)$ is a 2-element chain (for related results see [3]).

The above theorem will be obtained as a consequence of a result of [2] stating that if a quasivariety $\mathbf{K}$ of algebras of finite type contains an infinite family of finite algebras indexed by the set of all finite subsets of ω

and satisfying certain postulates denoted (P1)–(P4), then the ideal lattice of a free lattice with ω free generators is embeddable in the lattice of subquasivarieties of $Q(\mathbf{K})$. Not only is Theorem 1.1 a consequence, but it also follows that $Q(\widehat{B}_1)$ is Q-universal. A quasivariety $\mathbf{K}$ of algebras of finite type is Q-*universal* if, for every quasivariety $\mathbf{L}$ of algebras of finite type, the lattice of subquasivarieties of $\mathbf{L}$ is a homomorphic image of a sublattice of the lattice of subquasivarieties of $\mathbf{K}$. This notion was introduced by Sapir [12] where the first examples of Q-universal quasivarieties were given.

The postulates (P1)–(P4) will be stated in §2 and appropriate members of $Q(\widehat{B}_1)$ will be constructed in §3. Basic properties of these algebras will be established in §4, enabling the proof in §5 that they satisfy (P1)–(P4).

2. Preliminaries. Let $P_{\rm fin}(\omega)$ denote the set of all finite subsets of ω. We shall need the following.

PROPOSITION 2.1 ([2]). *Let* $\mathbf{K}$ *be a quasivariety of algebras of finite type that contains an infinite family* $(S_W : W \in P_{\rm fin}(\omega))$ *of finite algebras satisfying the following postulates*:

(P1) $S_\emptyset$ *is a trivial algebra*;
(P2) *for* $X \in P_{\rm fin}(\omega)$, *if* $X = Y \cup Z$, *then* $S_X \in Q(S_Y, S_Z)$;
(P3) *for* $X, Y \in P_{\rm fin}(\omega)$, *if* $X \neq \emptyset$ *and* $S_X \in Q(S_Y)$, *then* $X = Y$;
(P4) *for* $X \in P_{\rm fin}(\omega)$, *if* S_X *is a subalgebra of* $B \times C$ *for finite* $B, C \in Q(\{S_W : W \in P_{\rm fin}(\omega)\})$, *then there are* $Y, Z \in P_{\rm fin}(\omega)$ *with* $S_Y \in Q(B)$, $S_Z \in Q(C)$, *and* $X = Y \cup Z$.

Then the ideal lattice of a free lattice with ω *generators is embeddable in the lattice of all subquasivarieties of* $\mathbf{K}$. *In particular*, *the lattice of all subquasivarieties of* $\mathbf{K}$ *has a sublattice freely generated by* ω *elements and is of cardinality* 2^ω. ■

Before constructing a subfamily of $Q(\widehat{B}_1)$ satisfying (P1)–(P4), we note the following terminology.

Let S be a pseudocomplemented semilattice. The *skeleton* of S is the set $S^* = \{s^* : s \in S\}$. Because $s^{***} = s^*$ for all $s \in S$, the skeleton is the image of the *Glivenko endomorphism* $\gamma : S \to S$ given by $\gamma(s) = s^{**}$ for all $s \in S$. The kernel of γ is called the *Glivenko congruence* Γ of S, and hence for $s \in S$ the set $[s]\Gamma = \{t \in S : \gamma(t) = \gamma(s)\}$ is known as the *Glivenko class* of s. When $[s]\Gamma$ is finite it has a least element, which we denote as μs. Because S^* is a Boolean algebra (with complementation given by $*$ and join given by $p \vee q = (p^* \wedge q^*)^*$ for all $p, q \in S^*$), if $\varphi : S \to T$ is a homomorphism (where T is a pseudocomplemented semilattice), then $\varphi \restriction S^* : S^* \to T^*$ is a Boolean homomorphism.

Finally, we mention some notations concerning a partially ordered set $(S; \leq)$. For $s \in S$, the set $\{t \in S : t \leq s\}$ is denoted by $(s]$, and analogously

$\{t \in S : t \geq s\}$ is denoted by $[s)$. For $s, t \in S$, the notation $s \succ t$ will signify that s *covers* t in the sense that $s > t$ and there is no $u \in S$ satisfying $s > u > t$.

3. The construction. In this section we will construct an infinite family $(S_W : W \in P_{\text{fin}}(\omega))$ of finite pseudocomplemented semilattices. We begin by defining a finite pseudocomplemented semilattice $(S_m; \leq)$ for each $m < \omega$.

For $m < \omega$, distinguish three atoms of B_{m+4}, denoted by a, b, and c, and let $d = a \vee b$ and $e = a \vee c$. (We will refer to the element a immediately and to the element b later in this section. However, we shall not refer to the elements c, d, and e until §4.) Let

$$S_m = (B_{m+4} \times \mathbf{3}) \setminus (\{(0,1),(0,2),(a,2),(1,0),(1,1)\} \cup \{(p,1) : p \neq a \text{ and } p \not\succ a\})$$

where $\mathbf{3}$ denotes the 3-element chain $\{0 < 1 < 2\}$. It is to be shown that $(S_m; \leq)$, where $\leq$ denotes the restriction of the usual ordering of $B_{m+4} \times \mathbf{3}$ to S_m, is a pseudocomplemented semilattice. By way of example, $(S_0; \leq)$ is

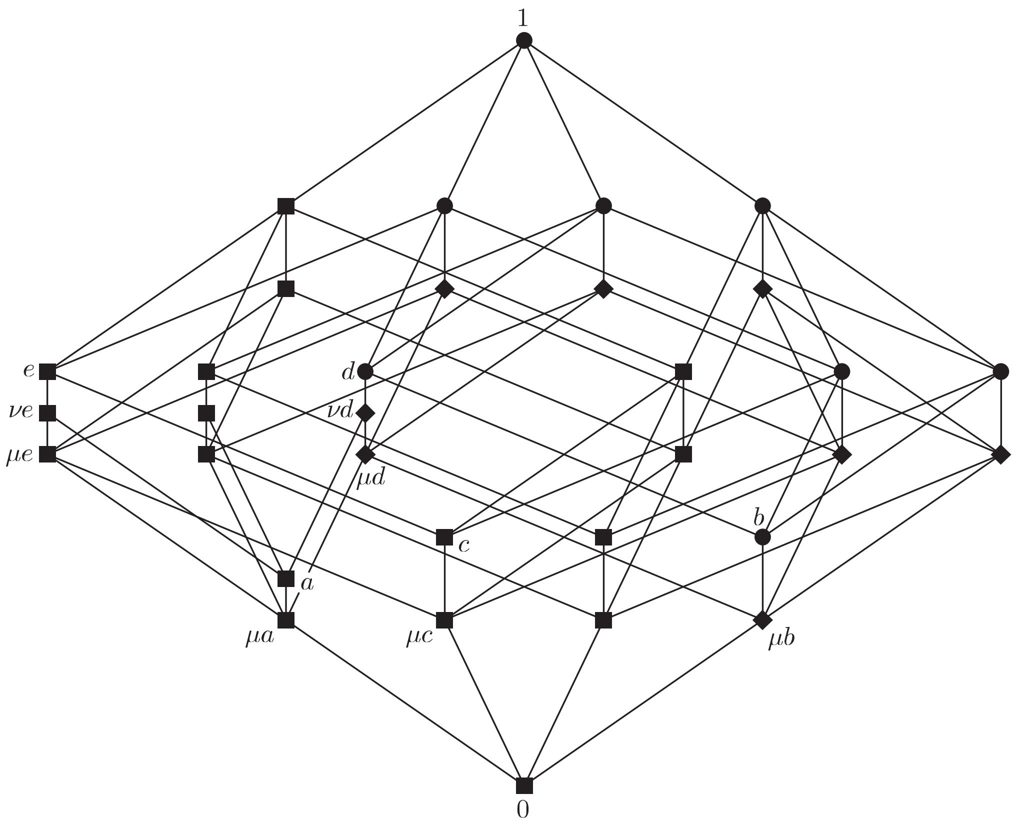

Fig. 1. S_0 with distinguished elements

diagrammed in Figure 1; for an explanation of the notation, see the paragraph following Lemma 3.2, and for an explanation of the different shapes used to denote elements, see the first paragraph of §4.

LEMMA 3.1. *For* $m < \omega$, $(S_m; \leq)$ *is a semilattice such that*, *for* (p, i), $(q, j) \in S_m$, $(p, i) \wedge (q, j) = (p \wedge q, k)$ *where* $k = 0$ *if* $p \wedge q = 0$, $k = 1$ *if* $p \wedge q = a$ *and* $i = j = 2$, *and* $i \wedge j$ *otherwise.*

Proof. For $(p, i), (q, j) \in S_m$, consider $(r, k) \in S_m$ such that $(r, k) \leq (p, i)$ and (q, j). Clearly, $(r, k) \leq (p \wedge q, i \wedge j)$ and, in the event that $(p \wedge q, i \wedge j) \in S_m$, $(p, i) \wedge (q, j) = (p \wedge q, i \wedge j)$. Suppose $(p \wedge q, i \wedge j) \notin S_m$.

By hypothesis, $p \wedge q \neq 1$.

If $p \wedge q = 0$, then $r = 0$ and it follows that, as required, $(r, k) = (0, 0)$.

If $p \wedge q = a$, then $i \wedge j = 2$ and, hence, $i = j = 2$. Consequently, $(r, k) = (0, 0)$, $(a, 0)$, or $(a, 1)$, whereupon $(p, i) \wedge (q, j) = (a, 1)$.

It remains to consider $p \wedge q \notin \{0, a, 1\}$. To complete the proof it is sufficient to show that (under the hypothesis $(p \wedge q, i \wedge j) \notin S_m$) this can never occur. Suppose $p \wedge q \notin \{0, a, 1\}$. Then, since $(p \wedge q, i \wedge j) \notin S_m$, $i \wedge j = 1$. Without loss of generality, we may assume $i = 1$. Since $(p, i) \in S_m$, either $p = a$ or $p \succ a$. Because $j \geq i$, it would follow that $(p \wedge q, i \wedge j) = (p, i) \in S_m$ were $q \geq p$. Thus, $q \ngeq p$ and, as a consequence, $p \wedge q = a$ or 0, which is absurd. ∎

LEMMA 3.2. *For* $m < \omega$, $(S_m; \leq)$ *is a pseudocomplemented semilattice where* $(1, 2)^* = (0, 0)$, $(a^*, 0)^* = (a^*, 2)^* = (a, 1)$, *and*, *for* $p \neq a^*$ *or* 1, $(p, i)^* = (p^*, 2)$.

Proof. By Lemma 3.1, for $(p, i), (q, j) \in S_m$, $(p, i) \wedge (q, j) = (0, 0)$ if and only if $p \wedge q = 0$ and, in particular, $q \leq p^*$. Consequently, $(p, i)^* = (p^*, 2)$, $(0, 0)$, or $(a, 1)$ for $p^* \neq 0$ or a, $p^* = 0$, or $p^* = a$, respectively. ∎

By Lemma 3.2, for $m < \omega$, $S_m^* = \{(0, 0), (a, 1)\} \cup \{(p, 2) : p \neq 0 \text{ or } a\}$, $|[(0, 0)]\Gamma| = |[(1, 2)]\Gamma| = 1$, $|[(a, 1)]\Gamma| = 2$, $|[(p, 2)]\Gamma| = 3$ if $p \succ a$, and $|[(p, 2)]\Gamma| = 2$ otherwise. Rather than consider n-tuples of ordered pairs, we will now, in the forthcoming interests of clarity, identify $(p, i) \in S_m^*$ with p. Thus, $(0, 0)$, $(a, 1)$, and $(p, 2)$ will be identified with 0, a, and p, respectively. Further, the smallest element of $[p]\Gamma$ will be denoted by μp. Thus, $(1, 2)$ and $(p, 0)$ for $p \neq 1$ will be identified with $\mu 1$ and μp, respectively. Finally, if $p \in S_m^*$ and $|[p]\Gamma| = 3$, then the remaining element of $[p]\Gamma$ will be denoted by νp. To summarize, for $p \in S_m^*$, $\mu p \leq p$ where the inequality is strict unless $p = 0$ or 1. Furthermore, $|[p]\Gamma| > 2$ if and only if $p \succ a$, in which case $[p]\Gamma = \{\mu p, \nu p, p\}$ where $\mu p < \nu p < p$.

We shall need the following result in order to show that, for $m < \omega$, $S_m \in Q(\widehat{B}_1)$.

Proposition 3.3 ([13]; cf. [14]). *For a finite pseudocomplemented semilattice* $(S; \wedge, *, 0, 1)$, $s \in S$, *and* μs *the smallest element of* $[s]\Gamma$, *if* $\theta \subseteq S \times S$ *is given by* $(t, r) \in \theta$ *if and only if both* t *and* r *belong to* $[s)$, $[\mu s) \setminus [s)$, *or* $S \setminus [\mu s)$ *and, in addition,* $t^{**} \wedge s = r^{**} \wedge s$ *also holds for the case* $t, r \in S \setminus [\mu s)$, *then* θ *is a congruence. Furthermore,* $S/\theta \cong \widehat{B}_m$ *for some* $m \geq 1$ *if and only if* $s \neq \mu s$. ∎

For $m < \omega$ and an atom p of S_m^*, let θ_p be the congruence given by Proposition 3.3 with $s = p$. Since $p \neq \mu p$ and $t^{**} \wedge p = 0$ for all $t \in S \setminus [\mu p)$, it follows that if p is an atom of S_m^*, then, for all $t, r \in S_m$, $(t, r) \in \theta_p$ if and only if both t and r belong to $[p)$, $[\mu p) \setminus [p)$, or to $S_m \setminus [\mu p)$. In other words, the equivalence classes of θ_p are $[p)$, $[\mu p) \setminus [p)$, and $S_m \setminus [\mu p)$ and, in particular, $S_m/\theta_p \cong \widehat{B}_1$.

Lemma 3.4. *For* $m < \omega$, $S_m \in Q(\widehat{B}_1)$.

Proof. For $q \in S_m^*$ distinct from 0 and 1, there exists an atom p of S_m^* such that $p \leq q$. In particular, $\mu q \not\equiv q\ (\theta_p)$. Furthermore, if $q \succ a$, then p may be chosen so that $q = a \vee p$. It follows that $\nu q \not\equiv q\ (\theta_p)$ and $\mu q \not\equiv \nu q$ (θ_a). Since S_m^* is a Boolean algebra and $S_m^* \cong S_m/\Gamma$, it follows that S_m is a subdirect product of suitably many copies of $\widehat{B}_0$ and $\widehat{B}_1$. ∎

We are now ready to define an infinite family $(S_W : W \in P_{\mathrm{fin}}(\omega))$ of finite pseudocomplemented semilattices. Recall that, for $m < \omega$, b is a distinguished atom of S_m^* distinct from a. For $W \in P_{\mathrm{fin}}(\omega)$ where $W = \{m_1, \ldots, m_n\}$, let S_W denote the subset of $\prod(S_{m_i} : m_i \in W)$ whose elements are precisely those $(s_1, \ldots, s_n)$ for which $s_i \in [b)$ for all $1 \leq i \leq n$, or $s_i \in [\mu b) \setminus [b)$ for all $1 \leq i \leq n$, or $s_i \in S_m \setminus [\mu b)$ for all $1 \leq i \leq n$. It is readily seen that S_W is a subalgebra of $\prod(S_{m_i} : m_i \in W)$. By Lemma 3.4, for every $W \in P_{\mathrm{fin}}(\omega)$, $S_W \in Q(\widehat{B}_1)$.

4. Homomorphisms of S_X. Let X be a non-empty member of $P_{\mathrm{fin}}(\omega)$. For each $i \in X$ the projection map $\pi_i : S_X \to S_i$ sends each member of S_X to its ith component. Recalling the elements $a, b, c, d, e, 0$ and 1 of each S_i we specify elements $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{e}, \mathbf{0}$, and $\mathbf{1}$ of S_X by stipulating that for each $s \in \{a, b, c, d, e, 0, 1\}$ the corresponding $\mathbf{s} \in S_X$ satisfies $\pi_i(\mathbf{s}) = s$ for all $i \in X$. Since S_X is the subalgebra of $\prod(S_i : i \in X)$ whose elements are those $\mathbf{r} \in \prod(S_i : i \in X)$ for which $\pi_i(\mathbf{r}) \in [b)$ for all $i \in X$, or $\pi_i(\mathbf{r}) \in [\mu b) \setminus [b)$ for all $i \in X$, or $\pi_i(\mathbf{r}) \in S_i \setminus [\mu b)$ for all $i \in X$, it is evident that, for $s \in \{a, b, c, d, e, 0, 1\}$, $\mathbf{s} \in S_X$. Clearly, for $\mathbf{r} \in S_X$, the conditions $\pi_i(\mathbf{r}) \in [b)$ for all $i \in X$, or $\pi_i(\mathbf{r}) \in [\mu b) \setminus [b)$ for all $i \in X$, or $\pi_i(\mathbf{r}) \in S_i \setminus [\mu b)$ for all $i \in X$ determine a partition of S_X. The classes of this partition will be referred to as the $\mathbf{b}$-*class*, $\mu\mathbf{b}$-*class*, and $\mathbf{0}$-*class*, respectively. Evidently, $\mathbf{b}$ and $\mathbf{0}$ are the smallest members of their respective classes. Since S_X is finite, for every $\mathbf{r} \in S_X$, the element $\mu\mathbf{r}$ exists in S_X. One may also observe that, for any $\mathbf{r} \in S_X$, the element $\mu\mathbf{r}$ satisfies $\pi_i(\mu\mathbf{r}) = \mu\pi_i(\mathbf{r})$ for all $i \in X$ unless $\pi_i(\mathbf{r}) = 1$ for some $i \in X$, in which case $\mu\mathbf{r} = \mathbf{r}$. In particular, $\mu\mathbf{b}$ is also the smallest member of its class. Note further that $\nu\mathbf{d} \in S_X$ exists where $\pi_i(\nu\mathbf{d}) = \nu d$ for all $i \in X$. (By way of example, observe that, of the remaining elements distinguished thus far, $\mathbf{d}$ and $\mathbf{1}$ belong to the $\mathbf{b}$-class, $\nu\mathbf{d}$ belongs to the $\mu\mathbf{b}$-class, and $\mathbf{a}$, $\mathbf{c}$, and $\mathbf{e}$ belong to the $\mathbf{0}$-class.) In S_m, the *b-class*, *μb-class*, and *0-class* are defined likewise. See Figure 1, where, for S_0, members of its b-class, μb-class, and 0-class are indicated by circles, diamonds, and boxes, respectively.

Further, for $\mathbf{r} \in S_X^*$ and $j \in X$ the element $\mu_j\mathbf{r} \in \prod(S_i : i \in X)$ is defined so that $\pi_j(\mu_j\mathbf{r}) = \mu\pi_j(\mathbf{r})$ and $\pi_i(\mu_j\mathbf{r}) = \pi_i(\mathbf{r})$ for $i \in X \setminus \{j\}$: since there exist $\mathbf{r} \in S_X^*$ such that $\mu_j(\mathbf{r}) \notin S_X$ (for example, $\mathbf{r} = \mathbf{b}$), this notation will be used only when $\mu_j\mathbf{r} \in S_X$. For any Boolean algebra B, let $\mathrm{At}(B)$ denote the set of atoms of B. Then, for any $j \in X$ and $p \in \mathrm{At}(S_j^*) \setminus \{b\}$, $\zeta_j\mathbf{p}$ denotes the element of S_X^* for which $\pi_j(\zeta_j\mathbf{p}) = p$ and $\pi_i(\zeta_j\mathbf{p}) = 0$ for $i \in X \setminus \{j\}$. Note that, for every $j \in X$ and $p \in \mathrm{At}(S_j^*) \setminus \{b\}$, $\zeta_j\mathbf{p}$ exists in S_X and that the atoms of S_X^* are precisely all elements of this form together with $\mathbf{b}$.

Finally, for each $i \in \omega$, set $C_i(a) = \{a \vee p : p \in \mathrm{At}(S_i^*) \setminus \{a\}\}$. If $\mathbf{r} \in S_X^*$ satisfies $\pi_j(\mathbf{r}) \in C_j(a) \setminus \{d\}$ for some $j \in X$, let $\nu_j\mathbf{r}$ denote the element of S_X defined by $\pi_j(\nu_j\mathbf{r}) = \nu\pi_j(\mathbf{r})$ and $\pi_i(\nu_j\mathbf{r}) = \pi_i(\mathbf{r})$ for $i \in X \setminus \{j\}$; since $\pi_j(\mathbf{r}) \in C_j(a) \setminus \{d\}$ for some $j \in X$, it is always the case that $\nu_j\mathbf{r}$ exists in S_X.

In this section we shall develop properties of homomorphisms defined on S_X that separate $\mathbf{e}$ and $\nu_j\mathbf{e}$. The key properties are expressed in Propositions 4.3, 4.5, and 4.6.

The first lemma is a clear consequence of the construction of S_m.

LEMMA 4.1. *For $m \in \omega$, $r \in S_m^*$, and $s \in S_m \setminus S_m^*$, the following hold*:

(i) *if $r \le s$, then $r \in \{0, a\}$*;
(ii) *if $s \ge a$ and $s \in [r]\Gamma$, then $r \in C_m(a)$.* ∎

LEMMA 4.2. *Let $X \in P_{\mathrm{fin}}(\omega)$, $j \in X$, and $m \in \omega$. If $\varphi : S_X \to S_m$ is a homomorphism such that $\varphi(\mathbf{e}) \ne \varphi(\nu_j\mathbf{e})$, then the following hold*:

(i) *$\varphi(\mathbf{a}) = a$, $\varphi(\mathbf{e}) \in C_m(a)$, and $\varphi(\mathbf{c}) \in \mathrm{At}(S_m^*)$*;

(ii) *if $\mathbf{r} \in S_X^*$ satisfies $\pi_i(\mathbf{r}) \in C_i(a) \setminus \{d\}$ for all $i \in X$, then $\varphi(\mathbf{r}) \ne \varphi(\nu_j\mathbf{r})$ and $\varphi(\mathbf{r}) \in C_m(a)$*;

(iii) *$\varphi(\mathbf{b}) \ne \varphi(\mu\mathbf{b})$, $\varphi(\mathbf{d}) \ne \varphi(\nu\mathbf{d})$, $\varphi(\mathbf{b}) \in \mathrm{At}(S_m^*)$, and $\varphi(\mathbf{d}) \in C_m(a)$*;

(iv) *$\varphi(\mu\zeta_j\mathbf{p}) \ne \varphi(\zeta_j\mathbf{p}) \in \mathrm{At}(S_m^*)$ for $p \in \mathrm{At}(S_j^*) \setminus \{b\}$, $\varphi(\zeta_i\mathbf{p}) = 0$ for $i \in X \setminus \{j\}$ and $p \in \mathrm{At}(S_i^*) \setminus \{b\}$, and, in particular, $\varphi(\zeta_j\mathbf{a}) = a$.*

Proof. (i) Because $\varphi(\mathbf{e})$ and $\varphi(\nu_j\mathbf{e})$ are distinct and share a Glivenko class, $\varphi(\nu_j\mathbf{e}) \in S_m \setminus S_m^*$. Thus, since $\mathbf{a} \le \nu_j\mathbf{e}$, Lemma 4.1(i) implies that $\varphi(\mathbf{a}) \in \{0, a\}$. Suppose $\varphi(\mathbf{a}) = 0$. Immediately $\varphi(\mathbf{a}^*) = 1$, whence the triviality of $[1]\Gamma$ yields $\varphi(\mu\mathbf{a}^*) = 1$, and therefore $\varphi(\mu\mathbf{c}) = \varphi(\mu\mathbf{a}^* \wedge \mathbf{c}) = \varphi(\mu\mathbf{a}^*) \wedge \varphi(\mathbf{c}) = \varphi(\mathbf{c})$, forcing $\varphi(\mu_j\mathbf{c}) = \varphi(\mathbf{c})$. Since $\mu_j\mathbf{c} < \nu_j\mathbf{e}$, it then follows that $\varphi(\mathbf{c}) = \varphi(\mu_j\mathbf{c}) \le \varphi(\nu_j\mathbf{e})$, whence $\varphi(\mathbf{c}) \in \{0, a\}$ by Lemma 4.1(i). Inasmuch as the hypothesis entails $0 < \varphi(\mathbf{e}) = \varphi(\mathbf{a}) \vee \varphi(\mathbf{c}) = \varphi(\mathbf{c})$, it follows that $\varphi(\mathbf{c}) = a = \varphi(\mathbf{e})$. Thus $a = \varphi(\mathbf{c}) = \varphi(\mu_j\mathbf{c}) \le \varphi(\nu_j\mathbf{e}) \le \varphi(\mathbf{e}) = a$, forcing $\varphi(\nu_j\mathbf{e}) = \varphi(\mathbf{e})$, contrary to hypothesis. Hence, $\varphi(\mathbf{a}) = a$. Because $\varphi(\mathbf{e})$ and $\varphi(\nu_j\mathbf{e})$ are distinct and share a Glivenko class, and $\varphi(\nu_j\mathbf{e}) \ge \varphi(\mathbf{a}) = a$, Lemma 4.1(ii) now implies that $\varphi(\mathbf{e}) \in C_m(a)$. Since $\varphi(\mathbf{e}) = \varphi(\mathbf{a}) \vee \varphi(\mathbf{c}) = a \vee \varphi(\mathbf{c})$ and $0 = \varphi(\mathbf{0}) = \varphi(\mathbf{a}) \wedge \varphi(\mathbf{c}) = a \wedge \varphi(\mathbf{c})$, it follows that $\varphi(\mathbf{c}) \in \mathrm{At}(S_m^*)$.

(ii) For each $i \in X$ choose $q_i \in \mathrm{At}(S_i^*) \setminus \{a\}$ such that $\pi_i(\mathbf{r}) = a \vee q_i$. Let $\mathbf{q}$ denote the element of S_X satisfying $\pi_i(\mathbf{q}) = q_i$ for all $i \in X$: notice that $\mathbf{q}$ exists in S_X. If $\varphi(\mathbf{r}) = \varphi(\nu_j\mathbf{r})$, then $\varphi(\mathbf{q}) = \varphi(\mathbf{r}\wedge\mathbf{q}) = \varphi(\mathbf{r})\wedge\varphi(\mathbf{q}) = \varphi(\nu_j\mathbf{r})\wedge \varphi(\mathbf{q}) = \varphi(\nu_j\mathbf{r} \wedge \mathbf{q}) = \varphi(\mu_j\mathbf{q})$. Thus, $\varphi(\mu_j\mathbf{b}^*) \ge \varphi(\mu_j\mathbf{q}) = \varphi(\mathbf{q})$. Either $\varphi(\mu_j\mathbf{b}^*) = \varphi(\mathbf{b}^*)$ or, by Lemma 4.1(i), $\varphi(\mathbf{q}) \in \{0, a\}$. If $\varphi(\mu_j\mathbf{b}^*) = \varphi(\mathbf{b}^*)$, then $\varphi(\mathbf{e}) = \varphi(\mathbf{e} \wedge \mathbf{b}^*) = \varphi(\mathbf{e}) \wedge \varphi(\mathbf{b}^*) = \varphi(\mathbf{e}) \wedge \varphi(\mu_j\mathbf{b}^*) = \varphi(\mathbf{e} \wedge \mu_j\mathbf{b}^*) = \varphi(\mu_j\mathbf{e})$ and, since $\mu_j\mathbf{e} \le \nu_j\mathbf{e} \le \mathbf{e}$, $\varphi(\mathbf{e}) = \varphi(\nu_j\mathbf{e})$, contrary to hypothesis. If $\varphi(\mathbf{q}) = a$, then, by (i), $0 = \varphi(\mathbf{0}) = \varphi(\mathbf{a} \wedge \mathbf{q}) = a \wedge a = a$. Thus, $\varphi(\mathbf{q}) = 0$. In particular, $\varphi(\mathbf{q}^*) = 1$, whence the triviality of $[1]\Gamma$ yields $\varphi(\mu\mathbf{q}^*) = 1$, and therefore $\varphi(\mathbf{b}^*) = \varphi(\mu\mathbf{q}^*) \wedge \varphi(\mathbf{b}^*) = \varphi(\mu\mathbf{q}^* \wedge \mathbf{b}^*) = \varphi(\mu\mathbf{q}^* \wedge \mu\mathbf{b}^*) = \varphi(\mu\mathbf{q}^*) \wedge \varphi(\mu\mathbf{b}^*) = \varphi(\mu\mathbf{b}^*)$. Since $\varphi(\mu\mathbf{b}^*) \le \varphi(\mu_j\mathbf{b}^*) \le \varphi(\mathbf{b}^*)$, $\varphi(\mu_j\mathbf{b}^*) = \varphi(\mathbf{b}^*)$, which we have already seen to be impossible. Thus, $\varphi(\mathbf{r}) \ne \varphi(\nu_j\mathbf{r})$, whereupon, as $\varphi(\nu_j\mathbf{r}) \ge \varphi(\mathbf{a}) = a$, Lemma 4.1(ii) implies that $\varphi(\mathbf{r}) \in C_m(a)$.

(iii) For each $i \in X$ choose $q_i \in \mathrm{At}(S_i^*)\setminus\{a, b, c\}$. Let $\mathbf{q}$ denote the element of S_X satisfying $\pi_i(\mathbf{q}) = q_i$ for all $i \in X$. If $\varphi(\mathbf{q}^*) = \varphi(\mu\mathbf{q}^*)$, then, as $\mathbf{q}^* \ge \mathbf{e}$, $\varphi(\mathbf{e}) = \varphi(\mathbf{q}^* \wedge \mathbf{e}) = \varphi(\mathbf{q}^*) \wedge \varphi(\mathbf{e}) = \varphi(\mu\mathbf{q}^*) \wedge \varphi(\mathbf{e}) = \varphi(\mu\mathbf{q}^* \wedge \mathbf{e}) = \varphi(\mu\mathbf{e})$, contradicting the hypothesis. Hence, $\varphi(\mathbf{q}^*) \ne \varphi(\mu\mathbf{q}^*)$. If $\varphi(\mathbf{b}) = \varphi(\mu\mathbf{b})$, then, as $\varphi(\mathbf{q}^*) > \varphi(\mu\mathbf{q}^*) \ge \varphi(\mu\mathbf{b}) = \varphi(\mathbf{b})$, Lemma 4.1(i) implies that $\varphi(\mathbf{b}) \in \{0, a\}$. Since $\varphi(\mathbf{a}) = a$ and $\mathbf{a} \wedge \mathbf{b} = \mathbf{0}$, it follows that $\varphi(\mathbf{b}) = 0$ and, hence, $\varphi(\mathbf{b}^*) = 1$. By the triviality of $[1]\Gamma$, $\varphi(\mu\mathbf{b}^*) = \varphi(\mathbf{b}^*)$ and, arguing as above, $\varphi(\mathbf{e}) = \varphi(\mu\mathbf{e})$, in contradiction to the hypothesis. Thus, $\varphi(\mathbf{b}) \ne \varphi(\mu\mathbf{b})$.

If $\varphi(\mathbf{d}) = \varphi(\nu\mathbf{d})$, then $\varphi(\mathbf{b}) = \varphi(\mathbf{b})\wedge\varphi(\mathbf{d}) = \varphi(\mathbf{b})\wedge\varphi(\nu\mathbf{d}) = \varphi(\mathbf{b}\wedge\nu\mathbf{d}) = \varphi(\mu\mathbf{b})$, contradicting the above conclusion. Consequently, $\varphi(\mathbf{d}) \ne \varphi(\nu\mathbf{d})$. Since $\varphi(\nu\mathbf{d}) \in S_m \setminus S_m^*$ and $\varphi(\nu\mathbf{d}) \ge \varphi(\mathbf{a}) = a$, Lemma 4.1(ii) implies that $\varphi(\mathbf{d}) \in C_m(a)$. Since $\varphi(\mathbf{a} \vee \mathbf{b}) = \varphi(\mathbf{d})$, $\varphi(\mathbf{a} \wedge \mathbf{b}) = 0$, $\varphi(\mathbf{a}) = a$, and $\varphi(\mathbf{d}) \in C_m(a)$, it follows that $\varphi(\mathbf{b}) \in \mathrm{At}(S_m^*)$.

(iv) Initially, suppose $p \in \mathrm{At}(S_j^*) \setminus \{a, b\}$ and, for $i \in X \setminus \{j\}$, choose $q_i \in \mathrm{At}(S_i^*) \setminus \{a, b\}$ and let $\mathbf{q}$ denote the element of S_X satisfying $\pi_j(\mathbf{q}) = 0$

and $\pi_i(\mathbf{q}) = q_i$ for all $i \in X \setminus \{j\}$. By (ii), $\varphi(\mathbf{a} \vee (\zeta_j \mathbf{p} \vee \mathbf{q})) \in C_m(a)$. Since $\varphi(\mathbf{a}) = a$ by (i) and $\varphi(\mathbf{a} \wedge (\zeta_j \mathbf{p} \vee \mathbf{q})) = 0$, it must be the case that $\varphi(\zeta_j \mathbf{p} \vee \mathbf{q}) \in \mathrm{At}(S_m^*) \setminus \{a\}$. Because $\varphi(\zeta_j \mathbf{p} \wedge \mathbf{q}) = 0$, it follows that either $\varphi(\zeta_j \mathbf{p}) = 0$ and $\varphi(\mathbf{q}) \in \mathrm{At}(S_m^*) \setminus \{a\}$ or $\varphi(\zeta_j \mathbf{p}) \in \mathrm{At}(S_m^*) \setminus \{a\}$ and $\varphi(\mathbf{q}) = 0$. Since $\nu_j(\mathbf{a} \vee (\zeta_j \mathbf{p} \vee \mathbf{q})) \geq \mathbf{q}$ and $\mathbf{q} = \bigvee\{\zeta_i \mathbf{q}_i : i \in X \setminus \{j\}\} \in S_X^*$, (ii) and Lemma 4.1(i) imply $\varphi(\mathbf{q}) \in \{0, a\}$, which, in turn, since $\varphi(\mathbf{a}) = a$ (by (i)) and $\mathbf{a} \wedge \mathbf{q} = \mathbf{0}$, yields $\varphi(\mathbf{q}) = 0$. Hence, for $p \in \mathrm{At}(S_j^*) \setminus \{a, b\}$, $\varphi(\zeta_j \mathbf{p}) \in \mathrm{At}(S_m^*) \setminus \{a\}$.

Let $\mathbf{q} \in S_X$ be given by $\pi_j(\mathbf{q}) = 0$ and $\pi_i(\mathbf{q}) = a$ for all $i \in X \setminus \{j\}$. Since $\varphi(\mathbf{a}) = a$, $\zeta_j \mathbf{a} \vee \mathbf{q} = \mathbf{a}$, and $\zeta_j \mathbf{a} \wedge \mathbf{q} = \mathbf{0}$, we have either $\varphi(\zeta_j \mathbf{a}) = a$ and $\varphi(\mathbf{q}) = 0$ or $\varphi(\zeta_j \mathbf{a}) = 0$ and $\varphi(\mathbf{q}) = a$. We wish to show that $\varphi(\mathbf{q}) \neq a$. Suppose not and that, to the contrary, $\varphi(\mathbf{q}) = a$. As argued in the proof of (ii), $\varphi(\mu_j \mathbf{b}^*) \neq \varphi(\mathbf{b}^*)$. However, $\mu_j \mathbf{b}^* \geq \mathbf{q}$ and, hence, $\varphi(\mu_j \mathbf{b}^*) \geq \varphi(\mathbf{q}) = a$. By Lemma 4.1(ii), $\varphi(\mathbf{b}^*) \in C_m(a)$. Since $\mathbf{b} \vee \mathbf{b}^* = \mathbf{1}$ and $\varphi(\mathbf{b}) \in \mathrm{At}(S_m^*)$ (see (iii)), it follows that the unit in S_m^* is a join of 3 atoms, which is impossible by the definition of S_m. Hence, $\varphi(\mathbf{q}) \neq a$ and, as required, $\varphi(\zeta_j \mathbf{a}) = a$. Thus, for $p \in \mathrm{At}(S_j^*) \setminus \{b\}$, $\varphi(\zeta_j \mathbf{p}) \in \mathrm{At}(S_m^*)$.

Let $i \in X \setminus \{j\}$ and $p \in \mathrm{At}(S_i^*) \setminus \{b\}$. If $p \neq a$, then choose $p' \in \mathrm{At}(S_j^*) \setminus \{a, b\}$ and $q_k \in \mathrm{At}(S_k^*) \setminus \{a, b\}$ for $k \in X \setminus \{i, j\}$. Let $\mathbf{q}$ be given by $\pi_i(\mathbf{q}) = p$, $\pi_j(\mathbf{q}) = 0$, and $\pi_k(\mathbf{q}) = q_k$ for $k \in X \setminus \{i, j\}$. Then, arguing as above, $\varphi(\zeta_j \mathbf{p}' \vee \mathbf{q}) \in \mathrm{At}(S_m^*)$, which in turn (see the above) implies $\varphi(\mathbf{q}) = 0$. Hence, $\zeta_i \mathbf{p} \leq \mathbf{q}$ implies $\varphi(\zeta_i \mathbf{p}) = 0$. Were $p = a$, let $\pi_j(\mathbf{q}) = 0$ and $\pi_i(\mathbf{q}) = a$ for all $i \in X \setminus \{j\}$. Since $\zeta_j \mathbf{a} \vee \mathbf{q} = \mathbf{a}$, $\zeta_j \mathbf{a} \wedge \mathbf{q} = \mathbf{0}$, and $\varphi(\mathbf{a}) = \varphi(\zeta_j \mathbf{a}) = a$, we have $\varphi(\mathbf{q}) = 0$. Thus, $\zeta_i \mathbf{a} \leq \mathbf{q}$ yields $\varphi(\zeta_i \mathbf{a}) = 0$. Hence, for $i \in X \setminus \{j\}$ and $p \in \mathrm{At}(S_i^*) \setminus \{b\}$, $\varphi(\zeta_i \mathbf{p}) = 0$.

As already seen, $\varphi(\mathbf{e}) \neq \varphi(\nu_j \mathbf{e})$ implies $\varphi(\mathbf{b}^*) \neq \varphi(\mu_j \mathbf{b}^*)$. Since $\mu \zeta_j \mathbf{p} \leq \mu_j \mathbf{b}^*$ for $p \in \mathrm{At}(S_j^*) \setminus \{b\}$, if $\varphi(\mu \zeta_j \mathbf{p}) = \varphi(\zeta_j \mathbf{p})$, then $\varphi(\mathbf{b}^*) \in C_m(a)$ by Lemma 4.1(i),(ii). As argued above, this is impossible since the unit of S_m^* is not the join of 3 atoms. Thus, for $p \in \mathrm{At}(S_j^*) \setminus \{b\}$, $\varphi(\mu \zeta_j \mathbf{p}) \neq \varphi(\zeta_j \mathbf{p})$. ■

PROPOSITION 4.3. *Let $X \in P_{\mathrm{fin}}(\omega)$, $j \in X$, and $m \in \omega$. If $\varphi : S_X \to S_m$ is a homomorphism such that $\varphi(\mathbf{e}) \neq \varphi(\nu_j \mathbf{e})$, then $j = m$ and there is an automorphism η of S_m such that $\varphi = \eta \pi_m$.*

Proof. Because $\mathrm{At}(S_X^*) = \{\mathbf{b}\} \cup \{\zeta_i \mathbf{p} : p \in \mathrm{At}(S_i^*) \setminus \{b\}$ and $i \in X\}$, $1 = \varphi(\mathbf{1}) = \varphi(\bigvee \mathrm{At}(S_X^*)) = \varphi(\mathbf{b}) \vee \bigvee\{\varphi(\zeta_i \mathbf{p}) : p \in \mathrm{At}(S_i^*) \setminus \{b\}$ and $i \in X\}$, which, by Lemma 4.2(iv), is $\varphi(\mathbf{b}) \vee \bigvee\{\varphi(\zeta_j \mathbf{p}) : p \in \mathrm{At}(S_j^*) \setminus \{b\}\}$. Also by Lemma 4.2(iii),(iv), each of $\varphi(\mathbf{b})$ and $\varphi(\zeta_j \mathbf{p})$ for $p \in \mathrm{At}(S_j^*) \setminus \{b\}$ is a distinct element of $\mathrm{At}(S_m^*)$. Thus, the unit element of S_m^* is a join of $|\mathrm{At}(S_j^*)|$ atoms, whence S_j^* and S_m^* have the same number of atoms, whereupon $j = m$.

We now show that, for $\mathbf{r}, \mathbf{s} \in S_X$, $\varphi(\mathbf{r}) = \varphi(\mathbf{s})$ if and only if $\pi_j \mathbf{r} = \pi_j \mathbf{s}$.

If $\mathbf{r} \in S_X^*$, then $\mathbf{r}$ is the join of all members of $\mathrm{At}(S_X^*)$ that lie below it. Let $I = \{p \in \mathrm{At}(S_j^*) \setminus \{b\} : \zeta_j \mathbf{p} \leq \mathbf{r}\}$. Then, as above, it follows from Lemma

4.2(iv) that $\varphi(\mathbf{r}) = \varphi(\mathbf{b}) \vee \bigvee\{\varphi(\zeta_j\mathbf{p}) : p \in I\}$ or $\bigvee\{\varphi(\zeta_j\mathbf{p}) : p \in I\}$ depending on whether $\mathbf{r}$ is a member of the $\mathbf{b}$-class or not. Similarly, for $\mathbf{s} \in S_X^*$, if $J = \{p \in \mathrm{At}(S_j^*) \setminus \{b\} : \zeta_j\mathbf{p} \le \mathbf{s}\}$, then $\varphi(\mathbf{s}) = \varphi(\mathbf{b}) \vee \bigvee\{\varphi(\zeta_j\mathbf{p}) : p \in J\}$ or $\bigvee\{\varphi(\zeta_j\mathbf{p}) : p \in J\}$ depending on whether $\mathbf{s}$ is a member of the $\mathbf{b}$-class or not. In particular, it follows from Lemma 4.2(iii), (iv) that $\varphi(\mathbf{r}) = \varphi(\mathbf{s})$ if and only if $I = J$ and $\mathbf{r} \ge \mathbf{b}$ is equivalent to $\mathbf{s} \ge \mathbf{b}$. Thus, for $\mathbf{r}, \mathbf{s} \in S_X^*$, $\varphi(\mathbf{r}) = \varphi(\mathbf{s})$ if and only if $\pi_j\mathbf{r} = \pi_j\mathbf{s}$.

Let $\mathbf{r}, \mathbf{s} \in S_X$. If $\pi_j(\mathbf{r}^{**}) \ne \pi_j(\mathbf{s}^{**})$, then, by the above, $\varphi(\mathbf{r}^{**}) \ne \varphi(\mathbf{s}^{**})$, which implies that $\varphi(\mathbf{r}) \ne \varphi(\mathbf{s})$. Assume $\pi_j(\mathbf{r}^{**}) = \pi_j(\mathbf{s}^{**})$. If $\pi_j(\mathbf{r}) \ne \pi_j(\mathbf{s})$ and, say, $\pi_j(\mathbf{r}) > \pi_j(\mathbf{s})$, then either $\mathbf{r} \ge \mathbf{b}$ and $\mathbf{s} \wedge \mathbf{b} = \mu\mathbf{b}$ or there exists $p \in \mathrm{At}(S_j^*) \setminus \{b\}$ such that $\mathbf{r} \ge \zeta_j\mathbf{p}$ and $\mathbf{s} \wedge \zeta_j\mathbf{p} = \mu\zeta_j\mathbf{p}$. By Lemma 4.2(iii) or (iv), respectively, $\varphi(\mathbf{r}) \ne \varphi(\mathbf{s})$. It remains to show that $\varphi(\mathbf{r}) = \varphi(\mathbf{s})$ whenever $\pi_j(\mathbf{r}) = \pi_j(\mathbf{s})$. Suppose $\pi_j(\mathbf{r}) = \pi_j(\mathbf{s})$ and $\varphi(\mathbf{r}) \ne \varphi(\mathbf{s})$. By the above, $\varphi(\mathbf{r}) \equiv \varphi(\mathbf{s})$ (Γ). Say, with no loss in generality, $\varphi(\mathbf{r}) > \varphi(\mathbf{s})$. Hence, there exists $p \in \mathrm{At}(S_m^*)$ such that $\varphi(\mathbf{r}) \ge p$ and $\varphi(\mathbf{s}) \wedge p = \mu p$. Since $S_X / \mathrm{Ker}(\pi_j) \cong S_j$ where $\mathrm{Ker}(\pi_j)$ denotes the congruence kernel of the homomorphism π_j, $j = m$, and, as seen above, $\mathrm{Ker}(\varphi) \le \mathrm{Ker}(\pi_j)$ where $\mathrm{Ker}(\varphi)$ denotes the congruence kernel of the homomorphism φ, it follows that φ is onto. Thus, $p = \varphi(\mathbf{t})$ for some $\mathbf{t} \in S_X^*$. Since $p \in \mathrm{At}(S_m^*)$ and $\mathbf{t}$ is the join of all members of $\mathrm{At}(S_X^*)$ that lie below it, $p = \varphi(\mathbf{u})$ for some $\mathbf{u} \in \mathrm{At}(S_X^*)$. This, by Lemma 4.2(iii), (iv), implies that either $\varphi(\mathbf{b}) = p$ or there exists $q \in \mathrm{At}(S_j^*)$ such that $\varphi(\zeta_j\mathbf{q}) = p$. Since $\pi_j(\mathbf{r}) = \pi_j(\mathbf{s})$, we have $\mathbf{r} \wedge \mathbf{b} = \mathbf{s} \wedge \mathbf{b}$ or $\mathbf{r} \wedge \zeta_j\mathbf{q} = \mathbf{s} \wedge \zeta_j\mathbf{q}$, respectively. Thus, $p = \varphi(\mathbf{r} \wedge \mathbf{b}) = \varphi(\mathbf{s} \wedge \mathbf{b}) = \mu p$ or $p = \varphi(\mathbf{r} \wedge \zeta_j\mathbf{q}) = \varphi(\mathbf{s} \wedge \zeta_j\mathbf{q}) = \mu p$, which is absurd.

Define $\eta : S_m \to S_m$ by $\eta(r) = \varphi(\mathbf{s})$, where $\mathbf{s}$ is any element of S_X with $\pi_m(\mathbf{s}) = r$. Since $m = j$ and, for all $\mathbf{r}$ and $\mathbf{s} \in S_X$, $\varphi(\mathbf{r}) = \varphi(\mathbf{s})$ if and only if $\pi_j(\mathbf{r}) = \pi_j(\mathbf{s})$, η is well-defined and indeed is an automorphism. Clearly, as required, $\varphi = \eta\pi_m$. ∎

In the following lemma the term *quasi-atom* refers to any $\mathbf{r} \in S_X$ such that $\pi_i(\mathbf{r}) \in \mathrm{At}(S_i^*)$ for all $i \in X$.

LEMMA 4.4. *Let $X, Y \in P_{\mathrm{fin}}(\omega)$, $j \in X$, and $n \in Y$. If $\psi : S_X \to S_Y$ is a homomorphism such that $\psi(\mathbf{e}) \ne \psi(\nu_j\mathbf{e})$, then the following hold*:

(i) *if $\mathbf{r}$ is a quasi-atom of S_X, then $\pi_n\psi(\mathbf{r}) \ne 0$ and either $\pi_n\psi(\mathbf{r}^*) \ne \pi_n\psi(\mu\mathbf{r}^*)$ or $\pi_n\psi(\mathbf{r}^*) \le b^*$*;

(ii) *there exists a quasi-atom $\mathbf{p}$ of S_X such that $\mathbf{p} \in (\mathbf{b}^*]$ and $\pi_n\psi(\mathbf{p}^*) \ne \pi_n\psi(\mu\mathbf{p}^*)$.*

Proof. (i) Because $\psi(\mathbf{e}) \ne \psi(\nu_j\mathbf{e})$, there exists $m \in Y$ such that $\pi_m\psi(\mathbf{e}) \ne \pi_m\psi(\nu_j\mathbf{e})$. Proposition 4.3, when applied to the homomorphism $\pi_m\psi : S_X \to S_m$, implies that $j = m$ and there is an automorphism η of S_m

such that $\pi_m\psi = \eta\pi_m$. (The indicated projection maps have domains S_Y and S_X, respectively.) In particular, $\pi_m\psi(\mu\mathbf{r}^*) = \eta\pi_m(\mu\mathbf{r}^*) \in S_m \setminus S_m^*$.

If $\pi_n\psi(\mathbf{r}) = 0$, then $\pi_n\psi(\mathbf{r}^*) = 1$, whence $\pi_n\psi(\mu\mathbf{r}^*) = 1$ by the triviality of $[1]\Gamma$. Thus, $\psi(\mu\mathbf{r}^*)$ belongs to the $\mathbf{b}$-class of S_Y, whence $\pi_m\psi(\mu\mathbf{r}^*)$ belongs to the b-class of S_m, which is a subset of S_m^*. Since this conclusion contradicts our finding that $\pi_m\psi(\mu\mathbf{r}^*) \in S_m \setminus S_m^*$, we have $\pi_n\psi(\mathbf{r}) \neq 0$.

If $\pi_n\psi(\mathbf{r}^*) = \pi_n\psi(\mu\mathbf{r}^*)$, then $\pi_n\psi(\mu\mathbf{r}^*) \in S_n^*$, and since $\pi_m\psi(\mu\mathbf{r}^*) \in S_m \setminus S_m^*$, it follows that $\psi(\mu\mathbf{r}^*)$ belongs to the $\mathbf{0}$-class of S_Y. That is, $\psi(\mu\mathbf{r}^*) \leq \mathbf{b}^*$, and therefore $\psi(\mathbf{r}^*) \leq \mathbf{b}^*$. Thus $\pi_n\psi(\mathbf{r}^*) \neq \pi_n\psi(\mu\mathbf{r}^*)$ or $\pi_n\psi(\mathbf{r}^*) \leq b^*$.

(ii) Because $\mathrm{At}(S_i^*) \setminus \{b\} \subseteq (b^*]$ for all $i \in X$, there exist in S_X quasi-atoms $\mathbf{r}, \mathbf{s} \in (\mathbf{b}^*]$ such that $\mathbf{r} \wedge \mathbf{s} = \mathbf{0}$. Since $1 = \pi_n\psi(\mathbf{1}) = \pi_n\psi(\mathbf{r}^*) \vee \pi_n\psi(\mathbf{s}^*)$, it is impossible to have both $\pi_n\psi(\mathbf{r}^*) \leq b^*$ and $\pi_n\psi(\mathbf{s}^*) \leq b^*$, whereupon (i) yields $\mathbf{p} \in \{\mathbf{r}, \mathbf{s}\}$ such that $\pi_n\psi(\mathbf{p}^*) \neq \pi_n\psi(\mu\mathbf{p}^*)$. ■

PROPOSITION 4.5. *Let $X, Y \in P_{\mathrm{fin}}(\omega)$ and $j \in X$. If there is a homomorphism $\psi : S_X \to S_Y$ such that $\psi(\mathbf{e}) \neq \psi(\nu_j\mathbf{e})$, then $j \in Y$ and $Y \subseteq X$.*

Proof. If $\psi(\mathbf{e}) \neq \psi(\nu_j\mathbf{e})$, then there exists $m \in Y$ such that $\pi_m\psi(\mathbf{e}) \neq \pi_m\psi(\nu_j\mathbf{e})$. Hence, for the homomorphism $\pi_m\psi : S_X \to S_m$, Proposition 4.3 implies that $j = m$ and, in particular, $j \in Y$. Likewise, to obtain $Y \subseteq X$ by means of Proposition 4.3 it suffices to prove that for every $n \in Y$ there exists $i \in X$ such that $\pi_n\psi(\mathbf{e}) \neq \pi_n\psi(\nu_i\mathbf{e})$ and, so, $i = n \in X$.

Suppose, on the contrary, that there exists $n \in Y$ such that $\pi_n\psi(\mathbf{e}) = \pi_n\psi(\nu_i\mathbf{e})$ for all $i \in X$. For notational convenience set $\varphi = \pi_n\psi$.

For $\nu\mathbf{e} = \bigwedge\{\nu_i\mathbf{e} : i \in X\}$, we have $\varphi(\mathbf{e}) = \varphi(\nu\mathbf{e})$. It follows that $\varphi(\mathbf{c}) \leq \varphi(\mu\mathbf{b}^*)$ because $\varphi(\mathbf{c}) = \varphi(\mathbf{e}) \wedge \varphi(\mathbf{c}) = \varphi(\nu\mathbf{e}) \wedge \varphi(\mathbf{c}) = \varphi(\mu\mathbf{c}) \leq \varphi(\mu\mathbf{b}^*)$.

Next we show that $\varphi(\mathbf{b}^*) = \varphi(\mu\mathbf{b}^*)$. Supposing the contrary, $\varphi(\mu\mathbf{b}^*) \in S_n \setminus S_n^*$. Since $\varphi(\mathbf{c}) \leq \varphi(\mu\mathbf{b}^*)$ it now follows from Lemma 4.1(i) that $\varphi(\mathbf{c}) \in \{0, a\}$, hence Lemma 4.4(i) yields $\varphi(\mathbf{c}) = a$, whereupon, by Lemma 4.1(ii), $\varphi(\mathbf{b}^*) \in C_n(a)$ and, hence, there exists $p \in \mathrm{At}(S_n^*) \setminus \{a\}$ such that $\varphi(\mathbf{b}^*) = a \vee p$. Since $\mathbf{a} \leq \mathbf{b}^*$, it follows that $\varphi(\mathbf{a}) \in \{0, a, p, a \vee p\}$. Inasmuch as $\varphi(\mathbf{a}) \neq 0$ by Lemma 4.4(i), and $0 = \varphi(\mathbf{a}) \wedge \varphi(\mathbf{c}) = \varphi(\mathbf{a}) \wedge a$, we conclude that $\varphi(\mathbf{a}) = p$. Choose a quasi-atom $\mathbf{r}$ of S_X such that, for all $i \in X$, $\pi_i(\mathbf{r}) \notin \{a, b, c\}$. It follows that $\mathbf{r} \leq \mathbf{b}^*$ and, hence, $\varphi(\mathbf{r}) \leq \varphi(\mathbf{b}^*)$. By Lemma 4.4(i), $\varphi(\mathbf{r}) \in \{a, p, a \vee p\}$. Since $\varphi(\mathbf{a}) = p$, $\varphi(\mathbf{c}) = a$, $\varphi(\mathbf{r}) \wedge \varphi(\mathbf{a}) = \varphi(\mathbf{r} \wedge \mathbf{a}) = 0$, and $\varphi(\mathbf{r}) \wedge \varphi(\mathbf{c}) = \varphi(\mathbf{r} \wedge \mathbf{c}) = 0$, this is absurd. Therefore $\varphi(\mathbf{b}^*) = \varphi(\mu\mathbf{b}^*)$.

For any $\mathbf{s} \in S_X$ such that $\mathbf{s} \leq \mathbf{b}^*$ we now have $\varphi(\mathbf{s}) = \varphi(\mu\mathbf{s})$. (Indeed, $\varphi(\mu\mathbf{s}) = \varphi(\mathbf{s}) \wedge \varphi(\mu\mathbf{b}^*) = \varphi(\mathbf{s}) \wedge \varphi(\mathbf{b}^*) = \varphi(\mathbf{s})$.) In particular, $\varphi(\mathbf{s}) = \varphi(\mu\mathbf{s})$ for $\mathbf{s} = \mathbf{p}$, where $\mathbf{p}$ is a quasi-atom obtained from Lemma 4.4(ii). This contradiction concludes the proof. ■

PROPOSITION 4.6. *Let $X \in P_{\mathrm{fin}}(\omega)$ and $m \in \omega$. If $\varphi : S_X \to S_m$ is a homomorphism such that $\varphi(\mathbf{e}) = \varphi(\nu_i\mathbf{e})$ for all $i \in X$, then $\Gamma_X \leq \mathrm{Ker}(\varphi)$*

where Γ_X denotes the Glivenko congruence on S_X and $\mathrm{Ker}(\varphi)$ *denotes the congruence kernel of the homomorphism* φ*.*

Proof. Suppose $\varphi(\mathbf{b}^*) \neq \varphi(\mu\mathbf{b}^*)$. Since $\mu\mathbf{b}^* = \bigwedge\{\mu_i\mathbf{b}^* : i \in X\}$, $\varphi(\mathbf{b}^*) \neq \varphi(\mu_i\mathbf{b}^*)$ for some $i \in X$. By hypothesis $\varphi(\mathbf{e}) = \varphi(\nu_i\mathbf{e})$ and, consequently, $\nu_i\mathbf{e}\wedge\mathbf{c} = \mu_i\mathbf{c}$ implies $\varphi(\mathbf{c}) = \varphi(\mathbf{e}\wedge\mathbf{c}) = \varphi(\mathbf{e})\wedge\varphi(\mathbf{c}) = \varphi(\nu_i\mathbf{e})\wedge\varphi(\mathbf{c}) = \varphi(\nu_i\mathbf{e}\wedge\mathbf{c}) = \varphi(\mu_i\mathbf{c})$. Since $\mu_i\mathbf{c} \leq \mu_i\mathbf{b}^*$, $\varphi(\mathbf{c}) = \varphi(\mu_i\mathbf{c}) \leq \varphi(\mu_i\mathbf{b}^*) \in S_m \setminus S_m^*$. By Lemma 4.1(i), $\varphi(\mathbf{c}) = 0$ or $\varphi(\mathbf{c}) = a$. Further, if $\varphi(\mathbf{c}) = a$, then it follows from Lemma 4.1(ii) that $\varphi(\mathbf{b}^*) \in C_m(a)$ and, since $\varphi(\mathbf{c}) \leq \varphi(\mu_i\mathbf{b}^*)$, $\varphi(\mu_i\mathbf{b}^*) = \nu\varphi(\mathbf{b}^*)$.

We show that neither $\varphi(\mathbf{c}) = 0$ nor $\varphi(\mathbf{c}) = a$ may occur.

If $\varphi(\mathbf{c}) = 0$, then $\varphi(\mathbf{c}^*) = 1$. The triviality of $[1]\Gamma$ reveals that $\varphi(\mathbf{c}^*) = \varphi(\mu\mathbf{c}^*)$. For every quasi-atom $\mathbf{r}$ such that $\pi_j(\mathbf{r}) \in \mathrm{At}(S_j^*) \setminus \{b, c\}$ for all $j \in X$, $\mathbf{r} \leq \mathbf{c}^*$ and $\mathbf{r} \wedge \mu\mathbf{c}^* = \mu\mathbf{r}$. Thus, $\varphi(\mathbf{r}) = \varphi(\mu\mathbf{r})$ and, in particular, $\varphi(\mu_i\mathbf{r}) = \varphi(\mathbf{r})$. From $\mathbf{b}^* \geq \mathbf{r}$ and $\mu_i\mathbf{b}^* \geq \mu_i\mathbf{r}$ it follows, by Lemma 4.1(i), that $\varphi(\mathbf{r}) = 0$ or $\varphi(\mathbf{r}) = a$. Further, in the event that $\varphi(\mathbf{r}) = a$, then, by Lemma 4.1(ii), $\varphi(\mathbf{b}^*) \in C_m(a)$. Since $\mathbf{b}^* = \mathbf{c} \vee \bigvee\{\mathbf{r} : \pi_j(\mathbf{r}) \in \mathrm{At}(S_j^*) \setminus \{b, c\}$ for all $j \in X\}$, either $\varphi(\mathbf{b}^*) = 0$ or $\varphi(\mathbf{b}^*) = a$. Because $\varphi(\mathbf{b}^*) \neq \varphi(\mu\mathbf{b}^*)$, it follows that $\varphi\mathbf{b}^* \neq 0$. Thus, $\varphi(\mathbf{b}^*) = a$. However, this is absurd since $\varphi(\mathbf{b}^*) = a$ only if $\varphi(\mathbf{r}) = a$ for some suitable $\mathbf{r}$, which, in turn, implies $\varphi(\mathbf{b}^*) \in C_m(a)$. Hence, contrary to the supposition, $\varphi(\mathbf{c}) \neq 0$.

If $\varphi(\mathbf{c}) = a$, $\varphi(\mathbf{b}^*) \in C_m(a)$, and $\varphi(\mu_i\mathbf{b}^*) = \nu\varphi(\mathbf{b}^*)$, then, for some $p \in \mathrm{At}(S_m^*)$, $\varphi(\mathbf{b}^*) = a \vee p$ and $\varphi(\mathbf{a}) \in \{0, a, p, a \vee p\}$. Since $\mathbf{a} \wedge \mathbf{c} = \mathbf{0}$, it follows that $\varphi(\mathbf{a}) \in \{0, p\}$. If $\varphi(\mathbf{a}) = 0$, then $\varphi(\mathbf{a}^*) = 1$. Arguing as above, $\varphi(\mathbf{a}^*) = \varphi(\mu\mathbf{a}^*)$, $\varphi(\mathbf{r}) = \varphi(\mu_i\mathbf{r})$ for every quasi-atom $\mathbf{r}$ such that $\pi_j(\mathbf{r}) \in \mathrm{At}(S_j^*) \setminus \{a, b\}$, and, hence, $\varphi(\mathbf{r}) = 0$ or a. As before, it follows that $\varphi(\mathbf{b}^*) = a$, which is absurd. Thus, $\varphi(\mathbf{a}) = p$ and, as $\mathbf{a} \wedge \mu_i\mathbf{b}^* = \mu_i\mathbf{a}$ and $\varphi(\mu_i\mathbf{b}^*) = \nu\varphi(\mathbf{b}^*)$, $\varphi(\mu_i\mathbf{a}) = \varphi(\mathbf{a}) \wedge \varphi(\mu_i\mathbf{b}^*) = p \wedge \nu\varphi(\mathbf{b}^*) = \mu p$. In particular, $\varphi(\mu_i\mathbf{a}) \neq \varphi(\mathbf{a})$. Choose a quasi-atom $\mathbf{r}$ such that $\pi_j(\mathbf{r}) \in \mathrm{At}(S_j^*) \setminus \{a, b, c\}$. Since $\mathbf{r} \leq \mathbf{b}^*$, $\varphi(\mathbf{r}) \in \{0, a, p, a \vee p\}$ and, as $\mathbf{a} \wedge \mathbf{r} = \mathbf{c} \wedge \mathbf{r} = \mathbf{0}$, $\varphi(\mathbf{r}) = 0$. Hence, $\varphi(\mathbf{r}^*) = 1$, $\varphi(\mu_i\mathbf{r}^*) = \varphi(\mathbf{r}^*)$ and, since $\mathbf{r}^* \geq \mathbf{a}$, it follows that $\varphi(\mu_i\mathbf{a}) = \varphi(\mathbf{a})$, which is absurd. Thus, contrary to hypothesis, $\varphi(\mathbf{c}) \neq a$.

We conclude that $\varphi(\mathbf{b}^*) = \varphi(\mu\mathbf{b}^*)$.

Since $\mathbf{r} \leq \mathbf{b}^*$ for every $\mathbf{r}$ in the $\mathbf{0}$-class, it remains to show that, for $\mathbf{r}$ in the $\mathbf{b}$-class, $\varphi(\mathbf{r}) = \varphi(\mu\mathbf{r})$. Let $\mathbf{r}$ in the $\mathbf{b}$-class be such that $\mathbf{r} \neq \mu\mathbf{r}$. Then, since $\pi_i(\mathbf{r}) < 1$ for every $i \in X$, there exists a quasi-atom $\mathbf{s}$ in the $\mathbf{0}$-class such that $\mathbf{r} \leq \mathbf{s}^*$. It is sufficient therefore to show that, for each quasi-atom $\mathbf{s}$ in the $\mathbf{0}$-class, $\varphi(\mathbf{s}^*) = \varphi(\mu\mathbf{s}^*)$. Suppose that for some such $\mathbf{s}$, $\varphi(\mathbf{s}^*) \neq \varphi(\mu\mathbf{s}^*)$. For every quasi-atom $\mathbf{t} \leq \mathbf{s}^*$ that belongs to the $\mathbf{0}$-class, we now know that $\varphi(\mathbf{t}) = \varphi(\mu\mathbf{t})$. Hence, by Lemma 4.1, either $\varphi(\mathbf{t}) = 0$ or else $\varphi(\mathbf{t}) = a$ and $\varphi(\mathbf{s}^*) \in C_m(a)$. Since there are distinct such quasi-atoms which meet to $\mathbf{0}$, we may assume that $\varphi(\mathbf{t}) = 0$ for some such $\mathbf{t}$. Thus,

$\varphi(\mathbf{t}^*) = 1$ and, so, $\varphi(\mathbf{t}^*) = \varphi(\mu\mathbf{t}^*)$, which implies that $\varphi(\mathbf{b}) = \varphi(\mu\mathbf{b})$. By Lemma 4.1, either $\varphi(\mathbf{b}) = 0$ or else $\varphi(\mathbf{b}) = a$ and $\varphi(\mathbf{s}^*) \in C_m(a)$. However, $\mathbf{s}^* = \mathbf{b} \vee \bigvee\{\mathbf{t} : \mathbf{t}$ is a quasi-atom in the $\mathbf{0}$-class and $\mathbf{t} \le \mathbf{s}^*\}$. Hence, either $\varphi(\mathbf{s}^*) = 0$ or else $\varphi(\mathbf{s}^*) = a$ and $\varphi(\mathbf{s}^*) \in C_m(a)$, each of which is absurd. ■

5. Verification of (P1)–(P4). We can now prove Theorem 1.1 by verifying that the family $(S_W : W \in P_{\mathrm{fin}}(\omega))$ satisfies the postulates (P1)–(P4) of Proposition 2.1.

Since (P1) and (P2) clearly hold, we need only establish (P3) to conclude the following.

PROPOSITION 5.1. *The family $(S_W : W \in P_{\mathrm{fin}}(\omega))$ of finite pseudocomplemented semilattices satisfies the postulates* (P1)–(P3).

Proof. Let $X, Y \in P_{\mathrm{fin}}(\omega)$ be such that $X \ne \emptyset$ and $S_X \in Q(S_Y)$. There is an embedding $\varphi : S_X \to (S_Y)^I$ for some finite set I. For each $j \in X$, $\varphi(\mathbf{e}) \ne \varphi(\nu_j\mathbf{e})$ and hence there exists $i \in I$ such that the projection map $\pi_i : (S_Y)^I \to S_Y$ satisfies $\pi_i\varphi(\mathbf{e}) \ne \pi_i\varphi(\nu_j\mathbf{e})$. Proposition 4.5 (applied to $\pi_i\varphi : S_X \to S_Y$) yields $j \in Y$ and $Y \subseteq X$. Since this holds for all $j \in X$, we have $Y = X$. ■

LEMMA 5.2. *If $X \in P_{\mathrm{fin}}(\omega)$, $\emptyset \ne Y \subseteq X$, and Θ is a congruence relation on S_X with $\Gamma_X \le \Theta$, then S_Y is embeddable into $S_X/(\Theta \wedge \mathrm{Ker}(\pi_Y))$, where Γ_X denotes the Glivenko congruence on S_X, π_Y denotes the projection map from S_X onto S_Y, and* $\mathrm{Ker}(\pi_Y)$ *denotes the congruence kernel of the homomorphism π_Y.*

Proof. Let Ψ denote the congruence relation on S_X where, for $\mathbf{r}, \mathbf{s} \in S_X$, $\mathbf{r} \equiv \mathbf{s}\ (\Psi)$ if and only if $\pi_Y(\mathbf{r}) \equiv \pi_Y(\mathbf{s})\ (\Gamma_Y)$ where Γ_Y denotes the Glivenko congruence on S_Y. We shall need the following:

LEMMA 5.3. $\Psi = \Gamma_X \circ \mathrm{Ker}(\pi_Y)$.

Proof. Since $\Gamma_X \le \Psi$ and $\mathrm{Ker}(\pi_Y) \le \Psi$, $\Gamma_X \circ \mathrm{Ker}(\pi_Y) \le \Psi$. To see that $\Psi \le \Gamma_X \circ \mathrm{Ker}(\pi_Y)$, suppose $\mathbf{r}, \mathbf{s} \in S_X$ and $\mathbf{r} \equiv \mathbf{s}\ (\Psi)$. It must be shown that there exists $\mathbf{t} \in S_X$ such that $\mathbf{r}\Gamma_X\mathbf{t}$ and $\mathbf{t}\,\mathrm{Ker}(\pi_Y)\mathbf{s}$.

Clearly, for any such $\mathbf{t}$, it must be the case that $\pi_j(\mathbf{t}) = \pi_j(\mathbf{s})$ for $j \in Y$. Thus, to determine a suitable $\mathbf{t}$, it is necessary to define $\pi_i(\mathbf{t})$ for $i \in X \setminus Y$. There are several cases depending on whether $\mathbf{r}$ and $\mathbf{s}$ belong to the $\mathbf{b}$-class, $\mu\mathbf{b}$-class, or $\mathbf{0}$-class of S_X:

Let $\mathbf{r}$ belong to the $\mathbf{b}$-class. If $\mathbf{s}$ belongs to the $\mathbf{b}$-class, then set $\pi_i(\mathbf{t}) = \pi_i(\mathbf{r})$. When $\mathbf{s}$ belongs to the $\mu\mathbf{b}$-class, then let $\pi_i(\mathbf{t}) = \mu\pi_i(\mathbf{r})$. For $\mathbf{s}$ to belong to the $\mathbf{0}$-class is not possible as $\pi_Y(\mathbf{r}) \equiv \pi_Y(\mathbf{s})\ (\Gamma_Y)$.

Let $\mathbf{r}$ belong to the $\mu\mathbf{b}$-class. If $\mathbf{s}$ belongs to the $\mathbf{b}$-class, then set $\pi_i(\mathbf{t}) = (\pi_i(\mathbf{r}))^{**}$. If $\mathbf{s}$ belongs to the $\mu\mathbf{b}$-class, then let $\pi_i(\mathbf{t}) = \pi_i(\mathbf{r})$. As above, it is not possible for $\mathbf{s}$ to belong to the $\mathbf{0}$-class.

Let $\mathbf{r}$ belong to the $\mathbf{0}$-class. As above, the only possibility is that $\mathbf{s}$ also belongs to the $\mathbf{0}$-class. In which case $\pi_i(\mathbf{t}) = \pi_i(\mathbf{r})$ will suffice. ∎

Proof of Lemma 5.2 continued. Clearly, the set

$$\{([\mathbf{r}]\Theta, [\mathbf{r}]\operatorname{Ker}(\pi_Y)) : \mathbf{r} \in S_X\}$$

is a subalgebra of $S_X/\Theta \times S_X/\operatorname{Ker}(\pi_Y)$. Denote this p-subsemilattice by S. Since $S_X/(\Theta \wedge \operatorname{Ker}(\pi_Y))$ is isomorphic to S via the isomorphism

$$[\mathbf{r}](\Theta \wedge \operatorname{Ker}(\pi_Y)) \mapsto ([\mathbf{r}]\Theta, [\mathbf{r}]\operatorname{Ker}(\pi_Y)),$$

to complete the proof of Lemma 5.2 it suffices to show that S_Y is embeddable into S.

Define $\varphi : S_Y \to S$ where, for $\mathbf{s} \in S_Y$, $\varphi(\mathbf{s}) = ([\mathbf{r}]\Theta, [\mathbf{r}]\operatorname{Ker}(\pi_Y))$ is such that $\mathbf{x}\Gamma_X\mathbf{r}$ and $\mathbf{r}\operatorname{Ker}(\pi_Y)\mathbf{x}'$ for suitably defined $\mathbf{x}$ and $\mathbf{x}' \in S_X$. The existence of such an $\mathbf{r}$ is guaranteed by Lemma 5.3 providing $\mathbf{x}\Psi\mathbf{x}'$. The choice of $\mathbf{x}$ and $\mathbf{x}'$ depends on whether $\mathbf{s}$ belongs to the $\mathbf{b}$-class, $\mu\mathbf{b}$-class, or $\mathbf{0}$-class of S_Y. There are three cases:

Let $\mathbf{s}$ belong to the $\mathbf{b}$-class of S_Y. Then $\mathbf{x}, \mathbf{x}' \in S_X$, where, for $i \in Y$, $\pi_i(\mathbf{x}) = \pi_i(\mathbf{s})^{**}$ and $\pi_i(\mathbf{x}') = \pi_i(\mathbf{s})$ and, for $i \in X \setminus Y$, $\pi_i(\mathbf{x}) = 1$ and $\pi_i(\mathbf{x}') = 1$.

Let $\mathbf{s}$ belong to the $\mu\mathbf{b}$-class of S_Y. Since $\mathbf{s}^{**}$ belongs to the $\mathbf{b}$-class, $\mathbf{x}, \mathbf{x}' \in S_X$ where, for $i \in Y$, $\pi_i(\mathbf{x}) = \pi_i(\mathbf{s})^{**}$ and $\pi_i(\mathbf{x}') = \pi_i(\mathbf{s})$ and, for $i \in X \setminus Y$, $\pi_i(\mathbf{x}) = 1$ and $\pi_i(\mathbf{x}') = \mu b$.

Let $\mathbf{s}$ belong to the $\mathbf{0}$-class of S_Y. Since $\mathbf{s}^{**}$ also belongs to the $\mathbf{0}$-class, $\mathbf{x}$, $\mathbf{x}' \in S_X$ where, for $i \in Y$, $\pi_i(\mathbf{x}) = \pi_i(\mathbf{s})^{**}$ and $\pi_i(\mathbf{x}') = \pi_i(\mathbf{s})$ and, for $i \in X \setminus Y$, $\pi_i(\mathbf{x}) = 0$ and $\pi_i(\mathbf{x}') = 0$.

In each case $\mathbf{x}\Psi\mathbf{x}'$. Thus, since $\Gamma_X \le \Theta$, φ is well-defined. It remains to show that φ is a one-to-one homomorphism.

To see that φ is one-to-one, suppose $\varphi(\mathbf{s}) = \varphi(\mathbf{t})$ for some $\mathbf{s}, \mathbf{t} \in S_Y$. Let $\varphi(\mathbf{s}) = ([\mathbf{r}]\Theta, [\mathbf{r}]\operatorname{Ker}(\pi_Y))$ and $\varphi(\mathbf{t}) = ([\mathbf{u}]\Theta, [\mathbf{u}]\operatorname{Ker}(\pi_Y))$. Thus, $[\mathbf{r}]\operatorname{Ker}(\pi_Y) = [\mathbf{u}]\operatorname{Ker}(\pi_Y)$. In particular, $\pi_Y(\mathbf{r}) = \pi_Y(\mathbf{u})$. However, by the definition of φ, $\pi_Y(\mathbf{r}) = \pi_Y(\mathbf{s})$ and $\pi_Y(\mathbf{u}) = \pi_Y(\mathbf{t})$. Hence, $\mathbf{s} = \mathbf{t}$ as required.

To show that φ is $\wedge$-preserving, we must establish that, for $\mathbf{s}, \mathbf{t} \in S_Y$, $\varphi(\mathbf{s} \wedge \mathbf{t}) = \varphi(\mathbf{s}) \wedge \varphi(\mathbf{t})$. Let $\varphi(\mathbf{s} \wedge \mathbf{t}) = ([\mathbf{r}]\Theta, [\mathbf{r}]\operatorname{Ker}(\pi_Y))$, $\varphi(\mathbf{s}) = ([\mathbf{u}]\Theta, [\mathbf{u}]\operatorname{Ker}(\pi_Y))$, and $\varphi(\mathbf{t}) = ([\mathbf{v}]\Theta, [\mathbf{v}]\operatorname{Ker}(\pi_Y))$, where $\mathbf{x}, \mathbf{x}', \mathbf{y}, \mathbf{y}', \mathbf{z}, \mathbf{z}' \in S_X$ determine appropriate $\mathbf{r}, \mathbf{u}, \mathbf{v} \in S_X$, respectively. Since $\Gamma_X \le \Theta$, it is sufficient to show that $\mathbf{r}\Gamma_X\mathbf{u} \wedge \mathbf{v}$ and $\mathbf{r}\operatorname{Ker}(\pi_Y)\mathbf{u} \wedge \mathbf{v}$ in order to conclude that $\varphi(\mathbf{s} \wedge \mathbf{t}) = \varphi(\mathbf{s}) \wedge \varphi(\mathbf{t})$. There are several cases depending on whether $\mathbf{s}$ and $\mathbf{t}$ belong to the $\mathbf{b}$-class, $\mu\mathbf{b}$-class, or $\mathbf{0}$-class of S_Y.

Let $\mathbf{s}$ belong to the $\mathbf{b}$-class of S_Y.

If $\mathbf{t}$ belongs to the $\mathbf{b}$-class, then so does $\mathbf{s} \wedge \mathbf{t}$. Thus, for $i \in Y$, $\pi_i(\mathbf{x}) = \pi_i(\mathbf{s} \wedge \mathbf{t})^{**}$, $\pi_i(\mathbf{y}) = \pi_i(\mathbf{s})^{**}$, and $\pi_i(\mathbf{z}) = \pi_i(\mathbf{t})^{**}$ and, for $i \in X \setminus Y$, $\pi_i(\mathbf{x}) = \pi_i(\mathbf{y}) = \pi_i(\mathbf{z}) = 1$. Since $(\mathbf{s} \wedge \mathbf{t})^{**} = \mathbf{s}^{**} \wedge \mathbf{t}^{**}$, $\mathbf{x} = \mathbf{y} \wedge \mathbf{z}$. Consequently, as

$\mathbf{x}\Gamma_X\mathbf{r}$, $\mathbf{y}\Gamma_X\mathbf{u}$, and $\mathbf{z}\Gamma_X\mathbf{v}$, it follows that $\mathbf{r}\Gamma_X\mathbf{u}\wedge\mathbf{v}$. Similarly, since in this case $\mathbf{x}=\mathbf{x}'$, $\mathbf{y}=\mathbf{y}'$, and $\mathbf{z}=\mathbf{z}'$, we have $\mathbf{r}\operatorname{Ker}(\pi_Y)\mathbf{u}\wedge\mathbf{v}$.

If $\mathbf{t}$ belongs to the $\mu\mathbf{b}$-class, then so does $\mathbf{s}\wedge\mathbf{t}$. Thus, for $i\in Y$, $\pi_i(\mathbf{x})=\pi_i(\mathbf{s}\wedge\mathbf{t})^{**}$, $\pi_i(\mathbf{y})=\pi_i(\mathbf{s})^{**}$, and $\pi_i(\mathbf{z})=\pi_i(\mathbf{t})^{**}$ and, for $i\in X\setminus Y$, $\pi_i(\mathbf{x})=\pi_i(\mathbf{y})=\pi_i(\mathbf{z})=1$. As before, it follows that $\mathbf{r}\Gamma_X\mathbf{u}\wedge\mathbf{v}$. For $i\in Y$, $\pi_i(\mathbf{x}')=\pi_i(\mathbf{s}\wedge\mathbf{t})$, $\pi_i(\mathbf{y}')=\pi_i(\mathbf{s})$, and $\pi_i(\mathbf{z}')=\pi_i(\mathbf{t})$ and, for $i\in X\setminus Y$, $\pi_i(\mathbf{x}')=\mu b$, $\pi_i(\mathbf{y}')=1$, $\pi_i(\mathbf{z}')=\mu b$. In particular, $\mathbf{x}'=\mathbf{y}'\wedge\mathbf{z}'$ and, as $\mathbf{x}'\operatorname{Ker}(\pi_Y)\mathbf{r}$, $\mathbf{y}'\operatorname{Ker}(\pi_Y)\mathbf{u}$, and $\mathbf{z}'\operatorname{Ker}(\pi_Y)\mathbf{v}$, it follows that $\mathbf{r}\operatorname{Ker}(\pi_Y)\mathbf{u}\wedge\mathbf{v}$.

If $\mathbf{t}$ belongs to the $\mathbf{0}$-class, then so does $\mathbf{s}\wedge\mathbf{t}$. Hence, for $i\in Y$, $\pi_i(\mathbf{x})=\pi_i(\mathbf{s}\wedge\mathbf{t})^{**}$, $\pi_i(\mathbf{y})=\pi_i(\mathbf{s})^{**}$, and $\pi_i(\mathbf{z})=\pi_i(\mathbf{t})^{**}$ and, for $i\in X\setminus Y$, $\pi_i(\mathbf{x})=0$, $\pi_i(\mathbf{y})=1$, and $\pi_i(\mathbf{z})=0$. As above, $\mathbf{r}\Gamma_X\mathbf{u}\wedge\mathbf{v}$. For $i\in Y$, $\pi_i(\mathbf{x}')=\pi_i(\mathbf{s}\wedge\mathbf{t})$, $\pi_i(\mathbf{y}')=\pi_i(\mathbf{s})$, and $\pi_i(\mathbf{z}')=\pi_i(\mathbf{t})$ and, for $i\in X\setminus Y$, $\pi_i(\mathbf{x}')=0$, $\pi_i(\mathbf{y}')=1$, and $\pi_i(\mathbf{z}')=0$. Once more, $\mathbf{r}\operatorname{Ker}(\pi_Y)\mathbf{u}\wedge\mathbf{v}$.

Let $\mathbf{s}$ belong to the $\mu\mathbf{b}$-class of S_Y.

Since $\wedge$ is commutative, it is now no longer necessary to consider $\mathbf{t}$ a member of the $\mathbf{b}$-class.

If $\mathbf{t}$ belongs to the $\mu\mathbf{b}$-class of S_Y, then so does $\mathbf{s}\wedge\mathbf{t}$. Consequently, for $i\in Y$, $\pi_i(\mathbf{x})=\pi_i(\mathbf{s}\wedge\mathbf{t})^{**}$, $\pi_i(\mathbf{y})=\pi_i(\mathbf{s})^{**}$, and $\pi_i(\mathbf{z})=\pi_i(\mathbf{t})^{**}$ and, for $i\in X\setminus Y$, $\pi_i(\mathbf{x})=\pi_i(\mathbf{y})=\pi_i(\mathbf{z})=1$. For $i\in Y$, $\pi_i(\mathbf{x}')=\pi_i(\mathbf{s}\wedge\mathbf{t})$, $\pi_i(\mathbf{y}')=\pi_i(\mathbf{s})$, and $\pi_i(\mathbf{z}')=\pi_i(\mathbf{t})$ and, for $i\in X\setminus Y$, $\pi_i(\mathbf{x}')=\pi_i(\mathbf{y}')=\pi_i(\mathbf{z}')=\mu b$. In particular, $\mathbf{x}=\mathbf{y}\wedge\mathbf{z}$ and $\mathbf{x}'=\mathbf{y}'\wedge\mathbf{z}'$. Hence $\mathbf{r}\Gamma_X\mathbf{u}\wedge\mathbf{v}$ and $\mathbf{r}\operatorname{Ker}(\pi_Y)\mathbf{u}\wedge\mathbf{v}$.

If $\mathbf{t}$ belongs to the $\mathbf{0}$-class, then so does $\mathbf{s}\wedge\mathbf{t}$. For $i\in Y$, $\pi_i(\mathbf{x})=\pi_i(\mathbf{s}\wedge\mathbf{t})^{**}$, $\pi_i(\mathbf{y})=\pi_i(\mathbf{s})^{**}$, $\pi_i(\mathbf{z})=\pi_i(\mathbf{t})^{**}$, $\pi_i(\mathbf{x}')=\pi_i(\mathbf{s}\wedge\mathbf{t})$, $\pi_i(\mathbf{y}')=\pi_i(\mathbf{s})$, and $\pi_i(\mathbf{z}')=\pi_i(\mathbf{t})$ and, for $i\in X\setminus Y$, $\pi_i(\mathbf{x})=0$, $\pi_i(\mathbf{y})=1$, $\pi_i(\mathbf{z})=0$, $\pi_i(\mathbf{x}')=0$, $\pi_i(\mathbf{y}')=\mu b$, and $\pi_i(\mathbf{z}')=0$. Clearly, once more $\mathbf{x}=\mathbf{y}\wedge\mathbf{z}$ and $\mathbf{x}'=\mathbf{y}'\wedge\mathbf{z}'$, as required.

Let $\mathbf{s}$ belong to the $\mathbf{0}$-class of S_Y.

By the commutativity of $\wedge$, it only remains to consider $\mathbf{t}$ a member of the $\mathbf{0}$-class. Obviously, $\mathbf{s}\wedge\mathbf{t}$ is also a member of the $\mathbf{0}$-class. For $i\in Y$, $\pi_i(\mathbf{x})=\pi_i(\mathbf{s}\wedge\mathbf{t})^{**}$, $\pi_i(\mathbf{y})=\pi_i(\mathbf{s})^{**}$, $\pi_i(\mathbf{z})=\pi_i(\mathbf{t})^{**}$, $\pi_i(\mathbf{x}')=\pi_i(\mathbf{s}\wedge\mathbf{t})$, $\pi_i(\mathbf{y}')=\pi_i(\mathbf{s})$, and $\pi_i(\mathbf{z}')=\pi_i(\mathbf{t})$ and, for $i\in X\setminus Y$, $\pi_i(\mathbf{x})=\pi_i(\mathbf{y})=\pi_i(\mathbf{z})=\pi_i(\mathbf{x}')=\pi_i(\mathbf{y}')=\pi_i(\mathbf{z}')=0$. Thus, in this case, as for every case, $\mathbf{r}\Gamma_X\mathbf{u}\wedge\mathbf{v}$ and $\mathbf{r}\operatorname{Ker}(\pi_Y)\mathbf{u}\wedge\mathbf{v}$. In particular, φ is $\wedge$-preserving.

To see that φ is $*$-preserving, it is necessary to establish that, for $\mathbf{s}\in S_Y$, $\varphi(\mathbf{s}^*)=\varphi(\mathbf{s})^*$. Let $\varphi(\mathbf{s}^*)=([\mathbf{r}]\Theta,[\mathbf{r}]\operatorname{Ker}(\pi_Y))$ and $\varphi(\mathbf{s})=([\mathbf{u}]\Theta,[\mathbf{u}]\operatorname{Ker}(\pi_Y))$, where $\mathbf{r}$ and $\mathbf{u}$ are determined by $\mathbf{x},\mathbf{x}',\mathbf{y},\mathbf{y}'\in S_X$. As above, it is sufficient to show that $\mathbf{r}\Gamma_X\mathbf{u}^*$ and $\mathbf{r}\operatorname{Ker}(\pi_Y)\mathbf{u}^*$.

If $\mathbf{s}$ is in the $\mathbf{b}$-class, then $\mathbf{s}^*$ is in the $\mathbf{0}$-class. Thus, for $i\in Y$, $\pi_i(\mathbf{x})=\pi_i(\mathbf{s}^*)^{**}$ and $\pi_i(\mathbf{y})=\pi_i(\mathbf{s})^{**}$ and, for $i\in X\setminus Y$, $\pi_i(\mathbf{x})=0$ and $\pi_i(\mathbf{y})=1$. Since $\mathbf{x}=\mathbf{y}^*$, $\mathbf{x}\Gamma_X\mathbf{r}$, and $\mathbf{y}\Gamma_X\mathbf{u}$, it follows that $\mathbf{r}\Gamma_X\mathbf{u}^*$. For $i\in Y$, $\pi_i(\mathbf{x}')=\pi_i(\mathbf{s}^*)$ and $\pi_i(\mathbf{y}')=\pi_i(\mathbf{s})$. Since $\mathbf{x}'\operatorname{Ker}(\pi_Y)\mathbf{y}'^*$, we have $\mathbf{r}\operatorname{Ker}(\pi_Y)\mathbf{u}^*$.

If $\mathbf{s}$ is in the $\mu\mathbf{b}$-class, then $\mathbf{s}^*$ is in the $\mathbf{0}$-class. Thus, for $i \in Y$, $\pi_i(\mathbf{x}) = \pi_i(\mathbf{s}^*)^{**}$ and $\pi_i(\mathbf{y}) = \pi_i(\mathbf{s})^{**}$ and, for $i \in X \setminus Y$, $\pi_i(\mathbf{x}) = 0$ and $\pi_i(\mathbf{y}) = 1$. Since $\mathbf{x} = \mathbf{y}^*$, we have $\mathbf{r}\Gamma_X\mathbf{u}^*$. For $i \in Y$, $\pi_i(\mathbf{x}') = \pi_i(\mathbf{s}^*)$ and $\pi_i(\mathbf{y}') = \pi_i(\mathbf{s})$. Since $\mathbf{x}' \operatorname{Ker}(\pi_Y)\mathbf{y}'^*$, we obtain $\mathbf{r} \operatorname{Ker}(\pi_Y)\mathbf{u}^*$.

If $\mathbf{s}$ is in the $\mathbf{0}$-class, then $\mathbf{s}^*$ is in the $\mathbf{b}$-class. Thus, for $i \in Y$, $\pi_i(\mathbf{x}) = \pi_i(\mathbf{s}^*)^{**}$ and $\pi_i(\mathbf{y}) = \pi_i(\mathbf{s})^{**}$ and, for $i \in X \setminus Y$, $\pi_i(\mathbf{x}) = 1$ and $\pi_i(\mathbf{y}) = 0$. Hence, $\mathbf{x} = \mathbf{y}^*$ and, so, $\mathbf{r}\Gamma_X\mathbf{u}^*$. For $i \in Y$, $\pi_i(\mathbf{x}') = \pi_i(\mathbf{s}^*)$ and $\pi_i(\mathbf{y}') = \pi_i(\mathbf{s})$. Hence, $\mathbf{x}' \operatorname{Ker}(\pi_Y)\mathbf{y}'^*$ which implies $\mathbf{r} \operatorname{Ker}(\pi_Y)\mathbf{u}^*$. Consequently, φ is $*$-preserving. ∎

PROPOSITION 5.4. *The family* $(S_W : W \in P_{\text{fin}}(\omega))$ *of finite pseudocomplemented semilattices satisfies the postulate* (P4).

P r o o f. Let $X \in P_{\text{fin}}(\omega)$ and $B, C \in Q(\{S_W : W \in P_{\text{fin}}(\omega)\})$ be finite algebras such that S_X is a subalgebra of $B \times C$. We must exhibit $Y, Z \in P_{\text{fin}}(\omega)$ with $S_Y \in Q(B)$, $S_Z \in Q(C)$, and $X = Y \cup Z$. Clearly, with no loss in generality, we may assume that either B or C is a non-trivial algebra.

Let π_B and π_C denote the projections of S_X into B and C, respectively. Define

$$Y = \{i \in X : \pi_B(\mathbf{e}) \neq \pi_B(\nu_i\mathbf{e})\}, \qquad Z = \{i \in X : \pi_C(\mathbf{e}) \neq \pi_C(\nu_i\mathbf{e})\}.$$

Since S_X is a subalgebra of $B \times C$, $X = Y \cup Z$. Observe, in passing, that $\operatorname{Ker}(\pi_B) \wedge \operatorname{Ker}(\pi_C)$ is the identity on S_X. Further, as $B \times C$ does not contain a trivial subalgebra, it follows that $X \neq \emptyset$ and, hence, either $Y \neq \emptyset$ or $Z \neq \emptyset$.

To begin with, suppose that either $Y = \emptyset$ or $Z = \emptyset$, say $Z = \emptyset$, and so $Y = X$. Since $S_\emptyset \in Q(C)$, it is sufficient to show that S_X is embeddable in B, which implies $S_Y = S_X \in Q(B)$. Because $B \in Q(\{S_W : W \in P_{\text{fin}}(\omega)\})$ and $Q(\{S_W : W \in P_{\text{fin}}(\omega)\}) = Q(\{S_i : i \in \omega\})$, there exists a set $W \in P_{\text{fin}}(\omega)$ and an embedding φ of B in $\prod(S_i : i \in W)$. Thus,

$$S_X \overset{\pi_B}{\hookrightarrow} B \overset{\varphi}{\longmapsto} \prod(S_i : i \in W).$$

Since $X = Y$, Proposition 4.3 implies that $X \subseteq W$ and, for each $i \in X$, the map $\pi_i \circ \varphi \circ \pi_B : S_X \to S_i$ satisfies $\operatorname{Ker}(\pi_i \circ \varphi \circ \pi_B) = \operatorname{Ker}(\pi_i')$ where π_i refers to the projection from $\prod(S_i : i \in W)$ into S_i and π_i' refers to the projection from S_X into S_i. Let $\mathbf{r}, \mathbf{s} \in S_X$ and $\mathbf{r} \equiv \mathbf{s}$ $(\operatorname{Ker}(\pi_B))$. Thus, $\pi_i \circ \varphi \circ \pi_B(\mathbf{r}) = \pi_i \circ \varphi \circ \pi_B(\mathbf{s})$ for all $i \in X$. Therefore $\pi_i(\mathbf{s}) = \pi_i(\mathbf{r})$ for all $i \in X$, which means that $\mathbf{r} = \mathbf{s}$. It follows that $\operatorname{Ker}(\pi_B)$ is the identity on S_X, and so S_X is embeddable in B. In particular, $S_X \in Q(B)$.

Suppose that both $Y \neq \emptyset$ and $Z \neq \emptyset$. As before, there exists a set W and an embedding φ of B in $\prod(S_i : i \in W)$ and, likewise, there exists a set V and an embedding ψ of C in $\prod(S_i : i \in V)$. It is sufficient to show that S_Y embeds in $S_X/\operatorname{Ker}(\pi_B)$ and that S_Z embeds in $S_X/\operatorname{Ker}(\pi_C)$ since this implies that $S_Y \in Q(B)$ and $S_Z \in Q(C)$. We only show that S_Y embeds in $S_X/\operatorname{Ker}(\pi_B)$; the proof that S_Z embeds in $S_X/\operatorname{Ker}(\pi_C)$ is similar. By Proposition 4.3, $Y \subseteq W$ and, for each $i \in Y$, the map $\pi_i \circ \varphi \circ \pi_B : S_X \to S_i$ satisfies $\operatorname{Ker}(\pi_i \circ \varphi \circ \pi_B) = \operatorname{Ker}(\pi_i')$ where π_i and π_i' are as above. In particular, $\bigwedge(\operatorname{Ker}(\pi_i \circ \varphi \circ \pi_B) : i \in Y) = \operatorname{Ker}(\pi_Y)$ where π_Y denotes the projection on S_X. Since $\bigwedge(\operatorname{Ker}(\pi_i \circ \varphi \circ \pi_B) : i \in W) = \operatorname{Ker}(\pi_B)$, it follows that

$$(\dagger) \qquad \operatorname{Ker}(\pi_B) = \bigwedge(\operatorname{Ker}(\pi_i \circ \varphi \circ \pi_B) : i \in W \setminus Y) \wedge \operatorname{Ker}(\pi_Y).$$

However, from Proposition 4.3 it follows that, for each $i \in W \setminus Y$, $\pi_i \circ \varphi \circ \pi_B(\mathbf{e}) = \pi_i \circ \varphi \circ \pi_B(\nu_j \mathbf{e})$ for all $j \in X$. Thus, by Proposition 4.6, $\Gamma_X \le \operatorname{Ker}(\pi_i \circ \varphi \circ \pi_B)$ for each $i \in W \setminus Y$, and so $\Gamma_X \le \bigwedge(\operatorname{Ker}(\pi_i \circ \varphi \circ \pi_B) : i \in W \setminus Y)$. By Lemma 5.2, S_Y is embeddable in $S_X/\bigwedge(\operatorname{Ker}(\pi_i \circ \varphi \circ \pi_B) : i \in W \setminus Y) \wedge \operatorname{Ker}(\pi_Y)$, which, by (†), implies S_Y is embeddable in $S_X/\operatorname{Ker}(\pi_B)$, as required. ∎

References

[1] M. E. Adams, *Implicational classes of pseudocomplemented distributive lattices*, J. London Math. Soc. 13 (1976), 381–384.

[2] M. E. Adams and W. Dziobiak, *Q-universal quasivarieties of algebras*, Proc. Amer. Math. Soc. 120 (1994), 1053–1059.

[3] —, —, *Lattices of quasivarieties of* 3*-element algebras*, J. Algebra 166 (1994), 181–210.

[4] M. E. Adams and M. Gould, *A construction for pseudocomplemented semilattices and two applications*, Proc. Amer. Math. Soc. 106 (1989), 899–905.

[5] S. Burris and H. P. Sankappanavar, *A Course in Universal Algebra*, Springer, New York, 1981.

[6] W. Dziobiak, *On subquasivariety lattices of some varieties related with distributive p-algebras*, Algebra Universalis 21 (1985), 62–67.

[7] G. Grätzer, *General Lattice Theory*, Birkhäuser, Basel, 1978.

[8] G. T. Jones, *Pseudocomplemented semilattices*, Ph.D. dissertation, U.C.L.A., 1972.

[9] A. I. Mal'cev, *On certain frontier questions in algebra and mathematical logic*, Proc. Internat. Congr. of Mathematicians, Moscow 1966, Mir, 1968, 217–231 (in Russian).

[10] —, *Algebraic Systems*, Grundlehren Math. Wiss. 192, Springer, New York, 1973.

[11] H. P. Sankappanavar, *Remarks on subdirectly irreducible pseudocomplemented semi-lattices and distributive pseudocomplemented lattices*, Math. Japon. 25 (1980), 519–521.

[12] M. V. Sapir, *The lattice of quasivarieties of semigroups*, Algebra Universalis 21 (1985), 172–180.

[13] J. Schmid, *Lee classes and sentences for pseudocomplemented semilattices*, ibid. 25 (1988), 223–232.

[14] —, *On amalgamation classes of pseudocomplemented semilattices*, ibid. 29 (1992), 402–418.

[15] A. Shafaat, *On implicational completeness*, Canad. J. Math. 26 (1974), 761–768.

[16] M. P. Tropin, *An embedding of a free lattice into the lattice of quasivarieties of distributive lattices with pseudocomplementation*, Algebra i Logika 22 (1983), 159–167 (in Russian).

[17] A. Wroński, *The number of quasivarieties of distributive lattices with pseudocomplementation*, Polish Acad. Sci. Inst. Philos. Sociol. Sect. Logic 5 (1976), 115–121.

DEPARTMENT OF MATHEMATICS
STATE UNIVERSITY OF NEW YORK
NEW PALTZ, NEW YORK 12561
U.S.A.
E-mail: ADAMSM@MATRIX.NEWPALTZ.EDU

INSTITUTE OF MATHEMATICS
NICHOLAS COPERNICUS UNIVERSITY
CHOPINA 12/18
87-100 TORUŃ, POLAND
E-mail: DZIOBIAK@PLTUMK11.BITNET

DEPARTMENT OF MATHEMATICS
VANDERBILT UNIVERSITY
NASHVILLE, TENNESSEE 37240
U.S.A.
E-mail: GOULDM00@VUCTRVX1.BITNET

INSTITUTE OF MATHEMATICS
UNIVERSITY OF BERN
CH-3012 BERN
SWITZERLAND
E-mail: SCHMID@MATH-STAT.UNIBE.CH

Current address of W. Dziobiak:
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF PUERTO RICO
MAYAGUEZ, PUERTO RICO 00681-5000
U.S.A.
E-mail: W_DZIOBIAK%RUMAC@UPR1.UPR.CLU.EDU

Received 28 February 1994