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Path differentiation: further unification

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Abstract. A. M. Bruckner, R. J. O'Malley, and B. S. Thomson introduced path differentiation as a vehicle for unifying the theory of numerous types of generalized differentiation of real valued functions of a real variable. Part of their classification scheme was based on intersection properties of the underlying path systems. Here, additional light is shed on the relationships between these various types of path differentiation and it is shown how composite differentiation and first return differentiation fit in to this scheme.

1. Introduction. In a 1984 paper, entitled "Path derivatives: a unified view of certain generalized derivatives" [1], A. M. Bruckner, R. J. O'Malley, and B. S. Thomson introduced path differentiation and showed it to be a valid synthesizing framework to encompass most of the fruitful notions of generalized differentiation of functions of a real variable which were under study at that time. Included among these types of differentiation were approximate, Peano, and selective differentiation. In that paper the authors introduced several intersection properties for the underlying path systems as a means of studying the resulting path derivatives and path differentiable functions. In a recent paper [3] by O'Malley and the present authors several of these types of path differentiation were shown to be equivalent under the hypothesis that the path derivative is a Baire 1 bilateral derivate function of the primitive function.

Subsequent to [1], the concept of composite differentiation was introduced by O'Malley and C. E. Weil in [5]. In [6] O'Malley showed that this type of differentiation can also be described in terms of path differentiation with the underlying path system satisfying a type of intersection condition different from those investigated in [1]. Recently, O'Malley [7] introduced the intriguing notion of first return differentiation. The purpose of the present paper is to observe where composite and first return differentiation fit in

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to the path differentiation scheme of generalized differentiation based on the intersection properties introduced in [1] and to shed additional light on the relationships between these various types of path differentiation. Specifically, a schematic outline of what we propose to show is illustrated in the following diagram:

$$
\text{(1)} \qquad \begin{array}{ccccccc}
 & & \text{EIC-PATH} & \longrightarrow & \text{EIC}[m]\text{-PATH} & & \\
 & \nearrow & & & & \searrow & \\
\text{COMPOSITE} & & & & & & \text{SELECTIVE} \\
 & \searrow & & & & \nearrow & \\
 & & \text{SPD} & \longrightarrow & 1^{\text{st}}\text{-RETURN} & &
\end{array}
$$

The remainder of this introductory section consists of introducing or reviewing the terminology necessary to interpret this diagram; the second section of the paper deals with the proofs of the indicated implications, or, more specifically, proofs for those which have not been proved elsewhere; and the final section presents examples to verify that no arrow missing from this chart, other than the obvious ones obtained by following two arrows, can be added.

Each node of diagram (1) is to be interpreted as a statement about an ordered pair (F, f) of real valued functions defined on $[0, 1]$. For example, the COMPOSITE node is intended to represent the statement "F has f as a bilateral derived function and is compositely differentiable to f on $[0, 1]$." Reviewing the notion of composite differentiation as defined in [5], we recall that a *decomposition* of $[0, 1]$ is a collection of closed sets A_n, $n = 1, 2, \ldots$, such that $\bigcup_{n=1}^{\infty} A_n = [0, 1]$, and that F is said to be *compositely differentiable* to f on $[0, 1]$ if there exists a decomposition $\{A_n\}$ of $[0, 1]$ such that for each n and each $y \in A_n$,

$$\lim_{\substack{t \to y \\ t \in A_n}} \frac{F(t) - F(y)}{t - y} = f(y).$$

Next, we review the notion of path differentiation. Let $x \in [0, 1]$. A *path leading to* x is a set $E_x \subseteq [0, 1]$ containing x and having x as an accumulation point. A *path system* is a collection $E = \{E_x : x \in [0, 1]\}$ such that each E_x is a path leading to x. If for each $x \in (0, 1)$, E_x has x as a bilateral limit point, E is called a *bilateral path system*. (We should point out the caveat that throughout this paper "bilateral" will be interpreted as unilateral at both 0 and 1 since we are only considering functions on $[0, 1]$.) We say that F is *path differentiable* to f if there is a path system E such that for each $x \in [0, 1]$,

$$\lim_{\substack{y \to x \\ y \in E_x}} \frac{F(y) - F(x)}{y - x} = f(x).$$

In [1] the following four types of intersection properties were investigated. A system of paths E is said to satisfy the condition listed below if there is associated with E a positive function δ on $[0,1]$ such that whenever $0 < y-x < \min\{\delta(x),\delta(y)\}$, then E_x and E_y intersect in the stated fashion:

- *intersection condition* (IC): $E_x \cap E_y \cap [x,y] \neq \emptyset$;
- *internal intersection condition* (IIC): $E_x \cap E_y \cap (x,y) \neq \emptyset$;
- *external intersection condition, parameter* $m \in \mathbb{N}$ (EIC[m]*)*:

$$E_x \cap E_y \cap (y,(m+1)y - mx) \neq \emptyset \quad \text{and}$$
$$E_x \cap E_y \cap ((m+1)x - my, x) \neq \emptyset;$$

- *one-sided external intersection condition, parameter* $m \in \mathbb{N}$ (*one-sided* EIC[m]):

$$E_x \cap E_y \cap (y,(m+1)y - mx) \neq \emptyset \quad \text{or}$$
$$E_x \cap E_y \cap ((m+1)x - my, x) \neq \emptyset.$$

A statement such as "F is IC path differentiable to f" is to be understood as indicating that F has f as a path derivative on $[0,1]$ with respect to a path system satisfying IC. As in [1], we let EIC denote the condition EIC[1], and we take the EIC-PATH node in diagram (1) to represent the statement "F is EIC path differentiable to f". The EIC[m]-PATH node is interpreted analogously.

A special type of path differentiation is first return differentiation as introduced in [7]. The path system in this instance is generated by a trajectory, where by a *trajectory* we mean any sequence $\{x_n\}_{n=0}^{\infty}$ of distinct points in $(0,1)$ which is dense in $[0,1]$. Let $\{x_n\}$ be a fixed trajectory. For a given interval $(a,b) \subset [0,1]$, $r(a,b)$ will be the first element of the trajectory in (a,b). For $0 \le y < 1$, the *right first return path* to y, R_y^+, is defined recursively via

$$y_1^+ = 1 \quad \text{and} \quad y_{k+1}^+ = r(y, y_k^+).$$

For $0 < y \le 1$, the *left first return path* to y, R_y^-, is defined similarly. For $0 < y < 1$, we set $R_y = R_y^+ \cup R_y^- \cup \{y\}$, and $R_0 = \{0\} \cup R_0^+$, $R_1 = R_1^- \cup \{1\}$. The collection $\mathcal{R} \equiv \{R_y : y \in [0,1]\}$ forms a path system. If there exists a trajectory $\{x_n\}$ such that F is path differentiable to f on $[0,1]$ with respect to the resulting first return path system, then we say that F is *first return differentiable* to f on $[0,1]$. Node 1$^{\text{st}}$-RETURN in our diagram represents the statement "F is first return differentiable to f on $[0,1]$".

Next, we review the notion of selective differentiation as introduced in [4]. We utilize the notation $[a,b]$, or (a,b), to denote the closed, or open, interval having endpoints a and b regardless of whether $a > b$ or $b > a$. A selection function is obtained by assigning to each closed interval $[a,b]$ in $[0,1]$ a point from (a,b) and labeling it $p_{[a,b]}$. The collection of p's thus obtained is called a *selection* S. We say that F is *selectively differentiable* to f on $[0,1]$ if there is a selection S such that for each $x \in [0,1]$,

$$\lim_{y\to x} \frac{F(p_{[x,y]}) - F(x)}{p_{[x,y]} - x} = f(x).$$

The SELECTIVE node in diagram (1) represents the conjunction of the statement "f is a bilateral Baire 1 derivate function of F" and any of the following equivalent statements:

"F is selectively differentiable to f on $[0,1]$."

"F is IIC path differentiable to f on $[0,1]$."

"F is one-sided EIC$[m]$ path differentiable to f on $[0,1]$."

"Every perfect set $M \subseteq [0,1]$ contains a dense G_δ set K such that F restricted to M is differentiable to f at each point of K."

The equivalence of these four statements under the assumption that f is a Baire 1 bilateral derivate function of F was established in [3], where the final condition was called Condition $\mathcal{B}$.

Lastly, we need to explain node SPD. Recently, we have been motivated by the problem of finding a satisfactory characterization of first return differentiation. The following notion of strong path differentiation does not accomplish this; however, it does yield a form of path differentiation which we shall show fits strictly between composite and first return differentiation. Although, contrived in appearance, we shall show (Corollary 1) that it yields a convenient tool for constructing first return differentiable functions. We say that the ordered pair of functions (F, f) satisfies *condition* SPD, or F is *strongly path differentiable* to f on $[0,1]$, if there exist

(a) a path system $\{E_t\}_{t\in[0,1]}$ such that F is path differentiable to f with respect to $\{E_t\}_{t\in[0,1]}$,

(b) a first category F_σ set N such that $F'(x) = f(x)$ for all $x \in [0,1] \setminus N$, and

(c) a Baire 1 function $\delta : N \to (0,\infty)$ such that for each closed subset M of N with $\operatorname{diam}(M) < \inf \delta(M)$, $\bigcap_{t\in M} E_t$ is bilaterally dense at each point of M (i.e., for every $x \in M$, arbitrarily close to x, there are points of $\bigcap_{t\in M} E_t$ to the right and left of x).

If this situation holds, we say that $\{E_t\}_{t\in[0,1]}$, N, $\delta : N \to (0,\infty)$ *witness* SPD for (F, f). The node SPD in diagram (1) represents the statement "(F, f) satisfies condition SPD". (We should clarify what we mean by the function $\delta : N \to (0,\infty)$ belonging to Baire 1, since N is only assumed to be an F_σ of first category. We mean that the inverse image of each open set is an F_σ.)

2. Implications in the diagram. Here we shall verify the implications noted in diagram (1). The statement EIC-PATH $\to$ EIC$[m]$-PATH is, of course, immediate. The implication EIC$[m] \to$ SELECTIVE results from the combination of the following three results: Corollary 6.3 and Theorem 3.4 of [1] and Corollary 2 of [3]. Theorem 2 of [7] established the implication 1^{st}-RETURN $\to$ SELECTIVE. The following theorem establishes the two implications COMPOSITE $\to$ EIC-PATH and COMPOSITE $\to$ SPD:

THEOREM 1. *If* $F : [0,1] \to \mathbb{R}$ *is compositely differentiable to one of its bilateral derivate functions* f *on* $[0,1]$, *then* F *is both* EIC *path differentiable to* f *and strongly path differentiable to* f.

Proof. Since F is compositely differentiable to f, utilizing Theorem 1 of [2] we may obtain an increasing sequence of perfect sets $\{M_n\}$ such that

(i) every point of M_n is a bilateral limit point of M_{n+1},
(ii) $\bigcup_{n=1}^{\infty} M_n = [0,1]$, and
(iii) $F|M_n$ is differentiable to $f|M_n$.

For each $x \in [0,1]$, let n_x be the smallest integer such that $x \in M_{n_x}$. For each x for which $n_x > 1$, let $\delta(x) = \operatorname{dist}(x, M_{n_x-1})$. If $x \in M_1$, let $\delta(x) = 1$. Let our path system $\{E_x\}_{x\in[0,1]}$ be defined by $E_x = M_{n_x+1}$. Clearly, properties (i)–(iii) guarantee that this is, indeed, a path system and that F is path differentiable to f with respect to this system. Furthermore, note that for all x and y, $\operatorname{dist}(x,y) < \min\{\delta(x), \delta(y)\}$ implies $n_x = n_y$. This condition, coupled with property (i) above, immediately yields that this path system satisfies EIC.

Next we shall show that (F, f) satisfies SPD. We first observe that δ is a Baire 1 function by noting that $\delta|P$ has a point of relative continuity for every perfect set P. To see this, let P be perfect and apply the Baire Category Theorem to find a smallest $n \in \mathbb{N}$ such that $P \cap M_n$ contains an open (relative to P) set U. If $n = 1$, then $\delta|P$ is continuous at each point of U. Otherwise, by the minimality of n, there exists an $x \in (M_n \setminus M_{n-1}) \cap U$. From the definition of δ it follows that $\delta(y) = \operatorname{dist}(y, M_{n-1})$ for all $y \in (x - \delta(x), x + \delta(x)) \cap U$, and, hence, $\delta|P$ is continuous at x.

It remains to show that condition (c) of the definition of SPD is satisfied. Since F is compositely differentiable to f, there is an open set O dense in $[0,1]$ such that F is differentiable to f on O. Let $N = [0,1] \setminus O$. Let $K \subseteq N$ be such that $\operatorname{diam}(K) < \inf \delta(K)$. Then there exists a positive integer t such that $n_x = t$ for every $x \in K$. Hence $E_x = M_{t+1}$ for every $x \in K$. Since every point of $K \subset M_t$ is a bilateral limit point of M_{t+1}, condition (c) holds and the proof is complete. ∎

The only remaining implication to be verified in our diagram is SPD $\to$ 1^{st}-RETURN.

THEOREM 2. *If* F *is strongly path differentiable to* f, *then* F *is first return differentiable to* f.

Proof. Let $\{E_x\}_{x\in[0,1]}$, N, $\delta : N \to (0,\infty)$ witness SPD for (F,f). We may assume without loss of generality that N is dense in $[0,1]$. (If N is not dense in $[0,1]$, we may find an F_σ first category set $N_1 \subset [0,1]\setminus \overline{N}$, where $\overline{N}$ represents the closure of N, such that the set $N^* = N \cup N_1$ is dense in $[0,1]$. Define $\delta^*(x) = \operatorname{dist}(x,\overline{N})$ for $x \in N_1$; otherwise let $\delta^*(x) = \delta(x)$. Define $E^*_x = [0,1]$ for all $x \in N_1$; otherwise let $E^*_x = E_x$. One can readily verify that $\{E^*_x\}_{x\in[0,1]}$, N^*, and δ^* witness SPD for (F,f).)

We now claim that N may be decomposed as $\{C_n\}_{n=1}^\infty$ such that

(1) C_n is closed and $\operatorname{diam}(C_n) < \inf \delta(C_n)$,
(2) $\bigcup_{n=1}^\infty C_n = N$, and
(3) $C_n \cap C_m = \emptyset$ for all $n \ne m$.

For each k, let $D_k = \{x \in N : \delta(x) > 1/k\}$. Since δ is Baire 1, D_k is F_σ and hence may be decomposed into countably many closed sets each of diameter less than $1/k$. We may do this for all k and obtain a collection of closed sets $\{F_n\}_{n=1}^\infty$ such that $N = \bigcup_{n=1}^\infty F_n$ and $\operatorname{diam} F_n < \inf \delta(F_n)$. For each $n > 1$, the set $F_n \setminus \bigcup_{k<n} F_k$ is F_σ and nowhere dense, so that it can be decomposed into a sequence of pairwise disjoint closed sets. These decompositions together give a decomposition of N with the desired properties.

Now we define our trajectory. We shall do this inductively by stages. At stage k we will select a partition $\mathcal{P}_k = \{p_k^i\}_{i=0}^{b_k}$ of $[0,1]$ such that

(A) $\operatorname{mesh}(\mathcal{P}_k) < 2^{-k}$,
(B) $\mathcal{P}_k$ properly refines $\mathcal{P}_{k-1}$, and
(C) $\{p_k^0, p_k^1, \ldots, p_k^{b_k}\} \cap (\bigcup_{n=1}^\infty C_n) = \emptyset$. (We are assuming that $0,1 \notin N$ to avoid dealing with cases. If 0 or 1 is in N, a small obvious modification of our proof will yield our theorem.)

Then we will select and order points from some of these partition intervals. At the end of the kth stage with $k \ge 1$, we want $\{x_l\}_{l=0}^{n_k}$, the trajectory defined up to this point, to satisfy the following properties:

(i) If $x \in C_j$ for some $j \le k$ and B is a partition interval from $\mathcal{P}_k$, then there are two points in $B \cap \{x_l\}_{l=0}^{n_k}$, one to the right of x, and the other to the left of x,

(ii) If $x \in C_j$ for some $j < k$, x_l is in the first return path to x, and $n_{k-1} < l \le n_k$, then $x_l \in E_x$.

To get us started, let $\mathcal{P}_0 = \{0,1\}$. At the first stage ($k=1$), we let $\mathcal{P}_1$ be any partition of $[0,1]$ of mesh less than $1/2$ such that condition (C) holds. Let A be a partitioning interval from $\mathcal{P}_1$. If $C_1 \cap A \ne \emptyset$, we select points $r_A \in A$ and $l_A \in A$, to the right of $\sup(C_1 \cap A)$ and to the left of $\inf(C_1 \cap A)$, such that $r_A, l_A \in \bigcap_{t\in C_1} E_t$. Do this for all partition intervals from $\mathcal{P}_1$ which intersect C_1. Put all of these points in one set and label them from left to right as $x_0, x_1, \ldots, x_{n_1}$. Note that both conditions (i) and (ii) are satisfied at this stage.

Assume that stage k has been completed, that the points $x_0, x_1, \ldots, x_{n_k}$ have been specified and conditions (i) and (ii) are satisfied at this stage. Choose a partition $\mathcal{P}_{k+1}$ so that it satisfies conditions (A)–(C) and

(a) each interval of $\mathcal{P}_{k+1}$ contains at most one of $\{x_0, x_1, \ldots, x_{n_k}\}$,
(b) each interval of $\mathcal{P}_{k+1}$ intersects at most one of $\{C_1, C_2, \ldots, C_{k+1}\}$.

We describe how to select the points to be added to the trajectory at this stage and then we shall explain how to order these newly selected points. Fix a partition interval A from $\mathcal{P}_{k+1}$. There exists at most one $1 \le i \le k+1$ such that $C_i \cap A \neq \emptyset$. If there is such i, select points $r_A \in A$ and $l_A \in A$, to the right of $\sup(C_i \cap A)$ and to the left of $\inf(C_i \cap A)$ such that $r_A, l_A \in \bigcap_{t \in C_i} E_t$. Now repeat this for each partition interval of $\mathcal{P}_{k+1}$. If a partition interval misses $\bigcup_{i=1}^{k+1} C_i$, we do not select any points from that interval at this stage.

We have now selected all the points which we wish to add to the trajectory at this stage, and have yet to describe how to order these points, or rather those which have not already appeared in the trajectory construction. We first define an ordering on the partitioning intervals of $\mathcal{P}_{k+1}$. Fix $1 \le j \le k+1$. Label all those partition intervals of $\mathcal{P}_{k+1}$ which intersect C_j as $\{A_i^j\}_{i=1}^{m_j}$. Do this for all $1 \le j \le k+1$. Note that each partition interval gets labeled in this scheme at most once because (b) is satisfied.

Now add the newly selected points at this stage $k+1$ to the trajectory in the following order: First look at the newly selected points of $\bigcup_{i=1}^{m_1} A_i^1$ and order them from left to right and label them as x_n's beginning with x_{n_k+1}. (Keep in mind that a point only gets listed in the trajectory once.) Then look at the newly selected points of $\bigcup_{i=1}^{m_2} A_i^2$ and order them from left to right, etc., continuing this until we have labeled the newly selected points from $\bigcup_{i=1}^{m_{k+1}} A_i^{k+1}$.

That condition (i) holds at stage $k+1$ follows from the construction. Let us now show that condition (ii) holds. Let $x \in C_j$ for some $1 \le j < k+1$ and x_l be in the first return sequence of x for some $n_k < l \le n_{k+1}$. Let A and B be the partition intervals from stages $k+1$ and k, respectively, such that $x \in A$ and $x \in B$. Note that $A \subset B$. Let A' be a partition interval from stage $k+1$ which contains x_l. By condition (i), $B \cap \{x_0, x_1, \ldots, x_{n_k}\}$ contains points to the right and left of x. Hence $x_l \in B$ and $A' \subset B$. Note that $A' \cap C_i = \emptyset$ for $i = 1, 2, \ldots, j-1, j+1, \ldots, k$ because B intersects only one of $C_1, C_2, \ldots, C_k$. Furthermore, $A' \cap C_{k+1} = \emptyset$ because $j < k+1$ and there are points selected to the right and left of x from A at stage $k+1$ and these points are labeled before the points from C_{k+1}. Therefore $A' \cap C_j \neq \emptyset$. This implies that $x_l \in E_x$, completing the verification of condition (ii).

This completes the construction of the trajectory $\{x_n\}$. (That $\{x_n\}$ is dense in $[0,1]$ follows from the fact that N is dense in $[0,1]$.)

Now we must show that F is first return differentiable to f. If $x \notin N$, then $F'(x) = f(x)$. If $x \in N$, then $x \in C_j$ for some j. From condition (ii) we deduce that all points picked in the first return path to x after the jth stage are in E_x, completing the proof of the theorem. ■

The following simple corollary is a convenient mechanism for constructing functions which are strongly path differentiable. We shall utilize this tool in the next section.

Corollary 1. *Let $f : [0,1] \to \mathbb{R}$ be a bilateral derivate of $F : [0,1] \to \mathbb{R}$. If $F'(x) = f(x)$ for all but countably many x, then (F, f) satisfies* SPD *and hence F is first return differentiable to f.*

Proof. Let $N = \{r_1, r_2, \ldots\}$ be the set where F is not differentiable. If $x \notin N$, let $E_x = [0,1]$. If $x \in N$, let E_x be a two-sided path through which $F'(x) = f(x)$. Define $\delta : N \to \mathbb{R}$ as $\delta(r_n) = \operatorname{dist}(r_n, \{r_1, \ldots, r_{n-1}\})$. It is easy to check that $\{E_x\}_{x \in [0,1]}$, N and δ witness SPD for (F, f). ■

4. Examples. We next wish to present examples to show that none of the arrows in diagram (1) can be reversed and that no additional arrows can be inserted between nodes, other than the obvious ones obtained by following two arrows.

Example 1. *There exists a function $F : [0,1] \to \mathbb{R}$ which is* EIC *path differentiable to f, but F is not first return differentiable to f.*

Proof. Let $C \subset [0,1]$ be a Cantor set of positive measure which contains a countable dense subset $\{r_n\}_{n=1}^{\infty}$ such that C has density 1 at each r_n. Label intervals contiguous to C as $\{S_n\}_{n=1}^{\infty}$. Let $F : [0,1] \to [-1,1]$ be a function which has the following properties:

(1) $F = 0$ on C,

(2) F oscillates between -1 and 1 in every neighborhood of each endpoint of each S_n, and

(3) F is differentiable on each S_n.

Let f be such that $f = F'$ on each S_n, $f = 0$ on $C \setminus \{r_1, r_2, \ldots\}$ and $f(r_n) = 2^{-n}$. It is clear that f is a Baire 1 bilateral derivate of F.

Let us first show that F is not first return differentiable to f. To obtain a contradiction assume that F is first return differentiable to f with respect to a trajectory $\{x_n\}_{n=0}^{\infty}$. For each n, let

$$A_n = \left\{ p \in C : \text{if } x_l \in R_p \text{ and } l > n, \text{ then } \left| \frac{F(p) - F(x_l)}{p - x_l} - f(p) \right| < 1 \right\}.$$

For some m, A_m is second category in C. Let U be an open interval such that $U \cap C \neq \emptyset$ and A_m is categorically dense in $U \cap C$. Let $r_j \in U$ such that $j > m$. Using the fact that C has density 1 at r_j and that $f(r_j) = 1/2^j$, we may obtain $x_k \in U$ such that $k > m$, $|r_j - x_k| < 1/m$, $x_k \in R_{r_j}$,

$$\left|\frac{F(r_j) - F(x_k)}{r_j - x_k}\right| > \frac{1}{2 \cdot 2^j} \quad \text{and} \quad \frac{\mu(I \cap C)}{\mu(I)} > 1 - \frac{1}{4 \cdot 2^j},$$

where I is the interval with endpoints r_j and x_k. Utilizing these properties of x_k and the fact that A_m is categorically dense in $U \cap C$, we may obtain $p \in A_m \setminus \{r_1, r_2, \ldots\}$ such that p is between r_j and x_k and $|(r_j - x_k)/(x_k - p)| > 2 \cdot 2^j$. However,

$$\left|\frac{F(p) - F(x_k)}{p - x_k}\right| = \left|\frac{F(r_j) - F(x_k)}{r_j - x_k}\right| \left|\frac{r_j - x_k}{p - x_k}\right| > \frac{1}{2 \cdot 2^j} \cdot 2 \cdot 2^j = 1,$$

contradicting that $p \in A_m$ and completing the proof of the fact that F is not first return differentiable to f.

Next, we show that F is EIC path differentiable to f. If $x \in [0,1] \setminus C$, let $E_x = [0,1]$. We will utilize a certain type of selection to construct a path to $x \in C$. First, for each n, let $0 < \varepsilon_n < \operatorname{dist}(r_n, \{r_1, \ldots, r_{n-1}\})$ be such that if $0 < h < 2\varepsilon_n$, then

$$\frac{\mu\{(r_n - h, r_n + h) \cap C\}}{2h} > \frac{7}{8}.$$

Suppose $x \in C \setminus \{r_1, r_2, \ldots\}$ and $|r_n - x| < \varepsilon_n$. For such x and r_n, we will pick $s^+[r_n, x]$ and $s^-[r_n, x]$ in the following manner: Without loss generality, assume that $r_n < x$. (If $x < r_n$ pick $s^+[r_n, x]$ and $s^-[r_n, x]$ in a symmetric fashion.) Let $s^-[r_n, x]$ be such that $2r_n - x < s^-[r_n, x] < r_n$ and

$$\frac{F(s^-[r_n, x]) - F(r_n)}{s^-[r_n, x] - r_n} = \frac{1}{2^n}.$$

Since $0 < x - r_n < \varepsilon_n$, the interval $(5x/4 - r_n/4, 7x/4 - 3r_n/4)$ intersects C in a set of positive measure and hence contains a portion (relatively open subset) of C. So we may pick $s^+[r_n, x] \in (5x/4 - r_n/4, 7x/4 - 3r_n/4)$ such that

$$\frac{F(s^+[r_n, x]) - F(r_n)}{s^+[r_n, x] - r_n} = \frac{1}{2^n}.$$

Now observe that

$$\left|\frac{F(s^-[r_n, x]) - F(x)}{s^-[r_n, x] - x}\right| < \frac{1}{2^n} \quad \text{and}$$

$$\left|\frac{F(s^+[r_n, x]) - F(x)}{s^+[r_n, x] - x}\right| = \left|\frac{F(s^+[r_n, x]) - F(r_n)}{s^+[r_n, x] - r_n}\right| \cdot \left|\frac{s^+[r_n, x] - r_n}{s^+[r_n, x] - x}\right| < \frac{1}{2^n} \cdot 7.$$

Now for each $x \in C \setminus \{r_1, r_2, \ldots\}$, let

$$E_x = F^{-1}(\{0\}) \cup \{s^+[r_n, x], s^-[r_n, x] : \mathrm{dist}(x, r_n) < \varepsilon_n \text{ for some } n\}$$

and if $x = r_m$, let

$$E_x = \{x\} \cup \{s^+[r_n, y], s^-[r_n, y] : \mathrm{dist}(y, r_m) < \varepsilon_m \text{ for some } y \in C\}.$$

Define $\delta : [0,1] \to \mathbb{R}^+$ as $\delta(x) = \mathrm{dist}(x, C)$ if $x \notin C$, $\delta(x) = 1$ if $x \in C \setminus \{r_1, r_2, \ldots\}$, and $\delta(r_n) = \varepsilon_n$.

Let us first observe that F is path differentiable to f with respect $\{E_x\}$. It is easy to check that $\{E_x\}_{x\in[0,1]}$ is a bilateral path system. We must show that for each $x \in [0,1]$,

$$\lim_{\substack{y\to x\\ y\in E_x}} \frac{F(y) - F(x)}{y - x} = f(x). \tag{2}$$

If $x \notin C$, then $F'(x) = f(x)$ and (2) is immediate. If $x \in C \setminus \{r_1, r_2, \ldots\}$ and $\varepsilon > 0$ then let $d = \mathrm{dist}(x, \{s^+[r_1, x], s^-[r_1, x], \ldots, s^+[r_N, x], s^-[r_N, x]\})$ where N is such that $7 \cdot 2^{-N} < \varepsilon$. From the estimates obtained above, it follows that for all $y \in E_x$ with $0 < |y-x| < d$, $|(F(x) - F(y))/(x - y)| < \varepsilon$, and, thus, (2) holds. If $x = r_m$, then (2) follows directly from the method by which s^+ and s^- were picked.

Next we show that $\{E_x\}_{x\in[0,1]}$ satisfies EIC. Let $x, y \in [0,1]$ be such that $|x - y| < \min\{\delta(x), \delta(y)\}$. Then either x and y are both in $[0,1] \setminus C$ or they are both in C. If $x, y \in [0,1] \setminus C$, then $E_x = E_y = [0,1]$ and we are done. If $x, y \in C \setminus \{r_1, r_2, \ldots\}$ then we are done because $F^{-1}(0) \subset E_x \cap E_y$. Next note that x and y cannot both be in $\{r_1, r_2, \ldots\}$. So the last case is that $x = r_m$ for some m and $y \in C \setminus \{r_1, r_2, \ldots\}$. Then $|r_m - y| = |x - y| < \delta(r_m) = e_m$, which implies that $s^+[r_m, y], s^-[r_m, y] \in E_x \cap E_y$, completing the proof for this example. ∎

We wish to acknowledge that the seed for the following example was planted during a conversation, concerning first return differentiation, between Richard O'Malley and the second author.

EXAMPLE 2. *There is a continuous function* $F : [0,1] \to \mathbb{R}$ *and a Baire* 1 *function* $f : [0,1] \to \mathbb{R}$ *such that*

(a) F *is path differentiable to* f *on* $[0,1]$.
(b) F *is differentiable except at countably many points.*
(c) F *is not* EIC$[m]$ *path differentiable to any function* g *on* $[0,1]$.

Furthermore, *the ordered pair of functions* (F, f) *satisfies* SPD.

Proof. First we define a certain symmetric Cantor set $C \subset [0,1]$. Let Σ denote the set of all finite sequences of 0's and 1's. If $\sigma \in \Sigma$ we denote the length of σ by $|\sigma|$. We set $\alpha_n = (n+1)/(n+2)$ for all $n = 0, 1, \ldots$ We shall utilize the sequence $\{\alpha_n\}$ to construct our Cantor set C by deleting open intervals of relative size α_n at the nth stage. More specifically, we identify the complementary intervals and the noncomplementary intervals to this Cantor set using subscripts from Σ in the standard way, i.e. $I_\emptyset = (1/2-\alpha_0/2, 1/2+\alpha_0/2)$, J_0 and J_1 are the left and right hand components of the complement of $I_\emptyset$ respectively; I_0 and I_1 are the open intervals of length $\alpha_1(1-\alpha_0)/2$ centered in J_0 and J_1 respectively, and so on. Our Cantor set is then

$$C = \bigcap_{n=1}^{\infty} \bigcup_{|\sigma|=n} J_\sigma.$$

Note that

$$|J_\sigma| = \prod_{n=0}^{|\sigma|-1} \left(\frac{1-\alpha_n}{2}\right) \quad \text{and} \quad |I_\sigma| = \alpha_{|\sigma|}|J_\sigma|,$$

where $|I|$ is used to denote the length of an interval I. For each $\sigma \in \Sigma$ we let l_σ and r_σ denote the left and right endpoints of I_σ, respectively. We set

$$H_\sigma = (l_\sigma + |\sigma||J_{\sigma 0}|, l_\sigma + (|\sigma|+1)|J_{\sigma 0}|),$$
$$K_\sigma = (r_\sigma - |J_{\sigma 0}|, r_\sigma),$$

and let h_σ denote the midpoint of H_σ.

Next, for each $\sigma \in \Sigma$ with the property that at least one term in σ is 1, we define $\tau_\sigma \in \Sigma$ to be such that I_{τ_σ} is the nearest interval of the collection $\{I_\tau : |\tau| < |\sigma|\}$ lying to the left of I_σ. If every term of σ is 0, then we set $\tau_\sigma = -1$, and define $|-1| = -1$, $I_{-1} = (-\infty, 0)$, and $r_{-1} = 0$.

Now we are ready to define the function $F : [0,1] \to \mathbb{R}$. First let $F(x) = 0$ for all $x \in G \equiv [0,1] \setminus \bigcup_{\sigma\in\Sigma}(H_\sigma \cup K_\sigma)$. For each $\sigma \in \Sigma$ we define F on $H_\sigma \cup K_\sigma$ to be a function which satisfies the following:

(i) F is differentiable on $H_\sigma \cup K_\sigma$,

(ii) F has right derivative 0 at the left endpoint of H_σ and at the left endpoint of K_σ,

(iii) F has left derivative 0 at the right endpoint of H_σ,

(iv) F has left derivative $1/2^{|\sigma|}$ at the right endpoint of K_σ,

(v) $F(h_\sigma) = (1/2^{|\tau_\sigma|})(h_\sigma - r_{\tau_\sigma})$,

(vi) $\forall x \in H_\sigma,\ 0 < F(x) \le (F(h_\sigma)/(h_\sigma - l_\sigma))(x - l_\sigma)$,

(vii) $\forall x \in K_\sigma,\ (1/2^{|\sigma|})(x - r_\sigma) \le F(x) < 0$.

It is then a straightforward exercise to verify that for all $\sigma \in \Sigma$, all $y \in I_\sigma$, and all $x \in J_{\sigma 0} \cup J_{\sigma 1}$ we have

$$\left|\frac{F(y)}{x-y}\right| \le \frac{1}{2^{|\tau_\sigma|-1}}. \tag{3}$$

Let N denote the set of all right endpoints of contiguous intervals of C. We first wish to observe that at each x in $C \setminus N$, F is differentiable with derivative 0. To see this, let $x \in C$. Then for each $n = 0, 1, \ldots,$ there is a unique $\sigma_n(x)$ of length n and a number $e_n(x) \in \{0, 1\}$ such that $x \in J_{\sigma_n(x)e_n(x)}$; that is,

$$\{x\} = \bigcap_{n=0}^{\infty} J_{\sigma_n(x)e_n(x)}.$$

Note that if $e_n(x) = 0$, then $\tau_{\sigma_n(x)} = \tau_{\sigma_{n-1}(x)}$, and if $e_n(x) = 1$, then $|\tau_{\sigma_n(x)}| \geq |\tau_{\sigma_{n-1}(x)}| + 1$. If $x \in C \setminus N$, then $e_n(x) = 1$ for infinitely many n and hence $|\tau_{\sigma_n(x)}| \to \infty$ as $n \to \infty$. This fact, inequality (3), and the fact that F is zero on C yield that $F'(x) = 0$ for each such x. (If x is a left endpoint of a contiguous interval, then the fact that F has right derivative zero at x, condition (iii), is also needed.) Thus, property (b) of the statement of this example holds.

Next, let

$$f(x) = \begin{cases} 1/2^{|\sigma|} & \text{if } x = r_\sigma, \\ F'(x) & \text{if } x \in H_\sigma \cup K_\sigma, \\ 0 & \text{otherwise.} \end{cases}$$

We define a path system $E = \{E_x : x \in [0, 1]\}$ as follows:

$$E_x = \begin{cases} [0, 1] & \text{if } x \notin N, \\ \{h_\sigma : \tau_\sigma = \tau\} \cup [0, r_\tau] & \text{if } x = r_\tau. \end{cases}$$

Clearly, F is path differentiable to f on $[0, 1]$ with respect to this path system E; that is, statement (a) holds.

Turning to statement (c), suppose that $E^* = \{E^*_x : x \in [0, 1]\}$ is *any* path system with respect to which F is path differentiable to some function g on $[0, 1]$, and suppose that there exists an m such that E^* satisfies EIC$[m]$. First, note that since F has $f(x)$ as an ordinary left derivative everywhere, we must have $g = f$. Let δ be the function noted in the definition of EIC$[m]$ associated with E^*. For each $k \in \mathbb{N}$ set

$$A_k = \{x \in C : \delta(x) > 1/k\}.$$

Using the Baire Category Theorem, we may find a k and an interval (a, b) such that $P \equiv (a, b) \cap C \neq \emptyset$ and A_k is dense in P. Choose $\tau \in \Sigma$ such that $r_\tau \in \overline{P}$. Choose $n \in \mathbb{N}$ greater than both $|\tau|$ and m. Let I_{σ^*} denote the closest interval to the right of r_τ belonging to the collection $\{I_\sigma : |\sigma| = n\}$, and let $\delta_1 = l_{\sigma^*} - r_\tau$. Next, choose $0 < \delta_2 < \min\{\delta_1, \delta(r_\tau)\}$ so that

$$\forall x \in (r_\tau, r_\tau + \delta_2) \cap E^*_{r_\tau}, \qquad \left| \frac{F(x) - F(r_\tau)}{x - r_\tau} - \frac{1}{2^{|\tau|}} \right| < \frac{1}{2^{|\tau|+1}}.$$

Based on the construction of F, it follows that

$$(r_\tau, r_\tau + \delta_2) \cap E^*_{r_\tau} \subseteq (r_\tau, r_\tau + \delta_2) \cap \bigcup \{H_\sigma : \tau_\sigma = \tau\}. \tag{4}$$

Let $p \in A_k$ with $0 < p - r_\tau < \min\{1/k, \delta_2/(m+1)\}$. Since $0 < p - r_\tau < 1/k$ it follows from EIC$[m]$ that

$$E^*_{r_\tau} \cap (p, (m+1)p - mr_\tau) \neq \emptyset.$$

The remainder of the proof of statement (c) consists in showing that this is impossible.

To this end, let $q \in E^*_{r_\tau} \cap (p, (m+1)p - mr_\tau)$. Since $0 < p - r_\tau < \delta_2/(m+1)$, it follows that $0 < q - r_\tau < \delta_2$, and hence, from (4), that

$$q \in (r_\tau, r_\tau + \delta_2) \cap \bigcup\{H_\sigma : \tau_\sigma = \tau\}. \tag{5}$$

Let σ_p be the longest sequence in Σ such that I_{σ_p} lies to the right of p and for which $H_{\sigma_p} \subset \bigcup\{H_\sigma : \tau_\sigma = \tau\}$. Since $0 < p - r_\tau < \delta_1$, we must have $|\sigma_p| > n$. Next, we note that

$$(p, r_{\sigma_p}) \cap \bigcup\{H_\sigma : \tau_\sigma = \tau\} = H_{\sigma_p}. \tag{6}$$

Consequently,

$$\operatorname{dist}(p, H_{\sigma_p}) < q - p < m(p - r_\tau) \le m(l_{\sigma_p} - r_\tau). \tag{7}$$

However, from the construction of the set C we have

$$\begin{aligned}\operatorname{dist}(p, H_{\sigma_p}) &> \operatorname{dist}(l_{\sigma_p}, H_{\sigma_p}) = |\sigma_p|\,|J_{\sigma_p 0}| \\ &= |\sigma_p|(l_{\sigma_p} - r_\tau) > n(l_{\sigma_p} - r_\tau) > m(l_{\sigma_p} - r_\tau).\end{aligned} \tag{8}$$

The contradiction resulting from (7) and (8) completes the proof of statement (c).

That (F, f) satisfies SPD follows immediately from Corollary 1. ∎

Our primary goal with the next example is to show that an EIC$[m]$ path differentiable function need not be EIC path differentiable. We show slightly more by constructing a function F with is both EIC[5] path differentiable and strongly path differentiable to a function f, but which is not EIC path differentiable.

EXAMPLE 3. *There is a continuous function $F : [0,1] \to \mathbb{R}$ and a Baire* 1 *function $f : [0,1] \to \mathbb{R}$ such that*

(a) *F is* EIC[5] *path differentiable to f on* $[0,1]$.
(b) *F is differentiable except at countably many points.*
(c) *F is not* EIC *path differentiable to any function g on* $[0,1]$.

Furthermore, the ordered pair of functions (F, f) satisfies SPD.

Proof. Our construction is a modification, actually a simplification, of the construction given in the previous example. We utilize the same notation with the following two exceptions:

- $\alpha_n = 3/5$ for each $n = 0, 1, 2, \ldots,$
- $H_\sigma = (l_\sigma + |J_{\sigma 0}|, l_\sigma + 2|J_{\sigma 0}|)$.

With these changes we define F, f, and the path system E exactly as before. For the same reasons as given in the previous example, it follows that F is differentiable to f at each point of $C\setminus N$ (that is, statement (b) holds) and F is path differentiable to f on $[0,1]$ with respect to the path system E.

Next, to complete the proof of statement (a), we need to observe that this path system E satisfies EIC[5]. To this end, define $\delta:[0,1]\to(0,\infty)$ by $\delta(x)=1$ if $x\in[0,1]\setminus N$, and $\delta(r_\tau)=\operatorname{dist}(r_\tau,\bigcup\{I_\sigma:|\sigma|<|\tau|\})/2$. Next, let x and y be any two points with $0<y-x<\min\{\delta(x),\delta(y)\}$. Clearly, x and y cannot both be in N, and if both are in $[0,1]\setminus N$, then $E_x=E_y=[0,1]$. Thus, the only situation to consider is where one of the points x,y is in N and the other is not. First, suppose that $x\in N$ and $y\in[0,1]\setminus N$. Let $x=r_\tau$ and let σ_y denote that unique element of $\{\sigma:\tau_\sigma=\tau\}$ for which $y\in[h_{\sigma_y0},h_\sigma)$. Then

$$y<h_{\sigma_y}=5h_{\sigma_y0}-4r_\tau<6h_{\sigma_y0}-5r_\tau\le 6y-5r_\tau,$$

and hence

$$E_x\cap E_y\cap(y,6y-5x)\neq\emptyset. \tag{9}$$

Furthermore, since $E_y=[0,1]$ and $E_x=[0,x]$, we clearly have

$$E_x\cap E_y\cap(6x-5y,x)\neq\emptyset. \tag{10}$$

The final situation is where $y\in N$ and $x\in[0,1]\setminus N$. However, since $E_x=[0,1]$, $[0,y]\subset E_y$, and y is a limit point from the right of E_y, we immediately obtain (9) and (10), completing the proof that E satisfies EIC[5] and, consequently, the proof of statement (a).

Turning to statement (c), we may proceed precisely as in the argument given in the previous example, replacing each occurrence of "m" by "1". All steps proceed as before with (7) now reading

$$\operatorname{dist}(p,H_{\sigma_p})<q-p<p-r_\tau\le l_{\sigma_p}-r_\tau. \tag{11}$$

In place of inequality (8), we now simply observe that based on the construction of C in this example we have

$$\operatorname{dist}(p,H_{\sigma_p})>\operatorname{dist}(l_{\sigma_p},H_{\sigma_p})=|J_{\sigma_p0}|=l_{\sigma_p}-r_\tau. \tag{12}$$

The contradiction resulting from (11) and (12) completes the proof of statement (c).

That (F,f) satisfies SPD again follows immediately from Corollary 1. ∎

EXAMPLE 4. *There exists (F,f) such that F is first return differentiable to f but (F,f) does not satisfy* SPD.

Proof. We utilize Example 2 from [2]. Let $C\subset[0,1]$ be the standard middle third Cantor set constructed in the standard fashion. Let $\{(a_i,b_i)\}_{i=1}^\infty$ be an enumeration of the intervals contiguous to C, listed in such a way that both of the sequences $\{a_{2j}\}_{j=1}^\infty$ and $\{b_{2j-1}\}_{j=1}^\infty$ are dense in C.

Let $F : [0,1] \to \mathbb{R}$ be such that

(1) $F(C) = \{0\}$,

(2) F is differentiable on $[0,1] \setminus C$, and

(3) for each even $i \in \mathbb{N}$, F is zero on $(a_i, (3a_i + b_i)/4] \cup [(a_i + 3b_i)/4, b_i)$, $F((a_i + b_i)/2) = 1$, F is increasing on $[(3a_i + b_i)/4, (a_i + b_i)/2]$, and F is decreasing on $[(a_i + b_i)/2, (a_i + 3b_i)/4]$; likewise, for each odd $i \in \mathbb{N}$, F is zero on $(a_i, (3a_i + b_i)/4] \cup [(a_i + 3b_i)/4, b_i)$, $F((a_i + b_i)/2) = -1$, F is decreasing on $[(3a_i + b_i)/4, (a_i + b_i)/2]$ and F is increasing on $[(a_i + b_i)/2, (a_i + 3b_i)/4]$.

Let $\{c_i\}_{i=1}^{\infty}$ be a dense subset of C, containing no endpoint of a contiguous interval, and neither 0 nor 1. Let $f : [0,1] \to \mathbb{R}$ be such that $f(c_i) = 2^{-i}$, f is the derivative of F on $[0,1] \setminus C$, and f is zero on $C \setminus \{c_1, c_2, \ldots\}$.

In [2], it was shown that F is first return differentiable to f. Here we will show that (F, f) does not satisfy SPD. In order to obtain a contradiction, assume that $\{E_x\}_{x \in [0,1]}$, N, and $\delta : N \to (0, \infty)$ witness SPD for (F, f). Since F is not differentiable on C, $C \subseteq N$. As δ is Baire 1, we may obtain an interval $U \subset C$ such that U is clopen relative to C and $\operatorname{diam}(U) < \inf \delta(U)$. For each positive integer n, let

$$A_n = \left\{ x \in U : \text{ if } y \in E_x \text{ and } 0 < |x - y| < \frac{1}{n}, \right. \qquad \left. \text{then } \left| \frac{F(x) - F(y)}{x - y} - f(x) \right| < 1 \right\}.$$

Let m be such that A_m is second category in U. Let $V \subset U$ be an open interval of C such that A_m is categorically dense in V. Then V contains c_j for some j. Let $0 < h < 1/m$ be such that if $t \in E_{c_j}$ and $0 < |c_j - t| < h$, then $|(F(c_j) - F(t))/(c_j - t)| > 1/(2 \cdot 2^j)$. Let $x \in (V \cap A_m) \setminus \{c_1, c_2, \ldots\}$ such that $|x - c_j| < h$. Since $x, c_j \in U$, there exists $y \in \bigcap_{t \in U} E_t$ such that y is between x and c_j and $|(y - c_j)/(x - y)| > 2 \cdot 2^j$. However, this yields a contradiction because $x \in A_m$, $y \in E_x$, $|x - y| < 1/m$, and

$$\left| \frac{F(x) - F(y)}{x - y} \right| = \left| \frac{F(c_j) - F(y)}{c_j - y} \right| \left| \frac{c_j - y}{x - y} \right| > \frac{1}{2 \cdot 2^j} \cdot 2 \cdot 2^j = 1. \ \blacksquare$$

These four examples accomplish the goal of this section; that is, they show that no nontrivial arrows can be added between the nodes of diagram (1).

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