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A theory of non-absolutely convergent integrals in $\mathbb{R}^n$ with singularities on a regular boundary

by

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Abstract. Specializing a recently developed axiomatic theory of non-absolutely convergent integrals in $\mathbb{R}^n$, we are led to an integration process over quite general sets $A \subseteq \mathbb{R}^n$ with a regular boundary. The integral enjoys all the usual properties and yields the divergence theorem for vector-valued functions with singularities in a most general form.

Introduction. Consider an n-dimensional vector field $\vec{v}$ which is differentiable everywhere on $\mathbb{R}^n$. We seek an integration process which integrates $\operatorname{div} \vec{v}$ over reasonable sets A ($\subseteq \mathbb{R}^n$) and expresses the integral $\int_A \operatorname{div} \vec{v}$ in terms of $\vec{v}$ on the boundary ∂A of A in the expected way. While the classical Denjoy–Perron integral (1912/14) solves this problem in dimension one, first solutions in higher dimensions were given for intervals A only in the eighties by [Maw], [JKS], [Pf 1].

More general sets were first discussed in [Jar-Ku 1], where the authors treat compact sets $A \subseteq \mathbb{R}^2$ with a smooth boundary, while in general (see [Jar-Ku 2, 3]) they take $A = \mathbb{R}^n$ and allow certain exceptional points where differentiability is replaced by weaker conditions.

Another approach, involving transfinite induction, is discussed in [Pf 2]. Here BV sets A (e.g., compact sets A with $|\partial A|_{n-1} < \infty$) are treated, and $(n-1)$-dimensional sets are allowed where $\vec{v}$ is only continuous or bounded.

In [Ju-No 1] we introduced a descriptive, axiomatic theory of non-absolutely convergent integrals in $\mathbb{R}^n$ which was specialized in [Ju-No 2] to the relatively simple ν_1-integral over compact intervals. This integral not only enjoys all the usual properties but yields a very general form of the divergence theorem including *exceptional points* where the vector field $\vec{v}$ is not differentiable but still bounded, as well as *singularities* where $\vec{v}$ is not bounded. At these singularities we assume $\vec{v}$ to be of Lipschitz type with a negative exponent

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$\beta > 1 - n$. Countably many types β are allowed, and the set of singularities of type β is assumed to have a finite outer $(\beta + n - 1)$-dimensional Hausdorff measure. Similar singularities were discussed in [Pf 1] but they were restricted to lie on hyperplanes. Also [Jar-Ku 3] discussed singularities, but only at isolated points.

In [Ju-No 3], using the ν_1-theory, we were able to treat this type of singularities in a corresponding divergence theorem on sets $A \in \mathcal{A}$, i.e. compact sets $A \subseteq \mathbb{R}^n$ with $|\partial A|_{n-1} < \infty$ (cf. also [No 1] where general BV sets A are discussed). Here we assumed the singularities to lie in the interior of A since otherwise the integral over ∂A (occurring in the divergence theorem) might not exist.

Imposing suitable *regularity conditions* on ∂A, balancing the magnitude of ∂A against the growth of the vector field, it is possible to relax this assumption. The involved ideas lead to a second specialization of our abstract theory which is presented in this paper. Here we fix an arbitrary set $S \subseteq \mathbb{R}^n$ (the set of potential singularities), and we treat sets $A \in \mathcal{A}$ which satisfy a simple (but very general) local regularity condition at each point $x \in S \cap \partial A$. In particular, the regularity condition is satisfied by any interval. The resulting $\nu(S)$-integral over such sets A again has all the usual properties (as additivity and extension of Lebesgue's integral), and in a corresponding divergence theorem, which in particular generalizes our results in [Ju-No 2, 3], we can now treat on A singularities of the type mentioned above lying in S.

The dependence of our $\nu(S)$-theory on S is as follows: if $S_1 \subseteq S_2$ $(\subseteq \mathbb{R}^n)$ then the $\nu(S_2)$-integral extends the $\nu(S_1)$-integral, and since the ν_1-integral extends any $\nu(S)$-integral all integrals discussed are compatible.

For $S = \emptyset$ and $S = \mathbb{R}^n$ we establish a substitution formula for bilipschitzian transformation maps by verifying the transformation axiom in our abstract theory [Ju-No 1].

Finally, we state without proof a directly constructive definition of the general $\nu(S)$-integral in terms of Riemann sums. The proof is provided in [No 2].

0. Preliminaries. We denote by $\mathbb{R}$ (resp. $\mathbb{R}^+$) the set of all real (resp. all positive real) numbers. Throughout this paper n is a fixed positive integer, and we work in $\mathbb{R}^n$ with the usual inner product $x \cdot y = \sum x_i y_i$ $(x = (x_i), y = (y_i) \in \mathbb{R}^n)$ and the associated norm $\|\cdot\|$. For $x \in \mathbb{R}^n$ and $r > 0$ we set $B(x,r) = \{y \in \mathbb{R}^n : \|x - y\| \leq r\}$.

If $x \in \mathbb{R}^n$ and $E \subseteq \mathbb{R}^n$ we denote by E°, $\overline{E}$, ∂E, $d(E)$ and $\operatorname{dist}(x, E)$ the interior, closure, boundary, diameter of E and the distance from the point x to the set E.

By $|\cdot|_s$ $(0 \leq s \leq n)$ we denote the s-dimensional normalized outer Hausdorff measure in $\mathbb{R}^n$ which coincides for integral s on $\mathbb{R}^s$ $(\subseteq \mathbb{R}^n)$ with the s-dimensional outer Lebesgue measure ($|\cdot|_0$ being the counting measure). Instead of $|\cdot|_{n-1}$ we also write $\mathcal{H}(\cdot)$, and terms like measurable and almost everywhere (a.e.) always refer to the Lebesgue measure $|\cdot|_n$ unless the contrary is stated explicitly. A set $E \subseteq \mathbb{R}^n$ is called *σ_s-finite* if it can be expressed as a countable union of sets with finite s-dimensional outer Hausdorff measure, and E is called an *s-null set* if $|E|_s = 0$.

An interval I in $\mathbb{R}^n$ is always assumed to be compact and non-degenerate.

1. The $\nu(S)$-integral and its basic properties. In this section we specialize the abstract quadruple $\nu = (\mathcal{B}, \mathcal{D}, \dot{\Gamma}, \Gamma)$ occurring in our axiomatic theory ([Ju-No 1]), and obtain a well-behaved n-dimensional integration process over quite general sets. The specialization will depend on an arbitrary set $S \subseteq \mathbb{R}^n$, the set of potential singularities (cf. Thm. 2.1). For the sake of completeness we will restate the basic properties of the associated $\nu = \nu(S)$-integral.

1a. *Definition of $\nu(S) = (\mathcal{B}, \mathcal{D}, \dot{\Gamma}, \Gamma)$.* By $\mathcal{A}$ we denote the system of all compact sets $A \subseteq \mathbb{R}^n$ such that $|\partial A|_{n-1}$ is finite.

Given $\varrho > 0$ we call a set $M \subseteq \mathbb{R}^n$ *ϱ-regulated* if $|B(x,r) \cap M|_{n-1} \le \varrho r^{n-1}$ for any $x \in \mathbb{R}^n$ and any $r > 0$.

Let S be a subset of $\mathbb{R}^n$ and let $\mathcal{A}(S)$ consist of those $A \in \mathcal{A}$ for which there is a $\varrho > 0$ such that for any $x \in S \cap \partial A$ there exists a neighborhood U of x with $U \cap \partial A$ being ϱ-regulated.

For $\varrho > 0$ we denote by $\mathcal{A}'_\varrho$ the system of all $A \in \mathcal{A}$ whose boundary is ϱ-regulated, and we let $\mathcal{A}_\varrho(S)$ consist of all sets $A \in \mathcal{A}(S)$ with $d(A)^n \le \varrho |A|_n$ and $|\partial A|_{n-1} \le \varrho d(A)^{n-1}$.

R e m a r k 1.1. (i) Note that there exists a positive constant ϱ^* ($\ge 2n^n$), depending only on n, such that each cube, i.e. an interval whose sides have equal length, belongs to $\mathcal{A}_{\varrho^*}(S)$, and each interval belongs to $\mathcal{A}'_{\varrho^*}$.

(ii) For any $\varrho > 0$ we have $\mathcal{A}'_\varrho \subseteq \mathcal{A}(S)$, and if $A \in \mathcal{A}'_\varrho$ then $|\partial A|_{n-1} \le (1+\varrho) d(A)^{n-1}$.

(iii) Observe that $\mathcal{A}(\emptyset) = \mathcal{A}$ and $\mathcal{A}(\mathbb{R}^n) = \bigcup_{\varrho > 0} \mathcal{A}'_\varrho$. For, if $A \in \mathcal{A}(\mathbb{R}^n)$ there exists a $\varrho > 0$ such that we can find for any $x \in \partial A$ a neighborhood $U(x)$ with $U(x) \cap \partial A$ being ϱ-regulated. Since ∂A is compact there are finitely many points $x_i \in \partial A$, $1 \le i \le m$, with $\partial A \subseteq \bigcup_{i=1}^m U(x_i)$, and if $x \in \mathbb{R}^n$ and $r > 0$ we see that

$$|B(x,r) \cap \partial A|_{n-1} \le \sum_{i=1}^m |B(x,r) \cap U(x_i) \cap \partial A|_{n-1} \le m\varrho r^{n-1}$$

and thus $A \in \mathcal{A}'_{m\varrho}$.

(iv) If $A, B \in \mathcal{A}(S)$ with corresponding parameters ϱ_A, ϱ_B (according to the definition of $\mathcal{A}(S)$) then $A \cap B$, $A \cup B$, $A - B^\circ \in \mathcal{A}(S)$ with (a possible) corresponding parameter $\varrho_A + \varrho_B$.

In what follows we assume S to be an arbitrary but fixed subset of $\mathbb{R}^n$.

Obviously (use Remark 1.1) $\mathcal{B} = \mathcal{A}(S)$ (resp. $\mathcal{D}(K) = \mathcal{A}_K(S)$ for $K > 0$) is a semi-ring (resp. differentiation class) according to [Ju-No 1, Sec. 1]. $\mathcal{D}$ associates with each positive K the class $\mathcal{D}(K)$.

Let $E \subseteq \mathbb{R}^n$ and $\delta : E \to \mathbb{R}^+$ be given. Then a finite sequence of pairs $\{(x_k, A_k)\}$ with $x_k \in A_k \in \mathcal{B}$, $A_i^\circ \cap A_j^\circ = \emptyset$ $(i \neq j)$, $x_k \in E$ and $d(A_k) < \delta(x_k)$ is called (E, δ)-*fine*. If in addition $E = \bigcup A_k$ we call $\{(x_k, A_k)\}$ a δ-*fine partition of* E.

The *control conditions* we want to use are defined as follows:

For $0 \leq \alpha < n - 1$ the control condition C_1^α (resp. C_2^α) associates with any positive numbers K and Δ the system of all finite sequences $\{A_k\}$ with $A_k \in \mathcal{A}'_K$ such that each $x \in S$ is contained in at most K of the A_k and such that $\sum d(A_k)^\alpha \leq K$ (resp. $\sum d(A_k)^\alpha \leq \Delta$). By $\mathcal{E}(C_1^\alpha)$ (resp. $\mathcal{E}(C_2^\alpha)$) we denote the system of all $E \subseteq S$ with $|E|_\alpha < \infty$ (resp. $|E|_\alpha = 0$).

The condition C_1^{n-1} (resp. C_2^{n-1}) associates with $K, \Delta > 0$ the system of all finite sequences $\{A_k\}$ with $A_k \in \mathcal{B}$ and $\sum |\partial A_k|_{n-1} \leq K$ (resp. $\sum |\partial A_k|_{n-1} \leq \Delta$), and we let $\mathcal{E}(C_1^{n-1})$ (resp. $\mathcal{E}(C_2^{n-1})$) be the system of all $E \subseteq \mathbb{R}^n$ with $|E|_{n-1} < \infty$ (resp. $|E|_{n-1} = 0$).

If $n - 1 < \alpha < n$ the control condition C_1^α (resp. C_2^α) associates with $K, \Delta > 0$ the system of all finite sequences $\{A_k\}$ with $A_k \in \mathcal{D}(K)$ and $\sum d(A_k)^\alpha \leq K$ (resp. $\sum d(A_k)^\alpha \leq \Delta$). $\mathcal{E}(C_1^\alpha)$ (resp. $\mathcal{E}(C_2^\alpha)$) consists of all $E \subseteq \mathbb{R}^n$ with $|E|_\alpha < \infty$ (resp. $|E|_\alpha = 0$).

Finally, the condition C^n associates with any positive K the system of all finite sequences $\{A_k\}$ with $A_k \in \mathcal{D}(K)$, and we let $\mathcal{E}(C^n) = \{E \subseteq \mathbb{R}^n : |E|_n = 0\}$.

Remark 1.2. The requirement that each $x \in S$ lies in at most K of the sets A_k in the definition of C_i^α $(0 \leq \alpha < n - 1)$ will be important when we give an equivalent constructive definition of our integral in terms of Riemann sums. Remember that if the A_k are intervals with disjoint interiors then each $x \in \mathbb{R}^n$ is contained in at most 2^n of them.

Set $\dot{\Gamma} = \{C^n\} \cup \{C_i^\alpha : n - 1 < \alpha < n,\ i = 1, 2\}$ (the requirements $(\dot{\Gamma}_1)$ and $(\dot{\Gamma}_2)$ in [Ju-No 1, Sec. 1] then obviously being satisfied) and $\Gamma = \{C_i^\alpha : 0 \leq \alpha \leq n - 1,\ i = 1, 2\}$ (disjoint from $\dot{\Gamma}$). We will prove that Γ is ordered by the relation $\succeq$ (see [Ju-No 1, Sec. 1]) and that $C^* = C_1^{n-1}$ is a minimal element of Γ. Analogously one then shows that $\dot{\Gamma}$ is ordered.

If $0 \leq \beta < \alpha < n - 1$ then $C_1^\beta \succeq C_2^\alpha$. For, given $K_1 > 0$ we let $K_2 = K_1$ and if $\Delta_2 > 0$ we set $\Delta_1 = \Delta_2$. If $x \in \mathbb{R}^n$ choose $\delta(x) > 0$ such that $\delta(x)^{\alpha-\beta} \leq \Delta_2/K_1$ (this defines $\delta : \mathbb{R}^n \to \mathbb{R}^+$), and let $\{(x_k, A_k)\}$ be any $(\mathbb{R}^n, \delta)$-fine sequence with $\{A_k\} \in C_1^\beta(K_1, \Delta_1)$. Since $\sum d(A_k)^\alpha \leq \sum \delta(x_k)^{\alpha-\beta} d(A_k)^\beta \leq \Delta_2$ we have $\{A_k\} \in C_2^\alpha(K_2, \Delta_2)$.

Furthermore, $C_1^\alpha \succeq C_2^{n-1}$ for $0 \leq \alpha < n - 1$. For, if $K_1 > 0$ set $K_2 = K_1$

and if $\varDelta_2 > 0$ let $\varDelta_1 = 1$. If $x \in \mathbb{R}^n$ we find $\delta(x) > 0$ such that $\delta(x)^{n-1-\alpha} \le \varDelta_2/K_1(1+K_1)$; this defines $\delta : \mathbb{R}^n \to \mathbb{R}^+$. Given any $(\mathbb{R}^n, \delta)$-fine sequence $\{(x_k, A_k)\}$ with $\{A_k\} \in C_1^\alpha(K_1, \varDelta_1)$ and recalling Remark 1.1(ii) we get

$$\begin{aligned}\sum |\partial A_k|_{n-1} &\le (1+K_1)\sum d(A_k)^{n-1} \\ &\le (1+K_1)\sum \delta(x_k)^{n-1-\alpha} d(A_k)^\alpha \le \varDelta_2\end{aligned}$$

and thus $\{A_k\} \in C_2^{n-1}(K_2, \varDelta_2)$.

Obviously $C_2^\alpha \succeq C_1^\alpha$ for $0 \le \alpha \le n-1$, and thus the transitivity property of the relation $\succeq$ shows that $\varGamma$ is ordered. Since $C_2^\alpha \succeq C_1^\alpha \succeq C_2^{n-1} \succeq C_1^{n-1} = C^*$ for $0 \le \alpha < n-1$ we furthermore see that C^* is a minimal element of $\varGamma$ which in addition satisfies conditions $(\varGamma_1)$ and $(\varGamma_2)$ since $\partial A \in \mathcal{E}(C^*)$ and $|A|_n \le d(A)|\partial A|_{n-1}$ for all $A \in \mathcal{A}$.

1b. *Verification of the decomposition and intersection axioms.* Before we can apply the results of our abstract theory it remains to verify the decomposition and intersection axioms ([Ju-No 1, Sec. 2]). The decomposition axiom is a direct consequence of the Decomposition Theorem in [Ju] which we state here in a slightly more general form.

Decomposition Theorem. *Suppose that an n-dimensional interval I is the disjoint union of countably many sets E_m with $|E_m|_{\alpha_m} < \infty$ ($0 \le \alpha_m \le n$) and that positive numbers ε_m and a function $\delta : I \to \mathbb{R}^+$ are given. Then there are finitely many intervals I_k, similar to I, and points x_k such that $\{(x_k, I_k)\}$ is a δ-fine partition of I and*

$$\sum_{x_k \in E_m} d(I_k)^{\alpha_m} \le \frac{c(n)}{r(I)^n}(|E_m|_{\alpha_m} + \varepsilon_m)$$

for all m, where $c(n)$ denotes a positive constant ($\ge n^{n/2}$) and $r(I)$ is the ratio of the smallest and the largest edges of I.

Recall that a *division* of a set $A \subseteq \mathbb{R}^n$ with $|\partial A|_n = 0$ consists of a set $\dot{E}$ and a sequence $(E_i, C_i)_{i\in\mathbb{N}}$ such that $\dot{E} \subseteq A^\circ$, $|A - \dot{E}|_n = 0$, $C_i \in \varGamma \cup \dot{\varGamma}$, $E_i \in \mathcal{E}(C_i)$ and A is the disjoint union of all the sets E_i and $\dot{E}$.

To verify the decomposition axiom let I be any interval in $\mathbb{R}^n$ and denote by $\dot{E}$, $(E_i, C_i)_{i\in\mathbb{N}}$ a division of I. Set $K^* = \varrho^* + (\sqrt{n}/r(I))^n$, where ϱ^* is the constant of Remark 1.1(i), and $K_i^* = K^* + 2nc(n)|E_i|_\alpha/r(I)^n$ (resp. $K_i^* = K^*$) depending on $C_i = C_1^\alpha$ ($0 \le \alpha < n$) (resp. $C_i = C^n$ or $C_i = C_2^\alpha$ ($0 \le \alpha < n$)). Then for any $\varDelta_i > 0$ and $\delta : I \to \mathbb{R}^+$, by the Decomposition Theorem, there is a δ-fine partition $\{(x_k, I_k)\}$ of I with $r(I_k) = r(I)$ and

$$\sum_{x_k\in E_i} d(I_k)^\alpha \le \begin{cases} \dfrac{K^*}{2n} + \dfrac{c(n)}{r(I)^n}|E_i|_\alpha & \text{if } C_i = C_1^\alpha \ (0 \le \alpha < n), \\ \dfrac{\varDelta_i}{2n} & \text{if } C_i = C_2^\alpha \ (0 \le \alpha < n). \end{cases}$$

Since in our situation all $I_k \in \mathcal{D}(K^*) \cap \mathcal{A}'_{K^*}$ and all $K_i^* \ge K^*$ the partition $\{(x_k, I_k)\}$ meets all requirements of the decomposition axiom.

The following remark will be needed when verifying the intersection axiom.

Remark 1.3. Let $E, M \subseteq \mathbb{R}^n$ with $|E|_{n-1} = 0$ and $|M|_{n-1} < \infty$. Then for any $\varepsilon > 0$ there is an open set G containing E such that $|G \cap M|_{n-1} < \varepsilon$. For, as is well known, we can find a set $G' \supseteq E$ with $|G'|_{n-1} = 0$ which is the countable intersection of a decreasing collection of open sets G_i. Since $0 = |G' \cap M|_{n-1} = \lim_{i\to\infty} |G_i \cap M|_{n-1}$ the result follows.

To verify the intersection axiom fix a control condition $C_i^\alpha \in \Gamma$ $(0 \le \alpha \le n-1,\ i = 1, 2)$, $E \in \mathcal{E}(C_i^\alpha)$ and $A \in \mathcal{B}$.

Assume first $0 \le \alpha < n-1$, recall that $E \subseteq S$ and let $\varrho > 0$ be a parameter coming from the condition $A \in \mathcal{B}$. Given $K_1 > 0$ set $K_2 = K_1 + \varrho$ and if $\Delta_2 > 0$ let $\Delta_1 = \Delta_2$. Set $\delta(x) = \operatorname{dist}(x, \mathbb{R}^n - A^\circ)$ if $x \in E \cap A^\circ$, and for $x \in E \cap \partial A$ find a neighborhood $U(x)$ of x and a $\delta(x) > 0$ such that $U(x) \cap \partial A$ is ϱ-regulated and $B(x, \delta(x)) \subseteq U(x)$. Then for any $(E \cap A, \delta)$-fine sequence $\{(x_k, A_k)\}$ with $\{A_k\} \in C_i^\alpha(K_1, \Delta_1)$ it follows that $\{A \cap A_k\} \in C_i^\alpha(K_2, \Delta_2)$, since for $x_k \in E \cap \partial A$ we have $\partial(A \cap A_k) \subseteq (A_k^\circ \cap \partial A) \cup \partial A_k \subseteq (U(x_k) \cap \partial A) \cup \partial A_k$ giving $A \cap A_k \in \mathcal{A}'_{K_2}$ for all k, and the other conditions to be checked are obvious.

Now assume $\alpha = n-1$ and look first at C_1^{n-1}: For given $K_1 > 0$ we set $K_2 = K_1 + |\partial A|_{n-1}$, and if $\Delta_2 > 0$ we let $\Delta_1 = \Delta_2$ and $\delta(\cdot) = 1$ on $E \cap A$. Then for any $(E \cap A, \delta)$-fine sequence $\{(x_k, A_k)\}$ with $\{A_k\} \in C_1^{n-1}(K_1, \Delta_1)$,

$$\sum |\partial(A \cap A_k)|_{n-1} \le \sum (|A_k^\circ \cap \partial A|_{n-1} + |\partial A_k|_{n-1}) \le |\partial A|_{n-1} + K_1 = K_2$$

and thus $\{A \cap A_k\} \in C_1^{n-1}(K_2, \Delta_2)$.

Finally, let us look at C_2^{n-1} and assume therefore $K_1 > 0$ to be given. Set $K_2 = K_1$ and for $\Delta_2 > 0$ let $\Delta_1 = \Delta_2/2$. Since $|E \cap \partial A|_{n-1} = 0$, by Remark 1.3 we can find an open set $G \supseteq E \cap \partial A$ with $|G \cap \partial A|_{n-1} < \Delta_1$, and for $x \in E \cap \partial A$ we choose a $\delta(x) > 0$ such that $B(x, \delta(x)) \subseteq G$ while for $x \in E \cap A^\circ$ we set $\delta(x) = \operatorname{dist}(x, \mathbb{R}^n - A^\circ)$. Thus $\delta : E \cap A \to \mathbb{R}^+$ is defined, and if $\{(x_k, A_k)\}$ denotes a $(E \cap A, \delta)$-fine sequence with $\{A_k\} \in C_2^{n-1}(K_1, \Delta_1)$ then

$$\begin{aligned}\sum |\partial(A \cap A_k)|_{n-1} &\le \sum_{x_k \in E \cap \partial A} |A_k^\circ \cap \partial A|_{n-1} + \sum |\partial A_k|_{n-1} \\ &\le |G \cap \partial A|_{n-1} + \Delta_1 \le \Delta_2\end{aligned}$$

and hence $\{A \cap A_k\} \in C_2^{n-1}(K_2, \Delta_2)$.

1c. *Integrability and properties of the integral.* We now define $\nu(S)$-integrability for point functions, and we summarize some of the results of [Ju-No 1, Sec. 5] for the associated $\nu(S)$-integral.

For $A \subseteq \mathbb{R}^n$ we denote by $\mathcal{B}(A)$ the system of all subsets B of A with $B \in \mathcal{B}$. Given a set function $F : \mathcal{B}(A) \to \mathbb{R}$ (on A) we call F *additive* if $F(B) = \sum F(B_k)$ for any $B \in \mathcal{B}(A)$ and every finite sequence $\{B_k\}$ with $B_k \in \mathcal{B}(A)$ having disjoint interiors and $B = \bigcup B_k$.

A set function $F : \mathcal{B}(A) \to \mathbb{R}$ is called *differentiable* at $x \in A^\circ$ if there exists a real number α such that for any $\varepsilon > 0$ and $K > 0$ there is a $\delta = \delta(x) > 0$ with $|F(B) - \alpha|B|_n| \le \varepsilon|B|_n$ for every $B \in \mathcal{B}(A)$ satisfying $B \in \mathcal{D}(K)$, $x \in B$ and $d(B) < \delta$. In this case α is uniquely determined and denoted by $\dot{F}(x)$.

Let $A \subseteq \mathbb{R}^n$, $E \subseteq A$, $C \in \varGamma \cup \dot{\varGamma}$ and let $F : \mathcal{B}(A) \to \mathbb{R}$ be a set function on A. We say that F satisfies the *null condition corresponding to* C *on* E (see [Ju-No 1, Sec. 3]), for short F satisfies $\mathcal{N}(C, E)$, if the following is true: $\forall \varepsilon > 0$, $K > 0$ $\exists \varDelta > 0$ $\exists \delta : E \to \mathbb{R}^+$ such that $\sum |F(A_k)| \le \varepsilon$ for any (E, δ)-fine sequence $\{(x_k, A_k)\}$ with $A_k \in \mathcal{B}(A)$ and $\{A_k\} \in C(K, \varDelta)$.

Given $A \subseteq \mathbb{R}^n$ we call an additive set function $F : \mathcal{B}(A) \to \mathbb{R}$ a $\nu(S)$-*integral on* A if there exists a division $\dot{E}$, $(E_i, C_i)_{i\in\mathbb{N}}$ of A such that F is differentiable on $\dot{E}$ and satisfies $\mathcal{N}(C_i, E_i)$ for all $i \in \mathbb{N}$, $\mathcal{N}(C^*, \dot{E})$ and $\mathcal{N}(C^*, E_i)$ if $C_i \in \dot{\varGamma}$.

Let $A \in \mathcal{B}$ and let f be a real-valued function defined on A. We call f $\nu(S)$-*integrable on* A if there exists a $\nu(S)$-integral F on A with $\dot{F} = f$ a.e. on A. In this case F is uniquely determined, and we write

$$^{\nu(S)}\!\!\int_A f = F(A) \qquad \text{(see [Ju-No 1, Remark 5.1(iii)]).}$$

The space of all $\nu(S)$-integrable functions on A is denoted by $\mathcal{I}_{\nu(S)}(A)$.

If there is no danger of misunderstanding we will often omit the index $\nu(S)$.

Proposition 1.1. *Let* $A \in \mathcal{B}$.

(i) $\mathcal{I}(A)$ *is a real linear space, and the map* $f \mapsto \int_A f$ *is a non-negative linear functional on* $\mathcal{I}(A)$.

(ii) *If* A *is the finite union of sets* $A_k \in \mathcal{B}$ *with disjoint interiors then* $f \in \mathcal{I}(A)$ *iff* $f \in \mathcal{I}(A_k)$ *for all* k, *and in that case*

$$\int_A f = \sum \int_{A_k} f.$$

(iii) *If for a measurable function* $f : A \to \mathbb{R}$ *a finite Lebesgue integral*

${}^{\mathcal{L}}\int_A |f|$ *exists, then f belongs to $\mathcal{I}_{\nu(S)}(A)$ and*

$$^{\nu(S)}\int_A f = {}^{\mathcal{L}}\int_A f.$$

Remark 1.4. In [Ju-No 2] we defined, also using our axiomatic theory, a relatively simple integral over n-dimensional compact intervals, the so-called ν_1-integral. Since any interval I is contained in $\mathcal{B} = \mathcal{A}(S)$ it follows immediately that every $\nu(S)$-integrable function $f : I \to \mathbb{R}$ is also ν_1-integrable and both integrals coincide.

1d. *Discussion.* Here we discuss the dependence of the integration theory induced by the quadruple $\nu(S) = (\mathcal{B}, \mathcal{D}, \dot{\Gamma}, \Gamma)$ on S. First, we extend the notion of $\nu(S)$-integrability to functions defined on quite arbitrary sets $A \subseteq \mathbb{R}^n$.

Assume in this subsection A to be a measurable and bounded subset of $\mathbb{R}^n$ and let f be a real-valued function defined at least on A. By f_A we denote the function $f_A : \mathbb{R}^n \to \mathbb{R}$ defined by $f_A(x) = f(x)$ if $x \in A$ and $f_A(x) = 0$ else.

Then, according to [Ju-No 1, Sec. 5a], we call f $\nu(S)$-integrable on A if there exists a $\nu(S)$-integral F on $\mathbb{R}^n$ with $\dot{F} = f_A$ a.e. In this case F is uniquely determined, and if I denotes any interval containing A the number $F(I)$ does not depend on I, and we set

$$^{\nu(S)}\int_A f = F(I).$$

Again we denote by $\mathcal{I}_{\nu(S)}(A)$ the set of all $\nu(S)$-integrable functions on A. (Note that in case of $A \in \mathcal{B} = \mathcal{A}(S)$ this definition of integrability coincides with the one given in Section 1c.)

Now suppose S_1 and S_2 to be subsets of $\mathbb{R}^n$ with $S_1 \subseteq S_2$. A glance shows that $\mathcal{A}(S_2) \subseteq \mathcal{A}(S_1)$, and any $\nu(S_1)$-integral on $\mathbb{R}^n$ also represents a $\nu(S_2)$-integral on $\mathbb{R}^n$ when restricted to $\mathcal{A}(S_2)$. Consequently, any $f \in \mathcal{I}_{\nu(S_1)}(A)$ also belongs to $\mathcal{I}_{\nu(S_2)}(A)$ and both integrals coincide. Thus all $\nu(S)$-integrals are compatible and, in particular, $\mathcal{I}_{\nu(\mathbb{R}^n)}(A) = \bigcup_{S \subseteq \mathbb{R}^n} \mathcal{I}_{\nu(S)}(A)$.

Remark 1.5. (i) Of particular interest are the extreme cases $S = \emptyset$ and $S = \mathbb{R}^n$ yielding $\mathcal{A}(\emptyset) = \mathcal{A}$ and $\mathcal{A}(\mathbb{R}^n) = \bigcup_{\varrho > 0} \mathcal{A}'_\varrho$ (see Remark 1.1), and the associated integral will also be called the ν_3-integral and ν_2-integral respectively. Furthermore, we set $\mathcal{I}_{\nu_3}(A) = \mathcal{I}_{\nu(\emptyset)}(A)$ and $\mathcal{I}_{\nu_2}(A) = \mathcal{I}_{\nu(\mathbb{R}^n)}(A)$.

(ii) By Remark 1.4, $\mathcal{I}_{\nu_3}(I) \subseteq \mathcal{I}_{\nu(S)}(I) \subseteq \mathcal{I}_{\nu_2}(I) \subseteq \mathcal{I}_{\nu_1}(I)$ for any interval I and any $S \subseteq \mathbb{R}^n$, and all integrals coincide.

2. The divergence theorem. Here we prove the divergence theorem for our $\nu(S)$-integral. The singularities, i.e. the points of unboundedness, of the vector-valued function $\vec{v}$ are assumed to lie in the set S, and we require $\vec{v}$ to satisfy Lipschitz conditions of suitable (negative) order at those points.

2a. *Formulation of the theorem.* Assume $A \subseteq \mathbb{R}^n, x \in A, 1-n \le \beta \le 1$ and let $\vec{v} : A \to \mathbb{R}^n$. Consider the following conditions:

(ℓ_1) there exists a real $n \times n$ matrix M such that

$$\vec{v}(y) - \vec{v}(x) - M(y-x) = o(1)\|y-x\| \quad (y \to x,\ y \in A),$$

(ℓ_β) $(\beta \ne 1)$ $\quad \vec{v}(y) - \vec{v}(x) = o(1)\|y-x\|^\beta \quad (y \to x,\ y \ne x,\ y \in A)$,

(L_β) $\quad \vec{v}(y) - \vec{v}(x) = O(1)\|y-x\|^\beta \quad (y \to x,\ y \ne x,\ y \in A)$.

If $x \in A^\circ$ and $\vec{v} = (v_i)_{1\le i\le n}$ is (totally) differentiable at x we set $\operatorname{div}\vec{v}(x) = \sum_{i=1}^n \frac{\partial v_i}{\partial x_i}(x)$, and at all other points $x \in A$ we set $\operatorname{div}\vec{v}(x) = 0$.

By [Fed], for each $A \in \mathcal{A}$ there exists an $\mathcal{H}$-measurable vector function $\vec{n}_A : \partial A \to \mathbb{R}^n$, the so-called exterior normal, with $\|\vec{n}_A\| \le 1$. Furthermore, for any $\vec{v}$ which is continuously differentiable in a neighborhood of A we have $\int_{\partial A} \vec{v}\cdot\vec{n}_A\, d\mathcal{H} = {}^{\mathcal{L}}\!\!\int_A \operatorname{div}\vec{v}$.

Theorem 2.1 (Divergence Theorem). *Suppose $A \in \mathcal{A}(S)$ and let $\vec{v} : A \to \mathbb{R}^n$. Denote by D the set of all points from the interior of A where $\vec{v}$ is differentiable, and write $A-D$ as a disjoint countable union of σ_{α_i}-finite sets M_i and α_i-null sets N_i with $0 < \alpha_i \le n$ $(i \in \mathbb{N})$ such that $\bigcup_{\alpha_i<n-1}(M_i \cup N_i)$ lies in S. If $\vec{v}$ satisfies the condition (ℓ_{α_i+1-n}) (resp. (L_{α_i+1-n})) at each point of M_i (resp. N_i) then $\vec{v}$ is continuous on A except for an $(n-1)$-null set, and for each subset $B \in \mathcal{A}(S)$ of A the integral $\int_{\partial B} \vec{v}\cdot\vec{n}_B\, d\mathcal{H}$ exists with a finite value, $\operatorname{div}\vec{v}$ is $\nu(S)$-integrable on B and*

$$\int_{\partial B} \vec{v}\cdot\vec{n}_B\, d\mathcal{H} = {}^{\nu(S)}\!\!\int_B \operatorname{div}\vec{v} \quad \Big(= {}^{\nu_2}\!\!\int_B \operatorname{div}\vec{v}\Big).$$

Remark 2.1. In the formulation of the theorem we have excluded the situation $\alpha_i = 0$ which in case of $n = 1$ is of course superfluous since $\vec{v}$ remains continuous on A. But for $n \ge 2$ the integral $\int_{\partial B} \vec{v}\cdot\vec{n}_B\, d\mathcal{H}$ can fail to exist. Anyhow, by redefining the condition (ℓ_{1-n}) it is possible to include the case $\alpha_i = 0$:

We say that $\vec{v} : A \to \mathbb{R}^n$ satisfies the condition (ℓ_{1-n}) $(n \ge 2)$ at $x \in A$ if there exists a decreasing function $g_x : \mathbb{R}^+ \to \mathbb{R}^+$ which is Lebesgue integrable on $[0,1]$ and

$$\vec{v}(y) - \vec{v}(x) = O(1) g_x(\|y-x\|)\|y-x\|^{2-n} \quad (y \to x,\ y \ne x,\ y \in A).$$

In the following proof of the theorem we will include this situation.

2b. *Proof of the theorem.* Observe that $|A-D|_n = 0$ since $\vec{v}$ satisfies (ℓ_1) on M_i with $\alpha_i = n$ and consequently $M_i \subseteq \partial A$. Furthermore, $\vec{v}$ is continuous on A except for an $(n-1)$-null set, and hence the $\mathcal{H}$-measurability of $\vec{v}$ on A follows.

Now fix $B \in \mathcal{B}(A)$, i.e. $B \subseteq A$ with $B \in \mathcal{B} = \mathcal{A}(S)$. We first show the existence of the finite integral $\int_{\partial B} \vec{v} \cdot \vec{n}_B \, d\mathcal{H}$; we closely follow [Ju-No 2, Sec. 2]. Note that for $n = 1$ there is nothing to prove since $\vec{v}$ is continuous on A, and we therefore assume $n \geq 2$. At each $x \in \partial B - \bigcup_{\alpha_i < n-1} (M_i \cup N_i)$ the function $\vec{v}$ is locally bounded, i.e. there is a positive number $K(x)$ and an open neighborhood $U(x)$ of x such that $\|\vec{v}(y)\| \leq K(x)$ for all $y \in U(x) \cap A$.

We denote by $\varrho > 0$ a parameter corresponding to $B \in \mathcal{A}(S)$. If $0 < \alpha_i < n-1$ and $x \in M_i \cap \partial B$ (resp. $x \in N_i \cap \partial B$) there is an open neighborhood $U(x)$ of x such that $U(x) \cap \partial B$ is ϱ-regulated and

$$\|\vec{v}(y) - \vec{v}(x)\| \leq \|y - x\|^{\alpha_i + 1 - n}$$

(resp.

$$\|\vec{v}(y) - \vec{v}(x)\| \leq K(x) \|y - x\|^{\alpha_i + 1 - n}$$

with some $K(x) > 0$) for all $y \in U(x) \cap A$, $y \neq x$.

Finally, if $\alpha_i = 0$ (note that $N_i = \emptyset$) and $x \in M_i \cap \partial B$ there is a decreasing function $g_x : \mathbb{R}^+ \to \mathbb{R}^+$ Lebesgue integrable on $[0, 1]$, a positive number $K(x)$ and an open neighborhood $U(x)$ of x with $d(U(x)) \leq 1$ such that $U(x) \cap \partial B$ is ϱ-regulated and

$$\|\vec{v}(y) - \vec{v}(x)\| \leq K(x) g_x(\|y - x\|) \|y - x\|^{2-n}$$

for all $y \in U(x) \cap A$, $y \neq x$.

Since ∂B is compact there are finitely many points $x_k \in \partial B$ with $\partial B \subseteq \bigcup U(x_k)$, and it suffices to prove that $\int_{U(x_k) \cap \partial B} \|\vec{v}\| \, d\mathcal{H}$ remains finite for all k. Since this is obvious for $x_k \notin \bigcup_{\alpha_i < n-1} (M_i \cup N_i)$, we first consider an $x_k \in M_i \cup N_i$ where $0 < \alpha_i < n - 1$.

We may assume $d(B) > 0$ since otherwise $|\partial B|_{n-1} = 0$ $(n \geq 2)$, and for $j = 0, 1, \ldots$ we let $C_j = \{x \in \mathbb{R}^n : d(B)/2^{j+1} < \|x - x_k\| \leq d(B)/2^j\}$. It suffices to observe that

$$\begin{aligned} \int_{U(x_k) \cap \partial B} \|y - x_k\|^{\alpha_i + 1 - n} \, d\mathcal{H}(y) &\leq \sum_{j=0}^{\infty} \int_{C_j \cap U(x_k) \cap \partial B} \|y - x_k\|^{\alpha_i + 1 - n} \, d\mathcal{H}(y) \\ &\leq \sum_{j=0}^{\infty} \left(\frac{d(B)}{2^{j+1}} \right)^{\alpha_i + 1 - n} |B(x_k, d(B)/2^j) \cap U(x_k) \cap \partial B|_{n-1} \\ &\leq \sum_{j=0}^{\infty} \left(\frac{d(B)}{2^{j+1}} \right)^{\alpha_i + 1 - n} \varrho \left(\frac{d(B)}{2^j} \right)^{n-1} = \frac{\varrho d(B)^{\alpha_i}}{2^{\alpha_i + 1 - n}} \sum_{j=0}^{\infty} \left(\frac{1}{2^{\alpha_i}} \right)^j, \end{aligned}$$

and so

$$(*)\qquad \int\limits_{U(x_k)\cap\partial B} \|y-x_k\|^{\alpha_i+1-n}\,d\mathcal{H}(y) \le \frac{\varrho 2^{n-1}}{2^{\alpha_i}-1} d(B)^{\alpha_i} \qquad (<\infty).$$

For $x_k \in M_i$ with $\alpha_i = 0$ the same arguments (use $U(x_k)\cap B$ instead of B in the definition of the C_j) combined with the properties of the function $g = g_{x_k}$ yield the inequality

$$(**)\qquad \int\limits_{U(x_k)\cap\partial B} g(\|y-x_k\|)\|y-x_k\|^{2-n}\,d\mathcal{H}(y) \le \varrho\beta(n)\int\limits_0^{\gamma} g(t)\,dt \qquad (<\infty),$$

where $\beta(n)$ denotes a positive absolute constant, and $\gamma = d(U(x_k)\cap B)$.

By what has just been proved, we can define an additive set function F on A by $F(B) = \int_{\partial B} \vec{v}\cdot\vec{n}_B\,d\mathcal{H}$ for $B \in \mathcal{B}(A)$. We will show that F is a $\nu(S)$-integral on A with $\dot{F} = \operatorname{div}\vec{v}$ a.e. on A, thus $\operatorname{div}\vec{v} \in \mathcal{I}_{\nu(S)}(A)$ and $\int_{\partial A} \vec{v}\cdot\vec{n}_A\,d\mathcal{H} = F(A) = {}^{\nu(S)}\!\int_A \operatorname{div}\vec{v}$. Of course the equality then also holds for each $B \in \mathcal{B}(A)$ (apply the theorem to B in place of A or use Thm. V(2) of [Ju-No 1]).

Without loss of generality we assume $|M_i|_{\alpha_i}$ to be finite ($i \in \mathbb{N}$), $M_i = \emptyset$ if $\alpha_i = n$ ($|M_i|_n = 0$), and we also assume the O-constant occurring in (L_{α_i+1-n}) to be bounded on N_i by $K_i > 0$ ($i \in \mathbb{N}$). Then a division of A is given by D, $(M_i, C_1^{\alpha_i})_{i\in\mathbb{N}}$, $(N_i, C_2^{\alpha_i})_{i\in\mathbb{N}}$ with the understanding that $C_1^{\alpha_i} = C_2^{\alpha_i} = C^n$ if $\alpha_i = n$.

• F is differentiable on D with $\dot{F} = \operatorname{div}\vec{v}$. Indeed, take $x \in D$, let $\varepsilon, K > 0$ and take a $\delta > 0$ such that $\|\vec{v}(y)-\vec{v}(x)-\vec{v}\,'(x)\cdot(y-x)\| \le \varepsilon\|y-x\|/K^2$ for all $y \in B(x,\delta)$ ($\subseteq A^\circ$), where $\vec{v}\,'(x)$ denotes the derivative of $\vec{v}$ at x. Then for each $B \in \mathcal{D}(K)$ with $x \in B$ and $d(B) < \delta$ we have

$$\begin{aligned} |F(B) - \operatorname{div}\vec{v}(x)|B|_n| &= \Big|\int\limits_{\partial B} (\vec{v}(y)-\vec{v}(x)-\vec{v}\,'(x)\cdot(y-x))\cdot\vec{n}_B\,d\mathcal{H}(y)\Big| \\ &\le \frac{\varepsilon}{K^2} d(B)|\partial B|_{n-1} \le \frac{\varepsilon}{K} d(B)^n \le \varepsilon|B|_n. \end{aligned}$$

• Similarly one proves that F satisfies the null conditions $\mathcal{N}(C_1^{\alpha_i}, M_i)$ and $\mathcal{N}(C_2^{\alpha_i}, N_i)$ if $n-1 \le \alpha_i \le n$ (cf. [Ju-No 2, proof of Thm. 2.1]). For example, let us show that F satisfies $\mathcal{N}(C_2^{\alpha_i}, N_i)$ if $n-1 < \alpha_i < n$.

Let $\varepsilon, K > 0$. For $x \in N_i$ find $K(x)$, $\delta(x) > 0$ such that $\|\vec{v}(y)-\vec{v}(x)\| \le K(x)\|y-x\|^{\alpha_i+1-n}$ for all $y \in B(x,\delta(x))\cap A$. By assumption, $K(x) \le K_i$ for all $x \in N_i$, and we set $\Delta = \varepsilon/(KK_i)$. Then for any (N_i,δ)-fine sequence

$\{(x_k, A_k)\}$ with $A_k \in \mathcal{B}(A)$ and $\{A_k\} \in C_2^{\alpha_i}(K, \Delta)$ we get

$$\begin{aligned}\sum |F(A_k)| &= \sum \Big| \int\limits_{\partial A_k} (\vec{v}(y) - \vec{v}(x_k)) \cdot \vec{n}_{A_k} \, d\mathcal{H}(y) \Big| \\ &\le K_i \sum d(A_k)^{\alpha_i + 1 - n} |\partial A_k|_{n-1} \\ &\le K K_i \sum d(A_k)^{\alpha_i} \le K K_i \Delta = \varepsilon.\end{aligned}$$

• Let us show that F satisfies $\mathcal{N}(C_1^{\alpha_i}, M_i)$ if $0 < \alpha_i < n-1$. Analogously one then proves that F also satisfies $\mathcal{N}(C_2^{\alpha_i}, N_i)$ for $0 < \alpha_i < n-1$.

Given $\varepsilon, K > 0$ we choose for $x \in M_i$ a $\delta(x) > 0$ such that $\|\vec{v}(y) - \vec{v}(x)\| \le \varepsilon' \|y - x\|^{\alpha_i + 1 - n}$ for all $y \in B(x, \delta(x)) \cap A$ with $y \ne x$, where $\varepsilon' = \varepsilon 2^{1-n}(2^{\alpha_i} - 1)/K^2$. Now let $\{(x_k, A_k)\}$ be an (M_i, δ)-fine sequence with $A_k \in \mathcal{B}(A)$ and $\{A_k\} \in C_1^{\alpha_i}(K)$. In particular, ∂A_k is K-regulated for all k, and thus we can use the inequality $(*)$ with $B = A_k$, $\varrho = K$ and $U(x_k) = B(x_k, \delta(x_k)) \supseteq A_k$ yielding

$$\begin{aligned}\sum |F(A_k)| &= \sum \Big| \int\limits_{\partial A_k} (\vec{v}(y) - \vec{v}(x_k)) \cdot \vec{n}_{A_k} \, d\mathcal{H}(y) \Big| \\ &\le \varepsilon' \sum \int\limits_{\partial A_k} \|y - x_k\|^{\alpha_i + 1 - n} \, d\mathcal{H}(y) \\ &\le \varepsilon' \sum \frac{K 2^{n-1}}{2^{\alpha_i} - 1} d(A_k)^{\alpha_i} \le \varepsilon.\end{aligned}$$

• F satisfies $\mathcal{N}(C_1^{\alpha_i}, M_i)$ if $\alpha_i = 0$. Indeed, given $\varepsilon, K > 0$ find for $x \in M_i$ a function $g_x : \mathbb{R}^+ \to \mathbb{R}^+$ and positive numbers $K(x)$ and $\delta(x)$ such that $\|\vec{v}(y) - \vec{v}(x)\| \le K(x) g_x(\|y - x\|) \|y - x\|^{2-n}$ for all $y \in B(x, \delta(x)) \cap A$, $y \ne x$. Without loss of generality we may assume $\delta(x) \le 1/2$ and $\int_0^{\delta(x)} g_x(t) \, dt \le \varepsilon / (\beta(n) K(x) K^2)$ by the Lebesgue integrability of g_x. Here $\beta(n)$ denotes the absolute constant occurring in $(**)$. Now let $\{(x_k, A_k)\}$ be an (M_i, δ)-fine sequence with $A_k \in \mathcal{B}(A)$ and $\{A_k\} \in C_1^{\alpha_i}(K)$. Using the inequality $(**)$ with $B = A_k$, $\varrho = K$ and $U(x_k) = B(x_k, \delta(x_k))$ we conclude

$$\begin{aligned}\sum |F(A_k)| &= \sum \Big| \int\limits_{\partial A_k} (\vec{v}(y) - \vec{v}(x_k)) \cdot \vec{n}_{A_k} \, d\mathcal{H}(y) \Big| \\ &\le \sum K(x_k) \int\limits_{\partial A_k} g_{x_k}(\|y - x_k\|) \|y - x_k\|^{2-n} \, d\mathcal{H}(y) \\ &\le \sum K(x_k) K \beta(n) \int\limits_0^{\delta(x_k)} g_{x_k}(t) \, dt \le \varepsilon.\end{aligned}$$

• Finally, the continuity of $\vec{v}$ directly implies that F satisfies $\mathcal{N}(C^*, D \cup \bigcup_{\alpha_i > n-1} (M_i \cup N_i))$, which completes the proof. ■

R e m a r k 2.2. (i) Since any interval is contained in $\mathcal{A}(\mathbb{R}^n)$ and since the ν_1-integral extends the ν_2-integral, our result contains the divergence theorem for the ν_1-integral of [Ju-No 2].

(ii) Furthermore, the divergence theorem of [Ju-No 3] can also be deduced from the theorem above: set $S = \bigcup_{\alpha_i < n-1}(M_i \cup N_i)$, and recall that the ν_1-integral extends any $\nu(S)$-integral.

3. The transformation formula. In this section we establish a quite general transformation formula for the ν_2-integral, i.e. the $\nu(S)$-integral with $S = \mathbb{R}^n$ (cf. Sec. 1.d), by verifying the transformation axiom in our abstract theory ([Ju-No 1, Sec. 7]).

Given a measurable subset A of $\mathbb{R}^n$ and a function $\phi : A \to \mathbb{R}^n$, we call ϕ a *transformation map* if it is one-to-one and if ϕ and its inverse ϕ^{-1} are Lipschitzian.

LEMMA 3.1. *Let A be a measurable subset of $\mathbb{R}^n$, assume $\phi : A \to \mathbb{R}^n$ to be a transformation map and denote by c_1 (resp. c_2) a positive Lipschitz constant of ϕ (resp. ϕ^{-1}).*

(i) *If $K > 0$ and $B \subseteq A$ with $B \in \mathcal{A}_K(\emptyset)$, then $\phi(B) \in \mathcal{A}_{\widetilde{K}}(\emptyset)$ with $\widetilde{K} = 1 + (c_1c_2)^n(1+K)^2$.*

(ii) *Assume $M \subseteq A$ to be ϱ-regulated ($\varrho > 0$). Then $\phi(M)$ is $\widetilde{\varrho}$-regulated with $\widetilde{\varrho} = \varrho(2c_1c_2)^{n-1}$.*

P r o o f. (i) Let $K > 0$ and $B \subseteq A$ with $B \in \mathcal{A}_K(\emptyset)$, i.e. $B \in \mathcal{A}(\emptyset) = \mathcal{A}$ and $d(B)^n \le K|B|_n$, $|\partial B|_{n-1} \le Kd(B)^{n-1}$. Since $\phi(B)$ is compact and $\phi(\partial B) = \partial\phi(B)$, we have $|\partial\phi(B)|_{n-1} \le c_1^{n-1}|\partial B|_{n-1}$ and thus $\phi(B) \in \mathcal{A}$. Furthermore, because ϕ and ϕ^{-1} are Lipschitzian we have

$$d(\phi(B))^n \le c_1^n d(B)^n \le Kc_1^n|B|_n \le K(c_1c_2)^n|\phi(B)|_n \le \widetilde{K}|\phi(B)|_n.$$

It remains to show that $|\partial\phi(B)|_{n-1} \le \widetilde{K}d(\phi(B))^{n-1}$. Since this is obvious if $d(\phi(B)) = 0$, we assume $d(\phi(B)) > 0$, yielding

$$\begin{aligned}|\partial\phi(B)|_{n-1} &\le c_1^{n-1}|\partial B|_{n-1} \le Kc_1^{n-1}d(B)^{n-1} \le Kc_1^n\frac{d(B)^n}{d(\phi(B))}\\ &\le K^2c_1^n\frac{|B|_n}{d(\phi(B))} \le (c_1c_2)^nK^2\frac{|\phi(B)|_n}{d(\phi(B))}\\ &\le (c_1c_2)^nK^2d(\phi(B))^{n-1}.\end{aligned}$$

(ii) To prove the $\widetilde{\varrho}$-regularity of $\phi(M)$ we first take a $y = \phi(x) \in \phi(M)$ and any $r > 0$, and we set $E = \phi^{-1}(B(y,r) \cap \phi(A))$, which is contained in

$B(x, rc_2)$. Consequently,

$$\begin{aligned}|B(y,r)\cap\phi(M)|_{n-1} &= |\phi(E\cap M)|_{n-1} \le c_1^{n-1}|E\cap M|_{n-1}\\ &\le c_1^{n-1}|B(x,rc_2)\cap M|_{n-1} \le c_1^{n-1}\varrho(rc_2)^{n-1}\\ &= \varrho(c_1c_2)^{n-1}r^{n-1}\end{aligned}$$

since M is ϱ-regulated.

If $y \in \mathbb{R}^n$ is arbitrary and if $r > 0$ we choose (if possible) a $z \in B(y, r) \cap \phi(M)$, which implies $B(y, r) \subseteq B(z, 2r)$, and thus

$$|B(y,r)\cap\phi(M)|_{n-1} \le |B(z,2r)\cap\phi(M)|_{n-1} \le \widetilde{\varrho}r^{n-1}. \ \blacksquare$$

To verify the transformation axiom for our ν_2-integral take a set $A \in \mathcal{A}(\mathbb{R}^n) = \bigcup_{\varrho>0}\mathcal{A}'_\varrho$ and a transformation map $\phi : A \to \mathbb{R}^n$.

If $B \subseteq A$ with $B \in \mathcal{A}'_\varrho$ for some $\varrho > 0$, Lemma 3.1 implies $\phi(B) \in \mathcal{A}(\mathbb{R}^n)$ since $\partial\phi(B) = \phi(\partial B)$, and this, combined with Lemma 3.1(i), yields the invariance of $\mathcal{B} = \mathcal{A}(\mathbb{R}^n)$ and $\mathcal{D}$ with respect to ϕ. Finally, one has to check the invariance of the control conditions under ϕ and this again is a simple consequence of Lemma 3.1. For example, take $C = C_1^\alpha$, $0 \le \alpha < n-1$, and let $K > 0$. Denote again by c_1 (resp. c_2) a Lipschitz constant of ϕ (resp. ϕ^{-1}) and set $\widetilde{K} = K(1 + c_1^\alpha + (2c_1c_2)^{n-1})$. For $\widetilde{\Delta} > 0$ let $\Delta = 1$ and assume $\{A_k\} \in C_1^\alpha(K, \Delta)$ with $A_k \subseteq A$. Since ∂A_k is K-regulated Lemma 3.1(ii) implies that $\partial\phi(A_k)$ is $\widetilde{K}$-regulated, $\sum d(\phi(A_k))^\alpha \le c_1^\alpha \sum d(A_k)^\alpha \le \widetilde{K}$, and since each $x \in \mathbb{R}^n$ is contained in at most K of the A_k the same is true for the sequence $\{\phi(A_k)\}$ and thus $\{\phi(A_k)\} \in C_1^\alpha(\widetilde{K}, \widetilde{\Delta})$. Furthermore, if $E \subseteq A$ with $E \in \mathcal{E}(C_1^\alpha)$ we have $|\phi(E)|_\alpha \le c_1^\alpha|E|_\alpha < \infty$ and therefore $\phi(E) \in \mathcal{E}(C_1^\alpha)$.

Now we can state the following

Theorem 3.1 (Transformation Formula). *Let $A \in \mathcal{A}(\mathbb{R}^n)$, $\phi : A \to \mathbb{R}^n$ be a transformation map and let $f : \phi(A) \to \mathbb{R}$. Then f is ν_2-integrable on $\phi(A)$ iff $(f\circ\phi)|\det\phi'|$ is ν_2-integrable on A, and in that case*

$$^{\nu_2}\int_{\phi(A)} f = {}^{\nu_2}\int_A (f\circ\phi)|\det\phi'|.$$

Remark 3.1. (i) Analogously one verifies the transformation axiom for the ν_3-integral, i.e. the $\nu(\emptyset)$-integral, and thus the corresponding transformation formula holds.

(ii) For $S = \emptyset$ and $S = \mathbb{R}^n$ we have seen the quadruple $\nu(S)$ to be invariant under transformation maps, and therefore a transformation formula holds within the $\nu(S)$-theory.

Of course for general S the semi-ring $\mathcal{A}(S)$ will no longer be invariant with respect to transformations, and thus no transformation formula can be stated within the $\nu(S)$-theory. Instead one also has to consider the transformed $\nu(\phi(S))$-theory, and then an analogue of Theorem 3.1 can be proved in which one of the integrals is a $\nu(S)$-integral and the other a $\nu(\phi(S))$-integral.

4. A constructive definition of the $\nu(S)$-integral. Here we assume $S \subseteq \mathbb{R}^n$ again to be arbitrary but fixed.

The definition of the $\nu(S)$-integral for a point function f given in Section 1 is of descriptive type, i.e. we associate with f a set function satisfying certain conditions. In contrast to this a constructive definition in the Riemann sense would associate with f only a single real number. Ideally, this seems to be the most natural way of defining an integration process, and our $\nu(S)$-integral indeed allows such an equivalent constructive definition.

THEOREM 4.1. *Let $A \in \mathcal{A}(S)$ and $f : A \to \mathbb{R}$. Then f is $\nu(S)$-integrable on A iff there exists a real number J and a division $\dot{E}$, $(E_i, C_i)_{i\in\mathbb{N}}$ of A with the following property*: $\forall \varepsilon > 0,\ K > 0,\ K_i > 0\ \exists \Delta_i > 0,\ \delta : A \to \mathbb{R}^+$ *such that*

$$\Big| J - \Big(\sum f(x_k)|A_k|_n + \sum f(x'_k)|A'_k|_n\Big)\Big| \le \varepsilon$$

for any δ-fine partition $\{(x_k, A_k)\} \cup \{(x'_k, A'_k)\}$ of A with

(i) *if $x_k \in \dot{E}$ then $A_k \in \mathcal{A}_K(S)$, $\{A_k : x_k \in E_i\} \in C_i(K_i, \Delta_i)$ $(i \in \mathbb{N})$,*
(ii) *$\{A'_k\} \in C^*(K)$ and $x'_k \in \dot{E} \cup \bigcup_{C_i \in \dot{\Gamma}} E_i$ for all k,*

and in that case J is uniquely determined and $J = {}^{\nu(S)}\!\int_A f$.

Since the control condition $C^* = C_1^{n-1}$ does not depend on Δ one part of the theorem, assuming the $\nu(S)$-integrability of f, is nothing but the concrete version of Corollary 6.1 of [Ju-No 1]. The other part of the theorem is much more involved and will be presented in a separate paper [No 2].

Remark 4.1. The analogous theorem for the ν_1-integral (cf. Remark 1.4) has been proved in [Ju-No 2, Thm. 3.1].

References

[Fed] H. Federer, *Geometric Measure Theory*, Springer, New York, 1969.

[Jar-Ku 1] J. Jarník and J. Kurzweil, *A non-absolutely convergent integral which admits C^1-transformations*, Časopis Pěst. Mat. 109 (1984), 157–167.

[Jar-Ku 2] —, —, *A non-absolutely convergent integral which admits transformation and can be used for integration on manifolds*, Czechoslovak Math. J. 35 (110) (1985), 116–139.

[Jar-Ku 3] —, —, *A new and more powerful concept of the PU integral*, ibid. 38 (113) (1988), 8–48.

[JKS] J. Jarník, J. Kurzweil and S. Schwabik, *On Mawhin's approach to multiple nonabsolutely convergent integral*, Časopis Pěst. Mat. 108 (1983), 356–380.

[Ju] W. B. Jurkat, *The Divergence Theorem and Perron integration with exceptional sets*, Czechoslovak Math. J. 43 (118) (1993), 27–45.

[Ju-No 1] W. B. Jurkat and D. J. F. Nonnenmacher, *An axiomatic theory of non-absolutely convergent integrals in* $\mathbb{R}^n$, Fund. Math. 145 (1994), 221–242.

[Ju-No 2] —, —, *A generalized n-dimensional Riemann integral and the Divergence Theorem with singularities*, Acta Sci. Math. (Szeged) 59 (1994), 241–256.

[Ju-No 3] —, —, *The Fundamental Theorem for the* ν_1*-integral on more general sets and a corresponding Divergence Theorem with singularities*, Czechoslovak Math. J., to appear.

[Maw] J. Mawhin, *Generalized multiple Perron integrals and the Green–Goursat theorem for differentiable vector fields*, ibid. 31 (106) (1981), 614–632.

[No 1] D. J. F. Nonnenmacher, *Sets of finite perimeter and the Gauss–Green Theorem with singularities*, J. London Math. Soc., to appear.

[No 2] —, *A constructive definition of the n-dimensional* $\nu(S)$*-integral in terms of Riemann sums*, preprint 1992, to appear.

[Pf 1] W. F. Pfeffer, *The divergence theorem*, Trans. Amer. Math. Soc. 295 (1986), 665–685.

[Pf 2] —, *The Gauss–Green Theorem*, Adv. in Math. 87 (1991), 93–147.

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