Co-H-structures on equivariant Moore spaces

by

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Abstract. Let $G$ be a finite group, $\mathcal{O}_G$ the category of canonical orbits of $G$ and $A : \mathcal{O}_G \to \text{Ab}$ a contravariant functor to the category of abelian groups. We investigate the set of $G$-homotopy classes of comultiplications of a Moore $G$-space of type $(A,n)$ where $n \geq 2$ and prove that if such a Moore $G$-space $X$ is a cogroup, then it has a unique comultiplication if $\dim X < 2n - 1$. If $\dim X = 2n - 1$, then the set of comultiplications of $X$ is in one-one correspondence with $\text{Ext}^{n-1}(A,A \otimes A)$. Then the case $G = \mathbb{Z}_p$ leads to an example of infinitely many $G$-homotopically distinct $G$-maps $\varphi_i : X \to Y$ such that $\varphi_i^H$, $\varphi_j^H : X^H \to Y^H$ are homotopic for all $i, j$ and all subgroups $H \subseteq G$.

1. Introduction. If $A$ is an abelian group and $n$ an integer $\geq 2$, then a Moore space of type $(A,n)$ is a space with a single nonvanishing homology group $A$ in dimension $n$. Moore spaces play a central role in homotopy theory and have been widely studied. In particular, the co-H-structures of a Moore space have been investigated. It is known that for $n > 2$ there is a unique co-H-structure (up to homotopy) on a Moore space, but that for $n = 2$ there may be several distinct co-H-structures (e.g., see [A–G]). In this paper we consider these results within the context of equivariant homotopy theory.

Throughout, $G$ denotes a finite group and all spaces, maps, homotopies and actions are pointed. We work in the category $G\text{-Top}_*$ of $G$-spaces which have the $G$-homotopy type of $G$-CW-complexes [Br]. We denote by $\mathcal{O}_G$ the category of canonical orbits of $G$ whose objects are the left cosets $G/H$ as $H$ ranges over all subgroups of $G$ and whose morphisms are the equivariant maps $G/H \to G/K$ with respect to left translation. An $O_G$-module is a contravariant functor from $\mathcal{O}_G$ into $\text{Ab}$, the category of abelian groups. For a pair $(X,Y)$ of $G$-spaces and an integer $n \geq 1$, an $O_G$-module $H_n(X,Y) : O_G \to \text{Ab}$ can be defined as follows: $H_n(X,Y)(G/H) = H_n(X^H,Y^H)$, where $H_n$ denotes the $n$th singular homology functor and $X^H$ is the $H$-fixedpoint subspace of $X$. Similarly, with $n \geq 3$ ($n \geq 2$ if $Y$ is the base point $*$) we define $\pi_n(X,Y) : O_G \to \text{Ab}$ using the $n$th homotopy functor $\pi_n$. For

Y = *, these $\mathcal{O}_G$-modules are denoted by $\widetilde{H}_n(X)$ and $\pi_n(X)$, respectively.

Now let $A : \mathcal{O}_G \to \text{Ab}$ be an $\mathcal{O}_G$-module and $n \geq 2$ an integer. Following Kahn [Ka1], we define a Moore $G$-space of type $(A, n)$ to be a $G$-space $X$ such that

1. $X^H$ is 1-connected for all subgroups $H$ of $G$,
2. $\widetilde{H}_n(X) \cong A$ as $\mathcal{O}_G$-modules,
3. $\widetilde{H}_i(X) = 0$ for $i \neq n$.

If the $H$-fixedpoint sets $X^H$ are disregarded when $H$ is a nontrivial subgroup, then we obtain a classical Moore $G$-space. More precisely, if $A$ is a $G$-module and $n \geq 2$, then a classical Moore $G$-space of type $(A, n)$ is a $G$-space $X$ such that

1. $X$ is 1-connected,
2. $\widetilde{H}_n(X) \cong A$ as $G$-modules,
3. $\widetilde{H}_i(X) = 0$ for $i \neq n$.

Moore $G$-spaces have been considered in several papers ([Do1], [Do2], [Ka3], [Ka2]) and shown to be important in equivariant homotopy theory (e.g., the construction of an equivariant homology decomposition [Ka2]). Furthermore, classical Moore spaces have been extensively studied in connection with the Steenrod problem (e.g., [Ca], [Ka3], [Sm]). Unlike the nonequivariant case, Moore $G$-spaces need not exist for any $\mathcal{O}_G$-module $A$, and when they exist, they need not be unique (see Section 2 for known existence and uniqueness results). This is so even for classical Moore $G$-spaces.

In this paper we extend the results of [A–G] to the equivariant case and investigate the set of $G$-homotopy classes of comultiplications of a Moore $G$-space. We begin with some generalities on closed model categories $\mathcal{C}$. We show that if $X$ is a cogroup object in $\text{Ho}\mathcal{C}$, the associated homotopy category of $\mathcal{C}$, then the collection of comultiplications of $X$ is in one-one correspondence with the set of morphisms $\text{Ho}\mathcal{C}(X, F)$, where $F$ is the fibre of the canonical morphism $X \vee X \to X \times X$. Next we introduce two closed model structures on $G-\text{Top}_*$, one to be used for Moore $G$-spaces and the other for classical Moore $G$-spaces. We then deduce in the next section that a Moore $G$-space $X$ of type $(A, n)$ which is a cogroup has a unique comultiplication if $\dim X < 2n - 1$. If $\dim X = 2n - 1$, we show that the set of comultiplications of $X$ is in one-one correspondence with $\text{Ext}^{n-1}(A, A \otimes A)$. Analogous results are established for classical Moore $G$-spaces. We then apply these considerations to the case $G = \mathbb{Z}_{p^k}$. This leads to an example of infinitely many $G$-homotopically distinct $G$-maps $\varphi_i : X \to Y$ such that $\varphi_i^H$, $\varphi_j^H : X^H \to Y^H$ are homotopic for all $i, j$ and all subgroups $H \subseteq G$. 
2. Background. The general reference here for category theory is [Qu].
Let $\mathbb{C}$ be a pointed category with finite products and coproducts. For objects $X$ and $Y$ of $\mathbb{C}$, morphisms are written $f : X \to Y$ or $f \in \mathbb{C}(X,Y)$. In particular, the zero morphism is $0 : X \to Y$ and the identity morphism is $1_X : X \to X$. Let $X \vee Y$ denote the coproduct of $X$ and $Y$ and $X \times Y$ the product of $X$ and $Y$. Then for an object $X$, there is a canonical morphism $j : X \vee X \to X \times X$ determined by two morphisms $(1_X, 0), (0, 1_X) : X \to X \times X$. Let $\Delta = (1_X, 1_X) : X \to X \times X$ be the diagonal morphism. A morphism $\varphi : X \to X \vee X$ such that $j \varphi = \Delta$ is called a comultiplication of $X$, and $X$ is said to have co-structure $\varphi$. If $(1 \vee \varphi) = (\varphi \vee 1) : X \to X \vee X \vee X$ then $\varphi$ is associative. If there exists a morphism $\eta : X \to X$ such that $\nabla(\eta \vee 1_X) = \nabla(1_X \vee \eta) \varphi = 0 : X \to X$, where $\nabla : X \vee X \to X$ is the folding morphism, we say that $\eta$ is an inverse. The triple $(X, \varphi, \eta)$ is then called a cogroup object in $\mathbb{C}$. If $(X, \varphi, \eta)$ is a cogroup object in $\mathbb{C}$ and $Y$ is any object, then $\varphi$ and $\eta$ induce a group structure on the set $\mathbb{C}(X, Y)$ such that for every morphism $g : Y \to Y'$, the induced map $g_* : \mathbb{C}(X, Y) \to \mathbb{C}(X, Y')$ is a homomorphism.

Now let $\mathbb{C}$ be a pointed closed model category. We localize $\mathbb{C}$ with respect to the class of weak equivalences and obtain the homotopy category $\text{Ho}\mathbb{C}$ [Qu]. A co-structure on an object in $\text{Ho}\mathbb{C}$ is called a co-H-structure and a cogroup object in $\text{Ho}\mathbb{C}$ is called a co-H-group. Quillen [Qu] has defined a suspension functor $\Sigma : \text{Ho}\mathbb{C} \to \text{Ho}\mathbb{C}$ such that $\Sigma X$ is a co-H-group. For any objects $X, Y$ in $\text{Ho}\mathbb{C}$, let us denote $\text{Ho}\mathbb{C}(X, Y) = [X, Y]$. Then if $f : X \to Y$, there exists an object $F$, called the fibre of $f$, such that for any object $Z$, the following sequence is exact [Qu]:

$$\cdots \to [\Sigma Z, X] \overset{f_*}{\to} [\Sigma Z, Y] \to [Z, F] \to [Z, X] \overset{f_*}{\to} [Z, Y].$$

Let $X$ be a co-H-group, $\mathcal{C}(X) \subseteq [X, X \vee X]$ the set of co-H-structures of $X$ and $F$ the fibre of the canonical morphism $j : X \vee X \to X \times X$. Then the set $\mathcal{C}(X)$ is an orbit of the action of the group $[X, F]$ on $[X, X \vee X]$ by (right) translation. So there is, in general, no natural group structure on $\mathcal{C}(X)$. However, if an element of $\mathcal{C}(X)$ is chosen as a base point it is possible to offer a direct interpretation of the group structure of $\mathcal{C}(X)$.

**Proposition 2.1.** For any co-H-group object $X$ in $\text{Ho}\mathbb{C}$, there is a group isomorphism

$$\mathcal{C}(X) \xrightarrow{\simeq} [X, F].$$

The proof follows from the above long exact sequence applied to $j$ together with the methods of [A–G].

Next let $\text{Top}_*$ be the category of pointed topological spaces. We give $\text{Top}_*$ the structure of a pointed closed model category by defining weak equivalences, fibrations and cofibrations in $\text{Top}_*$ in the usual way [Qu]. Let
$G$–$\text{Top}_*$ be the category with objects pointed $G$-spaces and morphisms $G$-maps. We define a closed model category structure $\mathbf{I}$ on $G$–$\text{Top}_*$ as follows:

\textbf{I-1.} A $G$-map $f : X \to Y$ is a \textit{weak equivalence} if the maps $f^H : X^H \to Y^H$ of $H$-fixedpoint subspaces are weak equivalences in $\text{Top}_*$ for all subgroups $H \subseteq G$.

\textbf{I-2.} A $G$-map $f : E \to B$ is a \textit{fibration} if $f^H : E^H \to B^H$ are fibrations in $\text{Top}_*$ for all subgroups $H \subseteq G$.

\textbf{I-3.} Cofibrations are determined by weak equivalences and fibrations by means of the lifting property [Qu, p. 5.1].

We also define a second closed model category structure $\mathbf{II}$ on $G$–$\text{Top}_*$:

\textbf{II-1.} A $G$-map $f : X \to Y$ is a \textit{weak equivalence} if $f$ is a weak equivalence in $\text{Top}_*$.

\textbf{II-2.} A $G$-map $f : E \to B$ is a \textit{fibration} if $f$ is a fibration in $\text{Top}_*$.

\textbf{II-3.} Cofibrations are determined by weak equivalences and fibrations as above.

One checks that $\mathbf{I}$ and $\mathbf{II}$ satisfy the axioms for a pointed closed model category (cf. [D–D–K]) and thus one obtains homotopy categories

\[ \text{Ho}^{\mathbf{I}} G$–$\text{Top}_* \quad \text{and} \quad \text{Ho}^{\mathbf{II}} G$–$\text{Top}_* \]

by localizing with respect to the weak equivalences of $\mathbf{I}$ and $\mathbf{II}$, respectively.

Finally, we summarize from [Ka$_1$] conditions for the existence and uniqueness of a Moore $G$-space $X$ of type $(A, n)$, where $A$ is an $\mathcal{O}_G$-module. We are especially interested in when $X$ is a cogroup in the appropriate category. If $\text{proj dim } A \leq 1$, then a Moore $G$-space $X$ of type $(A, n)$ exists and any two are $G$-equivalent (i.e., equivalent objects in $\text{Ho}^{\mathbf{I}} G$–$\text{Top}_*$). We denote $X$ by $M(A, n)$. Thus, for $\text{proj dim } A \leq 1$, $\Sigma M(A, n) \cong M(A, n + 1)$. Therefore, a Moore $G$-space of type $(A, n)$ with $n \geq 3$ and $\text{proj dim } A \leq 1$ is a co-$H$-group. This is also true for $n = 2$. For, following Kahn’s methods [Ka$_1$], we can find a $G$-space $K$ such that

\[ \overline{H}_i(K) = \begin{cases} A & \text{for } i = 1, \\ 0 & \text{for } i \neq 1. \end{cases} \]

By uniqueness, $M(A, 2) \cong \Sigma K$. Therefore, $M(A, n)$ is a cogroup object in $\text{Ho}^{\mathbf{I}} G$–$\text{Top}_*$ for $n \geq 2$ and $\text{proj dim } A \leq 1$.

If $A$ is a $G$-module and $\text{proj dim } A < \infty$, then by [Ka$_1$, p. 260] a classical Moore $G$-space of type $(A, n)$ exists and any two are equivalent (i.e., are equivalent objects in $\text{Ho}^{\mathbf{II}} G$–$\text{Top}_*$). This is seen by assigning an $\mathcal{O}_G$-module $\tilde{A}$ to $A$ as follows: let $\tilde{A}(G/H) = 0$ for $H \neq E$ and $\tilde{A}(G/E) = A$, where $E$ is the trivial subgroup of $G$. Then $\text{proj dim } \tilde{A} \leq 1$ and the existence of a classical Moore $G$-space follows from the previous paragraph. Uniqueness is also established and one concludes as above that a classical Moore $G$-space
There is a decreasing filtration of the group \( \{ A(n) \} \) for this category in the usual way as the right derived functor of the Hom functor. Using the above results and work of [Un], we conclude that a Moore \( G \)-space of type \((A, n)\) always exists. If, in addition, \( \text{proj dim } A < n \), then all such Moore \( G \)-spaces are equivalent. Thus if \( A \) is a rational \( \mathbb{O}_G \)-module of \( \text{proj dim } < n \), the Moore \( G \)-space of type \((A, n)\) is a cogroup object in \( \text{Ho}^1 G\text{-Top}_* \), \( n \geq 2 \). Similar considerations apply to classical Moore \( G \)-spaces.

3. Comultiplications. In this section we use Proposition 2.1 to determine the set \( C(X) \) of co-H-structures of \( X \), where \( X \) is a Moore \( G \)-space of type \((A, n)\), a co-group and \( \text{dim } X \leq 2n - 1 \). In preparation for this we need some results on Bredon cohomology.

For a given \( \mathbb{O}_G \)-module \( B \), Bredon [Br] and Illman [Il2] construct an equivariant cohomology theory \( H^G_*(-; B) \) defined on the category of pairs of \( G \)-spaces and \( G \)-maps. This cohomology theory satisfies all the Eilenberg–Steenrod axioms for cohomology suitably interpreted for equivariant spaces and maps. The category of \( \mathbb{O}_G \)-modules (i.e., the category whose objects are \( \mathbb{O}_G \)-modules and whose morphisms are natural transformations) contains sufficiently many projectives and injectives [Br]. Thus one can define \( \text{Ext}_G^p \) for this category in the usual way as the right derived functor of the Hom functor.

For a pair \((X, Y)\) of \( G\)-CW-complexes, Bredon [Br] derives a spectral sequence \( \{ E^{p,q}_r \} \) with

\[
E_2^{p,q} = \text{Ext}^p(X,Y; B) \Rightarrow H^{p+q}_G(X,Y; B).
\]

There is a decreasing filtration of the group \( H^{p+q}_G = H_G^{p+q}(X,Y; B), \)

\[
H^{p+q} = F^{-1}H^{p+q} \supseteq F^0H^{p+q} \supseteq \ldots \supseteq F^{p+q}H^{p+q} = 0,
\]

with

\[
F^pH^{p+q}/F^{p+1}H^{p+q} = E_{\infty}^{p,q}.
\]

Let now \( X \) be a Moore \( G \)-space of type \((A, n)\) for an \( \mathbb{O}_G \)-module \( A \) and \( n \geq 2 \). Then the Bredon spectral sequence degenerates, i.e., \( E_2^{p,q} = 0 \) for \( p \geq 0 \) and \( q \neq n \) and \( E_2^{p,n} = \text{Ext}^p(A,B) \). Thus

\[
0 = E_2^{p,q} = E_3^{p,q} = \ldots = E_{\infty}^{p,q} \text{ for } q \neq n \quad \text{and} \quad E_2^{p,n} = E_3^{p,n} = \ldots = E_{\infty}^{p,n}.
\]

Hence \( F_\infty^{p,q} = 0 \) for \( p \neq n \) and so (cf. [Ka1])

\[
H_2^G(X,B) = \text{Ext}^{p-n}(A,B).
\]

For a Moore \( G \)-space \( X \) of type \((A, n)\), let \( F \) denote the fibre of the map \( j : X \vee X \to X \times X \) in the category \( \text{Ho}^1 G\text{-Top}_* \) and let \( X \) be a cogroup in \( \text{Ho}^1 G\text{-Top}_* \). We denote by \([-,-]_G \) the set of morphisms in \( \text{Ho}^1 G\text{-Top}_* \).
THEOREM 3.2. Under the above assumptions, if $\dim X = d \leq 2n - 1$ then the set $\mathcal{C}(X)$ of co-$H$-structures of $X$ is in one-one correspondence with the group $\text{Ext}^{d-n}(A, \pi_d(F))$.

Proof. Since $H_i(X \times X, X \vee X) = 0$ for $i < 2n$ and $H_{2n}(X \times X, X \vee X) = H_n(X) \otimes H_n(X) = A \otimes A$, by the Hurewicz theorem, $\pi_i(X \times X, X \vee X) = 0$ for $i < 2n$ and $\pi_{2n}(X \times X, X \vee X) = A \otimes A$. Thus $\pi_i(F) = 0$ for $i < 2n - 1$ and $\pi_{2n-1}(F) = A \otimes A$. Let $F_d$ denote the $d$th term of the Postnikov $G$-tower of the $G$-space $F$ ([D–D–K], [Tr1]) and $f_d : F \to F_d$ the canonical map. Then the morphism $\pi_i(F) \to \pi_i(F_d)$ induced by $f_d$ is an isomorphism for $i \leq d$ and epimorphism for $i = d + 1$. Since $\dim X = d$, the equivariant Whitehead theorem ([H1], [Ma]) implies that $(f_d)_* : [X, F]_G \to [X, F_d]_G$ is a bijection. But $F_d \cong K(\pi_d(F), d)$, the Eilenberg–MacLane space of type $(\pi_d(F), d)$, since $d \leq 2n - 1$. Therefore

$$[X, F]_G \cong [X, F_d]_G \cong [X, K(\pi_d(F), d)]_G \cong H^d_G(X, \pi_d(F))$$

and this is $\text{Ext}^{d-n}(A, \pi_d(F))$ by (3.1). The result now follows from Proposition 2.1. \[\square\]

COROLLARY 3.3. If $\dim X < 2n - 1$, then $\mathcal{C}(X)$ has one element. If $\dim X = 2n - 1$, then $\mathcal{C}(X)$ is in one-one correspondence with $\text{Ext}^{n-1}(A, A \otimes A)$.

Now let $A$ be a $G$-module and $X$ a classical Moore $G$-space of type $(A, n)$ and a cogroup object in the category $\text{Ho}^H G\text{-Top}_\ast$. Then, by Proposition 2.1, $\mathcal{C}(X)$ is in one-one correspondence with the set $[X, F]_H$ of morphisms in $\text{Ho}^H G\text{-Top}_\ast$ of $X$ to $F$, where $F$ is the fibre of $j : X \vee X \to X \times X$. From the Hurewicz theorem we deduce that $\pi_i(F) = 0$ for $i < 2n - 1$ and $\pi_{2n-1}(F) \cong A \otimes A$ as $G$-modules. Suppose that $\dim X = d \leq 2n - 1$ and $F_d$ is the $d$th term of the Postnikov $G$-tower of $F$. Then as above $[X, F]_H$ is in one-one correspondence with $[X, F_d]_H$ and $F_d$ is an Eilenberg–MacLane $G$-space $K(\pi_d(F), d)$. Let $\tilde{\pi}_d(F)$ be the $O_G$-module defined by $\tilde{\pi}_d(F)(G/H) = 0$ for $H \neq E$ and $\tilde{\pi}_d(F)(G/E) = \pi_d(F)$, where $E$ is the trivial subgroup of $G$. Then $[X, F_d]_H \cong [X, K(\tilde{\pi}_d(F), d)]$, where $K(\tilde{\pi}_d(F), d)$ is the Eilenberg–MacLane $G$-space of type $(\tilde{\pi}_d(F), d)$. Hence by (3.1), $[X, F]_H \cong \text{Ext}^{d-n}_G(A, \pi_d(F))$, where $\text{Ext}^n_G$ denotes the $n$th Ext functor in the category of $G$-modules. Thus we obtain

COROLLARY 3.4. Let $A$ be a $G$-module and $X$ a classical Moore $G$-space of type $(A, n)$ and a cogroup object in $\text{Ho}^H G\text{-Top}_\ast$. If $\dim X < 2n - 1$, then $\mathcal{C}(X)$ has one element. If $\dim X = 2n - 1$, then $\mathcal{C}(X)$ is in one-one correspondence with $\text{Ext}^{n-1}_G(A, A \otimes A)$.

Remark 3.5. Corollary 3.2 (and 3.3) can also be proved by using a spectral sequence derived from an exact couple based on dual Puppe sequences.
obtained from the fibrations \( K(\pi_q, q) \to F_q \to F_{q-1} \) (cf. [M–T, Chap. 14]). In addition, this method shows, under the hypothesis of Corollary 3.3, that if \( \dim X > 2n - 1 \) and \( \text{Ext}^{n-1}(A, A \otimes A) \neq 0 \), then \( \mathcal{C}(X) \) has more than one element.

4. An example. Let \( \mathbb{Z}_{p^k} \) be the group of integers mod \( p^k \), where \( p \) is a prime, and let us denote \( \mathbb{O}_{\mathbb{Z}_{p^k}} \) by \( \mathbb{O}(\mathbb{Z}_{p^k}) \). Any \( \mathbb{O}(\mathbb{Z}_{p^k}) \)-module \( A \) determines a sequence

\[
A_0 \overset{m_1}{\to} A_1 \overset{m_2}{\to} \cdots \overset{m_i}{\to} A_k
\]

where \( A_i = A(\mathbb{Z}_{p^k}/\mathbb{Z}_{p^{k-i}}) \) and \( m_i = A(\pi_i) \), where \( \pi_i : \mathbb{Z}_{p^k}/\mathbb{Z}_{p^{k-i}} \to \mathbb{Z}_{p^k}/\mathbb{Z}_{p^{k-1}} \) are injections.

We restrict our considerations to rational \( \mathbb{O}(\mathbb{Z}_{p^k}) \)-modules, where all \( A_i \) are \( \mathbb{Q} \)-vector spaces and all \( m_i \) are linear maps. Triantafillou [Tr] shows that for any such \( \mathbb{O}(\mathbb{Z}_{p^k}) \)-module \( A \), \( \text{proj} \dim A \leq 1 \). Furthermore, \( A \) is projective if and only if all \( m_i \) are injections.

We define a rational \( \mathbb{O}(\mathbb{Z}_{p^k}) \)-module \( A \) to be null if all \( m_i = 0 \). With such a null \( \mathbb{O}(\mathbb{Z}_{p^k}) \)-module we associate the commutative diagram

\[
\begin{array}{cccccccc}
A_0 & \overset{0}{\to} & A_1 & \overset{0}{\to} & A_2 & \overset{0}{\to} & \cdots & \overset{0}{\to} & A_k \\
\| & & \uparrow m_1 & & \uparrow m_2 & & \cdots & & \uparrow m_k \\
A_0 & \overset{i_0}{\to} & A_0 \oplus A_1 & \overset{i_{0,1}}{\to} & A_0 \oplus A_1 \oplus A_2 & \overset{i_{0,1,2}}{\to} & \cdots & \overset{i_{0,1,\ldots,k-1}}{\to} & A_0 \oplus \cdots \oplus A_k \\
\| & & \uparrow i_0 & & \uparrow i_{0,1} & & \cdots & & \uparrow i_{0,1,\ldots,k-1} \\
0 & \to & A_0 & \overset{i_n}{\to} & A_0 \oplus A_1 & \overset{i_{n,1}}{\to} & \cdots & \overset{i_{n,1,\ldots,k-2}}{\to} & A_0 \oplus \cdots \oplus A_{k-1}
\end{array}
\]

where the arrows represent canonical projections and injections. Here the second horizontal line gives a rational \( \mathbb{O}(\mathbb{Z}_{p^k}) \)-module \( P_0 \) and the third horizontal line gives a rational \( \mathbb{O}(\mathbb{Z}_{p^k}) \)-module \( P_1 \) such that

\[
P_0(\mathbb{Z}_{p^k}/\mathbb{Z}_{p^{k-i}}) = A_0 \oplus \cdots \oplus A_i = P_1(\mathbb{Z}_{p^k}/\mathbb{Z}_{p^{k-1}}).
\]

Since all the maps in \( P_0 \) and \( P_1 \) are injective, \( P_0 \) and \( P_1 \) are projective \( \mathbb{O}(\mathbb{Z}_{p^k}) \)-modules. Therefore, we have a projective resolution

\[
0 \to P_1 \overset{d}{\to} P_0 \overset{\epsilon}{\to} A \to 0.
\]

If \( B \) is another null \( \mathbb{O}(\mathbb{Z}_{p^k}) \)-module, then the induced map

\[
d^* : \text{Hom}(P_0, B) \to \text{Hom}(P_1, B)
\]

is zero. Hence

\[
\text{Ext}^1(A, B) = \text{Hom}(P_1, B)/ \text{Im} d^* = \text{Hom}(P_1, B) = \bigoplus_{i=0}^{n-1} \text{Hom}(A_i, B_{i+1}).
\]

Thus we have proved
Proposition 4.1. If $A$ and $B$ are null $\mathbb{O}(\mathbb{Z}_{p^k})$-modules then

$$\text{Ext}^1(A, B) = \bigoplus_{i=0}^{k-1} \text{Hom}(A_i, B_{i+1}).$$

This leads to the following example.

Example 4.2. For the group $G = \mathbb{Z}_{p^k}$, there are $G$-spaces $X$ and $Y$ and $G$-maps $\varphi_i : X \to Y$, $i = 1, 2, \ldots$, such that $\varphi_i$ and $\varphi_j$ are not $G$-homotopic for all $i \neq j$ and $\varphi_i^H, \varphi_j^H : X^H \to Y^H$ are homotopic for all $i, j$ and all subgroups $H$ of $G$.

For this example we let $A$ be a null $\mathbb{O}(\mathbb{Z}_{p^k})$-module such that $\text{Hom}(A_i, A_{i+1} \otimes A_{i+1}) \neq 0$ for some $i \in \{0, 1, \ldots, k-1\}$, for example, $A_i = A_{i+1} = \mathbb{Q}$. Since $\text{proj dim } A \leq 1$, there is a Moore $G$-space $X$ of type $(A, 2)$ which is a co-H-group (see Section 2). Kahn [Ka1, p. 259] has shown how to construct $X$ such that $\dim X = 3$. By Corollary 3.3, $C(X)$ is in one-one correspondence with $\text{Ext}^1(A, A \otimes A)$. By Proposition 4.1, this latter group is isomorphic to $\bigoplus_{i=1}^{n-1} \text{Hom}(A_i, A_{i+1} \otimes A_{i+1}) \neq 0$. Thus $C(X)$ is an infinite set and so there are infinitely many co-H-structures $\varphi_i : X \to X \vee X = Y$ in $\text{Ho}^1 G \text{-Top}_*$. However, for any subgroup $H$, $X^H$ is the nonequivariant Moore space of type $(A(G/H), 2)$ and each $\varphi_i^H$ is a comultiplication of $X^H$. But by [A–G] the comultiplications of $X^H$ are in one-one correspondence with $\text{Ext}(A(G/H), A(G/H) \otimes A(G/H))$. This group is trivial since $A(G/H)$ is a $\mathbb{Q}$-vector space. Thus for each subgroup $H$ of $G$, $\varphi_i^H$ is homotopic to $\varphi_j^H$ for all $i, j = 1, 2, \ldots$.

Finally, we close with a problem suggested by [A–G]. Given an action of a finite group $G$ on $\mathbb{Z}_m$, the integers mod $m$. Suppose there is a classical Moore $G$-space $X$ of type $(\mathbb{Z}_m, 2)$ which is a co-H-group.

Problem 4.3. Describe the set $C(X)$ of all comultiplications of $X$.

References


Co-H-structures on equivariant Moore spaces


