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Iterated coil enlargements of algebras

by

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Abstract. Let Λ be a finite-dimensional, basic and connected algebra over an algebraically closed field, and $\operatorname{mod}\Lambda$ be the category of finitely generated right Λ-modules. We say that Λ has acceptable projectives if the indecomposable projective Λ-modules lie either in a preprojective component without injective modules or in a standard coil, and the standard coils containing projectives are ordered. We prove that for such an algebra Λ the following conditions are equivalent: (a) Λ is tame, (b) the Tits form q_Λ of Λ is weakly non-negative, (c) Λ is an iterated coil enlargement.

Introduction. Let k be an algebraically closed field, and Λ be a finite-dimensional, basic and connected k-algebra. We denote by $\operatorname{mod}\Lambda$ the category of finitely generated right Λ-modules.

The notion of admissible operations was introduced in [2]. There, a coil is defined as a translation quiver that is obtained from a stable tube by a sequence of admissible operations. A multicoil consists then of a finite set of coils glued together by some directed part. An algebra Λ is called a multicoil algebra if any cycle in $\operatorname{mod}\Lambda$ belongs to one standard coil of a multicoil in the Auslander–Reiten quiver of Λ. It is shown in [1, 1.4] that multicoil algebras are tame, and in [2, 4.6], that they are of polynomial growth. In fact, the class of multicoil algebras contains all the best understood examples of algebras of polynomial growth and finite global dimension. It also seems to be important for studying the simply connected algebras of polynomial growth, as seen in [13], where it is shown that a strongly simply connected algebra Λ is of polynomial growth if and only if Λ is a multicoil algebra.

In this paper, we are concerned with the study of a certain class of multicoil algebras, namely, those algebras obtained by an iteration of the process given in [4] for defining the tame coil enlargements of a tame concealed algebra. We call these algebras iterated coil enlargements, and we give a complete description of their module categories.

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Generalizing the definition given in [10], we say that an algebra Λ has acceptable projectives if the Auslander–Reiten quiver of Λ has components $\mathcal{P}, \mathcal{C}_1, \ldots, \mathcal{C}_r$ with the following properties:

(i) Any indecomposable projective Λ-module lies in $\mathcal{P}$ or in some $\mathcal{C}_i$.
(ii) $\mathcal{P}$ is a preprojective component without injective modules.
(iii) Each $\mathcal{C}_i$ is a standard coil.
(iv) If $\mathrm{Hom}_\Lambda(\mathcal{C}_i, \mathcal{C}_j) \neq 0$, then $i \leq j$.

An algebra Λ having acceptable projectives is triangular (that is, there is no oriented cycle in its ordinary quiver), and consequently, by [6], the Tits form q_Λ of Λ is defined (see (2.3)). The main result of our paper is the following generalization of [10, 3.4]:

THEOREM. *Let Λ be an algebra with acceptable projectives. Then the following conditions are equivalent*:

(a) *Λ is an iterated coil enlargement.*
(b) *Λ is tame.*
(c) *q_Λ is weakly non-negative.*

The paper is organized as follows. In Section 2, we fix the notations and recall some basic definitions. Section 3 contains the construction of the iterated coil enlargements and the description of their module categories. In Section 4 we prove our main theorem.

2. Preliminaries

2.1. *Notation.* Throughout this paper k denotes a fixed algebraically closed field. By an algebra we mean a finite-dimensional k-algebra, which we assume to be basic and connected. Such an algebra Λ can be written as a bound quiver algebra $\Lambda \cong kQ/I$, where Q is the quiver of Λ and I is an admissible ideal of the path algebra kQ of Q. We call an algebra *triangular* whenever its quiver has no oriented cycle. An algebra $\Lambda = kQ/I$ can be considered as a k-category with objects the vertices Q_0 of Q, and with the set of morphisms from x to y being the vector space $kQ(x,y)$ of all linear combinations of paths in Q from x to y modulo the subspace $I(x,y) = I \cap kQ(x,y)$ (see [7]). A full subcategory $\mathcal{C}$ of Λ is called *convex* in Λ if any path with source and sink in $\mathcal{C}$ lies entirely in $\mathcal{C}$.

By a Λ-module is always meant a finitely generated right Λ-module, and we denote their category by $\mathrm{mod}\,\Lambda$. We denote by $\mathrm{ind}\,\Lambda$ a full subcategory of $\mathrm{mod}\,\Lambda$ consisting of a set of representatives of the isomorphism classes of indecomposable Λ-modules. A *path* in $\mathrm{mod}\,\Lambda$ is a sequence of non-zero non-isomorphisms $M_0 \to M_1 \to \ldots \to M_t$, where the M_i are indecomposables. Such a path is said to be *sectional* if $M_{i-1} \ncong D\,\mathrm{tr}\, M_{i+1}$ for all $0 < i < t$;

and it is called a *cycle* if $M_0 \cong M_t$. An indecomposable Λ-module M is called *directing* if it lies in no cycle in $\operatorname{mod}\Lambda$.

For each vertex $x \in Q_0$, we denote by S_x the corresponding simple Λ-module, and by P_x (respectively, I_x) the projective cover (respectively, the injective envelope) of S_x. The *dimension vector* of a module M is the vector $\underline{\dim}\, M = (\dim_k \operatorname{Hom}_\Lambda(P_x, M))_{x\in Q_0}$. The *support* $\operatorname{Supp}(d)$ of a vector $d = (d_x)_{x\in Q_0}$ is the full subcategory of Λ with object class $\{x \in Q_0 \mid d_x \neq 0\}$. The *support* $\operatorname{Supp}(M)$ of a module M is the support of its dimension vector $\underline{\dim}\, M$. A module M is called *sincere* if its support is equal to Λ.

2.2. *Auslander–Reiten components*. For an algebra Λ we denote by Γ_Λ the Auslander–Reiten quiver of Λ, and by $\tau_\Lambda = D\operatorname{tr}$ and $\tau_\Lambda^- = \operatorname{tr} D$ the Auslander–Reiten translations. We identify the vertices of Γ_Λ with the corresponding indecomposable Λ-modules. Let $\mathcal{C}$ be a component of Γ_Λ. We denote by $\operatorname{ind}\mathcal{C}$ the full subcategory of $\operatorname{mod}\Lambda$ with objects the vertices of $\mathcal{C}$, and we say that $\mathcal{C}$ is *standard* if $\operatorname{ind}\mathcal{C}$ is equivalent to the mesh category $k(\mathcal{C})$ of $\mathcal{C}$ (see [7]). We denote by $\operatorname{add}\mathcal{C}$ the additive full subcategory of $\operatorname{mod}\Lambda$ consisting of the direct sums of indecomposable modules in $\mathcal{C}$.

A translation quiver without multiple arrows is called a *tube* if it contains a cyclic path and its underlying topological space is homeomorphic to $S^1 \times \mathbb{R}^+$, where S^1 is the unit circle and $\mathbb{R}^+$ the set of non-negative real numbers (cf. [11]).A tube has only two types of arrows: arrows pointing to infinity and arrows pointing to the mouth. An infinite sectional path consisting of arrows pointing to infinity (respectively, to the mouth) is called a *ray* (respectively, a *coray*). Tubes containing no projective or injective are called *stable*.

Let (Γ, τ) be a connected translation quiver and $X \in \Gamma_0$. In [3], three operations modifying (Γ, τ) to a new translation quiver (Γ', τ') are defined according to the shape of the support of $\operatorname{Hom}_{k(\Gamma)}(X, -)$. These three operations and their duals are called *admissible*. The point X is called the *pivot* of the operation. A translation quiver Γ is called a *coil* if there is a sequence of translation quivers $\Gamma_0, \Gamma_1, \ldots, \Gamma_m = \Gamma$ such that Γ_0 is a stable tube and, for each $0 \le i < m$, Γ_{i+1} is obtained from Γ_i by an admissible operation.

Coils share many properties with tubes, for instance, all but finitely many points in a coil belong to a cyclic path. The set of points in a coil Γ which are the starting or ending point of a mesh in Γ with a unique middle term is called the *mouth* of Γ. A coil contains a maximal tube as a cofinite full translation subquiver. Arrows of this tube may thus be subdivided into two classes: arrows pointing to the mouth and arrows pointing to infinity. A mesh in a coil has at most three middle terms. A mesh with exactly three middle terms is called *exceptional*, and it must have one of its middle terms on the mouth or projective-injective. A projective middle term of an exceptional mesh is called *exceptional projective*. Other meshes and projectives are called *ordinary*. An axiomatic description of coils is given in [3, 4.2].

Finally, a translation quiver (Γ, τ) is a *multicoil* if it contains a full translation subquiver Γ' such that Γ' is a disjoint union of coils and all points in $\Gamma \setminus \Gamma'$ are directing.

2.3. *The Tits and Euler forms.* For a triangular algebra Λ, the *Tits form* q_Λ of Λ was introduced in [6] as the quadratic form $q_\Lambda : \mathbb{Z}^{Q_0} \to \mathbb{Z}$ given by

$$q_\Lambda(z) = \sum_{i \in Q_0} z_i^2 - \sum_{i,j \in Q_0} z_i z_j \dim_k \operatorname{Ext}^1_\Lambda(S_i, S_j) + \sum_{i,j \in Q_0} z_i z_j \dim_k \operatorname{Ext}^2_\Lambda(S_i, S_j).$$

We denote by $(-,-)_\Lambda$ the symmetric bilinear form associated with q_Λ.

Assume that $Q_0 = \{1, \ldots, n\}$. The *Cartan matrix* C_Λ of Λ is the $n \times n$ matrix whose ij-entry is $\dim_k \operatorname{Hom}_\Lambda(P_i, P_j)$. If the global dimension of Λ is finite (for instance, if Λ is triangular), then C_Λ is invertible and we can define the *Euler characteristic* on $\mathbb{Z}^{Q_0}$ by

$$\langle x, y \rangle = x C_\Lambda^{-t} y^t.$$

It has the following homological interpretation:

$$\langle \underline{\dim} X, \underline{\dim} Y \rangle = \sum_{i=0}^{\infty} (-1)^i \dim_k \operatorname{Ext}^i_\Lambda(X, Y)$$

for any two Λ-modules X, Y. The *Euler form* χ_Λ of Λ is defined by $\chi_\Lambda(z) = \langle z, z \rangle_\Lambda$. Since the above homological formula only depends on $\underline{\dim}\, X$ and $\underline{\dim}\, Y$, we see that $q_\Lambda = \chi_\Lambda$ when $\operatorname{gl\,dim} \Lambda \le 2$, and $q_\Lambda(z) \ge \chi_\Lambda(z)$ for any vector z with non-negative coordinates when $\operatorname{gl\,dim} \Lambda \le 3$.

2.4. *One-point extensions.* The *one-point extension* of the algebra Λ by the Λ-module X is the algebra $\Lambda[X] = \left(\begin{smallmatrix} \Lambda & 0 \\ X & k \end{smallmatrix}\right)$ with the usual addition and multiplication of matrices. The quiver of $\Lambda[X]$ contains that of Λ as a full subquiver, and there is an additional vertex which is a source. The $\Lambda[X]$-modules are usually identified with the triples (V, M, γ), where V is a k-vector space, M is a Λ-module and $\gamma : V \to \operatorname{Hom}_\Lambda(X, M)$ is a k-linear map. A $\Lambda[X]$-homomorphism $(V, M, \gamma) \to (V', M', \gamma')$ is thus a pair (f, g), where $f : V \to V'$ is k-linear and $g : M \to M'$ is a Λ-homomorphism such that $\gamma' f = \operatorname{Hom}_\Lambda(X, g)\gamma$. One defines dually the *one-point coextension* $[X]\Lambda$ of Λ by X.

2.5. *Tame algebras.* Following [8], we call an algebra Λ *tame* if, for any dimension d, there is a finite number of $k[X]$-Λ-bimodules M_i which are finitely generated and free as left $k[X]$-modules, and such that every indecomposable Λ-module of dimension d is isomorphic to $k[X]/(X-\lambda)\otimes_{k[X]} M_i$ for some $\lambda \in k$ and some i.

Let $\mu_\Lambda(d)$ be the least number of bimodules M_i satisfying the above conditions. Then Λ is called of *polynomial growth* (respectively, *domestic*) if there is a natural number n such that $\mu_\Lambda(d) \le d^n$ (respectively, $\mu_\Lambda(d) \le n$) for all $d \ge 1$ (cf. [12]).

An algebra Λ is called a *multicoil algebra* if, for any cycle $M_0 \to M_1 \to \ldots \to M_t = M_0$ in modΛ, the indecomposable modules M_i belong to one standard coil of a multicoil in Γ_Λ.

It is shown in [2, 4.6] that multicoil algebras are of polynomial growth. The simplest examples of multicoil algebras are the coil enlargements of tame concealed algebras having tame coil type (see [4]). These contain Ringel's domestic tubular extensions and coextensions and tubular algebras (see [11, 4.9, 5.2]).

2.6. *Tame coil enlargements of tame concealed algebras*. Let Λ be an algebra and $\mathcal{T} = (T_i)_{i\in I}$ be a family of components of Γ_Λ. Then $\mathcal{T}$ is said to be *separating* (respectively, *weakly separating*) in mod Λ if the remaining indecomposable Λ-modules split into two classes $\mathcal{P}$ and $\mathcal{Q}$ such that:

(i) The components $T_i, i \in I$, are standard and pairwise orthogonal.
(ii) $\mathrm{Hom}_\Lambda(\mathcal{Q},\mathcal{P}) = \mathrm{Hom}_\Lambda(\mathcal{Q},\mathcal{T}) = \mathrm{Hom}_\Lambda(\mathcal{T},\mathcal{P}) = 0$.
(iii) Any morphism from $\mathcal{P}$ to $\mathcal{Q}$ factors through add T_i for any $i \in I$ (respectively, add $\mathcal{T}$).

Let A be a tame concealed algebra and $\mathcal{T} = (T_i)_{i\in I}$ be the stable separating tubular family of Γ_A. An algebra B is called a *coil enlargement* of A using modules from $\mathcal{T}$ if there is a sequence of algebras $A = A_0, A_1, \ldots, A_n = B$ such that for $0 \le j < n, A_{j+1}$ is obtained from A_j by an admissible operation with pivot either in a stable tube of $\mathcal{T}$ or in a coil of Γ_{A_j} obtained from a stable tube of $\mathcal{T}$ by means of the admissible operations done so far. If B is a coil enlargement of a tame concealed algebra A having a separating family $\mathcal{T} = (T_i)_{i\in I}$ of stable tubes, then the *coil type* $c_B = (c_B^-, c_B^+)$ of B is the pair of functions from I to $\mathbb{N}$ given by

$c_B^+(i) =$ rank of T_i + number of rays inserted in T_i by the sequence of admissible operations,
$c_B^-(i)$ is defined dually.

Moreover, there are a unique maximal branch extension B^+ of A and a unique maximal branch coextension B^- of A which are full convex subcategories of B, c_B^+ is the extension type of B^+ and c_B^- is the coextension type of B^-, (see [4, 3.5] and [11, 4.7]).

It is shown in [4, 4.2] that B is tame if and only if B^- and B^+ are tame, that is, if and only if B^- and B^+ are either domestic or tubular. In fact, if

$\operatorname{ind} A = \mathcal{P} \vee \mathcal{T} \vee \mathcal{Q}$, $\operatorname{ind} B^- = \mathcal{P}^{B^-} \vee \mathcal{T}^{B^-} \vee \mathcal{Q}$ and $\operatorname{ind} B^+ = \mathcal{P} \vee \mathcal{T}^{B^+} \vee \mathcal{Q}^{B^+}$, then $\operatorname{ind} B = \mathcal{P}^{B^-} \vee \mathcal{T}' \vee \mathcal{Q}^{B^+}$, where $\mathcal{T}' = (T'_i)_{i \in I}$ is the weakly separating family of coils obtained from $\mathcal{T} = (T_i)_{i \in I}$ by the sequence of admissible operations.

3. Construction of the iterated coil enlargements

3.1. Domestic tubular extensions and coextensions and tubular algebras are obtained from a tame concealed algebra by performing a sequence of admissible operations 1) or 1*) in the stable tubes of its separating tubular family. We call these algebras 0-iterated coil enlargements.

Let Λ_0 be a branch coextension of a tame concealed algebra A_0, and assume that Λ_0 is domestic or tubular. Then $\operatorname{ind} \Lambda_0 = \mathcal{P}_0 \vee \mathcal{T}_0 \vee \mathcal{Q}_0$, where $\mathcal{Q}_0$ is the preinjective component of Γ_{Λ_0}, and $\mathcal{T}_0$ is a separating tubular family separating $\mathcal{P}_0$ from $\mathcal{Q}_0$. Using admissible operations 1), 2), 3), we insert projectives in the coinserted and stable tubes of $\mathcal{T}_0$. We obtain a coil enlargement Λ_1 of A_0 with $(\Lambda_1)^- = \Lambda_0$. If $(\Lambda_1)^+$ is tame, we call Λ_1 a 1-iterated coil enlargement.

By [4, 4.1], $\operatorname{ind} \Lambda_1 = \mathcal{P}_0 \vee \mathcal{T}'_0 \vee \mathcal{Q}'_0$, where $\mathcal{T}'_0$ is the weakly separating family of coils obtained from $\mathcal{T}_0$, and $\mathcal{Q}'_0$ consists of $(\Lambda_1)^+$-modules. If $(\Lambda_1)^+$ is domestic, then $\mathcal{Q}'_0$ is the preinjective component of $\Gamma_{(\Lambda_1)^+}$ and the process stops.

If $(\Lambda_1)^+$ is tubular, then $(\Lambda_1)^+$ is a branch coextension of a tame concealed algebra A_1, and we can write

$$\operatorname{ind}(\Lambda_1)^+ = \mathcal{P}_0^1 \vee \mathcal{T}_0^1 \vee \bigvee_{\gamma \in \mathbb{Q}^+} \mathcal{T}_\gamma^1 \vee \mathcal{T}_\infty^1 \vee \mathcal{Q}_\infty^1,$$

where $\mathcal{Q}_\infty^1$ is the preinjective component of Γ_{A_1}, and $\mathcal{T}_\infty^1$ is the separating tubular family of $\operatorname{mod}(\Lambda_1)^+$ that is obtained from the family of stable tubes of $\operatorname{mod} A_1$ by coray insertions. Then $\mathcal{Q}'_0 = \bigvee_{\gamma \in \mathbb{Q}^+} \mathcal{T}_\gamma^1 \vee \mathcal{T}_\infty^1 \vee \mathcal{Q}_\infty^1$, and

$$\operatorname{ind} \Lambda_1 = \mathcal{P}_0 \vee \mathcal{T}'_0 \vee \bigvee_{\gamma \in \mathbb{Q}^+} \mathcal{T}_\gamma^1 \vee \mathcal{T}_\infty^1 \vee \mathcal{Q}_\infty^1.$$

LEMMA. *With the notation introduced above*,

(a) $\mathcal{T}_\infty^1$ *is a separating tubular family separating* $\mathcal{P}_0 \vee \mathcal{T}'_0 \vee \bigvee_{\gamma \in \mathbb{Q}^+} \mathcal{T}_\gamma^1$ *from* $\mathcal{Q}_\infty^1$.

(b) *For each* $\gamma \in \mathbb{Q}^+, \mathcal{T}_\gamma^1$ *is a separating tubular family separating* $\mathcal{P}_0 \vee \mathcal{T}'_0 \vee \bigvee_{\delta < \gamma} \mathcal{T}_\delta^1$ *from* $\bigvee_{\delta > \gamma} \mathcal{T}_\delta^1 \vee \mathcal{T}_\infty^1 \vee \mathcal{Q}_\infty^1$.

Proof. (a) We have to prove conditions (i), (ii) and (iii) of (2.6).

(i) follows from the fact that $\mathcal{T}_\infty^1$ is a separating tubular family in mod $(\Lambda_1)^+$.

(ii) follows from the fact that $\mathcal{T}'_0$ is weakly separating in $\operatorname{mod}\Lambda_1$, and $\mathcal{T}^1_\infty$ is separating in $\operatorname{mod}(\Lambda_1)^+$.

(iii) Let $f : M \to N$ be a non-zero morphism with $M \in \mathcal{P}_0 \vee \mathcal{T}'_0 \vee \bigvee_{\gamma\in\mathbb{Q}^+} \mathcal{T}^1_\gamma$ and $N \in \mathcal{Q}^1_\infty$. If $M \in \bigvee_{\gamma\in\mathbb{Q}^+} \mathcal{T}^1_\gamma$, then f factors through the additive category of any tube of $\mathcal{T}^1_\infty$. If $M \in \mathcal{P}_0$, then f factors through $\operatorname{add}\mathcal{T}'_0$. Finally, if $M \in \mathcal{T}'_0$ and M is a $(\Lambda_1)^+$-module, then $M \in \mathcal{T}^1_0$ and f factors through the additive category of any tube of $\mathcal{T}^1_\infty$. If M is not a $(\Lambda_1)^+$-module, then there is a commutative diagram

$$\begin{array}{ccc} M & \xrightarrow{\;f\;} & N \\ & \searrow \quad \nearrow_{f'} & \\ & M' = M|_{(\Lambda_1)^+} & \end{array}$$

where M' is the restriction of M to $(\Lambda_1)^+$; hence $M' \in \operatorname{add}\mathcal{T}^1_0$. Since f' factors through the additive category of any tube of $\mathcal{T}^1_\infty$, so does f. Therefore in all cases f factors through the additive category of any tube of $\mathcal{T}^1_\infty$.

(b) is proved similarly. ∎

Let $\mathcal{P}_1 = \mathcal{P}_0 \vee \mathcal{T}'_0 \vee \bigvee_{\gamma\in\mathbb{Q}^+} \mathcal{T}^1_\gamma$, $\mathcal{T}_1 = \mathcal{T}^1_\infty$ and $\mathcal{Q}_1 = \mathcal{Q}^1_\infty$. Then we can write $\operatorname{ind}\Lambda_1 = \mathcal{P}_1 \vee \mathcal{T}_1 \vee \mathcal{Q}_1$, where $\mathcal{T}_1$ is a separating tubular family in $\operatorname{mod}\Lambda_1$ consisting of coinserted and stable tubes, and $\mathcal{Q}_1$ is the preinjective component of Γ_{Λ_1}.

3.2. We are now able to iterate the process:

Using admissible operations 1), 2), 3), we insert projectives in the tubes of $\mathcal{T}_1$. We obtain a coil enlargement Λ^2 of A_1 with $(\Lambda^2)^- = (\Lambda_1)^+$. If $(\Lambda^2)^+$ is tame, we call the algebra Λ_2 obtained from Λ_1 by inserting projectives in the tubes of $\mathcal{T}_1$ a 2-iterated coil enlargement.

As in [4, 2.7], we know that $\operatorname{ind}\Lambda_2 = \mathcal{P}_1 \vee \mathcal{T}'_1 \vee \mathcal{Q}'_1$, where $\mathcal{T}'_1$ is the weakly separating family of coils obtained from $\mathcal{T}_1$. We want to describe $\mathcal{Q}'_1$. By [4, 4.1], $\operatorname{ind}\Lambda^2 = \mathcal{P}^2 \vee \mathcal{T}^2 \vee \mathcal{Q}^2$, where $\mathcal{T}^2 = \mathcal{T}'_1$ and $\mathcal{Q}^2$ consists of $(\Lambda^2)^+$-modules.

Lemma. *With the above notation, $\mathcal{Q}'_1 = \mathcal{Q}^2$.*

Proof. Let $X \in \mathcal{Q}^2$. Since the indecomposable projective Λ^2-modules lie in $\mathcal{P}^2 \vee \mathcal{T}^2$, and $\mathcal{T}^2$ is weakly separating in $\operatorname{mod}\Lambda^2$, $\operatorname{Hom}_{\Lambda^2}(\mathcal{T}^2, X) \neq 0$. Hence $\operatorname{Hom}_{\Lambda_2}(\mathcal{T}'_1, X) \neq 0$ and $X \in \mathcal{Q}'_1$.

Let $X \in \mathcal{Q}'_1$. We must show first that X is a Λ^2-module. Let z be a vertex in Q_{Λ_2} but not in Q_{Λ^2}. Then z is in Q_{Λ_1} but not in $Q_{(\Lambda_1)^+}$, and the indecomposable injective Λ_2-module I_z belongs to $\mathcal{T}'_0 \subset \mathcal{P}_1$. Therefore $\operatorname{Hom}_{\Lambda_2}(X, I_z) = 0$, and X is a Λ^2-module. As above, $X \in \mathcal{Q}^2$. ∎

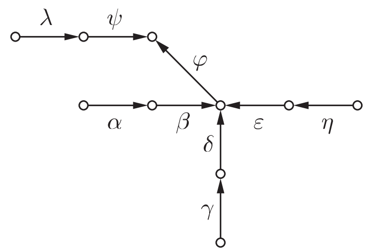

Fig. 1. Λ_0 bound by $\gamma\delta\varphi = 0$, $\varepsilon\varphi = 0$

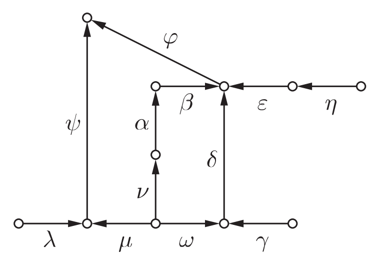

Fig. 2. Λ_1 bound by $\gamma\delta\varphi = 0$, $\varepsilon\varphi = 0$, $\nu\alpha\beta = \omega\delta$, $\mu\psi = \nu\alpha\beta\varphi$

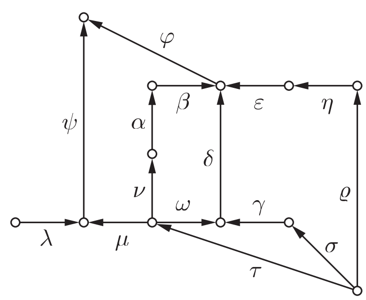

Fig. 3. Λ_2 bound by $\gamma\delta\varphi = 0$, $\varepsilon\varphi = 0$, $\nu\alpha\beta = \omega\delta$, $\mu\psi = \nu\alpha\beta\varphi$, $\tau\mu = 0$, $\tau\omega = \sigma\gamma$, $\sigma\gamma\delta = \varrho\eta\varepsilon$

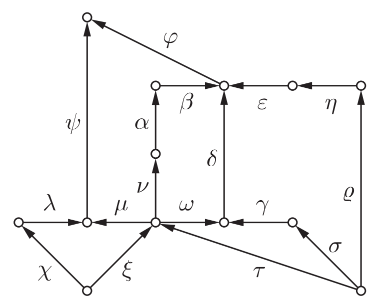

Fig. 4. Λ_3 bound by $\gamma\delta\varphi = 0$, $\varepsilon\varphi = 0$, $\nu\alpha\beta = \omega\delta$, $\mu\psi = \nu\alpha\beta\varphi$, $\tau\mu = 0$, $\tau\omega = \sigma\gamma$, $\sigma\gamma\delta = \varrho\eta\varepsilon$, $\chi\lambda = \xi\mu$, $\xi\nu = 0$, $\xi\omega = 0$

As before, if $(\Lambda^2)^+$ is domestic, $\mathcal{Q}_1'$ is the preinjective component of $\Gamma_{(\Lambda^2)^+}$ and the process stops. If $(\Lambda^2)^+$ is tubular, then it is a branch coextension of a tame concealed algebra A_2, and we can write

$$\operatorname{ind}(\Lambda^2)^+ = \mathcal{P}_0^2 \vee \mathcal{T}_0^2 \vee \bigvee_{\gamma\in\mathbb{Q}^+} \mathcal{T}_\gamma^2 \vee \mathcal{T}_\infty^2 \vee \mathcal{Q}_\infty^2,$$

where $\mathcal{Q}_\infty^2$ is the preinjective component of Γ_{A_2} and $\mathcal{T}_\infty^2$ is the separating tubular family of $\operatorname{mod}(\Lambda^2)^+$ that is obtained from the family of stable tubes of $\operatorname{mod} A_2$ by coray insertions. Then $\mathcal{Q}_1' = \bigvee_{\gamma\in\mathbb{Q}^+} \mathcal{T}_\gamma^2 \vee \mathcal{T}_\infty^2 \vee \mathcal{Q}_\infty^2$, and letting $\mathcal{P}_2 = \mathcal{P}_1 \vee \mathcal{T}_1' \vee \bigvee_{\gamma\in\mathbb{Q}^+} \mathcal{T}_\gamma^2$, $\mathcal{T}_2 = \mathcal{T}_\infty^2$ and $\mathcal{Q}_2 = \mathcal{Q}_\infty^2$, we can write $\operatorname{ind}\Lambda_2 = \mathcal{P}_2 \vee \mathcal{T}_2 \vee \mathcal{Q}_2$, where $\mathcal{T}_2$ is a separating tubular family in $\operatorname{mod}\Lambda_2$ consisting of coinserted and stable tubes, and $\mathcal{Q}_2$ is the preinjective component of Γ_{Λ_2}. We are now able to iterate the process once more.

We define the n-iterated coil enlargements inductively.

3.3. Let Λ be an iterated coil enlargement. From the description of $\operatorname{ind}\Lambda$ given above, we immediately obtain:

Proposition. *If Λ is an iterated coil enlargement, then*

(a) *Λ is of polynomial growth.*

(b) q_Λ *is weakly non-negative.*

Proof. (a) is [2, 4.6]. Since by [3, 3.5] Λ is triangular, (b) follows from (a) and [9, 1.2]. ■

3.4. In the examples of Figures 1–4, Λ_n is an n-iterated coil enlargement.

4. The main theorem

4.1. An algebra Λ has *acceptable projectives* if the Auslander–Reiten quiver Γ_Λ of Λ has components $\mathcal{P}, \mathcal{C}_1, \ldots, \mathcal{C}_r$ with the following properties:

(i) Any indecomposable projective Λ-module lies in $\mathcal{P}$ or in some $\mathcal{C}_i$.
(ii) $\mathcal{P}$ is a preprojective component without injective modules.
(iii) Each $\mathcal{C}_i$ is a standard coil.
(iv) If $\mathrm{Hom}_\Lambda(\mathcal{C}_i, \mathcal{C}_j) \neq 0$, then $i \leq j$.

Iterated coil enlargements have acceptable projectives.

LEMMA. *If Λ has acceptable projectives, then Λ is triangular.*

Proof. Let $\mathcal{P}, \mathcal{C}_1, \ldots, \mathcal{C}_r$ be as in the above definition, and assume that Λ is not triangular. Then there exists a cycle in $\mathrm{mod}\Lambda$ consisting of indecomposable projective modules none of which lies in $\mathcal{P}$, for otherwise $\mathcal{P}$ would contain a cycle. Hence the indecomposable projective modules in the cycle lie in the standard coils $\mathcal{C}_1, \ldots, \mathcal{C}_r$. From (iv), it follows that they all lie in one standard coil $\mathcal{C}_i$. Thus $\mathcal{C}_i$ contains a cycle of projectives, which contradicts [3, 3.2 or 4.5]. ■

4.2. Assume that Λ has acceptable projectives and let $\mathcal{P}, \mathcal{C}_1, \ldots, \mathcal{C}_r$ be as in the above definition. Consider the standard coil $\mathcal{C}_r$. Since by [3, 4.5] the mesh category $k(\mathcal{C}_r)$ has no oriented cycle of projectives, there is a projective $P = P_a$ in $\mathcal{C}_r$ such that P is a sink in the full subcategory of $k(\mathcal{C}_r)$ consisting of projectives, that is, the support of $\mathrm{Hom}_{k(\mathcal{C}_r)}(P, -)$ contains no projective. We shall consider three cases.

(a) Assume first that P is ordinary. Consider the wing $W(P)$ determined by P in $\mathcal{C}_r$ (that is, the full translation subquiver of $\mathcal{C}_r$ consisting of all modules M such that there is a sectional path $M \to \ldots \to Z$ for some module Z on the sectional path from P to the mouth). From the axioms (C1), (C4) for coils in [3, 4.3], it follows that all projectives in $W(P)$ are ordinary. Clearly, we may choose P so that all projectives in $W(P)$ lie in the ray passing through P and pointing to infinity. Let $e = \sum_{P_x \in W(P)} e_x$ and $\overline{\Lambda} = \Lambda/\Lambda e \Lambda$. Denote by R the set of points in $\mathcal{C}_r$ that lie in the rays passing through the support of $\mathrm{Hom}_{k(\mathcal{C}_r)}(P, -)$.

Let $\mathcal{C}'_r$ be the translation quiver obtained from $\mathcal{C}_r$ by deleting R and replacing the sectional paths $X_i \to Z_{i-1} \to \ldots \to X'_i \to \tau_\Lambda^{-1} X'_{i-1}$ (if they exist) by the respective compositions $X_i \to \tau_\Lambda^{-1} X'_{i-1}$ (see Fig. 5).

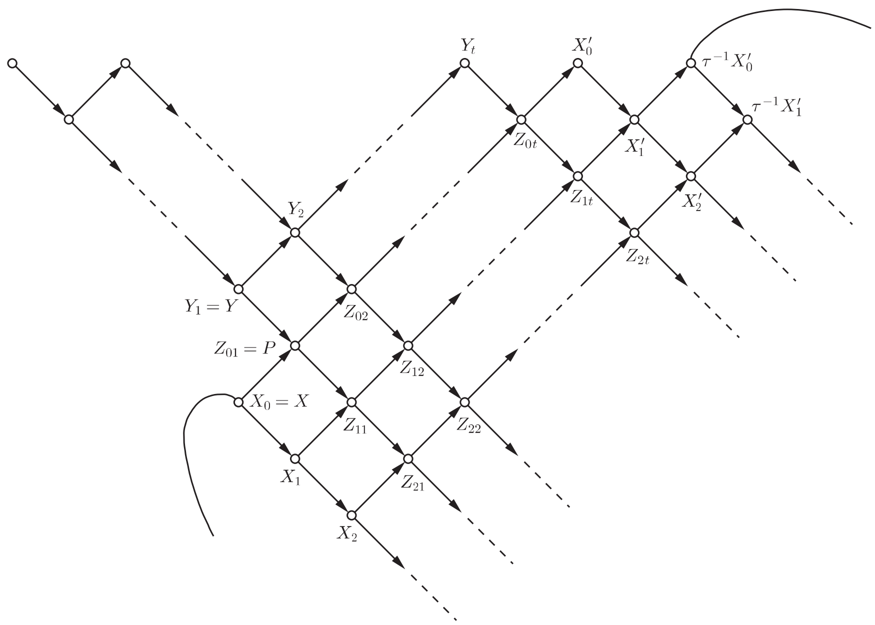

Fig. 5

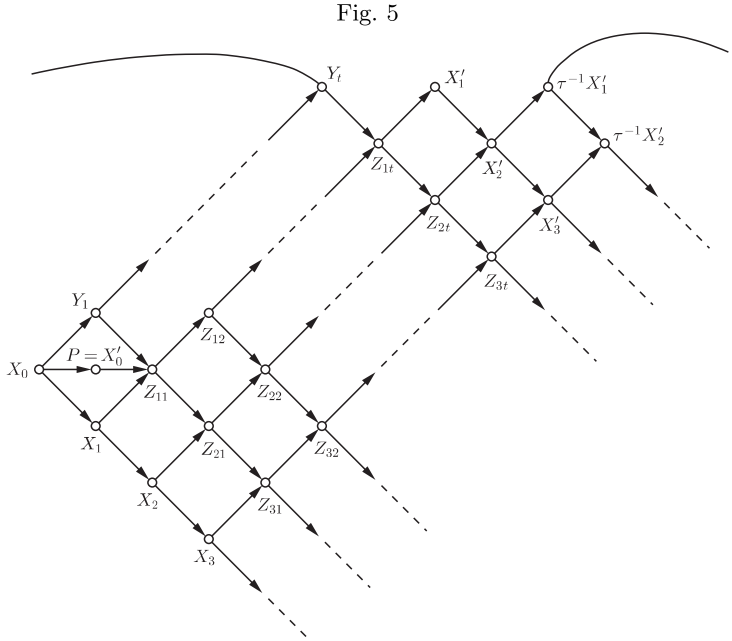

Fig. 6

Now assume that no sink in the full subcategory of $k(\mathcal{C}_r)$ consisting of projectives is ordinary. Then $P = P_a$ is exceptional. Let $\overline{\Lambda} = \Lambda/\Lambda e_a \Lambda$, and denote by R the set of points in the support of $\mathrm{Hom}_{k(\mathcal{C}_r)}(P, -)$.

(b) If P is injective, then R consists of the vertices X_i' and Z_{ij} of a mesh-complete translation subquiver of $\mathcal{C}_r$ as in Fig. 6.

Let $\mathcal{C}_r'$ be obtained from $\mathcal{C}_r$ by deleting R and replacing the sectional paths $X_i \to Z_{ij} \to \ldots \to X_i' \to \tau_\Lambda^{-1} X_{i-1}'$ (if they exist) by the respective compositions $X_i \to \tau_\Lambda^{-1} X_{i-1}'$.

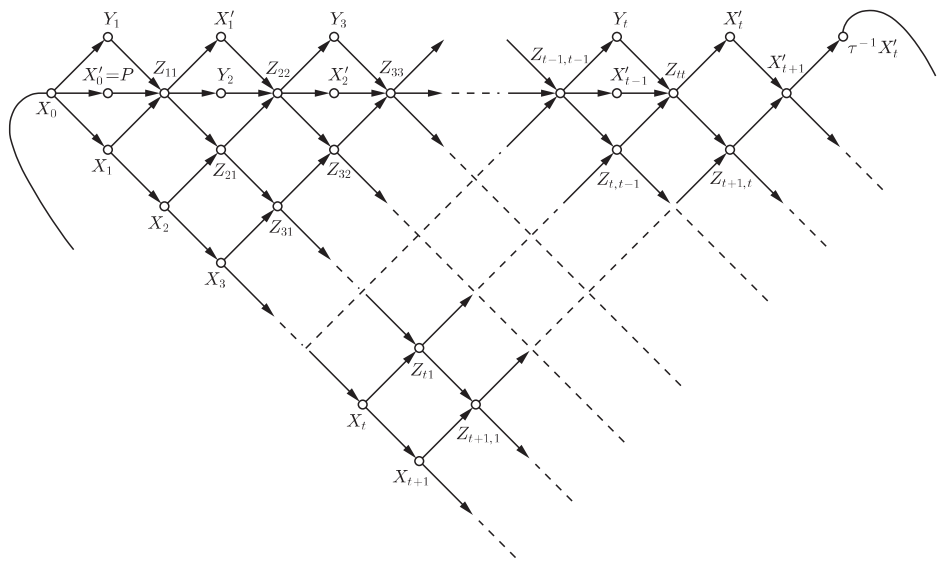

Fig. 7

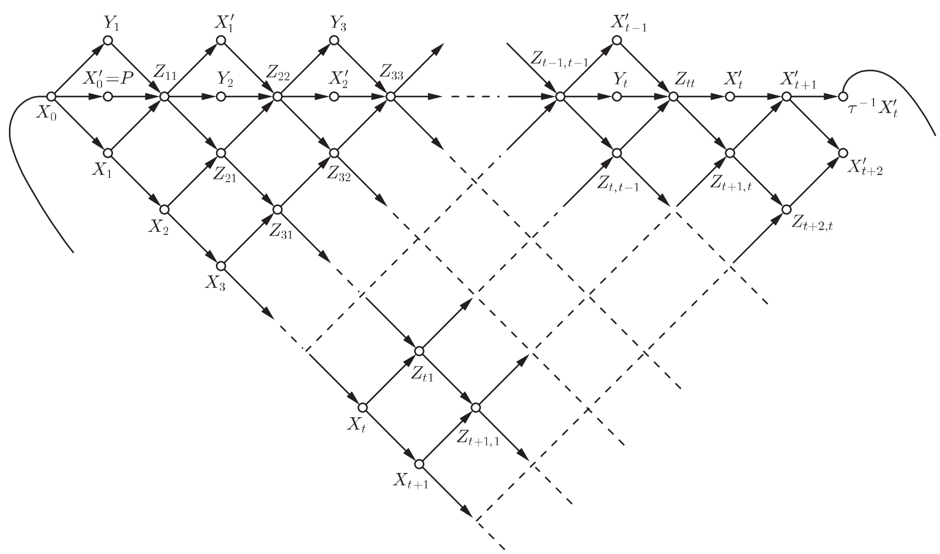

Fig. 8

(c) Finally, if P is not injective, then, by (C1), R consists of the vertices X_i' and Z_{ij} of a mesh-complete translation subquiver of $\mathcal{C}_r$ as in Figs. 7 or 8 according as t is odd or even (see [3, 4.4 Cor.]).

Let $\mathcal{C}_r''$ be obtained from $\mathcal{C}_r$ by deleting R and replacing the sectional paths $X_i \to Z_{i1} \to \ldots \to Y_{i+1}$ $(1 \le i < t)$ by the respective compositions $X_i \to Y_{i+1}$, the sectional paths $X_i \to Z_{i1} \to \ldots \to X_i' \to \tau_\Lambda^{-1} X_{i-1}'$ (if they exist) by the respective compositions $X_i \to \tau_\Lambda^{-1} X_{i-1}'$ and the sectional paths $Y_i \to Z_{ii} \to Y_{i+1}$ $(1 \le i < t)$ by the respective compositions $Y_i \to Y_{i+1}$.

PROPOSITION. *With the notation introduced above, we have*:

(a) *If P is ordinary, then $\Lambda = (\overline{\Lambda} \times D)[X \oplus Y]$ where $D = T_t(k)$ is the full $t \times t$ lower triangular matrix algebra, Y is the unique indecomposable projective-injective D-module and X is the indecomposable direct summand of* $\operatorname{rad} P$ *that belongs to* $\operatorname{mod} \overline{\Lambda}$. *Moreover*, $\mathcal{C}_r'$ *is a standard coil of* $\Gamma_{\overline{\Lambda}}$ *and* $\overline{\Lambda}$ *has acceptable projectives.*

(b) *If P is exceptional and injective, then $\Lambda = \overline{\Lambda}[X]$, where $X = \operatorname{rad} P$, $\mathcal{C}_r'$ is a standard coil of $\Gamma_{\overline{\Lambda}}$ and $\overline{\Lambda}$ has acceptable projectives.*

(c) *If P is exceptional and non-injective, then $\Lambda = \overline{\Lambda}[X]$, where $X =$* $\operatorname{rad} P$, *$\mathcal{C}_r''$ can be completed to a standard coil $\mathcal{C}_r'$ of $\Gamma_{\overline{\Lambda}}$ and $\overline{\Lambda}$ has acceptable projectives.*

Proof. (a) Since $P = P_a$ is a sink in the full subcategory of $\operatorname{ind} \Lambda$ consisting of projective objects, a is a source in Q_Λ. Let $x \in (Q_{\bar{\Lambda}})_0$ and assume there is an arrow $x \to y$ in Q_Λ. Then $y \in (Q_{\bar{\Lambda}})_0$, for if this were not the case, $P_y \in W(P) \subset \mathcal{C}_r$, and as $\operatorname{Hom}_\Lambda(P_y, P_x) \neq 0, P_x \in \mathcal{C}_r$. But $P_x \notin W(P)$, hence $\operatorname{Hom}_\Lambda(P_y, P_x) = 0$, a contradiction. Therefore $\overline{\Lambda}$ is convex in Λ and the bound quiver Q_Λ of Λ has the form shown in Fig. 9:

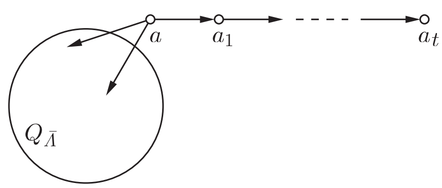

Fig. 9

where $a, a_1, \ldots, a_t$ are the vertices of Q_Λ corresponding to the projectives in $W(P)$. Let D be the full $t \times t$ lower triangular matrix algebra. Then $\Lambda = (\overline{\Lambda} \times D)[X \oplus Y]$, where $Y = P_{a_1}$ is the unique indecomposable projective-injective D-module and X is the indecomposable direct summand of $\operatorname{rad} P$ that belongs to $\operatorname{mod} \overline{\Lambda}$.

Since $\mathcal{C}_r$ satisfies all the axioms for coils in [3, 4.2], so does $\mathcal{C}_r'$. To prove that $\mathcal{C}_r'$ is actually a standard component of $\Gamma_{\bar{\Lambda}}$, we proceed as follows.

By [5, 3.5], if $0 \to L \to M \to N \to 0$ is the almost split sequence starting at L in $\mathrm{mod}\Lambda$, and L is a non-injective $\overline{\Lambda}$-module, then the almost split sequence starting at L in $\mathrm{mod}\overline{\Lambda}$ is the lower row of the exact commutative diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & K & = & K & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \to & L & \to & M' & \to & N' & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
0 & \to & L & \to & M'' & \to & N'' & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

where M' and N' are the largest quotients of M and N, respectively, to be $\overline{\Lambda}$-modules, and the morphisms $K \to M'$ and $K \to N'$ are sections. The assertion follows by applying this statement to compute the almost split sequences in $\mathrm{mod}\overline{\Lambda}$ starting at the indecomposable $\overline{\Lambda}$-modules in $\mathcal{C}_r$ (in particular, at the modules X_i, $i \geq 0$). Note that the fact that Λ has acceptable projectives implies that the kernels of the epimorphisms $M \to M'$ and $N \to N'$ belong to $\mathrm{add}\,\mathcal{C}_r$. Finally, the standardness of $\mathcal{C}'_r$ follows from that of $\mathcal{C}_r$. Since $\mathcal{P}, \mathcal{C}_1, \ldots, \mathcal{C}_{r-1}$ and $\mathcal{C}'_r$ (if it still has projectives) are the components of $\Gamma_{\bar{\Lambda}}$ where the projectives lie, we conclude that $\overline{\Lambda}$ has acceptable projectives.

(b) As $P = P_a$ is a sink in the full subcategory of $\mathrm{ind}\,\Lambda$ consisting of projectives, a is a source in Q_Λ. Hence $\Lambda = \overline{\Lambda}[X]$, where $X = \mathrm{rad}\,P$ is indecomposable in $\mathrm{mod}\overline{\Lambda}$. As in (a), $\mathcal{C}'_r$ satisfies the axioms for coils in [3, 4.2], and is in fact a standard component of $\Gamma_{\bar{\Lambda}}$. Moreover, $\overline{\Lambda}$ has acceptable projectives.

(c) As in (b), $\Lambda = \overline{\Lambda}[X]$, where $X = \mathrm{rad}\,P$ is indecomposable in $\mathrm{mod}\overline{\Lambda}$, and $\mathcal{C}''_r$ satisfies the axioms for coils. But in this case $\mathcal{C}''_r$ is not a component of $\Gamma_{\bar{\Lambda}}$. Using [5, 3.4, 3.5] we can compute the almost split sequences starting at the indecomposable $\overline{\Lambda}$-modules in $\mathcal{C}_r$ obtaining thus $\mathcal{C}''_r$ and also the inverse translates $\tau_{\bar{\Lambda}}^{-1} Y_i$ of the modules Y_i, $1 \leq i \leq t$. Let $S = \bigoplus_{i=1}^t Y_i$ and $B = \mathrm{Supp}(S)$. Using [5, 3.7, 3.8], we can also compute the translates $\tau_B Y_i$ of the modules $Y_i, 1 \leq i \leq t$. Then it is easy to show that $\mathcal{S} = \{Y_i \mid 1 \leq i \leq t\}$ is a slice in $\mathrm{mod}B$ (see [11, 4.2(3)]). Hence B is a tilted algebra of type $\mathbb{A}_n$, and $\mathcal{C}''_r$ can be completed to a standard coil $\mathcal{C}'_r$ of $\Gamma_{\bar{\Lambda}}$. As before, $\overline{\Lambda}$ has acceptable projectives. ■

4.3. Theorem. *Let Λ be an algebra with acceptable projectives. Then the following conditions are equivalent*:

(a) *Λ is an iterated coil enlargement.*

(b) *Λ is tame.*

(c) *q_Λ is weakly non-negative.*

Proof. (a)⇒(b)⇒(c) is (3.3).

(c)⇒(a). Let $\mathcal{P}, \mathcal{C}_1, \ldots, \mathcal{C}_r$ be the components of Γ_Λ where the projectives lie, with $\mathcal{P}$ preprojective without injective modules, and $\mathcal{C}_1, \ldots, \mathcal{C}_r$ standard coils such that $\mathrm{Hom}_\Lambda(\mathcal{C}_i, \mathcal{C}_j) \neq 0$ implies $i \leq j$. Let $\overline{\Lambda}$, $\mathcal{C}'_r$ and $P = P_a$ be as in (4.2, Proposition). Then $\overline{\Lambda}$ has acceptable projectives, and $\mathcal{P}, \mathcal{C}_1, \ldots, \mathcal{C}_{r-1}$, and $\mathcal{C}'_r$ (if it still has projectives) are the components of $\Gamma_{\bar{\Lambda}}$ where the projectives lie. We proceed by induction on the number p of projectives in the standard coils $\mathcal{C}_1, \ldots, \mathcal{C}_r$.

If $p = 0$, then $\mathcal{P}$ is a complete preprojective component. By [10, 1.3], Λ is a domestic tubular coextension of a tame concealed algebra, that is, a 0-iterated coil enlargement.

Now let $p > 0$. Since $\overline{\Lambda}$ is convex in Λ, $q_{\bar{\Lambda}}$ is weakly non-negative. By induction hypothesis, $\overline{\Lambda}$ is an n-iterated coil enlargement. Thus, $\overline{\Lambda} = \Lambda_n$, where Λ_n is obtained from an $(n-1)$-iterated coil enlargement Λ_{n-1} by inserting projectives using admissible operations 1), 2), or 3) in the last separating tubular family $\mathcal{T}_{n-1}$ of $\mathrm{mod}\Lambda_{n-1}$ (we may assume $n \geq 1$).

Using the notation introduced in Section 3, we see that if $\mathrm{mod}\Lambda_{n-1} = \mathcal{P}_{n-1} \vee \mathcal{T}_{n-1} \vee \mathcal{Q}_{n-1}$, then $\mathrm{mod}\Lambda_n = \mathcal{P}_{n-1} \vee \mathcal{T}'_{n-1} \vee \mathcal{Q}'_{n-1}$, where $\mathcal{T}'_{n-1}$ is the last weakly separating family of coils containing projectives in $\mathrm{mod}\Lambda_n$. Hence $\mathcal{C}'_r$ belongs to $\mathcal{T}'_{n-1} \vee \mathcal{Q}'_{n-1}$. Also, there is a coil enlargement Λ^n of a tame concealed algebra A_{n-1} such that $\mathrm{mod}\Lambda^n = \mathcal{P}^n \vee \mathcal{T}'_{n-1} \vee \mathcal{Q}'_{n-1}$, and the branch extension $(\Lambda^n)^+$ of A_{n-1} is either domestic or tubular.

If $(\Lambda^n)^+$ is domestic, then $\mathcal{Q}'_{n-1}$ is a preinjective component and $\mathcal{C}'_r$ belongs to $\mathcal{T}'_{n-1}$. By performing the admissible operation on $\mathcal{C}'_r$ to obtain Λ from $\overline{\Lambda}$, we get another coil enlargement of A_{n-1} which, being convex in Λ, has weakly non-negative Tits form. By [4, 4.2], it is tame and therefore Λ is also an n-iterated coil enlargement.

If $(\Lambda^n)^+$ is a tubular algebra, then it is a branch coextension of a tame concealed algebra A_n, and

$$\mathcal{Q}'_{n-1} = \bigvee_{\gamma \in \mathbb{Q}^+} \mathcal{T}^n_\gamma \vee \mathcal{T}^n_\infty \vee \mathcal{Q}^n_\infty,$$

where $\mathcal{Q}^n_\infty$ is the preinjective component of Γ_{A_n} and $\mathcal{T}^n_\infty$ is obtained from the separating tubular family of $\mathrm{mod}A_n$ by coray insertions. Then $\mathrm{mod}\,\Lambda_n = \mathcal{P}_n \vee \mathcal{T}_n \vee \mathcal{Q}_n$, where $\mathcal{P}_n = \mathcal{P}_{n-1} \vee \mathcal{T}'_{n-1} \vee \bigvee_{\gamma \in \mathbb{Q}^+} \mathcal{T}^n_\gamma$, $\mathcal{T}_n = \mathcal{T}^n_\infty$ and $\mathcal{Q}_n = \mathcal{Q}^n_\infty$. In this case $\mathcal{C}'_r$ must belong to $\mathcal{T}_n$, otherwise we can construct a vector z with non-negative coordinates such that $q_\Lambda(z) < 0$.

Indeed, if X is the indecomposable direct summand of $\mathrm{rad}\,P$ that lies in $\mathrm{mod}\,\overline{\Lambda}$ and $X \in \mathcal{T}'_{n-1}$, then $\Lambda_n[X]$ is also a coil enlargement of A_{n-1} which, being convex in Λ, has weakly non-negative Tits form and, by [4, 4.2], is tame. This contradicts the fact that $(\Lambda^n)^+$ is tubular. Therefore $X \in \mathcal{Q}'_{n-1}$. If $X \notin \mathcal{T}_n$, then by (2.1, Lemma), there exist $\gamma \in \mathbb{Q}^+$ and a module $E \in \mathcal{T}^n_\gamma$ such that $q_{(\Lambda^n)^+}(\underline{\dim}\, E) = 0$ and $\operatorname{Hom}_{(\Lambda^n)^+}(X, E) \neq 0$. Since $(\Lambda^n)^+[X]$ is convex in Λ and $\operatorname{gl\,dim}(\Lambda^n)^+[X] \leq 3$, we get

$$\begin{aligned} q_\Lambda(2\,\underline{\dim}\, E + e_a) &= q_{(\Lambda^n)^+[X]}(2\,\underline{\dim}\, E + e_a) \\ &= 2(\underline{\dim}\, E, e_a)_{(\Lambda^n)^+[X]} + 1 < 0 \end{aligned}$$

for

$$\begin{aligned} (\underline{\dim}\, E, e_a)_{(\Lambda^n)^+[X]} &\geq \langle \underline{\dim}\, E, e_a \rangle + \langle e_a, \underline{\dim}\, E \rangle \\ &= \langle \underline{\dim}\, E, \underline{\dim}\, I_a \rangle + \langle \underline{\dim}\, P_a - \underline{\dim}\, X, \underline{\dim}\, E \rangle \\ &= -\langle \underline{\dim}\, X, \underline{\dim}\, E \rangle = -\dim \operatorname{Hom}_{(\Lambda^n)^+}(X, E) < 0. \end{aligned}$$

Hence we obtain a coil enlargement Λ^{n+1} of A_n which, being convex in Λ, has weakly non-negative Tits form. By [4, 4.2], Λ^{n+1} is tame. Therefore Λ is an $(n+1)$-iterated coil enlargement. ■

COROLLARY. *Let Λ be a sincere algebra with acceptable projectives. Then the following conditions are equivalent*:

(a) *Λ is either a 0-iterated or a 1-iterated coil enlargement.*
(b) *Λ is tame.*
(c) *q_Λ is weakly non-negative.*

Proof. The assertion follows from the theorem and the fact that only 0-iterated and 1-iterated coil enlargements can have sincere modules. ■

Following [1], an algebra Λ is called *cycle-finite* if, for any cycle in mod Λ, no morphism on the cycle lies in the infinite power of the radical of modΛ.

COROLLARY. *Let Λ be an algebra with acceptable projectives. Then the following conditions are equivalent*:

(a) *Λ is an iterated coil enlargement.*
(b) *Λ is a multicoil algebra.*
(c) *Λ is cycle-finite.*
(d) *Λ is of polynomial growth.*
(e) *q_Λ is weakly non-negative.*

Proof. (a)⇒(b)⇒(c) and (d)⇒(e)⇒(a) are clear. (c)⇒(d) is [14, 4.3].■

References

[1] I. Assem and A. Skowroński, *Minimal representation-infinite coil algebras*, Manuscripta Math. 67 (1990), 305–331.

[2] I. Assem and A. Skowroński, *Indecomposable modules over multicoil algebras*, Math. Scand. 71 (1992), 31–61.
[3] —, *Multicoil algebras*, in: Proc. ICRA VI (Ottawa, 1992), Canad. Math. Soc. Conf. Proc. 14, 1993, 29–68.
[4] I. Assem, A. Skowroński and B. Tomé, *Coil enlargements of algebras*, Tsukuba J. Math., to appear.
[5] M. Auslander and S. O. Smalø, *Almost split sequences in subcategories*, J. Algebra 69 (1981), 426–454.
[6] K. Bongartz, *Algebras and quadratic forms*, J. London Math. Soc. 28 (1983), 461–469.
[7] K. Bongartz and P. Gabriel, *Covering spaces in representation theory*, Invent. Math. 65 (1981), 331–378.
[8] Yu. Drozd, *Tame and wild matrix problems*, in: Proc. ICRA II (Ottawa, 1979), Lecture Notes in Math. 832, Springer, Berlin, 1980, 240–258.
[9] J. A. de la Peña, *On the representation type of one-point extensions of tame concealed algebras*, Manuscripta Math. 61 (1988), 183–194.
[10] J. A. de la Peña and B. Tomé, *Iterated tubular algebras*, J. Pure Appl. Algebra 64 (1990), 303–314.
[11] C. M. Ringel, *Tame Algebras and Integral Quadratic Forms*, Lecture Notes in Math. 1099, Springer, Berlin, 1984.
[12] A. Skowroński, *Algebras of polynomial growth*, in: Topics in Algebra, Banach Center Publ. 26, Part I, PWN, Warszawa, 1990, 535–568.
[13] —, *Tame algebras with simply connected Galois coverings*, in preparation.
[14] —, *Cycle-finite algebras*, J. Pure Appl. Algebra, to appear.
[15] —, *Cycles in module categories*, in: Proc. Canad. Math. Soc. Annual Seminar-NATO Advanced Research Workshop on Representations of Algebras and Related Topics (Ottawa 1992).

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