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Rotation sets for subshifts of finite type

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Abstract. For a dynamical system (X, f) and a function $\varphi : X \to \mathbb{R}^N$ the rotation set is defined. The case when (X, f) is a transitive subshift of finite type and φ depends on the cylinders of length 2 is studied. Then the rotation set is a convex polyhedron. The rotation vectors of periodic points are dense in the rotation set. Every interior point of the rotation set is a rotation vector of an ergodic measure.

1. Introduction. In some dynamical systems, an important role is played by *rotation numbers, vectors* and *sets*. The classical example and the source of the name is the notion of the rotation number of an orientation preserving homeomorphism of a circle. It has been introduced by Poincaré [P] and has been used extensively since then. It has numerous applications, see e.g. [A].

The notion of rotation numbers has been generalized to the case of annulus homeomorphisms homotopic to the identity ([Bi]), the case of circle maps of degree one ([NPT]) and the case of N-dimensional torus maps homotopic to the identity ([KMG]). Those notions also have many applications (see e.g. [Ch], [NPT], [KMG]). In those generalizations we get more than one rotation number (vector) for a given system, so we think rather about the *rotation set*.

Quite recently, rotation sets have been introduced for interval maps ([Bl1], [Bl2]). In many other situations, for example for the systems considered in [MT], rotation sets can be defined in a natural way.

To understand the idea of the rotation set, let us consider a continuous map f of the N-dimensional torus $\mathbb{T}^N$ into itself, homotopic to the identity. We take a lifting $F : \mathbb{R}^N \to \mathbb{R}^N$ of f and study the limits of the expressions $(F^n(x)-x)/n$ as n goes to infinity (sometimes we allow x to vary, too). Those limits form the rotation set. The expression $(F^n(x) - x)/n$ can be rewritten as $n^{-1} \sum_{i=0}^{n-1} \varphi(f^i(y))$, where $y = \pi(x)$ (here $\pi : \mathbb{R}^N \to \mathbb{T}^N = \mathbb{R}^N/\mathbb{Z}^N$ is the

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natural projection), and $\varphi(\pi(z)) = F(z) - z$ for $z \in \mathbb{R}^N$. Notice that the condition that f is homotopic to the identity makes $\varphi(\pi(z))$ depend really on $\pi(z)$, not on z itself.

Thus, studying rotation sets boils down to studying limits of ergodic averages of the *displacement function* φ. In view of the ergodic theorem, the sets $\{\int \varphi\, d\mu : \mu$ is an f-invariant probability measure$\}$ ([He]) and $\{\int \varphi\, d\mu : \mu$ is an f-invariant ergodic probability measure$\}$ ([MZ1]) are closely related to the rotation set.

It is a quite natural idea to generalize the notion of the rotation set to the abstract situation when $f : X \to X$ is an arbitrary dynamical system and $\varphi : X \to \mathbb{R}^N$ an arbitrary function (a very similar approach has been developed independently by A. Blokh in [Bl2]). The set of limits of convergent sequences $(n_i^{-1} \sum_{i=0}^{n_i-1} \varphi(f^i(x_i)))_{i=1}^{\infty}$, where $\lim_{i\to\infty} n_i = \infty$ and $x_i \in X$, will be called the *general rotation set* of φ (or of (f, φ)). If for some $x \in X$ the limit $\lim_{n\to\infty} n^{-1} \sum_{i=0}^{n-1} \varphi(f^i(x))$ exists then we shall call it the *φ-rotation vector of x* (or simply the *rotation vector* of x if it is clear which φ we use) and denote it by $\varrho_\varphi(x)$. If $N = 1$ then instead of "vector" we can say "number". The set $\varrho(\varphi) = \{\varrho_\varphi(x) : x \in X\}$ will be called the *pointwise rotation set* of φ (or of (f, φ)). For an invariant probability measure μ on X we shall call the integral $\int \varphi\, d\mu$ the *rotation vector* of μ and denote it by $\varrho_\varphi(\mu)$.

In this general situation clearly the pointwise rotation set is contained in the general rotation set. Moreover, since for every ergodic measure there are generic points, the set of rotation vectors of ergodic measures is contained in the pointwise rotation set (when we speak of an ergodic measure, we mean an ergodic invariant probability measure). By the Krylov–Bogolyubov theorem, there exists an invariant probability measure for f, and since the closed convex hull of the set of ergodic measures is equal to the set of all invariant probability measures, there exists an ergodic measure for f. Therefore the pointwise and general rotation sets of φ are nonempty.

In the situation we shall consider throughout the main part of the paper, the general and pointwise rotation sets are equal. For historical reasons, and since it is slightly easier to use it, we will choose the pointwise rotation set to be more equal (see [O]). When using it, we will call it simply the *rotation set* of φ.

In this paper we want to show that to a great extent the properties of rotation sets and their relation to the underlying dynamics are independent of the concrete situation. Therefore we forget about the fact that the function φ has anything to do with displacement.

Since usually quite general results are few and weak, we have to make additional assumptions. Namely, we assume that our system is a transitive subshift of finite type and the function φ is constant on the cylinders of length 2. Those assumptions seem to be very strong, even stronger than assuming that the space is a torus and φ is displacement, but in fact they are not. Namely, they allow us to use symbolic dynamics, which is a natural and widely used tool. For circle maps it can be done by considering P-monotone maps (see e.g. [ALM]), and for homeomorphisms of surfaces by using Thurston's theory (see e.g. [LM]). In Section 2 we show why in those cases our assumptions on φ are not restrictive.

The main results of the paper are that under our assumptions the rotation set is a convex polyhedron (Theorem 3.4; here by polyhedron we mean the convex hull of a finite set), the rotation vectors of periodic points are dense in the rotation set (Theorem 3.5), and every interior point of the rotation set is the rotation vector of an ergodic measure (Theorem 4.6).

A reader interested in the properties of the rotation set and rotation vectors for non-transitive subshifts of finite type can easily derive them from the results of this paper.

The results of this paper can be used to reprove many known theorems on rotation sets for circle or torus maps. What is more important, they show which parts of the theory of rotation sets for those maps are independent of a specific situation. Next, the results of this paper are being used in further development of the theory of rotation sets for interval maps, started in [Bl1] and [Bl2]. Finally, in the situation considered in [MaT], by our Theorem 3.5 the weight-per-symbol polytope of [MaT] is equal to the rotation set, so Proposition 3.2 of [MaT] is a particular case of our results.

2. Using symbolic dynamics. Suppose that we are investigating a circle map of degree one or a 2-torus (or annulus) homeomorphism isotopic to the identity. We shall denote our space by X and our map by f. Let $\widetilde{X}$ be the universal cover of X, $\pi : \widetilde{X} \to X$ the natural projection, and $F : \widetilde{X} \to \widetilde{X}$ a lifting of f. To investigate the rotation set of f (or more precisely, of F), we have to look at the ergodic averages of the displacement function, as described in the preceding section. By standard methods—choosing one or more periodic orbits, calling their union P and then looking at the simplest map homotopic (isotopic in 2 dimensions) rel. P to f—we can often get a new map $g : X \to X$ that is easier to investigate. Namely, there is a Markov partition for g, and we can use coding to pass to a subshift of finite type. With a suitable choice of P, our new map g is really "simpler" than f. This happens if f is P-adjusted (see e.g. [MN]) in the one-dimensional case or pseudo-Anosov (see e.g. [LM]) in the two-dimensional case. Then every orbit of g has its counterpart for f. In particular, the rotation set of g is contained in the rotation set of f.

After coding, we get some subshift of finite type (Σ, σ). We also have a function $\varphi : \Sigma \to \mathbb{R}^N$ which is the composition of the projection from Σ to X

with the displacement function of g. Here $N=1$ for a circle and an annulus and $N=2$ for a torus. The rotation set of g is the same as the rotation set of (σ,φ). To be able to use the results of the next sections, we have to know that σ is transitive, and we have to have a function constant on the cylinders of length 2 that gives the same rotation set as φ. By a *cylinder of length* 2 we mean a set of the form $C_{AB}=\{(x_i)\in\Sigma : x_0=A \text{ and } x_1=B\}$.

Suppose that we replace φ by another function ψ such that $\varphi-\psi=\xi\circ\sigma-\xi$ for some bounded function ξ (that is, ψ is cohomologous to φ). Then

$$\frac{1}{n}\sum_{i=0}^{n-1}\varphi\circ\sigma^i-\frac{1}{n}\sum_{i=0}^{n-1}\psi\circ\sigma^i=\frac{1}{n}\sum_{i=0}^{n-1}(\xi\circ\sigma-\xi)\circ\sigma^i=\frac{1}{n}(\xi\circ\sigma^n-\xi).$$

Thus,

$$\left\|\frac{1}{n}\sum_{i=0}^{n-1}\varphi\circ\sigma^i-\frac{1}{n}\sum_{i=0}^{n-1}\psi\circ\sigma^i\right\|\le\frac{2}{n}\|\xi\|,$$

where $\|\cdot\|$ denotes the sup norm. Therefore the rotation sets of φ and ψ are the same. Hence, we want to show that φ is cohomologous to a function constant on cylinders of length 2.

Let $\{A_1,\ldots,A_s\}$ be our Markov partition of X. Choose one component B_i of every $\pi^{-1}(A_i)$ and set $B=\bigcup_{i=1}^s B_i$. Then (after removing from B a part of its boundary) we have $\pi(B)=X$ and $B\cap d(B)=\emptyset$ for every deck transformation $d:\widetilde{X}\to\widetilde{X}$ other than the identity. Therefore $x\in d_x(B)$ for each $x\in\widetilde{X}$, where the deck transformation d_x is uniquely determined by x. In our concrete situations, the deck transformations are translations by elements of $\mathbb{Z}^N$. In particular, d_x is translation by some $k_x\in\mathbb{Z}^N$.

Set $\widetilde{\psi}(x)=k_{F(y)}-k_y$ for $x\in X$, where $y\in\pi^{-1}(x)$. Since f is homotopic to the identity, $\widetilde{\psi}(x)$ does not depend on the choice of y, i.e. $\widetilde{\psi}$ is well defined. We may assume that our Markov partition is so fine that for each i and j the set $F(B_i)$ intersects at most one component of $\pi^{-1}(A_j)$. Then the composition $\psi:\Sigma\to\mathbb{R}^N$ of the projection from Σ to X with $\widetilde{\psi}$ is constant on cylinders of length 2. Moreover, ψ has values in $\mathbb{Z}^N$.

Set $\xi(x)=y-k_y$ for $x\in X$, where $y\in\pi^{-1}(x)$. By the same reasons as for $\widetilde{\psi}$, the function $\widetilde{\xi}$ is well defined. Since the set B is bounded, $\widetilde{\xi}$ is also bounded. If $\widetilde{\varphi}$ is the displacement function and $\pi(y)=x$ then

$$\begin{aligned}\widetilde{\varphi}(x)-\widetilde{\psi}(x)&=(F(y)-y)-(k_{F(y)}-k_y)\\&=(F(y)-k_{F(y)})-(y-k_y)=\widetilde{\xi}(f(x))-\widetilde{\xi}(x).\end{aligned}$$

Therefore we get $\varphi-\psi=\xi\circ\sigma-\xi$, where ξ is the composition of the projection from Σ to X with $\widetilde{\xi}$.

This is the result we wanted to prove. It means that in the cases described here one can use the techniques and results of the next sections. Moreover, the function ψ that we got has values in $\mathbb{Z}^N$, so even the results obtained under this additional assumption can be applied.

3. Rotation set is a polyhedron. In the remaining part of this paper (Σ, σ) will be a transitive subshift of finite type and $\varphi : \Sigma \to \mathbb{R}^N$ a function constant on cylinders of length 2. The subshift may be one-sided or two-sided—that will make no difference.

We can look at our system in the following way. There is a finite oriented graph G (a *Markov graph* of Σ) and elements of Σ are infinite paths in G (doubly infinite paths if we think of a two-sided subshift). Then we can think of φ as of a function on arrows of G. Transitivity of σ means that for any vertices A and B of G there is a finite path in G from A to B.

Notice that φ attains only finitely many values, and therefore it is bounded by some constant B (in the space $\mathbb{R}^N$ we use the Euclidean norm).

Let us look at finite paths in G. They correspond to cylinders in Σ. Denote by $\varrho_\varphi(\tau)$ the average of the values of φ on the arrows of such a path τ. A path τ with consecutive vertices $A_0, A_1, \ldots, A_n$ (written $\tau = (A_0, A_1, \ldots, A_n)$) is called a *loop* if $A_n = A_0$. To every loop there corresponds a periodic point of σ. Clearly, if τ is a loop then $\varrho_\varphi(\tau)$ is the rotation vector of this periodic point. Usually we think of a loop without specifying its beginning. Then we get a periodic orbit rather than just a periodic point. Of course, the rotation vectors of all elements of a given periodic orbit are the same.

We will say that a loop is *elementary* if it is not a concatenation of two shorter loops. Clearly, each loop that is not elementary can be written as a concatenation of two loops, at least one of which is elementary.

By the *length* of a path τ we will mean the number of arrows in τ. We will denote it by $|\tau|$. Thus, if $\tau = (A_0, A_1, \ldots, A_n)$ then $|\tau| = n$.

Now we start studying properties of the rotation set $\varrho(\varphi)$ of φ. Let $\tau_1, \ldots, \tau_k$ be all the elementary loops in the graph G, and let $\varrho_1, \ldots, \varrho_k$ be their rotation vectors.

LEMMA 3.1. *For each loop in the graph G its rotation vector belongs to* $\mathrm{Conv}(\varrho_1, \ldots, \varrho_k)$.

Proof. We shall prove this lemma by induction. Assume that L is a loop of length n in G and that for every loop in G of length smaller than n its rotation vector belongs to $\mathrm{Conv}(\varrho_1, \ldots, \varrho_k)$. If L is elementary then clearly $\varrho_\varphi(L) \in \mathrm{Conv}(\varrho_1, \ldots, \varrho_k)$. If L is not elementary then it is the concatenation of two loops τ and L', where τ is elementary. Therefore $\varrho_\varphi(\tau) \in \mathrm{Conv}(\varrho_1, \ldots, \varrho_k)$, and also $\varrho_\varphi(L') \in \mathrm{Conv}(\varrho_1, \ldots, \varrho_k)$ since

$|L'| < |L|$. Since $\varrho_\varphi(L)$ is a convex combination of $\varrho_\varphi(\tau)$ and $\varrho_\varphi(L')$, we get $\varrho_\varphi(L) \in \mathrm{Conv}(\varrho_1, \ldots, \varrho_k)$. ∎

LEMMA 3.2. *Let $P = (A_0, A_1, \ldots, A_n)$ be a path in G. Then there is a vector $v \in \mathrm{Conv}(\varrho_1, \ldots, \varrho_k)$ such that $\|\varrho_\varphi(P) - v\| \le 2Bs(s+1)/n$, where s is the number of vertices of G.*

Proof. If all $A_0, A_1, \ldots, A_n$ are different, we do nothing. Otherwise we take the smallest i such that there exists $j > i$ such that $A_i = A_j$. Then we take the largest j with those properties. In such a way we can write P as the concatenation of three paths: $(A_0, \ldots, A_i)$ of length smaller than s, a loop $(A_i, \ldots, A_j)$, and a path $(A_j, \ldots, A_n)$. Notice that in the last path the vertex A_i only appears in the first place. Then we repeat the same procedure starting from the last path, etc. Each time in the last path there is a new symbol that only appears in the first place. Therefore we can repeat our procedure at most s times. In such a way we write P as the concatenation of a path, a loop, a path, a loop, ..., a path (the first and the last paths perhaps are not there). In this decomposition there are at most $s+1$ paths, each of them of length at most s (we could give better estimates, but we do not need them). If we call those paths $P_1, \ldots, P_p$ and the loops $L_1, \ldots, L_q$ then we have

$$\varrho_\varphi(P) = \frac{l}{n} \sum_{i=1}^{p} \frac{|P_i|}{l} \varrho_\varphi(P_i) + \frac{t}{n} \sum_{i=1}^{q} \frac{|L_i|}{t} \varrho_\varphi(L_i),$$

where $l = \sum_{i=1}^{p} |P_i|$ and $t = \sum_{i=1}^{q} |L_i|$ (we have $l + t = n$). The vector

$$v = \sum_{i=1}^{q} \frac{|L_i|}{t} \varrho_\varphi(L_i)$$

belongs to $\mathrm{Conv}(\varrho_1, \ldots, \varrho_k)$ by Lemma 3.1. If there is no loop in our decomposition, then we take any $v \in \mathrm{Conv}(\varrho_1, \ldots, \varrho_k)$. The norm of each vector $\varrho_\varphi(P_i)$ is bounded by B, so the norm of the vector

$$u = \sum_{i=1}^{p} \frac{|P_i|}{l} \varrho_\varphi(P_i)$$

is also bounded by B. Similarly, $\|v\| \le B$. Moreover, $l \le s(s+1)$. Therefore

$$\|\varrho_\varphi(P) - v\| \le \frac{l}{n}(\|u\| + \|v\|) \le \frac{2Bs(s+1)}{n}. \ \blacksquare$$

LEMMA 3.3. *For every $v \in \mathrm{Conv}(\varrho_1, \ldots, \varrho_k)$, $\varepsilon > 0$, and every vertex A of G there is a loop L in G passing through A such that $\|\varrho_\varphi(L) - v\| \le \varepsilon$.*

Proof. We have $v = t_1\varrho_1 + \ldots + t_k\varrho_k$ for some $t_1, \ldots, t_k \ge 0$ such that $t_1 + \ldots + t_k = 1$. We approximate t_i's by rational numbers. There exist a positive integer m and non-negative integers $s_1, \ldots, s_k$ such that

$s_1 + \ldots + s_k = m$ and $|s_i/m - t_i| \le \varepsilon/(2kB)$ for $i = 1, \ldots, k$. Then we construct loops passing through A with rotation vectors close to ϱ_i. For each i we choose a vertex B_i through which τ_i passes. By transitivity of σ there exist paths P_i from A to B_i and R_i from B_i to A. Now we define a loop L_i as the concatenation of the path P_i, l_i repetitions of the loop τ_i (with the beginning and end at B_i) and the path R_i. By taking l_i sufficiently large, we can make the rotation vector of L_i as close to the rotation vector of τ_i as we want. For our purposes it suffices to have $\|\varrho_\varphi(L_i) - \varrho_i\| \le \varepsilon/(2k)$, so we take l_i large enough to get this inequality.

We take a natural number r divisible by the lengths of all loops L_i and set $r_i = r/|L_i|$ for $i = 1, \ldots, k$. Then we define L as the concatenation of $r_1 s_1$ copies of L_1, $r_2 s_2$ copies of L_2, $\ldots$, $r_k s_k$ copies of L_k. Let us estimate the difference between v and the rotation vector of L. We have

$$|L| = \sum_{i=1}^{k} r_i s_i |L_i| = \sum_{i=1}^{k} r s_i = rm,$$

and therefore

$$\varrho_\varphi(L) = \frac{1}{rm} \sum_{i=1}^{k} r_i s_i |L_i| \varrho_\varphi(L_i) = \sum_{i=1}^{k} \frac{s_i}{m} \varrho_\varphi(L_i).$$

Moreover,

$$\begin{aligned} \left\| \frac{s_i}{m} \varrho_\varphi(L_i) - t_i \varrho_i \right\| &\le \left\| \frac{s_i}{m} \varrho_\varphi(L_i) - \frac{s_i}{m} \varrho_i \right\| + \left\| \frac{s_i}{m} \varrho_i - t_i \varrho_i \right\| \\ &\le \|\varrho_\varphi(L_i) - \varrho_i\| + B \left\| \frac{s_i}{m} - t_i \right\| \le \frac{\varepsilon}{2k} + \frac{\varepsilon}{2k} = \frac{\varepsilon}{k}\,. \end{aligned}$$

Hence

$$\|\varrho_\varphi(L) - v\| = \left\| \sum_{i=1}^{k} \frac{s_i}{m} \varrho_\varphi(L_i) - \sum_{i=1}^{k} t_i \varrho_i \right\| \le \sum_{i=1}^{k} \left\| \frac{s_i}{m} \varrho_\varphi(L_i) - t_i \varrho_i \right\| \le k \frac{\varepsilon}{k} = \varepsilon. \ \blacksquare$$

Now we can prove the main result of this section.

Theorem 3.4. *Both the general rotation set and the pointwise rotation set of* (σ, φ) *are equal to* $\mathrm{Conv}(\varrho_1, \ldots, \varrho_k)$.

Proof. From Lemma 3.2 and the definition of the general rotation set it follows immediately that the general rotation set of (σ, φ) is contained in $\mathrm{Conv}(\varrho_1, \ldots, \varrho_k)$. As we noticed in the introduction, the pointwise rotation set is contained in the general rotation set. Therefore it remains to prove that $\mathrm{Conv}(\varrho_1, \ldots, \varrho_k)$ is contained in the pointwise rotation set. That is, we have to prove that for every $v \in \mathrm{Conv}(\varrho_1, \ldots, \varrho_k)$ there is an infinite path η in G with rotation vector v.

We apply Lemma 3.3. Fix some vertex A of G. For each positive integer m there exists a loop L_m in G beginning and ending at A, and such that $\|\varrho_\varphi(L_m) - v\| \le B/m$. Using induction we can find a sequence $(q_m)_{m=1}^{\infty}$ of positive integers such that for each m we have

$$\frac{|L_{m+1}|}{q_m|L_m|} \le \frac{1}{m} \quad \text{and} \quad \sum_{i=1}^{m-1} \frac{q_i|L_i|}{q_m|L_m|} \le \frac{1}{m}\,. \tag{3.1}$$

We define the path η as the concatenation of q_1 copies of L_1, then q_2 copies of L_2, etc. Let η_n be the finite path consisting of the first n arrows of η. Assume that $n > q_1|L_1|$. Observe that η_n is the concatenation of the loops L_i repeated q_i times for $i = 1, \ldots, m$ (for some m), then possibly the loop L_{m+1} repeated q'_{m+1} times (for some $q'_{m+1} < q_{m+1}$), and then possibly a path L'_{m+1} consisting of first few arrows of L_{m+1}. We have

$$n = \sum_{i=1}^{m} q_i|L_i| + q'_{m+1}|L_{m+1}| + |L'_{m+1}| \tag{3.2}$$

and

$$\begin{aligned} \varrho_\varphi(\eta_n) = \frac{1}{n}\Big(&\sum_{i=1}^{m} q_i|L_i|\varrho_\varphi(L_i) \\ &+ q'_{m+1}|L_{m+1}|\varrho_\varphi(L_{m+1}) + |L'_{m+1}|\varrho_\varphi(L'_{m+1})\Big). \end{aligned} \tag{3.3}$$

By (3.1) we get

$$\Big\|\frac{1}{n}|L'_{m+1}|\varrho_\varphi(L'_{m+1})\Big\| \le \frac{|L_{m+1}|B}{q_m|L_m|} \le \frac{B}{m} \tag{3.4}$$

and

$$\Big\|\frac{1}{n}\sum_{i=1}^{m-1} q_i|L_i|\varrho_\varphi(L_i)\Big\| \le \sum_{i=1}^{m-1} \frac{q_i|L_i|B}{q_m|L_m|} \le \frac{B}{m}\,. \tag{3.5}$$

Since $\|\varrho_\varphi(L_m) - v\| \le B/m$ and $\|\varrho_\varphi(L_{m+1}) - v\| \le B/(m+1) < B/m$, taking into account (3.1) and (3.2) we get

$$\begin{aligned} &\Big\|\frac{1}{n}(q_m|L_m|\varrho_\varphi(L_m) + q'_{m+1}|L_{m+1}|\varrho_\varphi(L_{m+1})) - v\Big\| \\ &\le \Big\|\frac{1}{n}(q_m|L_m|\varrho_\varphi(L_m) + q'_{m+1}|L_{m+1}|\varrho_\varphi(L_{m+1})) \\ &\qquad - \frac{1}{n}(q_m|L_m| + q'_{m+1}|L_{m+1}|)v\Big\| \\ &\qquad + \Big\|\frac{1}{n}(q_m|L_m| + q'_{m+1}|L_{m+1}|)v - v\Big\| \end{aligned} \tag{3.6}$$

$$\leq \frac{B}{mn}(q_m|L_m| + q'_{m+1}|L_{m+1}|) + \frac{1}{n}(n - q_m|L_m| - q'_{m+1}|L_{m+1}|)\|v\|$$
$$\leq \frac{B}{m} + \frac{1}{n}\Big(\sum_{i=1}^{m-1} q_i|L_i| + |L'_{m+1}|\Big)B \leq \frac{B}{m} + \frac{2q_m|L_m|B}{mn} \leq \frac{3B}{m}\,.$$

From (3.3)–(3.6) we obtain $\|\varrho_\varphi(\eta_n) - v\| \leq 5B/m$. As n goes to infinity, so does m, and therefore $\lim_{n\to\infty} \|\varrho_\varphi(\eta_n) - v\| = 0$. This proves that the rotation vector of η is v. ∎

From Lemma 3.3 and Theorem 3.4 we get immediately the following corollary.

THEOREM 3.5. *The set of the rotation vectors of periodic points of σ is dense in $\varrho(\varphi)$.*

Remark. In the case when $\varphi(X) \subset \mathbb{Z}^N$ all the rotation vectors of periodic points are rational (i.e. belong to $\mathbb{Q}^N$). In particular, the vertices of the polyhedron $\varrho(\varphi)$ are rational in this case.

4. Interior vectors of the rotation set. In this section we investigate the properties of the vectors from $\mathrm{int}(\varrho(\varphi))$, the topological interior of $\varrho(\varphi)$ as a subset of $\mathbb{R}^N$.

LEMMA 4.1. *Let $v \in \mathrm{int}(\varrho(\varphi))$ and let A be a vertex of G. Then there exist loops $L_1, \ldots, L_k$ in G passing through A such that $|L_1| = \ldots = |L_k|$ and $v \in \mathrm{int}(\mathrm{Conv}(\varrho_\varphi(L_1), \ldots, \varrho_\varphi(L_k)))$.*

Proof. By Lemma 3.3 for every $i \in \{1, \ldots, k\}$ and $\varepsilon > 0$ there exists a loop L'_i in G passing through A such that $\|\varrho_\varphi(L'_i) - \varrho_i\| \leq \varepsilon$. Since $v \in \mathrm{int}(\varrho(\varphi))$, if ε is sufficiently small then $v \in \mathrm{int}(\mathrm{Conv}(\varrho_\varphi(L'_1), \ldots, \varrho_\varphi(L'_k)))$. Let m be a common multiple of $|L'_1|, \ldots, |L'_k|$ and let L_i be the loop L'_i repeated $m/|L'_i|$ times, $i = 1, \ldots, k$. Then clearly $L_1, \ldots, L_k$ satisfy the conditions of the lemma. ∎

We start with rotation vectors from $\mathrm{int}(\varrho(\varphi))$ that are rational.

THEOREM 4.2. *If $\varphi(X) \subset \mathbb{Z}^N$, then for every $v \in \mathbb{Q}^N \cap \mathrm{int}(\varrho(\varphi))$ there exists a periodic point $x \in \Sigma$ with rotation vector v.*

Proof. Let $v \in \mathbb{Q}^N \cap \mathrm{int}(\varrho(\varphi))$ and let A be a vertex of G. By Lemma 4.1 there exist loops $L_1, \ldots, L_k$ in G passing through A such that $|L_1| = \ldots = |L_k|$ and $v \in \mathrm{Conv}(\varrho_\varphi(L_1), \ldots, \varrho_\varphi(L_k))$. Since v and $\varrho_\varphi(L_1), \ldots, \varrho_\varphi(L_k)$ are rational, there exist rational nonnegative numbers $r_1, \ldots, r_k$ such that

$$\sum_{i=1}^{k} r_i = 1 \quad \text{and} \quad \sum_{i=1}^{k} r_i \varrho_\varphi(L_i) = v. \tag{4.1}$$

Now we construct a loop L in G with rotation vector v. There is a positive integer q such that all qr_i are integers. We define L as the concatenation of qr_1 copies of L_1, qr_2 copies of L_2, ..., qr_k copies of L_k (each of those loops begins and ends at A, so we can do it). Since the lengths of all the loops L_i are equal, the rotation vector of L is

$$\frac{\sum_{i=1}^k qr_i|L_i|\varrho_\varphi(L_i)}{\sum_{i=1}^k qr_i|L_i|} = \frac{\sum_{i=1}^k r_i\varrho_\varphi(L_i)}{\sum_{i=1}^k r_i}\,.$$

By (4.1) this is equal to v.

Now the point x of Σ corresponding to L is periodic and its rotation vector is the same as the rotation vector of L. ∎

Remark. As we already know, the vertices of the polyhedron $\varrho(\varphi)$ are the rotation vectors of periodic points. Therefore in the case $N = 1$ in Theorem 4.2 the assumption $v \in \mathbb{Q}^N \cap \mathrm{int}(\varrho(\varphi))$ can be replaced by $v \in \mathbb{Q} \cap \varrho(\varphi)$.

To study arbitrary rotation vectors from $\mathrm{int}(\varrho(\varphi))$ we will need the following geometrical lemma. It is a generalization to N dimensions of a weak version of Lemma 1 from [MZ2].

LEMMA 4.3. *Let $w_1, \ldots, w_k \in \mathbb{R}^N$ and assume that $0 \in \mathrm{int}(\mathrm{Conv}(w_1, \ldots, w_k))$. Then there exists $R > 0$ such that if $z \in \mathbb{R}^N$ and $\|z\| \le R$ then there exists $j \in \{1, \ldots, k\}$ such that $\|z + w_j\| \le R$.*

Proof. If there is i such that $w_i = 0$ then we choose $j = i$ and we are done. Assume that $w_i \ne 0$ for every i.

For every unit vector $u \in \mathbb{R}^N$ and $i \in \{1, \ldots, k\}$ we define $\alpha_i(u)$ as the measure of the angle between the vectors u and w_i. Let $\alpha(u) = \min_i \alpha_i(u)$. Since 0 belongs to the interior of $\mathrm{Conv}(w_1, \ldots, w_k)$, we have $\alpha(u) < \pi/2$ for every unit vector u. The functions α_i are continuous, so α is also continuous. Therefore it assumes its maximum α_0 on the unit sphere, and $\alpha_0 < \pi/2$.

Set

$$R = \max_i \|w_i\| \cdot \max(2, 1/\cos\alpha_0).$$

Take $z \in \mathbb{R}^N$ such that $\|z\| \le R$. If $\|z\| \le R/2$ then we choose any j and we get

$$\|z + w_j\| \le \|z\| + \|w_j\| \le R/2 + R/2 = R.$$

Assume that $\|z\| > R/2$. There exists j such that the measure of the angle between $-z$ and w_j is smaller than or equal to α_0. Then

$$-z \cdot w_j \ge \|z\|\,\|w_j\|\cos\alpha_0 \ge \|z\|\,\|w_j\|\frac{\|w_j\|}{R} > \frac{1}{2}\|w_j\|^2.$$

Therefore

$$\|z + w_j\|^2 = \|z\|^2 + 2z \cdot w_j + \|w_j\|^2 < \|z\|^2 - \|w_j\|^2 + \|w_j\|^2 = \|z\|^2 \le R^2,$$

so $\|z + w_j\| < R$. This completes the proof. ∎

Lemma 4.4. *If $v \in \operatorname{int}(\varrho(\varphi))$ then there exist $x \in \Sigma$ and $M > 0$ such that*

$$\Big\| \sum_{i=0}^{m-1} \varphi(\sigma^i(x)) - mv \Big\| \le M \tag{4.2}$$

for all $m \ge 0$.

Proof. Choose a vertex A of G. By Lemma 4.1 there exist loops $L_1, \ldots, L_k$ in G passing through A such that $|L_1| = \ldots = |L_k|$ and $v \in \operatorname{int}(\operatorname{Conv}(\varrho_\varphi(L_1), \ldots, \varrho_\varphi(L_k)))$. Let K be the common length of L_i. For $i = 1, \ldots, k$ set $w_i = K\varrho_\varphi(L_i) - Kv$. Then $0 \in \operatorname{int}(\operatorname{Conv}(w_1, \ldots, w_k))$. Let R be the constant from Lemma 4.3.

We define a sequence $(j_n)_{n=1}^\infty$ in $\{1, \ldots, k\}$ and a sequence $(z_n)_{n=0}^\infty$ of vectors such that $\|z_n\| \le R$ by induction as follows. Set $z_0 = 0$. If z_{n-1} is defined and $\|z_{n-1}\| \le R$ then by Lemma 4.3 we choose j_n such that $\|z_{n-1} + w_{j_n}\| \le R$. Next we set $z_n = z_{n-1} + w_{j_n}$. This completes the induction step.

Now we define an infinite path η in G as the concatenation of $L_{j_1}, L_{j_2}, \ldots$ Let η_m be the finite path consisting of the first m arrows of η. Observe that η_m is the concatenation of the loops L_{j_i} for $i = 1, \ldots, n$ (for some n) and then possibly a path L' consisting of first few arrows of $L_{j_{n+1}}$. If x is the point of Σ corresponding to the path η then

$$\sum_{i=0}^{nK-1} \varphi(\sigma^i(x)) - nKv = \sum_{i=1}^{n} (K\varrho_\varphi(L_{j_i}) - Kv) = \sum_{i=1}^{n} w_{j_i} = z_n.$$

Therefore

$$\Big\| \sum_{i=0}^{m-1} \varphi(\sigma^i(x)) - nv \Big\| = \|z_n + |L'|\varrho_\varphi(L') - |L'|v\| \le R + 2KB.$$

Thus, (4.2) holds with $M = R + 2KB$. ∎

Using Lemma 4.4 we can prove the next of the main results of the paper.

Theorem 4.5. *If $v \in \operatorname{int}(\varrho(\varphi))$ then there exists a compact invariant set $Y \subset \Sigma$ such that $\varrho_\varphi(y) = v$ for every $y \in Y$.*

Proof. Let $v \in \operatorname{int}(\varrho(\varphi))$ and let x and M be as in Lemma 4.4. Let Y be the closure of the orbit of x (that is, of the set $Z = \{\sigma^i(x) : i = 0, 1, 2, \ldots\}$). Clearly, Y is compact and invariant. We will show that $\varrho_\varphi(y) = v$ for every $y \in Y$.

Fix $m \ge 0$. If $y \in Z$ then $y = \sigma^n(x)$ for some n. Then by (4.2) we have

$$\Big\| \sum_{i=0}^{m-1} \varphi(\sigma^i(z)) - mv \Big\| = \Big\| \sum_{i=n}^{n+m-1} \varphi(\sigma^i(x)) - mv \Big\|$$

$$= \Big\| \sum_{i=0}^{n+m-1} \varphi(\sigma^i(x)) - \sum_{i=0}^{n-1} \varphi(\sigma^i(x)) - (n+m)v + nv \Big\|$$

$$\le \Big\| \sum_{i=0}^{n+m-1} \varphi(\sigma^i(x)) - (n+m)v \Big\| + \Big\| \sum_{i=0}^{n-1} \varphi(\sigma^i(x)) - nv \Big\| \le 2M.$$

Since σ and φ are continuous, we get

$$\Big\| \sum_{i=0}^{m-1} \varphi(\sigma^i(y)) - mv \Big\| \le 2M$$

for every y from the closure of Z, i.e. from Y. This holds for every $m \ge 0$, and therefore we get $\varrho_\varphi(y) = v$ for every $y \in Y$. ∎

As a simple corollary to the above theorem we get the following result.

THEOREM 4.6. *If $v \in \operatorname{int}(\varrho(\varphi))$ then there exists an ergodic invariant probability measure μ on Σ such that $\varrho_\varphi(\mu) = v$.*

Proof. Let $v \in \operatorname{int}(\varrho(\varphi))$ and let Y be as in Theorem 4.5. Since Y is compact and invariant, there exists an ergodic invariant probability measure μ on Y. It can be considered as an ergodic invariant probability measure μ on Σ with support contained in Y. By the Ergodic Theorem, every generic point of μ has rotation vector $\varrho_\varphi(\mu)$. Since there are generic points of μ in Y, we get $\varrho_\varphi(\mu) = v$. ∎

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