FUNDAMENTA
MATHEMATICAE
146 (1995)

$\mathcal{M}$-rank and meager types

by

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Abstract. Assume T is superstable and small. Using the multiplicity rank $\mathcal{M}$ we find locally modular types in the same manner as U-rank considerations yield regular types. We define local versions of $\mathcal{M}$-rank, which also yield meager types.

0. Introduction. Throughout, T is superstable, small, and we work in $\mathfrak{C} = \mathfrak{C}^{\mathrm{eq}}$. In [Ne1] we defined the multiplicity rank $\mathcal{M}$ and proved that $\mathcal{M}$ has additivity properties similar to those of U-rank. In [Ne2] we defined the notion of meager regular type and proved that every such type is locally modular. It turns out that using $\mathcal{M}$-rank we can produce locally modular regular types of prescribed $\mathcal{M}$-rank. These types are either trivial or meager (the second case always holds when T has $< 2^{\aleph_0}$ countable models).

We use the following notation. If $s(x)$ is a (partial) type over $\mathfrak{C}$, then $[s]$ denotes the class of (partial) types over $\mathfrak{C}$ containing s. For $p \in S(A)$, $\mathrm{St}(p)$ is the set of stationarizations of p over $\mathfrak{C}$, $\mathrm{St}_A(p) = \{r|\mathrm{acl}(A) : r \in \mathrm{St}(p)\}$. For $B \supseteq A$, $S_p(B) = S(B) \cap [p]$ and $S_{p,\mathrm{nf}}(B) = \{q \in S(B) \cap [p] : q$ does not fork over $A\}$. We regard strong types over A as types over $\mathrm{acl}(A)$. We define $\mathrm{Tr}_A(s)$ (the trace of s on A) as the set of types $r(x) \in S(\mathrm{acl}(A))$ consistent with $s(x)$. Thus if $p \in S(A)$ then $\mathrm{Tr}_A(p) = \mathrm{St}_A(p)$. In general, $\mathrm{Tr}_A(s)$ is closed. $\mathrm{Tr}_A(a/B)$ abbreviates $\mathrm{Tr}_A(\mathrm{tp}(a/B))$. If $a \perp\!\!\!\perp B\,(A)$ then sometimes we use $\mathrm{St}_A(a/B)$ to denote $\mathrm{Tr}_A(a/B)$. Also, x_A denotes the tuple of variables x indexed by elements of A. We will often tacitly use the following easy fact.

Fact 0.1. *Assume $A \subseteq B \subseteq C$. Then either $\mathrm{Tr}_A(a/C)$ is open in $\mathrm{Tr}_A(a/B)$ or $\mathrm{Tr}_A(a/C)$ is nowhere dense in $\mathrm{Tr}_A(a/B)$.*

Proof. Suppose $\mathrm{Tr}_A(a/C)$ is not nowhere dense in $\mathrm{Tr}_A(a/B)$. Since $\mathrm{Tr}_A(a/C)$ is closed, this means that for some $\varphi = \varphi(x, c')$ with $c' \in \mathrm{acl}(A)$, $\emptyset \neq [\varphi] \cap \mathrm{Tr}_A(a/B) \subseteq \mathrm{Tr}_A(a/C)$. Thus we can choose a' realizing $\varphi(x, c')$

1991 *Mathematics Subject Classification*: Primary 03C45.

Research supported by KBN grant 2-1103-91-01.

with $a' \equiv a\,(C)$. Now let $r \in \mathrm{Tr}_A(a/C)$. So $r = \mathrm{stp}(a''/A)$ for some $a'' \equiv a\,(C)$. Choose c'' with $a''c'' \equiv a'c'\,(C)$. Clearly, $r \in [\varphi(x,c'')] \cap \mathrm{Tr}_A(a/B) \subseteq \mathrm{Tr}_A(a/C)$. This shows that $\mathrm{Tr}_A(a/C)$ is open in $\mathrm{Tr}_A(a/B)$.

We define the *multiplicity rank* $\mathcal{M}$ on complete types p over finite sets A, with values in $\mathrm{Ord} \cup \{\infty\}$, by the following conditions.

(1) $\mathcal{M}(p) \geq 0$.

(2) $\mathcal{M}(p) \geq \alpha + 1$ iff for some finite $B \supseteq A$ and $q \in S_{p,\mathrm{nf}}(B)$, $\mathcal{M}(q) \geq \alpha$ and $\mathrm{St}(q)$ is nowhere dense in $\mathrm{St}(p)$.

(3) For limit δ, $\mathcal{M}(p) \geq \delta$ iff $\mathcal{M}(p) \geq \alpha$ for every $\alpha < \delta$.

$\mathcal{M}(a/A)$ abbreviates $\mathcal{M}(\mathrm{tp}(a/A))$.

Notice that we have proved in [Ne2] that if $I(T,\aleph_0) < 2^{\aleph_0}$ then $\mathcal{M}(p) < \infty$ for every p.

Suppose P is a closed subset of $S(\mathrm{acl}(A))$ for some finite set A. We say that *forking is meager on* P if for every formula ψ forking over A, the set of types $r \in P$ consistent with ψ is nowhere dense in P, that is, $P \cap \mathrm{Tr}_A(\psi)$ is nowhere dense in P.

Suppose p is a stationary non-trivial regular type. Let φ be a p-simple formula over a finite set A such that the p-weight of φ is 1 and for each $a \in \varphi(\mathfrak{C})$, if $w_p(a/A) > 0$ then $\mathrm{stp}(a/A)$ is regular non-orthogonal to p. Let $P_\varphi = \{r \in S(\mathrm{acl}(A)) \cap [\varphi] : w_p(r) > 0\}$. Assume P_φ is closed in $S(\mathrm{acl}(A))$, and for each $a \in \varphi(\mathfrak{C})$ and $b \subseteq \mathfrak{C}$, if $w_p(a/b) = 0$ then for some formula $\psi(x,y)$ over $\mathrm{acl}(A)$ true of a, b, whenever $\psi(a',b')$ holds then $w_p(a'/b') = 0$. We call any φ with the above properties a *p-formula* (over A). Given p, a p-formula φ over some finite set A exists by [Hr-Sh]. We say that p is *meager* if forking is meager on P_φ. In [Ne2] we show that this definition does not depend on the choice of φ, and also that if p is meager then p is locally modular. A complete regular type p is meager iff every stationarization of p is meager.

In the next lemma we collect the properties of $\mathcal{M}$ we shall use.

Lemma 0.2. (1) $\mathcal{M}(ab/A) \leq \mathcal{M}(a/Ab) \oplus \mathcal{M}(b/A)$.

(2) *If $p \in S(A)$ and $A \subseteq B$ are finite, then $\mathcal{M}(p') = \mathcal{M}(p)$ for some $p' \in S_{p,\mathrm{nf}}(B)$.*

(3) *If $a \perp\!\!\!\perp b\,(A)$ then $\mathcal{M}(a/Ab) + \mathcal{M}(b/A) \leq M(ab/A)$.*

(4) *Assume $A \subseteq B$ and $a \perp\!\!\!\perp B\,(A)$. If $\mathrm{St}(a/B)$ is open in $\mathrm{St}(a/A)$ then $\mathcal{M}(a/B) = \mathcal{M}(a/A)$. If $\mathcal{M}(a/B) = \mathcal{M}(a/A) < \infty$ then $\mathrm{St}(a/B)$ is open in $\mathrm{St}(a/A)$.*

(5) (Symmetry) *If $a \perp\!\!\!\perp b\,(A)$ and $\mathrm{St}(a/Ab)$ is open in $\mathrm{St}(a/A)$ then $\mathrm{St}(b/Aa)$ is open in $\mathrm{St}(b/A)$ and $\mathcal{M}(ab/A) = \mathcal{M}(a/A) \oplus \mathcal{M}(b/A)$.*

Proof. (1)–(3) are proved in [Ne1]. (4) and (5) are easy.

The next theorem is a kind of open mapping theorem. It is an easy consequence of Shelah's finite equivalence relation theorem ([Sh]). However, it has non-trivial applications. In fact, the open mapping theorem of Lascar–Poizat ([Ba]) follows from the finite equivalence relation theorem in a similar way.

THEOREM 0.3. *Assume* $A \subseteq B \subseteq C$ *and* $f : \mathrm{Tr}_B(ab/C) \to \mathrm{Tr}_A(a/C)$ *is restriction to* $\mathrm{acl}(A)$ *and to formulas with free variable* x_a. *Then* f *is an open surjection.*

Proof. Clearly f is a surjection. To show that f is open, let $E(x_{ab}, x'_{ab}) \in \mathrm{FE}(B)$. Let $X = \{[E(x_{ab}, a'b')] \cap \mathrm{Tr}_B(ab/C) : a'b' \equiv ab\,(C)\}$. We see that X is finite, elements of X are closed, C-conjugate, and cover $\mathrm{Tr}_B(ab/C)$. Hence $\mathrm{Tr}_A(a/C) = \bigcup\{f(S) : S \in X\}$ and $f(S)$ is closed for each $S \in X$. It follows that for some (hence every) $S \in X$, $f(S)$ has non-empty interior in $\mathrm{Tr}_A(a/C)$. It is easy to see that in fact $f(S)$ is open in $\mathrm{Tr}_A(a/C)$. This shows that f is open.

In [Ne1] I pointed out that $\mathcal{M}$-rank may be defined in a way similar to the definition of Morley rank. It was puzzling whether there are local versions of $\mathcal{M}$-rank, just as there are local versions of Morley rank. Local Morley ranks measure (type-)definable sets by means of instances of formulas from some (finite) set Δ. A local $\mathcal{M}$-rank should measure $\mathrm{St}(p)$ with respect to some fixed (finite) set $C \subseteq \mathrm{Dom}(p)$. We define some local versions of $\mathcal{M}$-rank in §2. This must be done carefully, so that the resulting rank has the additivity properties of $\mathcal{M}$. We show that these local versions of $\mathcal{M}$-rank can also be used to produce meager types.

1. $\mathcal{M}$-rank and meager types. In this section we show how to produce meager types with the help of $\mathcal{M}$-rank. The next lemma improves Lemma 0.2(2).

LEMMA 1.1. *Assume* $A \subseteq B$ *are finite.*

(1) *If* $\mathrm{Tr}_A(a/B)$ *is open in* $\mathrm{St}_A(a/A)$ *then* $\mathcal{M}(a/B) \geq \mathcal{M}(a/A)$.

(2) *If* $p(x) \in S(A)$ *and* $s(x)$ *is a* (*partial*) *type over* B *and* $\mathrm{Tr}_A(s) \cap \mathrm{St}_A(p)$ *is open in* $\mathrm{St}_A(p)$ *then there is a* $p' \in S_p(B) \cap [s]$ *with* $\mathcal{M}(p') \geq \mathcal{M}(p)$.

Proof. (1) First we prove the following.

(a) For every finite $B' \supseteq A$ with $B' \mathop{\perp\!\!\!\perp} a\,(A)$ there is a B'' with $B'' \equiv B'\,(Aa)$, $B'' \mathop{\perp\!\!\!\perp} B\,(A)$, $a \mathop{\perp\!\!\!\perp} B''\,(B)$ and with $\mathrm{Tr}_{B''}(a/BB'')$ open in $\mathrm{St}_{B''}(a/B'')$.

Given B' as in (a) choose $B^* \equiv B'\,(Aa)$ with $B^* \mathop{\perp\!\!\!\perp} B\,(A)$. Let $p(x) = \mathrm{tp}(a/A)$, $p'(x) = \mathrm{tp}(a/B)$ and $q = \mathrm{tp}(a/B^*)$. Since $\mathrm{Tr}_A(p')$ is open in $\mathrm{St}_A(p)$ and $\mathrm{stp}(a/A) \in \mathrm{Tr}_A(q) \cap \mathrm{Tr}_A(p') \subseteq \mathrm{St}_A(p)$, we see that $\mathrm{Tr}_A(q) \cap \mathrm{Tr}_A(p')$ is non-empty and open in $\mathrm{Tr}_A(q)$. Let $s(x)$ be the type over BB^* saying that $x \mathop{\smile\hskip-0.9em^|\ } B^* \, (B)$. Let $f : \mathrm{St}_{B^*}(q) \to \mathrm{Tr}_A(q)$ be restriction and let $U = f^{-1}(\mathrm{Tr}_A(q) \cap \mathrm{Tr}_A(p'))$. Then U is open in $\mathrm{St}_{B^*}(q)$. Also we have

(b) $$U \subseteq \mathrm{Tr}_{B^*}(p'(x) \cup s(x)).$$

Indeed, let $r \in U$. Then $r|\mathrm{acl}(A) \in \mathrm{Tr}_A(p')$, hence we can choose c realizing $p' \cup r|\mathrm{acl}(A)$, and without loss of generality, $c \mathop{\smile\hskip-0.9em^|\ } B^* \, (B)$. Now, $B^* \mathop{\smile\hskip-0.9em^|\ } B \, (A)$ implies $c \mathop{\smile\hskip-0.9em^|\ } B^* \, (A)$, hence c realizes $r \cup p'(x) \cup s(x)$.

Since T is small, there is a $p'' \in S_q(BB^*) \cap [p' \cup s]$ such that $\mathrm{Tr}_{B^*}(p'') \cap U$ is open in $\mathrm{St}_{B^*}(q)$. By Fact 0.1 we see that $\mathrm{Tr}_{B^*}(p'')$ is open in $\mathrm{St}_{B^*}(q)$. Let a'' realize p''. We see that $a'' \equiv a \, (B)$, $a'' \mathop{\smile\hskip-0.9em^|\ } B^* \, (B)$, $\mathrm{Tr}_{B^*}(a''/BB^*)$ is open in $\mathrm{St}_{B^*}(a''/B^*)$ and $a''B^* \equiv aB'$. Choose B'' with $a''B^* \equiv aB'' \, (B)$. Clearly, B'' satisfies our demands in (a).

Now we prove by induction on α that $\mathcal{M}(a/A) \geq \alpha$ implies $\mathcal{M}(a/B) \geq \alpha$. We check only the successor step. Suppose $\mathcal{M}(a/A) \geq \alpha + 1$. So there is a finite $B' \supseteq A$ with $a \mathop{\smile\hskip-0.9em^|\ } B' \, (A)$, $\mathcal{M}(a/B') \geq \alpha$ and $\mathrm{St}(a/B')$ nowhere dense in $\mathrm{St}(a/A)$. By (a), without loss of generality, $a \mathop{\smile\hskip-0.9em^|\ } B' \, (B)$ and $\mathrm{Tr}_{B'}(a/BB')$ is open in $\mathrm{St}_{B'}(a/B')$. Thus by the inductive hypothesis, $\mathcal{M}(a/BB') \geq \alpha$. To finish, we must show

(c) $$\mathrm{St}(a/BB') \text{ is nowhere dense in } \mathrm{St}(a/B).$$

In the following diagram all the maps α_i are restrictions.

$$\begin{array}{ccc}
\mathrm{St}_{BB'}(a/BB') & \xrightarrow{\alpha_0} & \mathrm{St}_B(a/B) \\
{\scriptstyle\alpha_1}\big\downarrow & & \big\downarrow{\scriptstyle\alpha_2} \\
\mathrm{St}_{B'}(a/B') & \xrightarrow{\alpha_3} & \mathrm{St}_A(a/A)
\end{array}$$

This diagram commutes. Also, $\mathrm{Tr}_A(a/B) = \mathrm{Rng}\,\alpha_2$, hence by Theorem 0.3 and our assumptions, α_2 is open. By (a) and Theorem 0.3 also α_1 is open. $\mathrm{St}_{BB'}(a/BB')$ and $\mathrm{St}_B(a/B)$ are naturally homeomorphic to $\mathrm{St}(a/BB')$ and $\mathrm{St}(a/B)$ respectively. So to prove (c) it suffices to show that $\alpha_0(\mathrm{St}_{BB'}(a/BB'))$ is nowhere dense in $\mathrm{St}_B(a/B)$.

If not, then $\alpha_0(\mathrm{St}_{BB'}(a/BB'))$ is open in $\mathrm{St}_B(a/B)$, hence also $\alpha_2\alpha_0(\mathrm{St}_{BB'}(a/BB'))$ is open in $\mathrm{St}_A(a/A)$. We have

$$\alpha_2\alpha_0(\mathrm{St}_{BB'}(a/BB')) = \alpha_3\alpha_1(\mathrm{St}_{BB'}(a/BB')) \subseteq \alpha_3(\mathrm{St}_{B'}(a/B')).$$

Since $\mathrm{St}(a/B')$ is nowhere dense in $\mathrm{St}(a/A)$, we see that $\alpha_3(\mathrm{St}_{B'}(a/B'))$ is nowhere dense in $\mathrm{St}_A(a/A)$. Hence the more so $\alpha_2\alpha_0(\mathrm{St}_{BB'}(a/BB'))$ is nowhere dense in $\mathrm{St}_A(a/A)$, a contradiction.

(2) By the smallness of T, $S_p(B) \cap [s]$ is countable. The sets $\mathrm{Tr}_A(p')$, $p' \in S_p(B) \cap [s]$, cover $\mathrm{Tr}_A(s) \cap \mathrm{St}_A(p)$. Hence for some $p' \in S_p(B) \cap [s]$, $\mathrm{Tr}_A(p')$ is not nowhere dense in $\mathrm{St}_A(p)$. By Fact 0.1, $\mathrm{Tr}_A(p')$ is open in $\mathrm{St}_A(p)$. Hence (2) follows from (1).

The range of $\mathcal{M}$ is of the form $I \cup \{\infty\}$ or I, for some proper initial segment I of ω_1. Let $\alpha = \omega^\beta$ for some $\beta \in \mathrm{Ord}$, or $\alpha = \beta = \infty$. Assume α is in the range of $\mathcal{M}$. Let γ_β be the minimal γ such that $\mathcal{M}(p) \ge \alpha$ for some type p of ∞-rank γ. Notice that if $\beta' < \beta$ and $\alpha' = \omega^{\beta'}$ then $\gamma_{\beta'} \le \gamma_\beta$.

THEOREM 1.2. *Assume A is finite, $q \in S(A)$, $R_\infty(q) = \gamma_\beta$ and $\mathcal{M}(q) \ge \alpha$. Then q is regular and locally modular. If q is non-trivial then q is meager and for some isolated $q' \in S(A)$, $R_\infty(q') = \gamma_\beta$, $\mathcal{M}(q') \ge \alpha$ and q' is non-orthogonal to q.*

We begin the proof of Theorem 1.2 with two lemmas.

LEMMA 1.3. *q is regular and orthogonal to every type with ∞-rank $< \gamma_\beta$.*

Proof. Suppose not. Then for some finite $B \supseteq A$ and $r \in S(B)$ with $R_\infty(r) < \gamma_\beta$, r is not almost orthogonal to some nf extension q' of q over B. Choose $\psi \in r$ with $R_\infty(\psi) < \gamma_\beta$. Since $r \stackrel{a}{\not\perp} q'$, for some a, b realizing r, q' respectively, $a \not\perp b\,(B)$. This is witnessed by a formula $\delta(x,y)$ over B, true of a, b. Since forking means decreasing some local rank, without loss of generality we have

(a) if $\delta(a',b')$ holds and b' satisfies a nf extension of q over B, then $\psi(a')$ and $a' \not\perp b'\,(B)$.

Let $\delta'(y) = \exists x\, \delta(x,y)$. So $\delta'(y)$ does not fork over A. Let $s(y)$ be the type over B saying:

$$\text{“}\delta'(y) \text{ and } q(y) \text{ and } \mathrm{tp}(y/B) \text{ does not fork over } A.\text{”}$$

We see that $\mathrm{Tr}_A(s)$ is open in $\mathrm{St}_A(q)$. By Lemma 1.1(2) for some $q'' \in S_{q,\mathrm{nf}}(B) \cap [\delta']$, we have $\mathcal{M}(q'') \ge \alpha$. Let b'' realize q'', and choose a'' with $\delta(a'',b'')$. By (a), $R_\infty(b''/Ba'') < \gamma_\beta$ and $R_\infty(a''/B) < \gamma_\beta$ (as $R_\infty(\psi) < \gamma_\beta$). By the choice of γ_β, we have $\mathcal{M}(a''/B), \mathcal{M}(b''/Ba'') < \alpha$. By Lemma 0.2 we have $\mathcal{M}(b''/B) \le \mathcal{M}(a''b''/B) \le \mathcal{M}(a''/B) \oplus \mathcal{M}(b''/Ba'') < \alpha$, a contradiction.

Choose $p' \in S(A)$ non-orthogonal to q, with $R_\infty(p') = \gamma_\beta$, $\mathcal{M}(p') \ge \alpha$ and Cantor–Bendixson rank $\mathrm{CB}(p')$ minimal possible. Choose $\varphi \in p'$ with $R_\infty(\varphi) = \gamma_\beta$, $\mathrm{CB}(p') = \mathrm{CB}(\varphi)$ and CB-multiplicity of φ equal to 1. By Lemma 1.3, p' is regular.

LEMMA 1.4. *p' is orthogonal to every type in $S(A) \cap [\varphi] \setminus \{p'\}$ and forking is meager on $\mathrm{St}_A(p')$.*

Proof. Suppose $r \in S(A) \cap [\varphi] \setminus \{p'\}$ is non-orthogonal to p'. Then for some finite $B \supseteq A$ and a, b realizing over B nf extensions of r, p' respectively, we have $a \not\perp b\,(B)$. As in Lemma 1.3, for some b' realizing p', we have $b' \perp B\,(A)$, $\mathcal{M}(b'/B) \ge \alpha$ and for some $a' \in \varphi(\mathfrak{C}) \setminus p'(\mathfrak{C})$, $a' \not\perp b'\,(B)$.

If $\mathcal{M}(a'/A) \geq \alpha$ and $a' \mathrel{\perp\!\!\!\perp} B\,(A)$, then by the choice of p' and φ, $\mathrm{tp}(a'/A)$ is orthogonal to q, hence to p'. Thus also $\mathrm{tp}(a'/B)$ is orthogonal to p', a contradiction. If $a' \not\mathrel{\perp\!\!\!\perp} B\,(A)$ then $R_\infty(a'/B) < \gamma_\beta$, so again (by Lemma 1.3 applied to $q := p'$), $\mathrm{tp}(a'/B)$ is orthogonal to p'.

Hence we get $\mathcal{M}(a'/A) < \alpha$ and $a' \mathrel{\perp\!\!\!\perp} B\,(A)$. Thus $\mathcal{M}(a'/B) < \alpha$ and $\mathcal{M}(b'/Ba') < \alpha$, because $R_\infty(b'/Ba') < \gamma_\beta$. Hence $\mathcal{M}(b'/B) \leq \mathcal{M}(a'b'/B) \leq \mathcal{M}(a'/B) \oplus \mathcal{M}(b'/Ba') < \alpha$, a contradiction.

To prove the second clause, suppose forking is not meager on $\mathrm{St}_A(p')$. Then for some forking formula $\delta(x)$ over a finite set $B \supseteq A$, $\mathrm{Tr}_A(\delta) \cap \mathrm{St}_A(p')$ is open in $\mathrm{St}_A(p')$. By Lemma 1.1, for some $p'' \in S_{p'}(B) \cap [\delta]$, we have $\mathcal{M}(p'') \geq \mathcal{M}(p') \geq \alpha$. Since p'' forks over B, $R_\infty(p'') < \gamma_\beta$. Hence $\mathcal{M}(p'') < \alpha$, a contradiction.

Now we can conclude the proof of Theorem 1.2. Suppose q is non-trivial. So p' is non-trivial. Let p be any stationarization of p'. We shall prove that φ satisfies the conditions from the definition of a meager type, and forking is meager on P_φ, hence that p is meager. By Lemmas 1.3 and 1.4, every type in $S(A) \cap [\varphi] \setminus \{p'\}$ is hereditarily orthogonal to p. Also, p is orthogonal to any forking extension of p', and every type in $\mathrm{St}_A(p')$ is regular. Hence φ is p-simple of p-weight 1. By the claim in the proof of Lemma 1.6(2) in [Ne2], the set of $r \in \mathrm{St}_A(p')$ such that r is non-orthogonal to p is clopen in $\mathrm{St}_A(p')$. Thus P_φ is clopen in $\mathrm{St}_A(p')$. Since all stationarizations of p' have the same local ranks, as in Lemma 1.3 we see that the p-weight 0 is definable on $\varphi(\mathfrak{C})$. Thus φ is a p-formula. By Lemma 1.4, forking is meager on P_φ, hence p is meager.

Next notice that if some type in P_φ is modular, then every type in P_φ is, since they are all stationarizations of the complete type p'. But then if c realizes one of them, then they all have forking extensions over Ac, contradicting meagerness. It follows that every type in P_φ is not modular. By [Ne2, Corollary 1.8], P_φ is open in $S(\mathrm{acl}(A))$. Hence also $\mathrm{St}_A(p')$ is open in $S(\mathrm{acl}(A))$. This means that p' is isolated, proving the theorem.

Notice that by Lemma 1.4 and [Ne2, Lemma 1.6(1)], if $I(T, \aleph_0) < 2^{\aleph_0}$ then in the proof of Theorem 1.2 the type p' is non-trivial. It follows that if $I(T, \aleph_0) < 2^{\aleph_0}$ then the type q in Theorem 1.2 is non-trivial and meager.

Lemma 1.5. *Let $q \in S(A)$ be as in Theorem 1.2. Assume A' is finite, $q' \in S(A')$ is isolated, $R_\infty(q') = \gamma_\beta$ and q' is non-orthogonal to q. Then $\mathcal{M}(q') \geq \alpha$ and $\mathcal{M}(q) \leq \mathcal{M}(q') \oplus \alpha'$ for some $\alpha' < \alpha$.*

Proof. First we prove that $\mathcal{M}(q) \leq \mathcal{M}(q') \oplus \alpha'$ for some $\alpha' < \alpha$. For some finite $B \supseteq A \cup A'$ there are a, b realizing over B nf extensions of q, q' respectively, with $a \not\mathrel{\perp\!\!\!\perp} b\,(B)$. As in Lemma 1.3, by Lemma 0.2, changing a somewhat we can assume that $\mathcal{M}(a/B) \geq \mathcal{M}(q)$. This change affects b, but since q' is isolated, b still realizes q', and since q is orthogonal to any type with ∞-rank $< \gamma_\beta$, we have $b \perp B\,(A')$. Hence $\mathcal{M}(b/B) \le \mathcal{M}(q')$. Also, $a \not\perp b\,(B)$ yields $\mathcal{M}(a/Bb) < \alpha$. Let $\alpha' = \mathcal{M}(a/Ba)$. Hence we get

$$\mathcal{M}(q) \le \mathcal{M}(a/B) \le \mathcal{M}(ab/B) \le \mathcal{M}(q') \oplus \mathcal{M}(a/Bb) = \mathcal{M}(q') \oplus \alpha'.$$

If $\mathcal{M}(q') < \alpha$, then $\mathcal{M}(q) \le \mathcal{M}(q') \oplus \alpha' < \alpha$, a contradiction.

Suppose $\mathcal{M}(p) = n_1\omega^{\beta_1} \oplus \ldots \oplus n_k\omega^{\beta_k}$, where n_i are finite, $n_1 \ne 0$, and $\beta_1 > \ldots > \beta_t \ge \beta > \beta_{t+1} > \ldots > \beta_k = 0$. We define $\mathcal{M}_\beta(p)$ as $n_1\omega^{\beta_1} \oplus \ldots \oplus n_t\omega^{\beta_t}$. If $\mathcal{M}(p) = \infty$ then $\mathcal{M}_\beta(p) = \infty$. If $\beta = \infty$ then $\mathcal{M}_\beta(p) = 0$ if $\mathcal{M}(p) < \infty$ and $\mathcal{M}_\beta(p) = \infty$ otherwise.

Given β and $\alpha = \omega^\beta$, for isolated q over a finite set such that $R_\infty(q) = \gamma_\beta$ and $\mathcal{M}(q) \ge \alpha$, $\mathcal{M}_\beta(q)$ is determined by the non-orthogonality class of q. The next corollary clarifies the orthogonality relations between the locally modular types q obtained in Theorem 1.2.

Corollary 1.6. *Assume $\alpha = \omega^\beta$, $\alpha' = \omega^{\beta'}$ are in the range of $\mathcal{M}$, A, A' are finite and $q \in S(A)$, $q' \in S(A')$ are isolated with $R_\infty(q) = \gamma_\beta$, $R_\infty(q') = \gamma_{\beta'}$, $\mathcal{M}(q) \ge \alpha$ and $\mathcal{M}(q') \ge \alpha'$. If q, q' are non-orthogonal then $\gamma_\beta = \gamma_{\beta'}$ and $\mathcal{M}_\beta(q) = \mathcal{M}_{\beta'}(q) = \mathcal{M}_\beta(q') = \mathcal{M}_{\beta'}(q')$.*

In the case when T has $< 2^{\aleph_0}$ countable models, we can strengthen Theorem 1.2.

Theorem 1.7. *Assume $I(T, \aleph_0) < 2^{\aleph_0}$. For q and A as in Theorem* 1.2, *we have $\mathcal{M}_\beta(q) = \alpha$.*

Proof. By Theorem 1.2 there is an isolated p over A non-orthogonal to q, with $R_\infty(p) = \gamma_\beta$ and $\mathcal{M}(p) \ge \alpha$. By the claim in [Ne2, Lemma 1.6], extending A by an element of $\mathrm{acl}(A)$, we can assume that all types in $\mathrm{St}_A(p)$ are non-orthogonal. Let φ isolate p over A. By the proof of Theorem 1.2, φ and $P_\varphi = \mathrm{St}_A(p)$ witness that p is meager. By Lemma 1.5 it suffices to prove that $\mathcal{M}(p) < \alpha \oplus \alpha$.

We shall use Corollary 2.15 from [Ne2], which says that in our case, for every finite B extending A, for every a realizing p with $a \perp B\,(A)$, exactly one of the following holds.

(a) For some b realizing p with $b \perp B\,(A)$, $\mathrm{tp}(b/B)$ is isolated and $a \not\perp b\,(A)$.

(b) There are finitely many $a_0, \ldots, a_n$ realizing $r = \mathrm{tp}(a/B)$ such that for every b realizing r, for some $i \le n$ and $b' \underset{s}{\equiv} b\,(A)$, we have $b' \not\perp a_i\,(A)$.

Suppose for a contradiction that $\mathcal{M}(p) \ge \alpha \oplus \alpha$. Let a realize p. We can find a finite set $B \supseteq A$ with $a \perp B\,(A)$ and $\mathcal{M}(a/B) = \alpha$. The proof splits into two cases, depending on which of conditions (a), (b) holds.

Case 1: (a) holds. So choose b realizing p with $\mathrm{tp}(b/B)$ isolated, $b \mathop{\smile\!\!\!\!\!\!\,|}\, B\,(A)$ and $a \not\mathop{\smile\!\!\!\!\!\!\,|}\, b\,(A)$. It is easy to see that $\mathcal{M}(b/B) = \mathcal{M}(p)$, hence $\mathcal{M}(b/B) \geq \alpha \oplus \alpha$. Since $a \not\mathop{\smile\!\!\!\!\!\!\,|}\, b\,(B)$, by the choice of γ_β we have $\mathcal{M}(b/Ba) < \alpha$. Hence $\alpha \oplus \alpha \leq \mathcal{M}(b/B) \leq \mathcal{M}(ab/B) \leq \mathcal{M}(a/B) \oplus \mathcal{M}(b/Ba) < \alpha \oplus \alpha$, a contradiction.

Case 2: (b) holds. Extending B by an element from $\mathrm{acl}(B)$, we can assume that $n = 0$, that is, for every b realizing $r = \mathrm{tp}(a/B)$, for some $b' \underset{s}{\equiv} b\,(A)$, we have $b' \not\mathop{\smile\!\!\!\!\!\!\,|}\, a\,(A)$. In other words, every $s \in \mathrm{St}_B(r)$ has a forking extension over Ba. Since T is countable, there is a single formula $\psi(x)$ over Ba, forking over B, such that the set of $s \in \mathrm{St}_B(r)$ consistent with ψ has non-empty interior in $\mathrm{St}_B(r)$. Without loss of generality, $R_\infty(\psi) < \gamma_\beta$. Since $\mathcal{M}(r) = \alpha$, by Lemma 1.1 there is a type $r' \in S_r(Ba) \cap [\psi]$ with $\mathcal{M}(r') \geq \alpha$. Since $R_\infty(r') < \gamma_\beta$, this contradicts the choice of γ_β.

Corollary 1.8. *Assume* $I(T, \aleph_0) < 2^{\aleph_0}$.

(1) *Assume* $\beta < \beta'$, $\alpha = \omega^\beta$, $\alpha' = \omega^{\beta'}$ *and* α' *is in the range of* $\mathcal{M}$. *Then* $\gamma_{\beta'} > \gamma_\beta + 2$. *Also*, $\gamma_0 \geq 1$ *and* $\gamma_1 \geq \omega$.

(2) *Assume* A *is finite*, $p \in S(A)$ *and* $\mathcal{M}(p) \geq \omega^\beta$. *Then* $R_\infty(p) \geq 1 + 2\beta$.

Proof. (1) Without loss of generality, $\beta' = \beta + 1$. Choose a finite set A and an isolated $p \in S(A)$ with $R_\infty(p) = \gamma_{\beta'}$ and $\mathcal{M}(p) \geq \alpha'$. So p is non-trivial, regular and meager. Without loss of generality, all stationarizations of p are non-orthogonal, and the formula φ isolating p witnesses that p is meager. Again we apply Corollary 2.15 from [Ne2] (see the proof of Theorem 1.7 above). Let a realize p. Since $\beta < \beta'$, for some finite $B \supseteq A$ with $a \mathop{\smile\!\!\!\!\!\!\,|}\, B\,(A)$ we have $\mathcal{M}(a/B) = \alpha \oplus \alpha$. Since $\alpha \oplus \alpha < \alpha'$, as in Theorem 1.7 we see that in Corollary 2.15, case (b) holds, and (without loss of generality) $n = 0$ there. As in Theorem 1.7, we find a forking extension r' of $\mathrm{tp}(a/B)$ over Ba with $\mathcal{M}(r') \geq \alpha \oplus \alpha$. In particular, $R_\infty(r') \geq \gamma_\beta$. In fact, $R_\infty(r') \geq \gamma_\beta + 1$. If not, then $R_\infty(r') = \gamma_\beta$ and $\mathcal{M}(r') \geq \alpha \oplus \alpha$, which contradicts Theorem 1.7. Since r' forks over B, we get $\gamma_{\beta'} \geq R_\infty(r') + 1$. Altogether we get $\gamma_{\beta'} \geq \gamma_\beta + 2$.

We have $\gamma_0 \geq 1$ trivially, and $\gamma_1 \geq \omega$ because by [Ne1], every type with finite U-rank has finite $\mathcal{M}$-rank.

(2) Notice that $R_\infty(p) \geq \gamma_\beta$, and by (1), $\gamma_\beta \geq 1 + 2\beta$.

Corollary 1.8 shows that there is a bound on $\mathcal{M}$-rank, depending on the ∞-rank. Can we improve the bound obtained there? Assuming $I(T, \aleph_0) < 2^{\aleph_0}$, is it true that $\mathcal{M}(p) \leq U(p)$? This is proved in [Ne1] for types with finite U-rank.

Corollary 1.9. *Assume* $I(T, \aleph_0) < 2^{\aleph_0}$, $\beta < \beta'$, $\alpha = \omega^\beta$, $\alpha' = \omega^{\beta'}$ *and* α' *is in the range of* $\mathcal{M}$. *Suppose* A *and* A' *are finite*, $p \in S(A)$, $p' \in S(A')$,

$R_\infty(p) = \gamma_\beta$, $R_\infty(p') = \gamma_{\beta'}$, $\mathcal{M}(p) \geq \alpha$ and $\mathcal{M}(p') \geq \alpha'$. Then p and p' are meager, regular and orthogonal.

To illustrate our ideas we shall give an example of a superstable theory with a meager type p of $\mathcal{M}$-rank ω.

EXAMPLE 1.10. Let $V = {}^{\omega\times\omega}2$ and let $+$ be pointwise addition on V, modulo 2. We think of V as the product $\prod_n {}^{\{n\}\times\omega}2$. We define certain subgroups of V. For $n > 0$ let

$$P_n = \{f \in V : f|n \times n \equiv 0\} \quad \text{and} \quad G_n = \{f \in P_n : f|(\omega \setminus n) \times \omega \equiv 0\}.$$

Clearly, G_n and P_n are subgroups of $(V, +)$ and $G_n \cap P_{n+1} \subseteq G_{n+1}$. Moreover, for each $n > 1$, G_n/G_{n-1} is naturally isomorphic to $({}^{\{n-1\}\times(\omega\setminus n)}2, +)$, and $G_1 \cap P_n$ is isomorphic to ${}^{\{0\}\times(\omega\setminus n)}2$. Thus the mapping $g : {}^{\{n-1\}\times(\omega\setminus n)}2 \to {}^{\{0\}\times(\omega\setminus n)}2$ given by $g(f)((0,k)) = f((n-1,k))$ induces a group isomorphism $f_n : G_n/G_{n-1} \to G_1 \cap P_n$.

Let $M = (V; +, P_n, G_n, f_{n+1})_{0<n<\omega}$ and $T = \mathrm{Th}(M)$. Then T is a superstable 2-dimensional 1-based theory. Up to non-orthogonality, there are two regular types in T: p_0, the principal generic type of M, and p_1, the principal generic type of G_1. Both types are meager, p_1 is weakly minimal, $R_\infty(p_0) = U(p_0) = \omega$. The functions f_n, $n > 1$, ensure that the weakly minimal groups $G_n/G_{n-1}, n > 1$, are non-orthogonal to G_1.

Let $p \in S(\emptyset)$ be the type generated by $\neg P_1(x)$. Thus p is regular non-orthogonal to p_0, $R_\infty(p) = U(p) = \omega$ and $\mathrm{St}_\emptyset(p) = \mathrm{Str}(\emptyset) \cap [\neg P_1(x)]$. Let a realize p and let $\varphi_n(x,a)$ be $G_n(x-a)$, $n > 0$. We see that $\mathrm{Tr}_\emptyset(\varphi_n(x,a)) \subseteq \mathrm{St}_\emptyset(p)$ and $\mathrm{Tr}_\emptyset(\varphi_n(x,a)) \cap [P_{n+1}(x-a)]$ is nowhere dense in $\mathrm{Tr}_\emptyset(\varphi_{n+1}(x,a))$ (the formula $P_{n+1}(x-a)$ is almost over $\emptyset$). This is essentially because in V with the product topology, $G_n \cap P_{n+1}$ is nowhere dense in G_{n+1}.

Let $p_n \in S(a)$ be one of the countably many non-forking extensions of p such that $\mathrm{St}_\emptyset(p_n)$ is an open subset of $\mathrm{Tr}_\emptyset(\varphi_n(x,a))$. Of course every stationarization of p_n is modular. We see that $\mathcal{M}(p_n) = n$ and $\mathcal{M}(p) = \omega$. Also $\gamma_0 = 1$ and $\gamma_1 = \omega$ here.

Notice that γ_0 is the minimal γ such that some type without Morley rank has ∞-rank γ. By Theorem 1.7, if $I(T, \aleph_0) < 2^{\aleph_0}$ then for every type p of ∞-rank $\leq \gamma_0$, $\mathcal{M}(p) \leq 1$, and if $\mathcal{M}(p) = 1$ then p is meager (hence locally modular). This is related to Proposition 1.11 in [Pi]. Pillay proves there (under the few models assumption) that if G is a locally modular superstable group of rank γ_0, without Morley rank, then for every finite A and $a \in G$, if $\mathrm{tp}(a/A)$ is non-isolated then it has finite multiplicity. In general we cannot claim that much. The following example shows that it can happen that a type $p \in S(\emptyset)$ has ∞-rank γ_0, $\mathcal{M}(p) = 1$ and p is non-isolated (in this case necessarily $\gamma_0 \geq \omega$).

Example 1.11. Let $V = {}^{\omega\times\omega}2$ and $+$ be as in Example 1.10. We define some subgroups of V. For $n > 0$ let

$$P_n = \{f \in V : f|\{0\} \times n \equiv 0\},$$
$$G_n = \{f \in V : f|\{0\} \times \omega \equiv 0 \text{ and } f|(\omega \setminus (n+1)) \times \omega \equiv 0\},$$
$$H_n = \{f \in V : f|(\omega \setminus n) \times 1 \equiv 0\}.$$

Thus $G_n \subseteq G_{n+1} \subseteq \bigcap_n P_n$, $H_n \subseteq H_{n+1}$, $[G_{n+1} : G_n]$ is infinite and $[H_{n+1} : H_n] = 2$. Consider the structures $M_0 = (V; +, P_n, G_n)_{n>0}$ and $M_1 = (V; +, H_n)_{n>0}$ and their theories T_0, T_1 respectively. They are 1-based. T_0 is superstable, with ∞-rank ω and $\gamma_0 = \omega$, while T_1 has Morley rank 2 and ∞-rank 2. T_0 and T_1 have countably many countable models. Now we do not have to include into M_0 the functions f_n (as in Example 1.10) since G_n/G_{n-1} are strongly minimal and ω-categorical.

We shall define a structure M with universe $V' = V \times V$. Let E be the equivalence relation on V' defined by $(a, b)E(a', b')$ iff $b = b'$. We can naturally identify V'/E with V. Let $Q_n = \{(a, b) \in V' : a \in H_n\}$. For each $a \in V'$, a/E can be endowed with a structure isomorphic to M_1, uniformly in a. However, we do not want to have in M a full group structure on a/E. Let R be the 4-ary relation on V' defined by: $R((a_0, b_0), (a_1, b_1), (a_2, b_2), (a_3, b_3))$ iff $b_0 = b_1$, $b_2 = b_3$ and $a_1 - a_0 = a_3 - a_2$.

We see that R gives an affine group structure on each a/E, and additionally a group isomorphism between any two E-classes, after fixing single points in them.

Finally, let $M = (V'; E, Q_n, R; +, P_n, G_n)_{n>0}$, where $+, P_n, G_n$ are defined on V'/E via the identification of V'/E with V. Let $T = \mathrm{Th}(M)$. Then T is a superstable 1-based theory with countably many countable models. Here $\gamma_0 = \omega$. Let $p(x) \in S(\emptyset)$ be the type generated by $\{\neg Q_n(x), n < \omega\} \cup \{\neg P_1(x/E)\}$. Clearly p is non-isolated, $R_\infty(p) = \omega$ and $\mathcal{M}(p) = 1$. Also, p is regular and meager. Notice, however, that since M/E is bi-interpretable with M_0, M/E is a regular group of ∞-rank ω, non-orthogonal to p, with meager generics. If a realizes p, then a/E is generic in M/E and $\mathrm{tp}((a/E)/\emptyset)$ is isolated, as promised by Proposition 1.11 in [Pi]. Moreover, $\varphi(x) = (x = x)$ is a p-formula in M, and isolated types are dense in $P_\varphi = \mathrm{Str}(\emptyset) \cap \bigcap_{n>0}[\neg G_n(x/E)]$. This is a good illustration of the phenomena encountered in [Ne2].

2. Localization. There are several motivations for this section. One is that unfortunately possibly not all meager types (even up to non-orthogonality) are obtained by the minimization process from §1. Another is that if, for example, $S(\mathrm{acl}(\emptyset))$ is uncountable (and hence $\mathcal{M}(q) \geq 1$ for some $q \in S(\emptyset)$; remember that T is small), then the minimization procedure from §1 yields some meager (or trivial) $p \in S(A)$ with $\mathcal{M}(p) \geq 1$. However, p may have nothing in common with the rich topological structure of $S(\mathrm{acl}(\emptyset))$, that is, possibly $\mathrm{Tr}_\emptyset(p)$ is finite. My feeling was that we should be able to find a type p as above such that the topological structure of $\mathrm{St}(p)$ is related to $S(\mathrm{acl}(\emptyset))$. In other words, that the complicated structure of $S(\mathrm{acl}(\emptyset))$ really forces the existence of a locally modular type. In this section we will show that this is the case. To do this we need some local versions of $\mathcal{M}$-rank, which we define below. These local versions have many additivity properties of $\mathcal{M}$. Another reason for trying to define local $\mathcal{M}$-ranks is an attempt to find the most basic impact that the structure of $\mathrm{acl}(C)$ (for a given finite $C \subseteq \mathfrak{C}$) has on the structure of the whole theory. So local $\mathcal{M}$-ranks are intended to play the same role in the "theory of multiplicities" as local Morley ranks in the theory of forking. I leave it to the reader to judge how much I succeeded in this attempt.

In this section let C be a fixed finite subset of $\mathfrak{C}$. We define *local rank* $\mathcal{M}_C$ on types $p \in S(A)$ for finite sets $A \supseteq C$, as follows. Suppose $p = \mathrm{tp}(a/A)$. We define $\mathcal{M}_C(p) = \mathcal{M}_C(a/A)$ by the following conditions:

(1) $\mathcal{M}_C(a/A) \geq 0$.

(2) $\mathcal{M}_C(a/A) \geq \alpha + 1$ iff for some finite $B \supseteq A$ with $a \Cup B\,(A)$, $\mathcal{M}_C(a/B) \geq \alpha$ and $\mathrm{Tr}_C(a/B)$ is nowhere dense in $\mathrm{Tr}_C(a/A)$.

(3) $\mathcal{M}_C(a/A) \geq \delta$ for limit δ iff $\mathcal{M}_C(a/A) \geq \alpha$ for each $\alpha < \delta$.

From now on we usually assume that C is contained in the sets of parameters A, B we consider. $\mathcal{M}_C$ has the following properties.

Lemma 2.1. *Assume $C \subseteq A \subseteq B$, A, B are finite.*

(1) $\mathcal{M}_C(a/A) \leq \mathcal{M}(a/A)$.

(2) *If $a \Cup A\,(C)$ then $\mathcal{M}_C(a/A) = \mathcal{M}(a/A)$.*

(3) *If $\mathrm{Tr}_C(b/A) = \mathrm{Tr}_C(a/A)$ then $\mathcal{M}_C(b/A) = \mathcal{M}_C(a/A)$.*

(4) *For each a there is a b with $b \Cup A\,(C)$ such that $\mathrm{Tr}_C(b/A) = \mathrm{Tr}_C(a/A)$, hence $\mathcal{M}(b/A) = \mathcal{M}_C(b/A) = \mathcal{M}_C(a/A)$.*

(5) *If $a \Cup B\,(A)$ and $\mathcal{M}_C(a/A) < \infty$ then $\mathcal{M}_C(a/B) = \mathcal{M}_C(a/A)$ iff $\mathrm{Tr}_C(a/B)$ is open in $\mathrm{Tr}_C(a/A)$.*

(6) *For every a there is a b with $b \equiv a\,(A)$, $b \Cup B\,(A)$ and $\mathcal{M}_C(b/B) = \mathcal{M}_C(a/A)$.*

Proof. (1) We prove by induction on α that $\mathcal{M}_C(a/A) \geq \alpha$ implies $\mathcal{M}(a/A) \geq \alpha$. Let us check the successor step. Suppose $\mathcal{M}_C(a/A) \geq \alpha + 1$. Choose a finite $B \supseteq A$ with $a \Cup B\,(A)$, $\mathcal{M}_C(a/B) \geq \alpha$ and $\mathrm{Tr}_C(a/B)$ nowhere dense in $\mathrm{Tr}_C(a/A)$. Let $f_B : \mathrm{St}_B(a/B) \to \mathrm{Tr}_C(a/B)$, $f_A : \mathrm{St}_A(a/A) \to \mathrm{Tr}_C(a/A)$ and $g : \mathrm{St}_B(a/B) \to \mathrm{Tr}_A(a/B) \subseteq \mathrm{St}_A(a/A)$ be restrictions. By Theorem 0.3, f_A, f_B and g are open. Also, $f_B = g \circ f_A$. It follows that

$\mathrm{Tr}_A(a/B)$ is nowhere dense in $\mathrm{St}_A(a/A)$. Otherwise $\mathrm{Tr}_A(a/B)$ is open in $\mathrm{St}_A(a/A)$, hence $\mathrm{Tr}_C(a/B) = g(\mathrm{Tr}_A(a/B))$ is open in $\mathrm{Tr}_C(a/A)$, a contradiction.

As $\mathrm{Tr}_A(a/B)$ is nowhere dense in $\mathrm{St}_A(a/A)$, and $\mathrm{St}_A(a/A)$ and $\mathrm{St}_B(a/B)$ are canonically homeomorphic to $\mathrm{St}(a/A)$ and $\mathrm{St}(a/B)$ respectively, we see that $\mathrm{St}(a/B)$ is nowhere dense in $\mathrm{St}(a/A)$. By the inductive hypothesis, $\mathcal{M}(a/B) \geq \alpha$, hence by the definition of $\mathcal{M}$, $\mathcal{M}(a/A) \geq \alpha + 1$. The proofs of the other parts of the lemma are equally easy, so we omit them.

Unfortunately, $\mathcal{M}_C$ does not have the additivity properties of $\mathcal{M}$ from Lemma 0.2. One of the reasons for that is as follows. Suppose $b \subseteq A$. Then $\mathrm{St}(ab/A)$ is homeomorphic to $\mathrm{St}(a/A)$, hence $\mathcal{M}(ab/A) = \mathcal{M}(a/A)$. Let $p(x_{ab}) = \mathrm{tp}(ab/A)$. Consider $\mathrm{Tr}_C(a/A) = \mathrm{Tr}_C(p|x_a)$ and $\mathrm{Tr}_C(ab/A) = \mathrm{Tr}_C(p)$. Let $f : \mathrm{Tr}_C(ab/A) \to \mathrm{Tr}_C(a/A)$ be restriction to formulas with free variable x_a. Then f is continuous and open; however, f need not to be injective. In particular, it could happen that $\mathrm{Tr}_C(a/A)$ is finite and $\mathrm{Tr}_C(ab/A)$ is infinite. Then $\mathcal{M}_C(a/A) = 0$ and $\mathcal{M}_C(ab/A) \geq 1$, showing that $\mathcal{M}_C(ab/A) \leq \mathcal{M}_C(a/Ab) \oplus \mathcal{M}_C(b/A)$ fails.

It turns out that the situation described above is the only obstacle preventing $\mathcal{M}_C$ from having the additivity properties of $\mathcal{M}$. We correct this by defining an eventual version of $\mathcal{M}_C$, denoted by $\mathcal{M}_C^{\mathrm{e}}$. It is defined just like $\mathcal{M}_C$, but with (2) replaced by the following condition:

(2′) $\mathcal{M}_C^{\mathrm{e}}(a/A) \geq \alpha + 1$ iff for some finite B, D extending A with $B \subseteq D$ and $a \perp\!\!\!\perp D\,(A)$, we have $\mathcal{M}_C^{\mathrm{e}}(a/D) \geq \alpha$ and $\mathrm{Tr}_C(aB/D)$ is nowhere dense in $\mathrm{Tr}_C(aB/B)$.

Notice that $\mathcal{M}_C^{\mathrm{e}}$ is an eventual version of $\mathcal{M}_C$ just as the Morley rank is an "eventual" version of the Cantor–Bendixson rank. Also, in (2′) we can additionally assume that $\mathrm{Tr}_C(aA/B)$ is open in $\mathrm{Tr}_C(aA/A)$ (by Lemma 2.2(3) below and Theorem 0.3). We can give another, equivalent definition of $\mathcal{M}_C^{\mathrm{e}}$. For finite $B \subseteq \mathfrak{C}$ define an equivalence relation ${\sim}B$ on $S(\mathfrak{C})$ by $r \,{\sim}B\, r'$ iff $r|\mathrm{acl}(C)B = r'|\mathrm{acl}(C)B$. For $X \subseteq S(\mathfrak{C})$ let $X/{\sim}B$ be $\{r/{\sim}B : r \in X\}$. We can equivalently replace (2′) by the following condition:

(2″) $\mathcal{M}_C^{\mathrm{e}}(a/A) \geq \alpha + 1$ iff for some finite B, D extending A, with $B \subseteq D$ and $a \perp\!\!\!\perp D\,(A)$, we have $\mathcal{M}_C^{\mathrm{e}}(a/D) \geq \alpha$ and $\bigcup \mathrm{St}(a/D)/{\sim}B$ is nowhere dense in $\mathrm{St}(a/B)$.

We shall not use (2″), however. If in (2″) we required only that $\bigcup \mathrm{St}(a/D)/{\sim}B$ is nowhere dense in $\mathrm{St}(a/A)$, we would end up with $\mathcal{M}_C^{\mathrm{e}} = \mathcal{M}$. The role of B in (2′) and (2″) is puzzling. It would be natural to require in (2′), (2″) additionally that $\mathrm{St}(a/B)$ is open in $\mathrm{St}(a/A)$. If we did so, however, we might lose the property of $\mathcal{M}_C^{\mathrm{e}}$ that $\mathcal{M}_C^{\mathrm{e}}(p) \geq \mathcal{M}_C^{\mathrm{e}}(p')$ for any nf extension p' of p (see Lemma 2.2(3) below).

LEMMA 2.2. (1) $\mathcal{M}_C(a/A) \le \mathcal{M}^{\mathrm{e}}_C(a/A) \le \mathcal{M}(a/A)$.

(2) *If* $a \Downarrow A\,(C)$ *then* $\mathcal{M}_C(a/A) = \mathcal{M}^{\mathrm{e}}_C(a/A) = \mathcal{M}(a/A)$.

(3) *If* $A \subseteq B$ *and* $a \Downarrow B\,(A)$ *then* $\mathcal{M}^{\mathrm{e}}_C(a/B) \le \mathcal{M}^{\mathrm{e}}_C(a/A)$.

(4) *If* $A \subseteq B$, $a \Downarrow B\,(A)$ *and* $\mathrm{St}(a/B)$ *is open in* $\mathrm{St}(a/A)$ (*which implies* $\mathcal{M}(a/B) = \mathcal{M}(a/A)$), *then* $\mathcal{M}^{\mathrm{e}}_C(a/B) = \mathcal{M}^{\mathrm{e}}_C(a/A)$.

(5) *If* $A \subseteq B$, *then for every* a *there is an* $a' \equiv a\,(A)$ *with* $a' \Downarrow B\,(A)$ *and* $\mathcal{M}^{\mathrm{e}}_C(a'/B) = \mathcal{M}^{\mathrm{e}}_C(a'/A)$. *Also*, *if additionally* $B \subseteq \mathrm{acl}(A)$ *then* $\mathcal{M}^{\mathrm{e}}_C(a/A) = \mathcal{M}^{\mathrm{e}}_C(a/B)$.

(6) *In* (2′) *we can additionally assume that* $D \setminus A \subseteq \mathfrak{C}_=$ ($\mathfrak{C}_=$ *is the "real" sort of* $\mathfrak{C}^{\mathrm{eq}}$).

P r o o f. (1) To show $\mathcal{M}_C(a/A) \le \mathcal{M}^{\mathrm{e}}_C(a/A)$, we prove by induction on α that $\mathcal{M}_C(a/A) \ge \alpha$ implies $\mathcal{M}^{\mathrm{e}}_C(a/A) \ge \alpha$. Let us check the successor step. Assume $\mathcal{M}_C(a/A) \ge \alpha+1$. Choose a finite $B \supseteq A$ with $a \Downarrow B\,(A)$ such that $\mathcal{M}_C(a/B) \ge \alpha$ and $\mathrm{Tr}_C(a/B)$ is nowhere dense in $\mathrm{Tr}_C(a/A)$. It follows that $\mathrm{Tr}_C(aA/B)$ is nowhere dense in $\mathrm{Tr}_C(aA/A)$. By the inductive hypothesis, $\mathcal{M}^{\mathrm{e}}_C(a/B) \ge \alpha$, and by the definition of $\mathcal{M}^{\mathrm{e}}_C$, $\mathcal{M}^{\mathrm{e}}_C(a/A) \ge \alpha+1$. Similarly we prove $\mathcal{M}^{\mathrm{e}}_C(a/A) \le \mathcal{M}(a/A)$.

(2) follows from (1) and Lemma 2.1(2). (3) is easy.

(4) By (3), $\mathcal{M}^{\mathrm{e}}_C(a/B) \le \mathcal{M}^{\mathrm{e}}_C(a/A)$. By Lemma 2.5(1) below, also $\mathcal{M}^{\mathrm{e}}_C(a/A) \le \mathcal{M}^{\mathrm{e}}_C(a/B)$ (and no vicious circle arises).

(5) Since T is small, we can find an $a' \equiv a\,(A)$ with $a' \Downarrow B\,(A)$ and $\mathrm{St}(a'/B)$ open in $\mathrm{St}(a/A)$. Hence (5) follows from (4).

(6) Suppose $\mathcal{M}^{\mathrm{e}}_C(a/D) \ge \alpha$ and $\mathrm{Tr}_C(aB/D)$ is nowhere dense in $\mathrm{Tr}_C(aB/B)$ (for some B, D as in (2′)). Choose finite B', D' with $B' \subseteq D' \subseteq \mathfrak{C}_=$, $B \setminus A \subseteq \mathrm{acl}(B')$, $D \setminus A \subseteq \mathrm{acl}(D')$ and $a \Downarrow D'\,(A)$. As in (4) we can choose B', D' so that $\mathcal{M}^{\mathrm{e}}_C(a/DD') \ge \alpha$ and $\mathrm{Tr}_C(aBB'/DD')$ is nowhere dense in $\mathrm{Tr}_C(aBB'/BB')$. As in (4) and (5), we see that $\mathcal{M}^{\mathrm{e}}_C(a/AD') \ge \alpha$ and $\mathrm{Tr}_C(aAB'/AD')$ is nowhere dense in $\mathrm{Tr}_C(aAB'/AB')$, so we are done.

Notice that Lemma 2.2(4) naturally corresponds to the following property of forking: If $A \subseteq B$ and $U(a/B) = U(a/A)$ then $R(a/B) = R(a/A)$ for any (local) rank R. This shows again a similarity between our treatment of multiplicities and the theory of forking.

To evaluate $\mathcal{M}^{\mathrm{e}}_C(ab/A)$ in terms of $\mathcal{M}^{\mathrm{e}}_C(a/Ab)$ and $\mathcal{M}^{\mathrm{e}}_C(b/A)$ we need the following technical lemma.

LEMMA 2.3. *Assume* $C \subseteq A \subseteq B$ *and* $b \Downarrow B\,(A)$.

(1) *If* $\mathrm{Tr}_C(a/Bb)$ *is open in* $\mathrm{Tr}_C(a/Ab)$ *and* $\mathrm{Tr}_C(bA/B)$ *is open in* $\mathrm{Tr}_C(bA/A)$ *then* $\mathrm{Tr}_C(a/B)$ *is open in* $\mathrm{Tr}_C(a/A)$.

(2) *If* $\mathrm{Tr}_C(ab/B)$ *is open in* $\mathrm{Tr}_C(ab/A)$ *then* $\mathrm{Tr}_C(a/B)$ *is open in* $\mathrm{Tr}_C(a/A)$.

Proof. (1) Let $\varphi(x_a; B, y_b)$ be a formula over B such that for some $E(x_a, x'_a) \in \mathrm{FE}(C)$, for each b', $\varphi(\mathfrak{C}; B, b')$ is a union of some E-classes, and

(a) $$\mathrm{Tr}_C(a/Ab) \cap [\varphi(x_a; B, b)] = \mathrm{Tr}_C(a/Bb).$$

Let $\varphi'(x_a, x_b)$ be a formula over some $c \in \mathrm{acl}(A)$, saying: for $B' \underset{s}{\equiv} B\,(A)$ with $B' \mathrel{\unicode{x2AEB}} x_a x_b\,(A)$, $\varphi(x_a; B', y_b)$ holds. We can also assume that $c \in \mathrm{dcl}(B)$. Now, $\varphi'(x_a, b)$ is equivalent to $\varphi(x_a; B, b)$.

Indeed, $\varphi'(a', b)$ implies that $\varphi(a'; B', b)$ holds for some $B' \mathrel{\unicode{x2AEB}} a'b\,(A)$ with $B' \underset{s}{\equiv} B\,(A)$. This gives $B'b \underset{s}{\equiv} Bb\,(A)$, hence $\varphi(\mathfrak{C}; B, b) = \varphi(\mathfrak{C}; B', b)$, and $\varphi(a'; B, b)$ holds. The other direction is similar.

So we can assume $\varphi = \varphi'$, that is, $\varphi(x_a; B, y_b)$ is both over B and over $c \in \mathrm{dcl}(B) \cap \mathrm{acl}(A)$. We need the following claim.

Claim. *Suppose $C \subseteq A \subseteq B$ and $c \in \mathrm{acl}(b) \cap A$. If $\mathrm{Tr}_C(b/B)$ is open in $\mathrm{Tr}_C(b/A)$ then $\mathrm{Tr}_C(bc/B)$ is open in $\mathrm{Tr}_C(bc/A)$.*

Proof. For $b' \equiv b\,(A)$ let $X_{b'} = \{\mathrm{stp}(b''c/C) : b'' \underset{s}{\equiv} b'\,(C)$ and $b'' \equiv b\,(A)\}$. Since $c \in \mathrm{acl}(b) \cap A$, $X_{b'}$ is finite. Indeed, let $C_{b'} = \{c' : c' \equiv c\,(b')$ and for some $b'' \equiv b\,(A)$, $b''c \underset{s}{\equiv} b'c'\,(C)\}$. Since $c \in \mathrm{acl}(b) \cap A$, $C_{b'}$ is finite and Ab'-definable. Moreover, if $\mathrm{stp}(b''c/C) \in X_{b'}$ then for some c', $\mathrm{stp}(b''c/C) = \mathrm{stp}(b'c'/C)$, and $c' \in C_{b'}$, hence $X_{b'} = \{\mathrm{stp}(b'c'/C) : c' \in C_{b'}\}$. Also, if $b'' \equiv b' \equiv b\,(A)$ then $X_{b''}$ is the b''-copy of $X_{b'}$ over A, hence $|X_{b''}| = |X_{b'}|$.

Similarly, for $b' \equiv b\,(B)$ let

$$\begin{aligned} Y_{b'} &= \{\mathrm{stp}(b''c/C) : b'' \underset{s}{\equiv} b'\,(C) \text{ and } b'' \equiv b\,(B)\} \\ &= \{\mathrm{stp}(b'c'/C) : c' \equiv c\,(b') \text{ and for some } b'' \equiv b\,(B),\ b''c \underset{s}{\equiv} b'c'\,(C)\}. \end{aligned}$$

We see that $\mathrm{stp}(b'c/C) \in Y_{b'} \subseteq X_{b'}$. Notice that

$$\mathrm{Tr}_C(bc/A) = \bigcup\{X_{b'} : b' \equiv b\,(A)\} \quad \text{and} \quad \mathrm{Tr}_C(bc/B) = \bigcup\{Y_{b'} : b' \equiv b\,(B)\}.$$

Choose a formula $\chi(y_{bc})$ over $\mathrm{acl}(C)$ with $X_b \cap [\chi(y_{bc})] = \{\mathrm{stp}(bc/C)\}$. Clearly,

$$\begin{gathered} \text{if } b' \underset{s}{\equiv} b\,(C) \text{ and } b' \equiv b\,(A) \text{ then } X_{b'} = X_b, \quad \text{and} \\ \text{if } b' \underset{s}{\equiv} b\,(C) \text{ and } b' \equiv b\,(B) \text{ then } Y_{b'} = Y_b. \end{gathered}$$

So by compactness there is a formula $\alpha(y_b)$ over $\mathrm{acl}(C)$, true of b, such that:

$$\begin{gathered} \text{if } b' \equiv b\,(A) \text{ and } \alpha(b') \text{ then } X_{b'} \cap [\chi(y_{bc})] \text{ has size 1}, \quad \text{and} \\ \text{if } b' \equiv b\,(B) \text{ and } \alpha(b') \text{ then } Y_{b'} \cap [\chi(y_{bc})] \text{ has size 1}. \end{gathered}$$

Thus if $b' \equiv b\,(B)$ and $\alpha(b')$ then moreover $X_{b'} \cap [\chi(y_{bc})] = Y_{b'} \cap [\chi(y_{bc})]$.

Since $\mathrm{Tr}_C(b/B)$ is open in $\mathrm{Tr}_C(b/A)$, for some $\beta(y_b)$ over $\mathrm{acl}(C)$,

$$\mathrm{Tr}_C(b/B) = \mathrm{Tr}_C(b/A) \cap [\beta(y_b)].$$

Let $\gamma(y_{bc})$ be $\alpha(y_b)\&\beta(y_b)\&\chi(y_{bc})$. So $\gamma(bc)$ holds. To finish, it suffices to show that

$$\mathrm{Tr}_C(bc/A) \cap [\gamma(y_{bc})] \subseteq \mathrm{Tr}_C(bc/B).$$

So suppose $b'c' \equiv bc\,(A)$ and $\gamma(b'c')$ holds (and necessarily $c' = c$). We want to find b'' with $b''c \underset{s}{\equiv} b'c\,(C)$ and $b'' \equiv b\,(B)$. Now, $\beta\,(b')$ yields a $b'' \underset{s}{\equiv} b'\,(C)$ with $b'' \equiv b\,(B)$. So $\alpha(b'')$ holds, which gives that $X_{b''} \cap [\chi(y_{bc})]$ equals $Y_{b''} \cap [\chi(y_{bc})]$ and has size 1. By the definition of $Y_{b''}$, there is a $b^* \underset{s}{\equiv} b''\,(C)$ with $b^* \equiv b\,(B)$ and $\{\mathrm{stp}(b^*c/C)\} = Y_{b''} \cap [\chi(y_{bc})]$. Without loss of generality, $b^* = b''$. Since $b'' \underset{s}{\equiv} b'\,(C)$ and $b'' \equiv b'\,(A)$, we get $X_{b'} = X_{b''}$. So $\mathrm{stp}(b'c/C) \in X_{b'} \cap [\chi(y_{bc})] = X_{b''} \cap [\chi(y_{bc})] = Y_{b''} \cap [\chi(y_{bc})] = \{\mathrm{stp}(b''c/C)\}$. It follows that $b''c \underset{s}{\equiv} b'c\,(C)$ and $b'' \equiv b\,(B)$, hence $\mathrm{stp}(b'c/C) \in \mathrm{Tr}_C(bc/B)$. This proves the claim.

Returning to the proof of the lemma, since $\mathrm{Tr}_C(bA/B)$ is open in $\mathrm{Tr}_C(bA/A)$, and $c \in \mathrm{dcl}(B)$ gives $\mathrm{Tr}_C(bA/B) \subseteq \mathrm{Tr}_C(bA/Ac)$, we deduce that $\mathrm{Tr}_C(bA/B)$ is open in $\mathrm{Tr}_C(bA/Ac)$. Applying the claim to $b := bA$, $A := Ac$ and $B := Bc$, we see that $\mathrm{Tr}_C(bAc/B)$ is open in $\mathrm{Tr}_C(bAc/Ac)$. Moreover, $c \in \mathrm{acl}(A)$ gives that $\mathrm{Tr}_C(bAc/Ac)$ is open in $\mathrm{Tr}_C(bAc/A)$. Altogether we conclude that $\mathrm{Tr}_C(bAc/B)$ is open in $\mathrm{Tr}_C(bAc/A)$. Hence there is a formula $\delta(y_{bAc}; B)$ almost over C such that

(b) $$\mathrm{Tr}_C(bAc/A) \cap [\delta(y_{bAc}; B)] = \mathrm{Tr}_C(bAc/B).$$

Let $S = S_B = \{\mathrm{stp}(a'/C) :$ for some b', $a'b' \equiv ab\,(Ac)$ and $\delta(b'Ac; B)$ and $\varphi(a'; B, b')$ hold$\}$.

Then S_B is a closed subset of $\mathrm{Tr}_C(a/A)$. Since $c \in \mathrm{dcl}(B)$, S_B is definable over B (that is, $\mathrm{Aut}_B(\mathfrak{C})$-invariant), and for $B' \equiv B\,(A)$ we can define $S_{B'}$ as the B'-copy of S_B over A. Since δ and φ are almost over A, the set $X = \{S_{B'} : B' \equiv B\,(A)\}$ is finite and $\mathrm{Tr}_C(a/A) = \bigcup X$. It follows that S is open in $\mathrm{Tr}_C(a/A)$ and $\mathrm{stp}(a/C) \in S$. So to show that $\mathrm{Tr}_C(a/B)$ is open in $\mathrm{Tr}_C(a/A)$ it suffices to prove

(c) $$S \subseteq \mathrm{Tr}_C(a/B).$$

So suppose $\mathrm{stp}(a'/C) \in S$. Thus for some b' we have $a'b' \equiv ab\,(Ac)$ and $\delta(b'Ac; B)$ and $\varphi(a'; B, b')$ hold. We must find a'' with $a'' \underset{s}{\equiv} a'\,(C)$ and $a'' \equiv a\,(B)$. By (b), for some b'' with $b''Ac \underset{s}{\equiv} b'Ac\,(C)$, we have $b'' \equiv b\,(B)$. Hence we can find a_0 and a_1 with

(d) $$a_0b''Ac \underset{s}{\equiv} a'b'Ac\,(C), \qquad ab \equiv a_1b''\,(B).$$

In particular, we have

(e) $$a_0b'' \equiv a'b' \equiv ab \equiv a_1b''\,(Ac).$$

By (a) and (d) we have

(f) $$\mathrm{Tr}_C(a_1/Ab'') \cap [\varphi(x_a; B, b'')] = \mathrm{Tr}_C(a_1/Bb'').$$

By (e), since $\varphi(x_a; B, y_b)$ is over c, we get $\varphi(a_0; B, b'')$. Thus by (e), (f), for some a'' we have $a'' \underset{s}{\equiv} a_0\,(C)$ and $a''b'' \equiv a_1 b''\,(B)$. By (d) we get $a'' \underset{s}{\equiv} a'\,(C)$ and $a'' \equiv a\,(B)$, as needed.

(2) Let $f : \mathrm{Tr}_C(ab/A) \to \mathrm{Tr}_C(a/A)$ be restriction to formulas with free variable x_a. By Theorem 0.3, f is an open surjection and $f(\mathrm{Tr}_C(ab/B)) = \mathrm{Tr}_C(a/B)$. Since $\mathrm{Tr}_C(ab/B)$ is open in $\mathrm{Tr}_C(ab/A)$, we see that $\mathrm{Tr}_C(a/B)$ is open in $\mathrm{Tr}_C(a/A)$.

Corollary 2.4. (1) $\mathcal{M}^{\mathrm{e}}_C(a/A) \le \mathcal{M}^{\mathrm{e}}_C(ab/A) \le \mathcal{M}^{\mathrm{e}}_C(a/Ab) \oplus \mathcal{M}^{\mathrm{e}}_C(b/A)$. *If* $a \mathop{\perp\!\!\!\perp} b\,(A)$ *then also* $\mathcal{M}^{\mathrm{e}}_C(a/Ab) + \mathcal{M}^{\mathrm{e}}_C(b/A) \le \mathcal{M}^{\mathrm{e}}_C(ab/A)$ *and if additionally* $\mathrm{St}(a/Ab)$ *is open in* $\mathrm{St}(a/A)$ *then* $\mathcal{M}^{\mathrm{e}}_C(ab/A) = \mathcal{M}^{\mathrm{e}}_C(a/A) \oplus \mathcal{M}^{\mathrm{e}}_C(b/A)$.

(2) $\mathcal{M}_C(a/A) \le \mathcal{M}_C(ab/A) \le \mathcal{M}_C(ab/Ab) \oplus \mathcal{M}^{\mathrm{e}}_C(b/A)$. *If* $a \mathop{\perp\!\!\!\perp} b\,(A)$ *then* $\mathcal{M}_C(a/Ab) + \mathcal{M}_C(b/A) \le \mathcal{M}_C(ab/A)$.

(3) $\mathcal{M}_C(a/A) \le \mathcal{M}_C(a/Ab) \oplus \mathcal{M}^{\mathrm{e}}_C(b/A)$.

(4) *If* $\mathcal{M}_C(bA/A) = \mathcal{M}_C(a/Ab) = 0$, *then* $\mathcal{M}_C(a/A) = 0$.

Proof. It is standard, and relies on Lemma 2.3 (see also the proofs in [Ne1]). For example we shall prove that

$$\mathcal{M}^{\mathrm{e}}_C(ab/A) \ge \alpha \quad \text{implies} \quad \mathcal{M}^{\mathrm{e}}_C(a/Ab) \oplus \mathcal{M}^{\mathrm{e}}_C(b/A) \ge \alpha.$$

We proceed by induction on α. Suppose $\mathcal{M}^{\mathrm{e}}_C(ab/A) \ge \alpha + 1$. Then for some finite $A' \subseteq B'$ extending A, with $ab \mathop{\perp\!\!\!\perp} B'\,(A)$, we see $\mathrm{Tr}_C(abA'/B')$ is nowhere dense in $\mathrm{Tr}_C(abA'/A')$ and $\mathcal{M}^{\mathrm{e}}_C(ab/B') \ge \alpha$. By the inductive hypothesis, $\mathcal{M}^{\mathrm{e}}_C(a/B'b) \oplus \mathcal{M}^{\mathrm{e}}_C(b/B') \ge \alpha$. By Lemma 2.3(1) (applied to $a := abA'$, $b := b$, $A := A'$, $B := B'$) either $\mathrm{Tr}_C(abA'/B'b)$ is nowhere dense in $\mathrm{Tr}_C(abA'/A'b)$ or $\mathrm{Tr}_C(bA'/B')$ is nowhere dense in $\mathrm{Tr}_C(bA'/A')$. Hence $\mathcal{M}^{\mathrm{e}}_C(a/Ab) \oplus \mathcal{M}^{\mathrm{e}}_C(b/A) \ge \alpha + 1$.

The next lemma strengthens Lemma 1.1.

Lemma 2.5. *Assume* $C \subseteq A \subseteq B$ *are finite.*

(1) *If* $\mathrm{Tr}_A(a/B)$ *is open in* $\mathrm{St}_A(a/A)$ *then* $\mathcal{M}(a/B) \ge \mathcal{M}(a/A)$, $\mathcal{M}_C(a/B) \ge \mathcal{M}_C(a/A)$ *and* $\mathcal{M}^{\mathrm{e}}_C(a/B) \ge \mathcal{M}^{\mathrm{e}}_C(a/A)$.

(2) *Assume* $p(x) \in S(A)$, $s(x)$ *is a (partial) type over* B *and* $\mathrm{Tr}_A(s) \cap \mathrm{St}_A(p)$ *is open in* $\mathrm{St}_A(p)$. *Then there is* $p' \in S_p(B) \cap [s]$ *with* $\mathcal{M}(p') \ge \mathcal{M}(p)$, $\mathcal{M}_C(p') \ge \mathcal{M}_C(p)$ *and* $\mathcal{M}^{\mathrm{e}}_C(p') \ge \mathcal{M}^{\mathrm{e}}_C(p)$.

Proof. (2) follows from (1) and smallness of T. So it suffices to prove (1). The following condition is proved in Lemma 1.1.

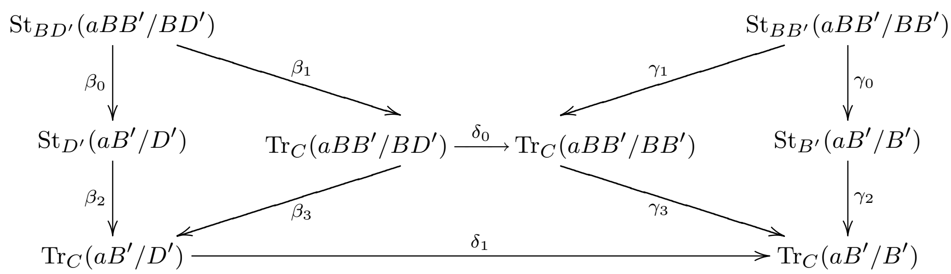
(a) For every finite $B' \supseteq A$ with $B' \mathop{\perp\!\!\!\perp} a\,(A)$ there is a B'' with $B'' \equiv B'\,(Aa)$, $B'' \mathop{\perp\!\!\!\perp} B\,(A)$, $a \mathop{\perp\!\!\!\perp} B''\,(B)$ and with $\mathrm{Tr}_{B''}(a/BB'')$ open in $\mathrm{St}_{B''}(a/B'')$.

We prove that $\mathcal{M}^{\mathrm{e}}_C(a/B) \ge \mathcal{M}^{\mathrm{e}}_C(a/A)$ as in Lemma 1.1. We prove by induction on α that $\mathcal{M}^{\mathrm{e}}_C(a/A) \ge \alpha$ implies $\mathcal{M}^{\mathrm{e}}_C(a/B) \ge \alpha$. We check the successor step.

So suppose $\mathcal{M}^{\mathrm{e}}_C(a/A) \ge \alpha + 1$. Thus there are finite B' and D' with $A \subseteq B' \subseteq D'$, $a \mathrel{\smile\!\!\!\!\!\mid} D'\,(A)$, $\mathcal{M}^{\mathrm{e}}_C(a/D') \ge \alpha$ and $\mathrm{Tr}_C(aB'/D')$ nowhere dense in $\mathrm{Tr}_C(aB'/B')$. Applying (a) twice we can assume additionally that $D' \mathrel{\smile\!\!\!\!\!\mid} B\,(A)$, $a \mathrel{\smile\!\!\!\!\!\mid} D'\,(B)$, $\mathrm{Tr}_{B'}(a/BB')$ is open in $\mathrm{St}_{B'}(a/B')$ and $\mathrm{Tr}_{D'}(a/BD')$ is open in $\mathrm{St}_{D'}(a/D')$. Since $\mathcal{M}^{\mathrm{e}}_C(a/D') \ge \alpha$, by the inductive hypothesis we get $\mathcal{M}^{\mathrm{e}}_C(a/BD') \ge \alpha$. Hence to show that $\mathcal{M}^{\mathrm{e}}_C(a/B) \ge \alpha + 1$ it suffices to prove that

(b) $\mathrm{Tr}_C(aBB'/BD')$ is nowhere dense in $\mathrm{Tr}_C(aBB'/BB')$.

Notice that e.g. $\mathrm{St}_{BD'}(aBB'/BD')$ is naturally homeomorphic to $\mathrm{St}_{BD'}(a/BD')$. Hence it makes sense to speak of restriction from $\mathrm{St}_{BD'}(aBB'/BD')$ to $\mathrm{St}_{D'}(aB'/D')$. In the following diagram all the functions β_i, γ_i are natural restrictions and δ_i's are inclusions.



Notice that this diagram commutes.

Since $\mathrm{Tr}_{D'}(a/BD')$ is open in $\mathrm{St}_{D'}(a/D')$ and $\mathrm{Tr}_{D'}(a/BD') = \mathrm{Rng}\, f$, where $f : \mathrm{St}_{BD'}(a/BD') \to \mathrm{St}_{D'}(a/D')$ is restriction, and by Theorem 0.3, $f : \mathrm{St}_{BD'}(a/BD') \to \mathrm{Tr}_{D'}(a/BD')$ is open, it follows that also $f : \mathrm{St}_{BD'}(a/BD') \to \mathrm{St}_{D'}(a/D')$ is open. It follows that β_0 is open. By Theorem 0.3, β_1 and β_2 are also open. Since the diagram commutes, we see that β_3 is open. Symmetrically, we deduce that all γ_i's are open.

Suppose for a contradiction that $\mathrm{Tr}_C(aBB'/BD')$ is open in $\mathrm{Tr}_C(aBB'/BB')$. Then $\gamma_3\delta_0(\mathrm{Tr}_C(aBB'/BD'))$ is open in $\mathrm{Tr}_C(aB'/B')$. On the other hand, by the choice of B' and D', $\delta_1(\mathrm{Tr}_C(aB'/D'))$ is nowhere dense in $\mathrm{Tr}_C(aB'/B')$. Since the diagram commutes, we get

$$\gamma_3\delta_0(\mathrm{Tr}_C(aBB'/BD')) = \delta_1\beta_3(\mathrm{Tr}_C(aBB'/BD')) \subseteq \delta_1(\mathrm{Tr}_C(aB'/D')).$$

Hence $\gamma_3\delta_0(\mathrm{Tr}_C(aBB'/BD'))$ is nowhere dense in $\mathrm{Tr}_C(aB'/B')$, a contradiction.

We leave the proof that $\mathcal{M}_C(a/B) \ge \mathcal{M}_C(a/A)$ as an exercise.

Now using the properties of $\mathcal{M}^{\mathrm{e}}_C$ proved above we can repeat most of the arguments from §1. Let $\alpha = \omega^\beta$ for some $\beta \in \mathrm{Ord}$ or $\alpha = \beta = \infty$. Assume that $\mathcal{M}(p) \ge \alpha$ for some $p \in S_1(C)$. Let γ_β be the minimal γ such that for some finite $A \supseteq C$ and some $p \in S_1(A)$ of ∞-rank γ we have $\mathcal{M}^{\mathrm{e}}_C(p) \ge \alpha$. The following theorem has the same proof as the corresponding items in §1.

THEOREM 2.6. *Assume $A \supseteq C$ is finite, $q \in S_1(A)$, $R_\infty(q) = \gamma_\beta$ and $\mathcal{M}^{\mathrm{e}}_C(q) \ge \alpha$. Then q is regular and locally modular. If q is non-trivial (which is the case if $I(T,\aleph_0) < 2^{\aleph_0}$) then q is meager and non-orthogonal to some isolated $q' \in S_1(A)$ with $R_\infty(q') = \gamma_\beta$ and $\mathcal{M}^{\mathrm{e}}_C(q') \ge \alpha$. Moreover, for any such q', $\mathcal{M}^{\mathrm{e}}_C(q) \le \mathcal{M}^{\mathrm{e}}_C(q') \oplus \alpha'$ for some $\alpha' < \alpha$ and if $q'' \in S(A'')$ (for some finite $A'' \supseteq C$) is another isolated type non-orthogonal to q, with $R_\infty(q'') = \gamma_\beta$, then $(\mathcal{M}^{\mathrm{e}}_C)_\beta(q'') = (\mathcal{M}^{\mathrm{e}}_C)_\beta(q')$. If $I(T,\aleph_0) < 2^{\aleph_0}$ then $(\mathcal{M}^{\mathrm{e}}_C)_\beta(q) = \alpha$.*

Here $(\mathcal{M}^{\mathrm{e}}_C)_\beta$ is defined analogously to $\mathcal{M}_\beta$ in §1.

Also, assuming $I(T,\aleph_0) < 2^{\aleph_0}$ we see that the types q obtained in Theorem 2.6 for distinct β, β' are orthogonal.

Theorem 2.6 has the advantage over Theorem 1.2 that the locally modular type we get here is strongly related to $S(\mathrm{acl}(C))$. Let us illustrate this by an example. Suppose $p \in S(C)$ is isolated and of infinite multiplicity (that is, $\mathcal{M}(p) \ge 1$). Theorem 2.6 yields a finite set $A \supseteq C$ and a locally modular $q \in S(A)$ of infinite multiplicity extending p. Say, $q = \mathrm{tp}(a/A)$. Still it may happen that $\mathcal{M}_C(a/A) = 0$, that is, $\mathrm{Tr}_C(a/A)$ is finite. By Theorem 2.6, $\mathcal{M}^{\mathrm{e}}_C(q) \ge 1$, that is, for some finite b with $a \Cup b\,(A)$, $\mathrm{Tr}_C(ab/Ab)$ is infinite. Notice that from the point of view of forking and meagerness, the types $\mathrm{tp}(ab/Ab)$ and $\mathrm{tp}(a/A)$ are practically the same.

Corollary 2.4(4) enables us to produce meager types with the use of $\mathcal{M}_C$ in one special case.

THEOREM 2.7. *Assume some $p \in S(C)$ has infinite multiplicity and γ^* is the minimal γ such that for some type $q \in S(A)$ (for some finite $A \supseteq C$), $\mathcal{M}_C(q) > 0$ and $R_\infty(q) = \gamma$. Then any $q \in S(A)$ (for some finite $A \supseteq C$) with $\mathcal{M}_C(q) > 0$ and $R_\infty(q) = \gamma^*$ is locally modular, and if q is non-trivial then q is meager.*

Proof. We give a sketch only. Suppose $q = \mathrm{tp}(a/A)$, $\mathcal{M}_C(q) > 0$ and $R_\infty(q) = \gamma^*$. It is easy to see that q is regular. Suppose q is non-trivial. Notice that for any $q' = \mathrm{tp}(b/A)$ non-orthogonal to q, with $R_\infty(q') = \gamma^*$ and $\mathrm{CB}(q')$ minimal, for some finite $B \supseteq A$ with $b \Cup B\,(A)$ we have $\mathrm{Tr}_C(bB/B) > 0$. Then choose such q' and B with $\mathrm{CB}(bB/B)$ minimal possible, and let $\varphi(x_{bB}, B) \in \mathrm{tp}(bB/B)$ witness the CB-rank of $\mathrm{tp}(bB/B)$ and have CB-multiplicity 1. As in §1 we see that φ also witnesses that $\mathrm{tp}(bB/B)$ is meager. Hence $\mathrm{tp}(b/B)$ and $\mathrm{tp}(b/A)$ are meager and isolated, and q is meager.

Notice that the types p, q in Theorem 2.7 may have different arity. Now suppose p is a fixed stationary meager type. Suppose φ is a p-formula over a finite set A. By [Ne2, Corollary 1.8], after adding to A an element of $\mathrm{acl}(A)$ there is a meager isolated type $q \in S(A) \cap [\varphi]$ such that each stationarization of q is non-orthogonal to p. We cannot claim in general that q is obtained by the minimization process from §1. Thus we cannot say what $\mathcal{M}(q)$ is. Suppose φ' is another p-formula over a finite set A' and $q' \in S(A') \cap [\varphi']$ is another meager isolated type non-orthogonal to p. Then using Corollary 2.4 we can compare $\mathcal{M}(q)$ and $\mathcal{M}(q')$. Let us say that a term ω^β *appears in* γ if $\gamma = \gamma' \oplus \omega^\beta$ for some γ'.

COROLLARY 2.8. *If* $\mathcal{M}(q') < \omega^{\beta'+1}$ *then in* $\mathcal{M}(q)$ *there appears a term* ω^β *for some* $\beta \le \beta'$.

P r o o f. Since q and q' are non-orthogonal, for some finite $C \supseteq A \cup A'$ and a, b realizing over C nf extensions of q, q' respectively, we have $a \not\perp b\,(C)$. As in Lemma 1.3, by Lemma 1.1 we can assume that $\mathcal{M}(a/C) = \mathcal{M}(q)$ and $\mathrm{tp}(a/C)$ is isolated. We also have $\mathcal{M}_C(a/C) = \mathcal{M}(a/C)$ and $\mathcal{M}_C(b/C) = \mathcal{M}(b/C) \le \mathcal{M}(b/A')$. Suppose no term ω^β with $\beta \le \beta'$ appears in $\mathcal{M}(q)$. By Corollary 2.4(3) we have $\mathcal{M}(a/C) \le \mathcal{M}_C(a/Cb) \oplus \mathcal{M}(b/C)$. Since $\mathrm{tp}(a/C)$ is isolated, forking is meager on $\mathrm{St}_C(a/C)$, which implies $\mathcal{M}_C(a/Cb) < \mathcal{M}_C(a/C) = \mathcal{M}(a/C)$. This quickly leads to a contradiction.

For example, Corollary 2.8 gives that if $\mathcal{M}(q) = \omega^\beta$ then $\mathcal{M}(q') > \omega^\beta$.

References

[Ba] J. T. Baldwin, *Fundamentals of Stability Theory*, Springer, 1988.

[Hr-Sh] E. Hrushovski and S. Shelah, *A dichotomy theorem for regular types*, Ann. Pure Appl. Logic 45 (1989), 157–169.

[Ne1] L. Newelski, *A model and its subset*, J. Symbolic Logic 57 (1992), 644–658.

[Ne2] —, *Meager forking*, Ann. Pure Appl. Logic, to appear.

[Pi] A. Pillay, *Certain locally modular regular superstable groups*, preprint, 1992.

[Sh] S. Shelah, *Classification Theory*, 2nd ed., North-Holland, 1990.

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Received 5 May 1993;
in revised form 29 April 1994