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Borel partitions of unity and lower Carathéodory multifunctions

by

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Abstract. We prove the existence of Carathéodory selections and representations of a closed convex valued, lower Carathéodory multifunction from a set A in $\mathcal{A}(\mathcal{E} \otimes \mathcal{B}(X))$ into a separable Banach space Y, where $\mathcal{E}$ is a sub-σ-field of the Borel σ-field $\mathcal{B}(E)$ of a Polish space E, X is a Polish space and $\mathcal{A}$ is the Suslin operation. As applications we obtain random versions of results on extensions of continuous functions and fixed points of multifunctions. Such results are useful in the study of random differential equations and inclusions and in mathematical economics.

As a key tool we prove that if A is an analytic subset of $E \times X$ and if $\{U_n : n \in \omega\}$ is a sequence of Borel sets in A such that $A = \bigcup_n U_n$ and the section $U_n(e)$ is open in $A(e)$, $e \in E$, $n \in \omega$, then there exist Borel functions $p_n : A \to [0,1]$, $n \in \omega$, such that for every $e \in E$, $\{p_n(e,\cdot) : n \in \omega\}$ is a locally Lipschitz partition of unity subordinate to $\{U_n(e) : n \in \omega\}$.

1. Introduction. In [F, J, Kuc, KPY, Ri, Ry etc.] the following problem has been considered: if E is a measurable space, X Polish, Y a separable Banach space and $F : E \times X \to Y$ a closed convex valued, lower Carathéodory multifunction then does there exist a Carathéodory selection $f : E \times X \to Y$ of F? (Definitions and notation are given in the next section.) In these papers it is shown that if E is a complete measure space then such a selection f of F exists. Though we do not have a counterexample the result is probably false for a general measurable space. In Section 4 we prove the following selection theorem.

THEOREM 1.1. *Let E, X be Polish spaces, Y a separable Banach space, $\mathcal{E}$ a sub-σ-field of the Borel σ-field $\mathcal{B}(E)$ and $A \in \mathcal{A}(\mathcal{E} \otimes \mathcal{B}(X))$, where $\mathcal{A}(\mathcal{C})$ is the set of all sets obtained as the result of the Suslin operation on a system*

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$\{C(\alpha) : \alpha \in \omega^{<\omega}\}$ of sets in $\mathcal{C}$, $\mathcal{C}$ a family of sets. If $F : A \to Y$ is a lower Carathéodory multifunction then F admits a Carathéodory selection.

To prove our main selection theorem we follow the approach of Kim–Prikry–Yannelis [KPY] and Rybiński [Ry]. As a main tool they prove the following interesting theorem.

THEOREM 1.2. *Let $(E, \mathcal{E}, \mu)$ be a complete measure space, X a Polish space and $\{U_n : n \in \omega\}$ a sequence of sets in $\mathcal{E} \otimes \mathcal{B}(X)$ such that $E \times X = \bigcup_{n=0}^{\infty} U_n$ and $U_n(e)$ is open for every $n \in \omega$ and $e \in E$. Then there exists a sequence $\{p_n : n \in \omega\}$ of $\mathcal{E} \otimes \mathcal{B}(X)$-measurable maps from $E \times X$ to $[0,1]$ such that for every $e \in E$, $\{p_n(e, \cdot) : n \in \omega\}$ is a locally Lipschitz partition of unity subordinate to $\{U_n(e) : n \in \omega\}$.*

It turns out that this is a key result to the study of Carathéodory multifunctions (see also [S2]). In Section 3, we study the existence of random partitions of unity (in the sense of Theorem 1.2) and prove the following theorem.

THEOREM 1.3. *Let E and (X, d) be Polish spaces and A an analytic subset of $E \times X$. If $\{U_n : n \in \omega\}$ is a sequence of subsets of A such that for $n \in \omega$ and $e \in E$,*

(i) *U_n is Borel in A,*
(ii) *$U_n(e)$ is open in $A(e)$ and*
(iii) *$\bigcup_{n=0}^{\infty} U_n = A$*

then there exist Borel measurable functions $p_n : A \to [0,1]$ such that for every $e \in E$, $\{p_n(e, \cdot) : n \in \omega\}$ is a locally Lipschitz partition of unity subordinate to $\{U_n(e) : n \in \omega\}$.

As a simple consequence of Theorem 1.3, we obtain

COROLLARY 1.4. *Let E and X be Polish spaces, $\mathcal{E}$ a sub-σ-field of $\mathcal{B}(E)$, $A \in \mathcal{A}(\mathcal{E} \otimes \mathcal{B}(X))$ and $\{U_n : n \in \omega\}$ a sequence in $(\mathcal{E} \otimes \mathcal{B}(X))|A$ satisfying conditions* (ii) *and* (iii) *of Theorem* 1.3*. Then there exist $(\mathcal{E} \otimes \mathcal{B}(X))|A$-measurable functions $p_n : A \to [0,1]$ such that for every $e \in E$, $\{p_n(e, \cdot) : n \in \omega\}$ is a locally Lipschitz partition of unity subordinate to $\{U_n(e) : n \in \omega\}$.*

To prove Theorem 1.2 the main fact used is that if $C \in \mathcal{E} \otimes \mathcal{B}(X)$ then the map $(e, x) \to \mathrm{dist}(x, C(e))$, $(e, x) \in E \times X$, where $\mathrm{dist}(\cdot, \cdot)$ is with respect to a fixed complete metric on X, is $\mathcal{E} \otimes \mathcal{B}(X)$-measurable.

However, this is not necessarily true if $\mathcal{E}$ is not complete.

EXAMPLE 1.1 [SS1]. Fix a complete metric d on ω^ω and let $\alpha, \beta \in \omega^\omega$ and $\alpha \neq \beta$. Let U be a clopen subset of ω^ω contained in $S(\beta, \frac{1}{2} d(\alpha, \beta))$.

Take a closed set B in $[0,1]\times U$ such that $\pi(B)$ is non-Borel and put $C = B\cup([0,1]\times\{\alpha\})$. Then

$$e\in\pi(B)\Leftrightarrow \operatorname{dist}(\beta, C(e))\le \tfrac{1}{2}d(\alpha,\beta).$$

In Section 4 we also prove the following representation theorem for a multifunction satisfying the hypothesis of Theorem 1.2. This result generalizes a representation theorem of S. Łojasiewicz, Jr. [Łoj] (see also [AF, Theorem 9.6.2]).

Theorem 1.5. *Under the hypothesis of Theorem* 1.1, *there exists an* $\mathcal{E}\otimes\mathcal{B}(X\times\omega^\omega)$*-measurable function* $f: A\times\omega^\omega\to Y$ *such that*

(i) $f(e,x,\omega^\omega)=F(e,x)$ *for every* $(e,x)\in A$, *and*
(ii) *for every* $e\in E$, $f(e,\cdot,\cdot): A(e)\times\omega^\omega\to Y$ *is continuous.*

In Section 5, we give some random versions of results on extensions of continuous functions and a random fixed point theorem for multifunctions.

Results of the kind proved in this paper are useful in game theory and economics, random differential equations and inclusions etc. [AF, KPY, Y].

2. Definitions and preliminaries. For standard concepts and results in descriptive set theory we refer the reader to Kuratowski [Kur] or Moschovakis [Mo]. The set of natural numbers $0,1,2,\ldots$ will be denoted by ω and $\omega^{<\omega}$ will denote the set of all finite sequences of elements of ω of positive length.

If $s\in\omega^{<\omega}$ and $k\in\omega$ then sk will denote the concatenation of s and k. The set ω^ω of all sequences of natural numbers is equipped with the product of discrete topologies on ω. Then ω^ω is a Polish space (a completely metrizable second countable topological space). For $\alpha=(\alpha(0),\alpha(1),\ldots)\in\omega^\omega$ and $k\in\omega$, $a|k=(\alpha(0),\alpha(1),\ldots,\alpha(k))$. For $s\in\omega^{<\omega}$, $\omega^\omega(s)=\{\alpha\in\omega^\omega:\alpha \text{ extends } s\}$.

If (X,d) is a metric space, $x\in X$ and r a positive real then $S(x,r)$ denotes the open sphere in X with centre x and radius r. For $A\subseteq X$, $d(x,A)=\inf\{d(x,y):y\in A\}$. *Unless otherwise specified, a metric space is equipped with its Borel* σ*-field* $\mathcal{B}(X)$.

A *multifunction* $F: E\to X$ is a map with domain E and values nonempty subsets of X. For $U\subseteq X$,

$$F^{-1}(U)=\{e\in E: F(e)\cap U\ne\emptyset\}.$$

Also, the *graph* of F, denoted by $G(F)$, is the set

$$\{(e,x)\in E\times X: x\in F(e)\}.$$

A map $f: E\to X$ is called a *selector* of F if $f(e)\in F(e)$ for every $e\in E$. If X is a topological space then a sequence $\{f_i : i\in\omega\}$ of selectors of F is called a *dense sequence of selectors* of F if $F(e) = \overline{\{f_i(e) : i \in \omega\}}$ for every $e \in E$.

If $(E, \mathcal{E})$ is a measurable space and X a topological space then $F : E \to X$ is called $\mathcal{E}$-measurable or simply *measurable* if $F^{-1}(U) \in \mathcal{E}$ for every open set U in X. It is well known that if X is Polish and $F : E \to X$ a closed valued $\mathcal{E}$-measurable multifunction then F admits a dense sequence of measurable selectors [MR]. We shall use the following result which is stated here for easy reference.

LEMMA 2.1 [S1]. *If $(E, \mathcal{E})$ is a measurable space, $F : E \to X$ a closed valued, measurable multifunction, $f : E \to X$ a measurable function and $\varepsilon > 0$ then the multifunction*

$$e \to F(e) \cap S(f(e), \varepsilon)$$

is measurable.

If E and X are topological spaces then $F : E \to X$ is called *lower semicontinuous* (l.s.c.) if $F^{-1}(U)$ is open in E for every open subset U of X. We have

LEMMA 2.2 [Mic]. *Let E and X be metrizable spaces, $F : E \to X$ a l.s.c. multifunction, $f : E \to X$ a continuous map and ε a positive real such that $F(e) \cap S(f(e), \varepsilon) \neq \emptyset$ for every e. Then the multifunction $e \to F(e) \cap S(f(e), \varepsilon)$ is l.s.c.*

If E and X are sets, $A \subseteq E \times X$, $e \in E$ then $\pi(A)$ denotes the projection of A onto E and $A(e)$ the section $\{x \in X : (e, x) \in A\}$.

Now assume that $(E, \mathcal{E})$ is a measurable space, and X, Y Polish spaces, and let $A \subseteq E \times X$ be equipped with the σ-field $\mathcal{E} \otimes \mathcal{B}(X)|A$. A point map $f : A \to Y$ is called *Carathéodory* if f is measurable and for every $e \in E$, $F(e, \cdot) : A(e) \to Y$ is continuous. A multifunction $F : A \to Y$ is *lower Carathéodory* if F is measurable and $F(e, \cdot) : A(e) \to Y$ is l.s.c. for every $e \in E$.

We close this section by stating some known results for easy reference.

LEMMA 2.3. *Let $\mathcal{G}$ be a collection of subsets of a set E and let A belong to the σ-field $\sigma(\mathcal{G})$ generated by $\mathcal{G}$. Then there exists a countable collection $\mathcal{G}' \subseteq \mathcal{G}$ such that $a \in \sigma(\mathcal{G}')$.*

LEMMA 2.4 [RR]. *Let $\mathcal{E}$ be a countably generated σ-field of subsets of E and $\{E_n : n \in \omega\}$ a generator of $\mathcal{E}$. Let $I : E \to [0, 1]$ be the map*

$$I(e) = \sum_{n=0}^{\infty} \frac{2}{3^{n+1}} I_{E_n}(e), \qquad e \in E,$$

where $I_A(\cdot)$ is the indicator function of $A \subseteq E$. If $T = I(E)$ then $I : (E, \mathcal{E}) \to (T, \mathcal{B}(T))$ is bimeasurable.

LEMMA 2.5. *Let $(T,\mathcal{T})$ be a metrizable space and $\{B_n : n \in \omega\}$ a sequence of Borel subsets of T. Then there exists a metrizable topology $\mathcal{T}'$ on T such that*

(i) *$\mathcal{T}'$ is finer than $\mathcal{T}$,*
(ii) *$B_n \in \mathcal{T}'$, $n \in \omega$, and*
(iii) *$\sigma(\mathcal{T}) = \sigma(\mathcal{T}')$.*

Proof. Define a map $h : T \to T \times \{0,1\}^\omega$ by

$$h(t) = (t, I_{B_0}(t), I_{B_1}(t), \ldots), \qquad t \in T,$$

and put $\mathcal{T}' = \{h^{-1}(U) : U \text{ is open in } h(T)\}$.

Remark. If $(T,\mathcal{T})$ is Polish then we can choose $\mathcal{T}'$ such that $(T,\mathcal{T}')$ is, moreover, Polish [Mil].

LEMMA 2.6 [Lou]. *Let E and X be Polish spaces, and A and B analytic subsets of $E \times X$ such that for every $e \in E$ there exists an open subset U of X such that*

$$A(e) \subseteq U \quad \textit{and} \quad B(e) \cap U = \emptyset.$$

Then there exist a sequence $\{B_n : n \in \omega\}$ of Borel subsets of E and a sequence $\{U_n : n \in \omega\}$ of open subsets of X such that

$$A \subseteq \bigcup_{n=0}^{\infty} (B_n \times U_n) \quad \textit{and} \quad B \cap \bigcup_{n=0}^{\infty} (B_n \times U_n) = \emptyset.$$

LEMMA 2.7. *Let E and X be Polish spaces, $\mathcal{E}$ a sub-σ-field of $\mathcal{B}(E)$ and $B \in \mathcal{E} \otimes \mathcal{B}(X)$ with non-empty compact sections. Then $\pi(B) \in \mathcal{E}$ and there exists an $\mathcal{E}$-measurable map $s : E \to X$ such that $s(e) \in B(e)$ for all e.*

This is an easy generalization of Novikov's uniformization theorem [Mo, Theorem 4F.12] and the proof is omitted.

3. Borel partitions of unity

Proof of Theorem 1.3. Fix $n \in \omega$. By Lemma 2.6, we get a sequence $\{B_{nk} : k \in \omega\}$ of Borel subsets of E and a sequence $\{U_{nk} : k \in \omega\}$ of open subsets of X such that

$$U_n \subseteq \bigcup_{k=0}^{\infty} (B_{nk} \times U_{nk}) \quad \text{and} \quad (A \setminus U_n) \cap \bigcup_{k=0}^{\infty} (B_{nk} \times U_{nk}) = \emptyset.$$

By Lemma 2.5, we get a finer second countable metrizable topology $\mathcal{T}$ on E such that

(i) $B_{nk} \in \mathcal{T}$, $n, k \in \omega$, and
(ii) $\sigma(\mathcal{T}) = \mathcal{B}(E)$.

Let d' be a metric on E inducing $\mathcal{T}$ and ϱ be the metric on $E\times X$ defined by

$$\varrho((e,x),(e',x')) = \max\{d'(e,e'), d(x,x')\}.$$

Now note that $\{U_n : n\in\omega\}$ is an open cover of the metric space (A,ϱ). Therefore, by [AC, Theorem 2, pp. 10–12] there exist locally Lipschitz maps $p_n : (A,\varrho)\to[0,1]$, $n\in\omega$, such that $\{p_n(e,\cdot) : n\in\omega\}$ is a partition of unity subordinate to $\{U_n(e) : n\in\omega\}$. It is clear that $\{p_n : n\in\omega\}$ has all the desired properties.

Proof of Corollary 1.4. By Lemma 2.3, we can assume that $\mathcal{E}$ is countably generated. Fix a countable generator $\{E_n : n\in\omega\}$ of $\mathcal{E}$ and let $I : E\to[0,1]$ be the map defined by

$$I(e) = \sum_{n=0}^{\infty}\frac{2}{3^{n+1}} I_{E_n}(e),\qquad e\in E.$$

Further, let $I' : E\times X\to[0,1]\times X$ be the map

$$I'(e,x) = (I(e),x),\qquad (e,x)\in E\times X,$$

and set $A' = I'(A)$ and $U'_n = I'(U_n)$. By Lemma 2.4 and Theorem 1.3, we get Carathéodory maps $p'_n : A'\to[0,1]$ such that for every $e\in E$, $\{p'_n(e,\cdot) : n\in\omega\}$ is a locally Lipschitz partition of unity subordinate to $\{U'_n(e) : n\in\omega\}$. Put $p_n = p'_n\circ I'$, $n\in\omega$. The proof is complete.

4. Selection and representation of lower Carathéodory multifunctions. *Throughout this section we assume that* E, $\mathcal{E}$, X *and* A *are as in Theorem* 1.1.

Lemma 4.1. *Let* Y *be a separable normed linear space*, $F : A\to Y$ *a convex valued*, *lower Carathéodory multifunction and* $\varepsilon>0$. *Then there exists a Carathéodory map* $f_\varepsilon : A\to Y$ *such that for every* $(e,x)\in A$, $F(e,x)\cap S(f_\varepsilon(e,x),\varepsilon)\neq\emptyset$.

Proof. Fix a countable dense set $\{y_n : n\in\omega\}$ in Y and $n\in\omega$, let

$$U_n = \{(e,x)\in A : F(e,x)\cap S(y_n,\varepsilon)\neq\emptyset\}.$$

By Corollary 1.4, we get Carathéodory maps $p_n : A\to[0,1]$, $n\in\omega$, such that for every $e\in\omega$, $\{p_n(e,\cdot) : n\in\omega\}$ is a partition of unity subordinate to $\{U_n(e) : n\in\omega\}$. Put

$$f_\varepsilon(e,x) = \sum_{n=0}^{\infty} y_n p_n(e,x),\qquad (e,x)\in A.$$

The proof is complete.

Proof of Theorem 1.1. For each $n\in\omega$, we define a Carathéodory map $f_n : A\to Y$ such that for every $(e,x)\in A$ and $n\in\omega$,

(i) $F(e,x) \cap S(f_n(e,x), 1/2^{n+1}) \neq \emptyset$, and
(ii) $\|f_n(e,x) - f_{n+1}(e,x)\| \leq 1/2^n$.

We proceed by induction on n. By Lemma 4.1, we get $f_0 : A \to Y$ satisfying (i). Suppose $f_0, \ldots, f_n$ satisfying (i) and (ii) have been defined. By Lemmas 2.1 and 2.2, the multifunction

$$F_n(e,x) = F(e,x) \cap S(f_n(e,x), 1/2^{n+1})$$

defined on A is lower Carathéodory. Apply Lemma 4.1 to the multifunction F_n with $\varepsilon = 1/2^{n+1}$ to get f_{n+1} satisfying (i) and (ii). Now, put

$$f(e,x) = \lim_n f_n(e,x), \qquad (e,x) \in A.$$

The proof of Theorem 1.1 is complete.

A relevant example here is the following.

Example 4.1. Let $E = X = [0,1]$ and $y = \mathbb{R}$. Let $\{f_\alpha : \alpha < \mathrm{c}\}$ be an enumeration of all Borel maps from $E \times X$ into $[0,1] \subseteq Y$ and $\{t_\alpha : \alpha < \mathrm{c}\}$ be an enumeration of $[0,1]$. For each $\alpha < \mathrm{c}$, choose $y_\alpha \in [0,1] \setminus \{f_\alpha(t_\alpha, t_\alpha)\}$. Now define a multifunction $F : E \times X \to Y$ by

$$F(e,x) = \begin{cases} \{y_\alpha\} & \text{if } (e,x) = (t_\alpha, t_\alpha) \text{ for some } \alpha < \mathrm{c}, \\ [0,1] & \text{otherwise.} \end{cases}$$

Then for every e and every x, $F(e,\cdot)$ and $F(\cdot,x)$ are l.s.c. but F does not admit even a Borel selection.

Proposition 4.2. *Under the hypothesis of Theorem* 1.1, F *admits a dense sequence of Carathéodory selections.*

Proof. Fix a countable base $\{W_n : n \in \omega\}$ for Y and $n \in \omega$. Let

$$U_n = \{(e,x) \in A : F(e,x) \cap W_n \neq \emptyset\}.$$

Using Lemma 2.6 and the idea contained in the proof of Corollary 1.4 we get sets $B_{nk} \in \mathcal{E}$ and open sets U_{nk} in X, $k \in \omega$, such that

$$U_n \subseteq \bigcup_{k=0}^{\infty} (B_{nk} \times U_{nk}) \quad \text{and} \quad (A \setminus U_n) \cap \bigcup_{k=0}^{\infty} (B_{nk} \times U_{nk}) = \emptyset.$$

Let $U_{nk} = \bigcup_{l=0}^{\infty} C_{nkl}$, where C_{nkl} are closed in X, $l \in \omega$. Now define the multifunction $F_{nkl} : A \to Y$ by

$$F_{nkl}(e,x) = \begin{cases} \overline{F(e,x) \cap W_n} & \text{if } (e,x) \in A \cap (B_{nk} \times C_{nkl}), \\ F(e,x) & \text{otherwise.} \end{cases}$$

Clearly $F_{nkl} : A \to Y$ is a closed convex valued, lower Carathéodory multifunction. By Theorem 1.1, we get a Carathéodory selector f_{nkl} of F_{nkl}. It is obvious that $\{f_{nkl} : n, k, l \in \omega\}$ is a dense sequence of selectors of F.

Remark. If we adapt the proof of [KPY, Theorem 3.2] we have: If E, $\mathcal{E}$, X, A and Y are as in Theorem 1.1 and if $F : A \to Y$ is a convex valued, lower Carathéodory multifunction then either F admits a Carathéodory selection if Y is finite-dimensional, or $F(e,x)$ has non-empty interior for every $(e,x) \in A$.

Proof of Theorem 1.5. For every $s \in \omega^{<\omega}$, we define Carathéodory selectors $f_s : A \to Y$ of F such that

(i) $\{f_n : n \in \omega\}$ is a dense sequence of selectors of F, and

(ii) if s is of length k, then $\{f_{sm} : m \in \omega\}$ is a dense sequence of selectors of the multifunction

$$F_s(e,x) = \overline{F(e,x) \cap S(f_s(e,x), 1/2^k)}, \qquad (e,x) \in A.$$

We proceed by induction on the length of s. By Proposition 4.2, we get a dense sequence $\{f_n : n \in \omega\}$ of Carathéodory selections of F. Let f_t be defined for all $t \in \omega^{<\omega}$ of length $\le k$ and $s \in \omega^{<\omega}$ be of length k. By Lemmas 2.1, 2.2 and Proposition 4.2, we get a dense sequence of Carathéodory selectors $\{f_{sm} : m \in \omega\}$ of the multifunction F_s as defined in condition (ii).

Now put

$$f(e,x,\alpha) = \lim_k f_{\alpha|k}(e,x), \qquad (e,x,\alpha) \in A \times \omega^\omega.$$

It is easy to check that $f(e,x,\omega^\omega) = F(e,x)$ for every $(e,x) \in A$. To show that f satisfies the rest of the conclusion, fix an open set U in Y and write $\bigcup_{k=0}^\infty U_k$ where U_k is a non-decreasing sequence of open sets such that $\overline{U}_k \subseteq U$. Further, for $k, l \in \omega$, let

$$U_{kl} = \{y \in Y : d(x, X \setminus U_k) > 1/2^l\}.$$

For $(e,x,\alpha) \in A \times \omega^\omega$, note that

$$f(e,x,\alpha) \in U \;\Leftrightarrow\; \exists k\ \exists l\ \exists m > l\ (f_{\alpha|m}(e,x) \in U_{kl}).$$

The proof is complete.

5. Extensions of Carathéodory maps and random fixed points of multifunctions. In this section we prove results on extensions of Carathéodory maps which substantially generalize the results proved in [SS1, SS2].

THEOREM 5.1. *Let E, $\mathcal{E}$, X and Y be as in Theorem* 1.1 *and let $B \in \mathcal{E} \otimes \mathcal{B}(X)$ with sections $B(e)$ closed in X for every $e \in E$. Further, suppose $H : E \to Y$ is a closed convex valued, $\mathcal{E}$-measurable multifunction. If $f : B \to Y$ is a Carathéodory map such that $f(e, B(e)) \subseteq H(e)$ for every $e \in E$ then there exists a Carathéodory extension $g : E \times X \to Y$ of f such that $g(e,X) \subseteq H(e)$ for all e.*

Proof. Consider the multifunction $G : E \times X \to Y$ defined by

$$G(e,x) = \begin{cases} \{f(e,x)\} & \text{if } (e,x) \in B, \\ H(e) & \text{otherwise.} \end{cases}$$

By Theorem 1.1, we get a Carathéodory selection $g : E \times X \to Y$ of G. The map g has all the desired properties.

THEOREM 5.2. *Let E, X, Y and $\mathcal{E}$ be as in Theorem* 4.1 *and B the graph of an $\mathcal{E}$-measurable closed valued multifunction from E to X. If $f : B \to Y$ is a Carathéodory map then there exists a Carathéodory extension $g : E \times X \to Y$ of f such that $g(e,X) \subseteq \overline{\mathrm{co}(f(e,B(e)))}$ for every $e \in E$, where* $\mathrm{co}(A)$ *is the convex hull of $A \subseteq Y$. Moreover, if Y is finite-dimensional then we can get the Carathéodory extension g to satisfy $g(e,X) \subseteq \mathrm{co}(f(e,B(e)))$ for every $e \in E$.*

Proof. Consider the multifunction $H : E \to Y$ defined by

$$H(e) = \mathrm{co}(f(e,B(e))), \qquad e \in E.$$

Then H is $\mathcal{E}$-measurable. To see this fix a dense sequence of measurable selectors $\{s_n : n \in \omega\}$ of B. For any open set U in Y,

$H(e) \cap U \neq \emptyset \Leftrightarrow$ there exist positive rational numbers $t_0, \ldots, t_k$ with $\sum_{i=0}^k t_i = 1$ and natural numbers $n_0, \ldots, n_k$ such that $\sum_{i=0}^k t_i f(e, s_{n_i}(e)) \in U$.

This shows that the multifunctions H and $e \to \overline{H(e)}$ are measurable. The result follows by Theorem 4.1. Moreover, if Y is finite-dimensional, we get g such that $g(e,X) \subseteq \mathrm{co}(f(e,B(e)))$, $e \in E$, by using the observation made in the remark following the proof of Proposition 4.2 in the proof of Theorem 5.1.

THEOREM 5.3. *Let E be a Polish space, $\mathcal{E}$ a sub-σ-field of $\mathcal{B}(E)$, X a separable Banach space and $B \in \mathcal{E} \otimes \mathcal{B}(X)$ with sections $B(e)$ compact, non-empty and convex. Suppose $F : B \to X$ is a convex valued, lower Carathéodory map such that $F(e,x) \subseteq B(e)$ for every $(e,x) \in B$. If either X is finite-dimensional or $F(e,x)$ is closed for all (e,x) then there exists an $\mathcal{E}$-measurable $s : E \to X$ such that $s(e) \in F(e,s(e))$ for all e.*

Proof. By Theorem 1.1 or the remark following the proof of Proposition 4.2, we get a Carathéodory selection $f : B \to X$ of F. Let

$$S = \{(e,x) \in B : f(e,x) = x\}.$$

Clearly $S \in \mathcal{E} \otimes \mathcal{B}(X)$, $S(e)$ is compact and by Schauder's fixed point theorem, $S(e) \neq \emptyset$ for all e. Hence, by Lemma 2.7, there is an $\mathcal{E}$-measurable map $s : E \to X$ such that $s(e) \in S(e)$ for all e. Then $s(e) \in F(e,s(e))$ for all e.

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