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Properly homotopic nontrivial planes are isotopic

by

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Abstract. It is proved that two planes that are properly homotopic in a noncompact, orientable, irreducible 3-manifold that is not homeomorphic to $\mathbb{R}^3$ are isotopic. The end-reduction techniques of E. M. Brown and C. D. Feustal and M. G. Brin and T. L. Thickstun are used.

Introduction. In this paper it is proved that two planes that are properly homotopic in a noncompact, orientable, irreducible 3-manifold that is not homeomorphic to $\mathbb{R}^3$ are isotopic. The end-reduction techniques of Brown–Feustal and Brin–Thickstun are used.

It is not uncommon among those who study noncompact 3-manifolds to consider the end-irreducible and eventually end-irreducible cases as a starting point. These cases, while quite far from being general, do occur often enough to be useful. In recent years a technique known as "end-reduction" has been used to extend from the eventually end-irreducible case to the general case.

The technique of end-reduction was used by Brown and Feustel in [BF] to prove that if there is a "nontrivial" mapping of $\mathbb{R}^2$ in a noncompact 3-manifold W, then W must contain a "nontrivial" embedded plane as well. This result had been proved in [BBF] for eventually end-irreducible W.

In [BT] Brin and Thickstun recognized that given a noncompact 3-manifold W and a compact $K \subset W$ an eventually end-irreducible 3-manifold W_K could be associated with (W, K). By using the properties of the eventually end-irreducible 3-manifold W_K for increasingly large K and piecing things together nicely, they were able to obtain results in a more general case.

Following the approach of [BT], this author proved in [W] that if W is a noncompact, orientable, irreducible 3-manifold that is not homeomorphic to $\mathbb{R}^3$, and P and Q are planes that are nontrivial in W with P properly

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homotopic to Q and $P \cap Q = \emptyset$, then P and Q are parallel in W. This paper, which is a sequel to [W], follows the methods of [BF] more closely, however.

Definitions. A *plane* (*annulus*, *circle*, *arc*, 2-*sphere*, *disk*) is a space homeomorphic to $\mathbb{R}^2$ ($\mathbb{S}^1 \times I$, $\mathbb{S}^1$, I, $\mathbb{S}^2$, $\mathbb{D}^2$).

Suppose that X, Y, and Z are topological spaces.

If $f : X \to Y$ is a map and $f^{-1}(K)$ is compact for every compact $K \subset Y$, then f is said to be *proper*. If $X \subset Y$ and $X \cap K$ is compact for every compact $K \subset Y$, then X is *proper* in Y. (This is equivalent to the inclusion map $X \to Y$ being proper.) If $X \subset Y$ and X and Y are n- and $(n+1)$- manifolds, respectively, then we say that X is *properly embedded* in Y when X is proper in Y and $X \cap \partial Y = \partial X$.

Suppose that X and Y are subspaces of Z. We say that X *traps* Y if there is no proper homotopy $h : Y \times I \to Z$ such that $h(y,0) = y$ for every $y \in Y$ and $h(Y \times 1) \cap X = \emptyset$.

Let $h : X \times I \to X$ be a proper map. For $t \in I$, define $h_t : X \to X$ by $h_t(x) = h(x,t)$. If $h_0 = 1_W$ and h_t is a homeomorphism for every $t \in I$, then we say that h is an *isotopy* of X.

We let $\sharp(X)$ denote the number of path components of the space X.

Suppose that W is a noncompact 3-manifold.

If $P \subset W$ is a plane, then we say that P is *nontrivial* in W when P is proper in W and there is a compact subset of W that traps P.

An *exhaustion* or *exhausting sequence* for W is a function V from $\mathbb{N}$ to the set of compact 3-submanifolds of W such that $W = \bigcup_{n\in\mathbb{N}} V(n)$ and $V(n) \subset V(n+1) - \mathrm{Fr}(V(n+1))$. Traditionally we put $V_n = V(n)$.

If there is an exhausting sequence V for W such that $\mathrm{Fr}(V_n)$ is incompressible in W for every $n \in \mathbb{N}$, then we say that W is *end-irreducible.*

If there is a compact subset $K \subset W$ and an exhaustion V for W such that $K \subset V_1 - \mathrm{Fr}(V_1)$ and $\mathrm{Fr}(V_n)$ is incompressible in $W - K$ for every $n \in \mathbb{N}$, then W is said to be *end-irreducible rel* K. If W is end-irreducible rel K for some compact $k \subset W$, then W is said to be *eventually end-irreducible.*

Some results about compact 3-manifolds

LEMMA 1. *Let M be a compact 3-manifold and let $T \subset \partial M$ be a compact 2-manifold with at least two components. For $i = 1, 2$, let A_i be an annulus that is properly embedded and incompressible in M with each component of ∂A_i contained in a different component of T. Suppose that if J is a component of $A_1 \cap A_2$, then J is a circle in $M - \partial M$ that is noncontractible in M. If D is a compressing disk for T in M, then there is a compressing disk D' for T in M such that $D' \cap (A_1 \cup A_2) = \emptyset$.*

Proof. Suppose that D is a compressing disk for T in M. By Lemma 2 of [W] (in which the assumption of irreducibility in the hypothesis may be dropped), we may assume that $D \cap A_1 = \emptyset$. Now choose such a disk D with $\sharp(D \cap A_2)$ minimal. By standard arguments involving innermost disks and incompressibility, we may assume that no component of $D \cap A_2$ is a circle.

To get a contradiction, suppose that α is an arc component of $D \cap A_2$. Let D_α be a disk that is separated off D by α. We may choose α so that $D_\alpha \cap A_2 = \alpha$. Since each component of ∂A_2 is in a different component of T, it follows that α is a separating arc of A_2. Let E_α be the disk separated off A_2 by α. Then $D_\alpha \cup E_\alpha$ is a disk. Let $J = \partial(D_\alpha \cup E_\alpha)$.

If J is noncontractible in T, then there is a compressing disk D' for T in M that is parallel to $D_\alpha \cup E_\alpha$ with $D' \cap (A_1 \cup A_2) = \emptyset$. On the other hand, suppose that there is a disk $E \subset T$ with $\partial E = J$. Then we may slide ∂D along E to move it past A_2 and remove an arc of $D \cap A_2$ while introducing a circle of $D \cap A_2$. This circle can be removed by standard methods. We have reduced $\sharp(D \cap A_2)$, which contradicts minimality. This ends the proof. ■

LEMMA 2. *Let $M \subset N$ be compact, irreducible 3-manifolds such that each component of* $\mathrm{cl}(N - M)$ *meets ∂N. Suppose that A_0 and A_1 are annuli that are incompressible and properly embedded in M with $\partial A_0 = \partial A_1$. If A_0 and A_1 are homotopic in N, then A_0 and A_1 are isotopic in M by an isotopy fixed on ∂M.*

Proof. Move A_1 by an isotopy of M fixed on ∂M so that $\sharp(A_0 \cap A_1)$ is minimal and A_0 meets A_1 transversely.

Suppose that J is a component of $(A_0 \cap A_1) - \partial A_0$ that is contractible in either A_0 or A_1. It is easy to argue using the incompressibility of A_0 and A_1 that J must be contractible in both A_0 and A_1. Let $D \subset A_1$ be a disk with $J = \partial D$. Without loss of generality, we may assume that $(D - \partial D) \cap A_0 = \emptyset$. Let $E \subset A_0$ be the disk with $\partial E = \partial D$. Since M is irreducible and $D \cup E$ is a 2-sphere, there is a 3-ball $B \subset M$ with $\partial B = D \cup E$. We may use B to reduce $\#(A_0 \cap A_1)$ by an isotopy of M. Therefore, we may assume that no component of $(A_0 \cap A_1)$ is contractible in either A_0 or A_1. Consequently, if F is the closure of a component of $A_0 - A_1$ or $A_1 - A_0$, then F is not a disk.

Suppose that F is the closure of a component of $A_0 - A_1$ or $A_1 - A_0$.

By Proposition 5.4 of [Wa], there is a product $F \times I \subset N$ such that $F \times 0 \subset A_0$, $(\partial F \times I) \cup (F \times 1) \subset A_1$, and $(F \times (0,1)) \cap (A_0 \cup A_1) = \emptyset$.

Since $\partial(F \times I) \subset M$, either $F \times I \subset M$ or the interior of $F \times I$ contains a component of $\mathrm{cl}(N - M)$. Since each component of $\mathrm{cl}(N - M)$ meets ∂N, it follows that $F \times I \subset M$.

Since $\sharp(A_0 \cap A_1)$ is minimal, we have $F \times 0 = A_0$ and $(\partial F \times I) \cup (F \times 1) = A_1$. Therefore A_1 is isotopic in M to A_0 by an isotopy fixed on ∂M. ■

Handle moves respecting planes. Beginning now and for the rest of the paper, let W be a connected, noncompact, orientable, irreducible 3-manifold that is not homeomorphic to $\mathbb{R}^3$, and let P and Q be planes that are nontrivial in W.

In what follows, we will at times need to do handle moves along compressing 1-, 2-, and 3-handles. In particular, suppose that $M \subset W$.

First suppose that there is a properly embedded disk $D \subset M$ with ∂D a noncontractible circle in $\mathrm{Fr}(M)$. Let H be a regular neighborhood of D in M. Then H is a *compressing* 1*-handle* for M. Let $M(H) = \mathrm{cl}(M - H)$. We say that $M(H)$ is *obtained from* M *by removing the* 1*-handle* H.

Suppose that $D \subset \mathrm{cl}(W - M)$ is a properly embedded disk with ∂D noncontractible in $\mathrm{Fr}(M)$. Let H be a regular neighborhood of D in $\mathrm{cl}(W - M)$. Then we say that H is a *compressing* 2*-handle* for M. Let $M(H) = M \cup H$. We say that $M(H)$ is *obtained from* M *by adding the* 2*-handle* H.

Suppose that S is a 2-sphere in $\mathrm{Fr}(M)$ that bounds a 3-ball H in $\mathrm{cl}(W - M)$. Then H is a *compressing* 3*-handle* for M. Let $M(H) = M \cup H$. Then $M(H)$ is said to be *obtained from* M *by adding the* 3*-handle* H.

When H is a compressing 1-, 2-, or 3-handle for M, then we say that H is a *compressing handle* for M.

Suppose that H_1 is a compressing handle for M. Suppose that H_2 is a compressing handle for $M(H_1)$. Define $M(H_1, H_2) = M(H_1)(H_2)$. Assume that $M(H_1, \ldots, H_{k-1})$ has been defined and let H_k be a compressing handle for $M(H_1, \ldots, H_{k-1})$. Define $M(H_1, \ldots, H_k) = M(H_1, \ldots, H_{k-1})(H_k)$. Then we say that $H_1, \ldots, H_k, \ldots$ is a *sequence of compressing handles* in W for M.

Let $K \subset M$ and $F \subset W$. Suppose that $H_1, \ldots, H_\nu$ is a sequence of disjoint compressing 1-handles in $W - K$ for M such that

(1) $H_i \cap F = \emptyset$ and $H_i \cap K = \emptyset$ for $1 \le i \le \nu$, and
(2) $\mathrm{Fr}(M(H_1, \ldots, H_\nu))$ is incompressible in $W - K$.

Then we say that M *can be compressed in* $W - K$ *to* $M(H_1, \ldots, H_\nu)$ *by removing* 1*-handles that miss* F.

LEMMA 3. *Suppose that* $K \subset W$ *is a compact* 3*-manifold that traps* P *and meets* P *in a single disk. Suppose that* $L \subset W$ *is compact. Then there is a compact*, *connected* 3*-manifold* $M \subset W$ *with* $K \cup L \subset M - \mathrm{Fr}(M)$ *such that* $P \cap M$ *is a single disk and* M *can be compressed in* $W - K$ *by removing* 1*-handles that miss* P.

Proof. The proof is essentially the first two paragraphs of the proof of Lemma 3 of [W], which owes much to Lemma 1.1 of [BF]. We repeat it here for the convenience of the reader.

Let $M \subset W$ be a compact, connected 3-manifold with $K \cup L \subset M - \mathrm{Fr}(M)$ such that $P \cap M$ is a single disk. There is a sequence $H_1, \ldots, H_\nu$ of compressing 1-, 2-, and 3-handles in $W - K$ for M such that if $M^* = M(H_1, \ldots, H_\nu)$, then $\mathrm{Fr}(M)^*$ is incompressible in $W - K$. We may argue using Lemma 1 and the fact that $W - K$ is irreducible that $H_1, \ldots, H_\nu$ may be chosen so as to not intersect P. We choose M so that, with respect to the indicated properties, $H_1, \ldots, H_\nu$ contains the fewest possible 2-handles.

We claim that $H_1, \ldots, H_\nu$ has no 2-handles (and therefore no 3-handles). Let k be the least integer such that H_k is a 2-handle. We may choose H_k so that $H_k \cap H_i$ is a subproduct of the 1-handle structure of H_i for $1 \le i \le k-1$. Let $H = \bigcup_{i=1}^{k-1} H_i$ and let $H_1', \ldots, H_\mu'$ be the components of $\mathrm{cl}(H - H_k)$. Let $M' = M \cup H_k$. Then H_i' is a 1-handle for M' for $1 \le i \le \mu$. Note that $M'(H_1', \ldots, H_\mu', H_{k+1}, \ldots, H_\nu) = M^*$. Since $H_1', \ldots, H_\mu', H_{k+1}, \ldots, H_\nu$ has fewer 2-handles, this contradicts the minimality assumption. This ends the proof. ∎

Let V be an exhaustion for W. For $n \in \mathbb{N}$, let $G_n = \mathrm{cl}(V_{n+1} - V_n)$, $A_n = P \cap G_n$, and $B_n = Q \cap G_n$. Suppose that A_n and B_n are incompressible annuli that are properly embedded in G_n and meet both $\mathrm{Fr}(V_{n+1})$ and $\mathrm{Fr}(V_n)$, and suppose that each component of $A_n \cap B_n$ is a circle in $G_n - \partial G_n$ that is noncontractible in G_n. Note that $P \cap V_1$ and $Q \cap V_1$ are necessarily disks.

LEMMA 4. *Suppose that $L \subset W$ is compact. There is a compact, connected 3-manifold $M \subset W$ such that $V_1 \cup L \subset M - \mathrm{Fr}(M)$, $P \cap M$ and $Q \cap M$ are both disks, and M can be compressed in $W - V_1$ by removing 1-handles that miss $P \cup Q$.*

Remark. The proof that follows is a modification of the first two paragraphs of the proof of Lemma 3 of [W], which itself owes much to Lemma 1.1 of [BF].

Proof of Lemma 4. Note that, for $n \in \mathbb{N}$, $P \cap V_n$ and $Q \cap V_n$ are both single disks.

Let M be a compact, connected 3-manifold with $V_1 \cup L \subset M - \mathrm{Fr}(M)$ such that $P \cap M$ and $P \cap Q$ are single disks, and $P \cap \mathrm{cl}(M - V_1)$ and $Q \cap \mathrm{cl}(M - V_1)$ are annuli that are incompressible in $\mathrm{cl}(M - V_1)$, meet both $\mathrm{Fr}(M)$ and $\mathrm{Fr}(V_1)$, and intersect one another in circles that are noncontractible in $\mathrm{cl}(M - V_1)$. So far $M = V_m$ for some $m \ge 2$ would satisfy these conditions.

Let $H_1, \ldots, H_\nu$ be a compressing sequence of 1-, 2-, and 3-handles for M which miss $P \cup Q$. By Lemma 1, such a compressing sequence exists. Choose M so that $H_1, \ldots, H_\nu$ has the fewest possible 2-handles. The rest of the proof proceeds as in the latter part of the proof of Lemma 3. ∎

Proper homotopies between planes. Beginning now and for the rest of the paper, let $f : \mathbb{R}^2 \times I \to W$ be a proper map such that $f|\mathbb{R}^2 \times i$ is an embedding for $i \in \partial I$, and $f(\mathbb{R}^2 \times 0) = P$ and $f(\mathbb{R}^2 \times 1) = Q$.

For this section, suppose that $K \subset W$ is a compact, connected 3-manifold that traps both P and Q and meets P in a single disk. Also assume that no component of $\mathrm{Fr}(K)$ is a 2-sphere. Since K traps P, it can be argued that there is no 3-ball $B \subset W$ with $K \subset B - \partial B$. It now follows that $W - K$ is irreducible and that $\pi_2(W - K) = 0$. We may also argue that $P - K$ is incompressible in $W - K$.

Let $\Delta \subset \mathbb{R}^2$ be a disk and let $\Lambda = \mathrm{cl}(\mathbb{R}^2 - \Delta)$; then Λ is homeomorphic to $\mathbb{S}^1 \times [0, \infty)$, a half open annulus. Since f is proper, we may choose Δ so that $f(\Lambda \times I) \subset W - K$. It follows that $P \cap K \subset f(\Delta \times 0) - f(\partial\Delta \times 0)$. So $f(\partial\Lambda \times 0) = f(\partial\Delta \times 0)$ is noncontractible in $W - K$.

Let N be a compact, connected 3-manifold in W such that $K \subset N - \mathrm{Fr}(N)$ and $\mathrm{Fr}(N)$ is in general position with respect to P and Q. Suppose that $f(\Delta \times I) \subset N - \mathrm{Fr}(N)$ and that $P \cap N$ is a single disk. Let M' be a compact 3-manifold obtained from N by removing 1-handles that miss P and are transverse to Q. Suppose that there is a component M of M' such that $K \subset M - \mathrm{Fr}(M)$ and $\mathrm{Fr}(M)$ is incompressible in $W - K$. Let $D = \mathrm{cl}(\mathrm{Fr}(M) - \mathrm{Fr}(N))$. Then each component of D is a disk.

Since $\mathrm{Fr}(M)$ is incompressible in $W - K$, it follows that

$$(*) \qquad \ker(\pi_1(\mathrm{Fr}(M)) \to \pi_1(W - K))$$

is trivial. It is also easy to argue that

$$(**) \qquad \pi_2(\mathrm{Fr}(M)) = \pi_2((W - K) - \mathrm{Fr}(M)) = 0.$$

In Lemma 5, we make use of the techniques used by Hempel in the proof of Lemma 6.5 of [He]. For the convenience of the reader, we reproduce the part of Hempel's language that we need here without proof.

Let $g : \Lambda \times I \to W - K$ be a proper map. We will wish at times in the proof of Lemma 5 to obtain a proper map $g_1 : \Lambda \times I \to W - K$ that agrees with g except on the interior of some closed 3-ball contained in $\Lambda \times I$. These modifications are in the from of three "moves" listed below. We refer the reader to the body of the proof of Lemma 6.5 of [He] for the proof and more specific details of the respective modifications. However, note that $(*)$ and $(**)$ above satisfy all of the algebraic hypotheses that the proof requires.

Without loss of generality, we may assume that $g^{-1}(\mathrm{Fr}(M))$ is a 2-sided, compact 2-manifold. Let F be a component of $g^{-1}(\mathrm{Fr}(M))$.

Move 1. Suppose that F is a 2-sphere that bounds a 3-ball $B' \subset \Lambda \times I$. Then we change the definition of g on a regular neighborhood $B \subset (\Lambda \times I) - \partial(\Lambda \times I)$ of B' to obtain a new map g_1 that agrees with g off $B - \partial B$ and $g_1^{-1}(\mathrm{Fr}(M)) = g^{-1}(\mathrm{Fr}(M)) - B$.

M o v e 2. Suppose that F is a disk that is parallel in $\Lambda \times I$ to a disk in $\partial(\Lambda \times I)$ by a parallelism B'. We may change the definition of g on a regular neighborhood B of B' to obtain a map g_1 that agrees with g off $B - \partial B$ such that $g_1^{-1}(\mathrm{Fr}(M)) = g^{-1}(\mathrm{Fr}(M)) - B$.

M o v e 3. Suppose that there is a disk D in $\Lambda \times I$ with $\partial D = D \cap g^{-1}(\mathrm{Fr}(M)) \subset F$ such that ∂D is contractible in F. Then there is a ball $B \subset (\Lambda \times I) - \partial(\Lambda \times I)$ which contains D and intersects $g^{-1}(\mathrm{Fr}(M))$ precisely in a regular neighborhood of ∂D in F. We may change the definition of g on B to obtain a map g_1 that agrees with g off $B - \partial B$ such that $g_1^{-1}(\mathrm{Fr}(M))$ may be obtained from $g^{-1}(\mathrm{Fr}(M))$ by removing the interior of an annulus regular neighborhood of ∂D in F and capping off the two resulting circles with a pair of disjoint disks in $\Lambda \times I$ which intersect $g^{-1}\mathrm{Fr}(M)$ precisely in the boundaries of the disks.

We will also borrow from Hempel the measure of complexity of maps that he uses. For $i = 2, 1, 0, -1, \ldots$, let $c_i(g)$ be the number of components of $g^{-1}(\mathrm{Fr}(M))$ that have Euler characteristic equal to i. Let $c(g) = (\ldots, c_{-1}, c_0(g), c_1(g), c_2(g))$.

LEMMA 5. (1) *There is a proper map* $g : \Lambda \times I \to W - K$ *such that* $g|\partial(\Lambda \times I) = f|\partial(\Lambda \times I)$ *and if* F *is a component of* $g^{-1}(\mathrm{Fr}(M))$, *then* F *is* 2-*sided*, *is not a* 2-*sphere*, *and the inclusion induced map* $\pi_1(F) \to \pi_1(\Lambda \times I)$ *is injective.*

(2) *There is a proper map* $h : \Lambda \times I \to W - K$ *such that* $h|\Lambda \times \partial I = f|\Lambda \times \partial I$ *and if* F *is a component of* $h^{-1}(\mathrm{Fr}(M))$, *then* F *is* 2-*sided*, *is not a* 2-*sphere*, *the inclusion induced map* $\pi_1(F) \to \pi_1(\Lambda \times I)$ *is injective*, *and no component of* $F \cap (\partial\Lambda \times I)$ *is a circle.*

P r o o f. We mimic the proof of Lemma 6.5 of [He] with obvious modifications.

To prove part (1), let $g : \Lambda \times I \to W - K$ be a proper map such that $g|\partial(\Lambda \times I) = f|\partial(\Lambda \times I)$ and let $F = g^{-1}(\mathrm{Fr}(M))$. We may choose g so that F is 2-sided and g differs from f only by repeated modifications by Move 1 and Move 3. It may be that $g = f$. Choose g so that $c(g)$ is minimal when taken in lexicographic order. It is easy to check that no component F' of F is a 2-sphere and that if F' is a component of F, then $\pi_1(F') \to \pi_1(\Lambda \times I)$ is injective.

To prove part (2), let $h : \Lambda \times I \to W - K$ be a proper map such that $h|\Lambda \times \partial I = f|\Lambda \times \partial I$. Let $F = h^{-1}(\mathrm{Fr}(M))$. We may assume by part (1) that if F' is a component of F, then F' is not a 2-sphere and $\pi_1(F') \to \pi_1(\Lambda \times I)$ is injective. We allow h to differ from f by successive modifications by Move 2. Choose h among such maps so that $\#(F)$ is minimal.

We claim that no component of $F \cap (\partial\Lambda \times I)$ is a circle. To get a contradiction, suppose that J is such a component. Let F_J be the component of F such that $J \subset \partial F_J$. We claim that J is contractible in $\partial\Lambda \times I$. Since $h(\partial\Lambda \times I) \subset N - \mathrm{Fr}(N)$, it follows that $h(J) \subset D$. Hence $h(J)$ is contractible in $W - K$. Since $P - K$ is incompressible in $W - K$, it follows that $h(J)$ is not homotopic in $W - K$ to $P \cap \mathrm{Fr}(N)$. Therefore J is not parallel in $\partial\Lambda \times I$ to $\partial\Lambda \times 0$. It follows that J is contractible in $\partial\Lambda \times I$.

Since J is contractible in $\Lambda \times I$ and since $\pi_1(F_J) \to \pi_1(\Lambda \times I)$ is injective, it follows that F_J is a disk. There is a 3-ball B_J in $\Lambda \times I$ such that $\partial B_J = F_J \cup (B_J \cap (\partial\Lambda \times I))$. We may choose J so that $B_J \cap F = F_J$. By using case (2) of Lemma 6.5 of [He], we may reduce $\sharp(F)$. This is a contradiction. ■

Lemma 6. *Let $g : \Lambda \times I \to W - K$ be a proper map that agrees with f on $\Lambda \times \partial I$. Let $F = g^{-1}(\mathrm{Fr}(M))$. Suppose that F is properly embedded and 2-sided in $\Lambda \times I$. Suppose that if F' is a component of F, then F' is not a 2-sphere and $\pi_1(F') \to \pi_1(\Lambda \times I)$ is injective.*

(1) *If F' is a component of F, then F' is either a disk or an annulus. Furthermore, at least one component A of F is an annulus that meets $\Lambda \times 0$ in a single circle that is noncontractible in $\Lambda \times 0$.*

(2) *If no component of $F \cap (\partial\Lambda \times I)$ is a circle, then either $g(\partial\Lambda \times I) \subset M - \mathrm{Fr}(M)$ or $\sharp(\partial \mathrm{cl}(Q - f(\Delta \times 1)) \cap D)$ can be reduced by an isotopy of W that is fixed on $K \cup \mathrm{cl}(W - N)$.*

(3) *If $g(\partial\Lambda \times I) \subset M - \mathrm{Fr}(M)$, then $\partial F \subset \Lambda \times \partial I$ and either $F = A$ or $\sharp(Q \cap \mathrm{Fr}(M))$ can be reduced by an isotopy of compact support fixed on $K \cup g(\partial\Lambda \times I)$.*

Proof. To prove (1), let F' be a component of F. Since $\pi_1(\Lambda \times I) = \mathbb{Z}$, it follows that F' is either a disk, annulus, or Möbius band. Note that $\Lambda \times I$ contains no 2-sided Möbius band. Let A be the component of F that contains the unique component of $F \cap (\Lambda \times 0)$. Since $P - K$ is incompressible, it follows that A is not a disk.

To prove (2), suppose that no component of $F \cap (\partial\Lambda \times I)$ is a circle and that $g(\partial\Lambda \times I)$ is not contained in $M - \mathrm{Fr}(M)$. Since $g(\partial\Lambda \times I)$ is not contained in $M - \mathrm{Fr}(M)$, it follows that $F \cap (\partial\Lambda \times I)$ is nonempty. Let α be a component of $F \cap (\partial\Lambda \times I)$. Then α is an arc. Since $g(\partial\Lambda \times 0) \subset M - \mathrm{Fr}(M)$, it follows that $\partial\alpha \subset \partial\Lambda \times 1$. Let $D_\alpha \subset \partial\Lambda \times I$ be the disk that is separated off by α. Let $\beta = D_\alpha \cap (\partial\Lambda \times 1)$ and let $\beta' = g(\beta)$. We may choose α so that $D_\alpha \cap F = \alpha$.

Now $g(\alpha) \subset D$. Let $h = g|D_\alpha$. Then exactly one of the following manifolds contains $h(D_\alpha)$:

(1) $M - K$ or
(2) $\mathrm{cl}(N - M)$.

Let Ω be whichever of these two manifolds contains $h(D_\alpha)$. Let Θ be the result of splitting Ω along $Q \cap \Omega$.

Recall that $D_\alpha \subset \partial\Lambda \times I$. We lose nothing by assuming that g is such that, for some $\varepsilon > 0$, $g(\partial\Lambda \times [1-\varepsilon, 1])$ lies all on one side of a regular neighborhood of Q in W. Consequently, $h^{-1}(Q)$ is the union of β and circles that are in the interior of D_α. We may modify h so that $h^{-1}(Q) = \beta$. Consequently, we may assume that $h(D_\alpha)$ is contained in Θ. Note that $h|\alpha$ is fixed endpoint homotopic in D to an arc. We may therefore assume that $h(\partial D_\alpha)$ is a circle in $\partial\Theta$. Let B be a regular neighborhood of $h(D_\alpha)$ in Θ. Then, by the Loop Theorem, there is a disk $D' \subset B$ such that $\partial D'$ is nontrivial in $B \cap \partial\Theta$. Since $B \cap \partial\Theta$ is an annulus, we may we may assume that $\partial D' = h(\partial D_\alpha)$. Consequently, $D' \cap D = \alpha$ and $D' \cap Q = \beta'$. Let B' be a regular neighborhood of D' in W such that $B' \cap \Omega = B$.

We may use B' to reduce $\sharp(\partial \mathrm{cl}(Q - f(\Delta \times 1)) \cap D)$ by an isotopy of W fixed on $K \cup \mathrm{cl}(W - N)$.

To prove (3), suppose that $g(\partial\Lambda \times I) \subset M - \mathrm{Fr}(M)$. Then $F \cap (\partial\Lambda \times I) = \emptyset$ and so $F - (\Lambda \times 0) \subset \Lambda \times 1$. Now suppose that $F \neq A$. Let G be a component of $F - A$. Then $\partial G \subset \Lambda \times 1$.

Suppose that G is a disk. Then $g(\partial G)$ is contractible in $\mathrm{Fr}(M)$. Therefore there is a disk $G' \subset \mathrm{Fr}(M)$ with $\partial G' = g(\partial G)$. We may assume that $G' \cap Q = \partial G'$. Let G'' be the unique disk in Q with $\partial G'' = \partial G'$. Since $Q - K$ is incompressible in $W - k$, it follows that $G'' \subset W - K$. So $G' \cup G''$ is a 2-sphere in $W - K$ that bounds a 3-ball U' in $W - K$.

On the other hand, suppose that G is an annulus. By standard arguments, there is a parallelism $U \subset \Lambda \times I$ with $\partial U = G \cup (U \cap (\Lambda \times 1))$. Then, by Proposition 5.4 of [Wa], there is a parallelism U' in $W - K$ between $g(U \cap (\Lambda \times 1))$ and a 2-manifold in $\mathrm{Fr}(M)$.

In either case, we may use U' to reduce $\sharp(Q \cap \mathrm{Fr}(M))$ by an isotopy of compact support fixed on $K \cup g(\partial\Lambda \times 1)$. ■

The main theorem

THEOREM 7. *There is an isotopy $q_t : W \to W$ such that $q_1(Q) = P$.*

Proof. By Lemma 1 of [W], there is a compact, connected 3-manifold V_1 that traps both P and Q and is such that $P \cap V_1$ is a single disk and no component of $\mathrm{Fr}(V_1)$ is a 2-sphere. Let $\Delta \subset \mathbb{R}^2$ be a disk and let $\Lambda = \mathrm{cl}(\mathbb{R}^2 - \Delta)$. Choose Δ so that $f(\Lambda \times I)$ is contained in $W - V_1$. By Lemma 3 there is a compact, connected 3-manifold $V_1' \subset W$ with $V_1 \subset V_1' - \mathrm{Fr}(V_1')$ such that $f(\Delta \times I) \subset V_1' - \mathrm{Fr}(V_1')$, $V_1' \cap P$ is a disk, and V_1' can be compressed in $W - V_1$ to a compact 3-manifold X_1' by removing 1-handles that miss P. Let X_1 be the component of X_1' that contains V_1.

Then $V_1 \subset X_1 - \mathrm{Fr}(X_1)$ and $\mathrm{Fr}(X_1)$ is incompressible in $W - V_1$. Let $D = \mathrm{cl}(\mathrm{Fr}(X_1) - \mathrm{Fr}(V_1'))$.

Let h be an isotopy of W that has compact support. Let $\widehat{f} : \mathbb{R}^2 \times I \to W$ be defined by $\widehat{f}(x,t) = h_t(f(x,t))$. Then $\widehat{f}(\mathbb{R}^2 \times 0) = P$; put $\widehat{f}(\mathbb{R}^2 \times 1) = Q_1$. Let $a = \sharp(\partial \mathrm{cl}(Q_1 - \widehat{f}(\varDelta \times 1)) \cap D)$ and let $b = \sharp(Q_1 \cap \mathrm{Fr}(X_1))$. Suppose that h is fixed on V_1 and that $h_t(f(\varDelta \times I)) \subset V_1' - \mathrm{Fr}(V_1')$ for every $t \in I$. Choose h among such isotopies so that (a,b) is minimal in lexicographic order.

Let $g : \varLambda \times I \to W - V_1$ be a proper map that agrees with $\widehat{f}$ on $\varLambda \times \partial I$. Let $F = g^{-1}(\mathrm{Fr}(X_1))$. By Lemma 5 we may choose g so that if F' is a component of F, then F' is not a 2-sphere and the inclusion induced map $\pi_1(F') \to \pi_1(\varLambda \times I)$ is injective, and so that no component of $F \cap (\partial\varLambda \times I)$ is a circle.

By Lemma 6(2) and the minimality of a, it follows that $g(\partial\varLambda \times I) \subset X_1 - \mathrm{Fr}(X_1)$. By Lemma 6(3) and the minimality of b, it follows that F is a single annulus with $F \cap (\varLambda \times i)$ a single circle that is noncontractible in $\varLambda \times i$ for $i \in \partial I$. Therefore $g|F$ is a homotopy in $\mathrm{Fr}(X_1)$ between $P \cap \mathrm{Fr}(X_1)$ and $Q_1 \cap \mathrm{Fr}(X_1)$, which are both single circles. Therefore $P \cap \mathrm{Fr}(X_1)$ is isotopic in $\mathrm{Fr}(X_1)$ to $Q_1 \cap \mathrm{Fr}(X_1)$. Without loss of generality, we may assume that $P \cap \mathrm{Fr}(X_1)$ is parallel to $Q_1 \cap \mathrm{Fr}(X_1)$ in $\mathrm{Fr}(X_1)$ by applying an isotopy of W that is fixed off a product neighborhood of $\mathrm{Fr}(X_1)$.

Let V_2 be a compact, connected 3-manifold that contains V_1' and the support of h. We may choose V_2 so that $P \cap V_2$ is a single disk, so that $\mathrm{Fr}(V_2)$ contains no 2-spheres, and so that V_2 contains any prechosen compact subset of W. As before, we may construct a plane Q_2 that is isotopic to Q_1 by an isotopy of compact support fixed on V_2 and a compact, connected 3-manifold X_2 such that $Q_2 \cap \mathrm{Fr}(X_2)$ and $P \cap \mathrm{Fr}(X_2)$ are single circles that are parallel in $\mathrm{Fr}(X_2)$.

Continuing in this fashion, we may construct an exhaustion X and a plane Q' isotopic to Q so that, for $n \in \mathbb{N}$, $P \cap \mathrm{Fr}(X_n)$ and $Q' \cap \mathrm{Fr}(X_n)$ are single circles that are parallel to one another in $\mathrm{Fr}(X_n)$. Let $h_t' : W \to W$ be the isotopy that takes Q to Q'. Define $f' : \mathbb{R}^2 \times I \to W$ by $f'(x,t) = h_t'(f(x,t))$. To conserve notation, put $f = f'$ and $Q = Q'$.

For $n \in \mathbb{N}$, let $M_n = \mathrm{cl}(X_{n+1} - X_n)$, let $A_n = P \cap M_n$ and $B_n = Q \cap M_n$. By standard arguments, there is an isotopy of M_n fixed on ∂M_n that takes B_n to an annulus B_n' such that each component of $A_n \cap B_n'$ is a circle in $M_n - \mathrm{Fr}(M_n)$ that is noncontractible in M_n. We may compose these isotopies for each $n \in \mathbb{N}$ so that we may assume that each component of $A_n \cap B_n$ is a circle that is noncontractible in M_n.

We may construct another exhaustion Y for W as follows. Let $Y_1 = X_1$. By Lemma 4, there is a compact, connected 3-manifold Y_2 that contains X_2, meets P and Q in single disks whose boundaries are parallel in $\mathrm{Fr}(Y_2)$, and can be compressed in $W - Y_1$ to a compact 3-manifold Z_1' by removing 1-handles that miss both P and Q. Let Z_1 be the component of Z_1' that contains Y_1. Since $P \cap \mathrm{Fr}(Y_2)$ is noncontractible in $W - Y_1$, it follows that none of the compressing 1-handles removed from Y_2 to obtain Z_1' intersected the parallelism in $\mathrm{Fr}(Y_2)$ between $P \cap \mathrm{Fr}(Y_2)$ and $Q \cap \mathrm{Fr}(Y_2)$.

Continuing in the obvious way, we may construct an exhaustion Y for W such that for $n \in \mathbb{N}$,

(1) Y_n is connected,

(2) $P \cap Y_n$ and $Q \cap Y_n$ are disks whose boundaries are parallel in $\mathrm{Fr}(Y_n)$,

(3) Y_{n+1} can be compressed in $W - Y_1$ by removing 1-handles that miss P and Q to obtain Z_n', and

(4) if Z_n is the component of Z_n' that contains Y_1, we may assume that $Z_n \subset Z_{n+1} - \mathrm{Fr}(Z_{n+1})$. (This is because $\mathrm{Fr}(Z_n)$ is incompressible in $W - Y_1$.)

Note that properties (1)–(4) are preserved under the taking of subsequences.

Let $\Delta \subset \mathbb{R}^2$ be a disk and let $\Lambda = \mathrm{cl}(\mathbb{R}^2 - \Delta)$. Choose Δ so that $f(\Lambda \times I) \subset W - Y_1$. By taking a subsequence of Y and the corresponding subsequence of Z, we may assume that $f(\Delta \times I) \subset Y_2 - \mathrm{Fr}(Y_2)$.

Let $n \geq 1$ be an integer. By Lemma 5, there is a map $g : \Lambda \times I \to W - Y_1$ that agrees with $f|\Lambda \times I$ on $\partial(\Lambda \times I)$ such that if $F_n = g^{-1}(\mathrm{Fr}(Z_n))$ and if F' is a component of F_n, then F' is not a 2-sphere and $\pi_1(F') \to \pi_1(\Lambda \times I)$ is injective. By parts (1) and (2) of Lemma 6, it follows that F_n is a single annulus.

It is not difficult to see that we may warp the product structure of $\Lambda \times I$ so that $F_n = J \times I$ for some circle $J \subset \Lambda$. Let Λ' be the closure of the component of $\Lambda - J$ that has noncompact closure. Arguing as before, there is a map $g' : \Lambda' \times I \to \mathrm{cl}(W - Z_n)$ that agrees with $g|\Lambda' \times I$ on $\partial(\Lambda' \times I)$ such that $(g')^{-1}(\mathrm{Fr}(Z_{n+1}))$ is a single annulus F_{n+1}. We may warp the product structure of $\Lambda' \times I$ off $\partial\Lambda' \times I$ so that $F_{n+1} = J' \times I$ for some circle $J' \subset \Lambda'$ that is parallel in Λ to J. Let $A \subset \Lambda$ be the annulus with $\partial A = J \cup J'$. Put $N_n = \mathrm{cl}(Z_{n+1} - Z_n)$ and $M_n = \mathrm{cl}(Z_{n+1} - Y_{n+1})$. Then $g'|A \times I : A \times I \to N_n$ is a homotopy from $P \cap N_n$ to $Q \cap N_n$.

By an isotopy of W fixed off a product neighborhood of $\bigcup_{n\in\mathbb{N}} \mathrm{Fr}(Y_n)$, we may assume that $P \cap \mathrm{Fr}(Y_n) = Q \cap \mathrm{Fr}(Y_n)$. By composing this isotopy with the appropriate homotopies, we retain that $P \cap \mathrm{Fr}(Y_n)$ is homotopic to $Q \cap N_n$ in N_n for every $n \in \mathbb{N}$. By Lemma 2, it follows that $P \cap N_n$ is isotopic in M_n by an isotopy fixed on ∂M_n. By piecing together these isotopies, we may assume that $P \cap \mathrm{cl}(W - Z_1) = Q \cap \mathrm{cl}(W - Z_1)$ and that $P \cap Z_1$ and $Q \cap Z_1$ are disks that share a common boundary. Since W is irreducible, $(P \cap Q) \cap Z_1$ bounds a ball B. Use B to finish isotoping Q onto P. ∎

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