FUNDAMENTA
MATHEMATICAE
146 (1995)

Rational Hopf G-spaces with two nontrivial homotopy group systems

by

Ryszard Doman (Poznań)

Abstract. Let G be a finite group. We prove that every rational G-connected Hopf G-space with two nontrivial homotopy group systems is G-homotopy equivalent to an infinite loop G-space.

1. Introduction. It is known that a rational H-space X of the homotopy type of a connected CW-complex is homotopy equivalent to a weak product of Eilenberg–MacLane spaces [3], and thus to an infinite loop space. In this paper we study the question of whether there is an analogous result for X admitting a finite group action compatible with the H-structure.

Let G be a finite group. G-spaces, G-maps and G-homotopies considered in this paper will be pointed. We shall work in the category of G-spaces having the G-homotopy type of G-CW-complexes [1]. In case of need, we shall tacitly replace G-spaces by their G-CW-substitutes. We shall also assume all G-spaces to be G-connected in the sense that all the fixed point spaces X^H are connected for all subgroups H of G.

Definition. A *Hopf G-space* is a Hopf space X on which G acts in such a way that the multiplication $m : X \times X \to X$ is G-equivariant, and the composite $X \vee X \subset X \times X \xrightarrow{m} X$ is G-homotopic to the folding map.

For example, if Y is a G-space, then the loop space ΩY is a Hopf G-space, where the action of G is defined by $(gf)(t) = g(f(t))$.

Let X be a G-simple G-space (i.e. each X^H is simple). We shall call X *rational* if the homotopy groups $\pi_i(X^H)$ are $\mathbb{Q}$-vector spaces for each subgroup H of G. Note that, by [4], every G-simple G-space, in particular Hopf G-space, can be rationalized. Moreover, in the latter case, the resulting G-space is a Hopf G-space.

1991 *Mathematics Subject Classification*: Primary 55P45, 55P62, 55P91.
Supported by Polish scientific grant RP.I.10.

Let O_G be the category of canonical orbits of G and G-maps between them. A *coefficient system* for G is a contravariant functor from O_G to the category of abelian groups. We shall call a coefficient system *rational* if its range is the category of $\mathbb{Q}$-vector spaces. For a G-space X, the homotopy and homology group systems $\underline{\pi}_n(X)$ and $\underline{\widetilde{H}}_n(X)$ are defined by

$$\underline{\pi}_n(X)(G/H) = \pi_n(X^H), \quad \underline{\widetilde{H}}_n(X)(G/H) = \widetilde{H}_n(X^H),$$

where $\widetilde{H}_n$ denotes the reduced singular homology with $\mathbb{Z}$-coefficients.

Given a coefficient system M for G and an integer $m \geq 1$, an *Eilenberg–MacLane G-space* of type (M, n) is a G-space X such that $\underline{\pi}_m(X) = M$ and $\underline{\pi}_q(X) = 0$ for $q \neq m$. Such G-spaces always exist and are unique up to G-homotopy equivalence [1].

We shall call a G-space X^0 an *infinite loop G-space* if there exist a sequence $X^0, X^1, X^2, \ldots$ of G-spaces and a sequence $f_n : X^n \to \Omega X^{n+1}$, $n \geq 0$, of G-homotopy equivalences.

Eilenberg–MacLane G-spaces ere examples of infinite loop G-spaces. Moreover, using the G-obstruction argument [1], it can be shown that a G-Hopf structure on an Eilenberg–MacLane G-space is unique up to G-homotopy. We shall denote an Eilenberg–MacLane G-space of type (M, m) by $K(M, m)$. Recall also that Eilenberg–MacLane G-spaces represent Bredon cohomology [1]: $\widetilde{H}^m_G(X, M) = [X, K(M, m)]_G$, where $[\ ,\ ]_G$ denotes the G-homotopy classes of G-maps. The relation between $\widetilde{H}^m_G(X, M)$ and $\underline{\widetilde{H}}_*(X)$ is given by a spectral sequence with $E_2^{p,q} = \operatorname{Ext}^p(\underline{\widetilde{H}}_q(X), M) \Rightarrow \widetilde{H}^{p+q}_G(X, M)$ [1]. We shall refer to it as the *Bredon spectral sequence*.

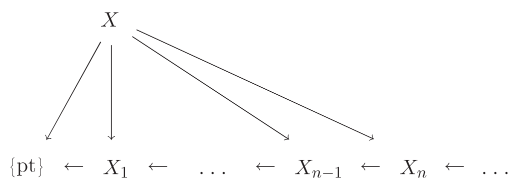
Let $\mathbb{Z}/p^k$ denote the cyclic group of order p^k, where p is prime and k is a positive integer. A theorem by G. Triantafillou [5] states that each rational $\mathbb{Z}/p^k$-connected Hopf $\mathbb{Z}/p^k$-space is $\mathbb{Z}/p^k$-homotopy equivalent to a weak product of Eilenberg–MacLane $\mathbb{Z}/p^k$-spaces, and hence to an infinite loop $\mathbb{Z}/p^k$-space. In contrast to the nonequivariant case, however, rational Hopf G-spaces do not split equivariantly into a product of Eilenberg–MacLane G-spaces in general [5]. Nevertheless, the counterexamples given in [5] are still infinite loop G-spaces. This gives rise to the question of whether every rational Hopf G-space X is G-homotopy equivalent to an infinite loop G-space.

In the present paper we answer the above question affirmatively in the case where X has only two nontrivial homotopy group systems. Thus we prove the following

Theorem. *Let X be a rational G-connected Hopf G-space having only two nontrivial homotopy group systems. Then X is G-homotopy equivalent to an infinite loop G-space.*

2. Equivariant k-invariants of Hopf G-spaces. Let X be a Hopf G-space with a G-multiplication $m : X \times X \to X$ and let N be a coefficient system for G. An element u of $\widetilde{H}^n_G(X, N)$ is called *primitive* if $m^*(u) = p_1^*(u) + p_2^*(u)$ in $\widetilde{H}^n_G(X \times X, N)$, where p_1 and p_2 are the two projections.

Now, let



be an equivariant Postnikov decomposition of X (see [4]). We shall use the following results of [5]:

Proposition 2.1. *Each X_n is a Hopf G-space.*

Proposition 2.2. *The equivariant k-invariant $k^{n+1} \in \widetilde{H}^n_G(X_{n-1}, \underline{\pi}_n(X))$ is primitive for all $n \geq 1$.*

3. Primitive elements in the Bredon cohomology of rational Eilenberg–MacLane G-spaces. For a Hopf G-space Y with a G-multiplication $\mu : Y \times Y \to Y$ and a coefficient system N for G, consider the homomorphism $t^* = \mu^* - p_1^* - p_2^* : \widetilde{H}^n_G(Y, N) \to \widetilde{H}^n_G(Y \times Y, N)$, where p_1 and p_2 are the two projections. Let $\widetilde{H}^n_G(Y, N) = J^{0,n} \supset \ldots \supset J^{n,0} = 0$ and $\widetilde{H}^n_G(Y \times Y, N) = F^{0,n} \supset \ldots \supset F^{n,0} = 0$ be the filtrations corresponding to the Bredon spectral sequences converging to $\widetilde{H}^n_G(Y, N)$ and $\widetilde{H}^n_G(Y \times Y, N)$, respectively. By our assumptions about G-spaces, the G-cellular approximation theorem [1], and the construction of the Bredon spectral sequence, the homomorphism t^* preserves the filtrations, and the induced homomorphism $E^{p,q}_\infty(Y) \to E^{p,q}_\infty(Y \times Y)$ can be identified with the one induced by $\mathrm{Ext}^p(t_*, N) : \mathrm{Ext}^p(\underline{\widetilde{H}}_q(Y), N) \to \mathrm{Ext}^p(\underline{\widetilde{H}}_q(Y \times Y), N)$, where $t_* = \mu_* - p_{1*} - p_{2*} : \underline{\widetilde{H}}_q(Y \times Y) \to \underline{\widetilde{H}}_q(Y)$ is a morphism of coefficient systems. We can summarize the above in

Proposition 3.1. *The homomorphism $t^* : \widetilde{H}^n_G(Y, N) \to \widetilde{H}^n_G(Y \times Y, N)$ is the limit of a morphism of the Bredon spectral sequences whose $E_2^{p,q}$-term is* $\mathrm{Ext}^p(t_*, N) : \mathrm{Ext}^p(\underline{\widetilde{H}}_q(Y), N) \to \mathrm{Ext}^p(\underline{\widetilde{H}}_q(Y \times Y), N)$. We are now going to examine the morphism $t_* : \underline{\widetilde{H}}_q(Y \times Y) \to \underline{\widetilde{H}}_q(Y)$ for Y being a rational Eilenberg–MacLane G-space.

Proposition 3.2. *Let Y be an Eilenberg–MacLane G-space of type (M, m). Then the morphism $t_* : \underline{\widetilde{H}}_q(Y \times Y) \to \underline{\widetilde{H}}_q(Y)$ has a right inverse for each $q \neq m$.*

Proof. For each subgroup H of G, the fixed point space Y^H is a rational Eilenberg–MacLane G-space of type $(M(G/H), m)$. Thus, by [2, Appendix], the Pontryagin algebra $H_*(Y^H)$ is the free graded commutative algebra generated by $\widetilde{H}_m(Y^H)$. In particular, the multiplication $\mu_*^H : H_*(Y^H) \otimes H_*(Y^H) \to H_*(Y^H)$ is a graded algebra homomorphism. Now suppose that $a_1, \ldots, a_k$ belong to $\widetilde{H}_m(Y^H)$, and let $a_1 \ldots a_k \in \widetilde{H}_{km}(Y^H)$ be their product. Let $\Delta^H : Y^H \to Y^H \times Y^H$ be the diagonal map. Since every element of $H_m(Y^H)$ is primitive, we have

$$\begin{aligned}(\mu_*^H - p_{1*}^H - p_{2*}^H)\Delta_*^H(a_1 \ldots a_k) &= \mu_*^H \Delta_*^H(a_1 \ldots a_k) - 2a_1 \ldots a_k \\ &= \mu_*^H((a_1 \otimes 1 + 1 \otimes a_1) \ldots (a_k \otimes 1 + 1 \otimes a_k)) - 2a_1 \ldots a_k \\ &= (2^k - 2)a_1 \ldots a_k.\end{aligned}$$

This implies that $(1/(2^k-2))\Delta_* : \underline{\widetilde{H}}_{km}(Y) \to \underline{\widetilde{H}}_{km}(Y \times Y)$, where $\Delta : Y \to Y \times Y$ is the diagonal, is a right inverse of t_* for each $k \neq 1$. Since $\underline{\widetilde{H}}_q(Y) = \underline{\widetilde{H}}_q(Y \times Y) = 0$ for $q \neq km$, the desired result follows. ∎

For each element $u \in \widetilde{H}_G^n(Y, N)$, define the *weight* $w(u)$ of u to be the greatest lower bound of the integers q such that $u \in J^{n-q,q}$, where $\widetilde{H}_G^n(Y, N) = J^{0,n} \supset \ldots \supset J^{n,0} = 0$ is the filtration corresponding to the Bredon spectral sequence.

PROPOSITION 3.3. *Suppose that Y is an Eilenberg–MacLane G-space of type (M, m), where M is a rational coefficient system for G. Then $w(u) \le m$ for every primitive element u of $\widetilde{H}_G^n(Y, N)$.*

Proof. Let $\{J^{p,n-p}(Y)\}$ and $\{J^{p,n-p}(Y \times Y)\}$ be the filtrations of $\widetilde{H}_G^n(Y, N)$ and $\widetilde{H}_G^n(Y \times Y)$, respectively, which correspond to the Bredon spectral sequences. Suppose that $u \in \widetilde{H}_G^n(Y, N)$ is primitive and set $w(u) = q$. Consider the commutative diagram

$$\begin{array}{ccc} J^{n-q,q}(Y) & \xrightarrow{\ \gamma\ } & E_\infty^{n-q,q}(Y) \\ \downarrow \alpha & & \downarrow \beta \\ J^{n-q,q}(Y \times Y) & \longrightarrow & E_\infty^{n-q,q}(Y \times Y) \end{array}$$

where α is the restriction of $t^* : \widetilde{H}_G^n(Y, N) \to \widetilde{H}_G^n(Y \times Y)$, β is induced by $\mathrm{Ext}^{n-q}(t_*, N) : \mathrm{Ext}^{n-q}(\underline{\widetilde{H}}_q(Y), N) \to \mathrm{Ext}^{n-q}(\underline{\widetilde{H}}_q(Y \times Y), N)$, and γ is the projection. If $w(u) > m$ then, by Proposition 3.2, β is a monomorphism. Thus $\beta\gamma(u) \neq 0$. Consequently, u cannot be primitive. ∎

4. Proof of Theorem. Let X be a rational Hopf G-space having only two nontrivial homotopy group systems $\underline{\pi}_m(X) = M$ and $\underline{\pi}_n(X) = N$, $m < n$. Then X is determined by its equivariant k-invariant$k(X) \in$

$\widetilde{H}_G^{n+1}(K(M,m),N)$, which, by Proposition 2.2, is primitive. The cohomology suspension $\sigma^*:\widetilde{H}_G^{q+1}(K(M,r+1),N)\to\widetilde{H}_G^{q}(K(M,r),N)$, which corresponds to the map $\Omega:[K(M,r+1),K(N,q+1)]_G\to[K(M,r),K(N,q)]_G$, is, by Lemma 3.3 of [5], the limit of a morphism of spectral sequences with E_2-term

$$\mathrm{Ext}^i(\sigma_*,N):\mathrm{Ext}^i(\underline{\widetilde{H}}_{j+1}(K(M,r+1)),N)\to\mathrm{Ext}^i(\underline{\widetilde{H}}_{j}(K(M,r)),N),$$

where $\sigma_*:\underline{\widetilde{H}}_{j}(K(M,r))\to\underline{\widetilde{H}}_{j+1}(K(M,r+1))$ is determined by homology suspension.

In order to prove the Theorem we only need to show that the equivariant k-invariant $k(X)$ belongs to the image of the composite

$$\widetilde{H}_G^{n+k}(K(M,m+k-1),N)\to\widetilde{H}_G^{n+k-1}(K(M,m+k-2),N)\\ \to\ldots\to\widetilde{H}_G^{n+1}(K(M,m),N)$$

of cohomology suspensions for each $k>1$.

By Proposition 3.3, we know that $w(k(X))\le m$. Thus the proof of the Theorem will be completed if we prove the following

PROPOSITION 4.1. *Let* $\widetilde{H}_G^{q+1}(K(M,r+1),N)=F^{0,q+1}\supset\ldots\supset F^{q+1,0}=0$ *and* $\widetilde{H}_G^{q}(K(M,r),N)=J^{0,q}\supset\ldots\supset J^{q+1,0}=0$ *be the filtrations corresponding to the Bredon spectral sequences*, *where* $q\ge n+1$ *and* $r=m+q-n-1$. *Then the cohomology suspension* $\sigma^*:\widetilde{H}_G^{q+1}(K(M,r+1),N)\to\widetilde{H}_G^{q}(K(M,r),N)$ *restricted to* $F^{q-r,r+1}$ *gives an isomorphism* $\widetilde{\sigma}^*:F^{q-r,r+1}\to J^{q-r,r}$.

Proof. Denote by $E_*^{*,*}$ the Bredon spectral sequence converging to $\widetilde{H}_G^{q+1}(K(M,r),N)$, and by $'E_*^{*,*}$ the one converging to $\widetilde{H}_G^{q}(K(M,r),N)$. We have

$$E_2^{q-1,r+1}=\mathrm{Ext}^{q-1}(\underline{\widetilde{H}}_{r+1}(K(M,r+1)),N)$$

and

$$'E_2^{q-r,r}=\mathrm{Ext}^{q-r}(\underline{\widetilde{H}}_{r+1}(K(M,r)),N).$$

Hence $E_\infty^{q-r,r+1}\cong F^{q-r,r+1}$ and $'E_\infty^{q-r,r}\cong J^{q-r,r}$. Under the above identification, $\widetilde{\sigma}^*$ is induced by $\sigma_*:\underline{\widetilde{H}}_{r}(K(M,r))\to\underline{\widetilde{H}}_{r+1}(K(M,r+1))$. Since, evidently, σ_* is an isomorphism, so is $\widetilde{\sigma}^*$. ∎

Remark 4.2. Since we have not used the assumption that the coefficient system N is rational, the conclusion of the Theorem is valid for a Hopf G-space X having only two nontrivial homotopy group systems $\underline{\pi}_m(X)$ and $\underline{\pi}_n(X)$, $m<n$, with $\underline{\pi}_m(X)$ rational.

References

[1] G. E. Bredon, *Equivariant Cohomology Theories*, Lecture Notes in Math. 34, Springer, 1967.

[2] J. W. Milnor and J. C. Moore, *On the structure of Hopf algebras*, Ann. of Math. 81 (1965), 211–264.

[3] H. Scheerer, *On rationalized H- and co-H-spaces*, Manuscripta Math. 51 (1984), 63–87.

[4] G. W. Triantafillou, *Equivariant minimal models*, Trans. Amer. Math. Soc. 274 (1982), 509–532.

[5] —, *Rationalization of Hopf G-spaces*, Math. Z. 182 (1983), 485–500.

FACULTY OF MATHEMATICS AND COMPUTER SCIENCE
ADAM MICKIEWICZ UNIVERSITY
MATEJKI 48/49
60-769 POZNAŃ, POLAND

Received 2 September 1992;
in revised form 15 March 1994