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Co-H-structures on equivariant Moore spaces

by

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Abstract. Let G be a finite group, $\mathbb{O}_G$ the category of canonical orbits of G and $\boldsymbol{A} : \mathbb{O}_G \to \mathbb{A}\mathrm{b}$ a contravariant functor to the category of abelian groups. We investigate the set of G-homotopy classes of comultiplications of a Moore G-space of type $(\boldsymbol{A}, n)$ where $n \geq 2$ and prove that if such a Moore G-space X is a cogroup, then it has a unique comultiplication if $\dim X < 2n-1$. If $\dim X = 2n-1$, then the set of comultiplications of X is in one-one correspondence with $\mathrm{Ext}^{n-1}(\boldsymbol{A}, \boldsymbol{A} \otimes \boldsymbol{A})$. Then the case $G = \mathbb{Z}_{p^k}$ leads to an example of infinitely many G-homotopically distinct G-maps $\varphi_i : X \to Y$ such that $\varphi_i^H, \varphi_j^H : X^H \to Y^H$ are homotopic for all i, j and all subgroups $H \subseteq G$.

1. Introduction. If A is an abelian group and n an integer ≥ 2, then a *Moore space of type* (A,n) is a space with a single nonvanishing homology group A in dimension n. Moore spaces play a central role in homotopy theory and have been widely studied. In particular, the co-H-structures of a Moore space have been investigated. It is known that for $n > 2$ there is a unique co-H-structure (up to homotopy) on a Moore space, but that for $n = 2$ there may be several distinct co-H-structures (e.g., see [A–G]). In this paper we consider these results within the context of equivariant homotopy theory.

Throughout, G denotes a finite group and all spaces, maps, homotopies and actions are pointed. We work in the category G-$\mathbb{T}\mathrm{op}_*$ of G-spaces which have the G-homotopy type of G-CW-complexes [Br]. We denote by $\mathbb{O}_G$ the *category of canonical orbits* of G whose objects are the left cosets G/H as H ranges over all subgroups of G and whose morphisms are the equivariant maps $G/H \to G/K$ with respect to left translation. An $\mathbb{O}_G$*-module* is a contravariant functor from $\mathbb{O}_G$ into $\mathbb{A}\mathrm{b}$, the category of abelian groups. For a pair (X,Y) of G-spaces and an integer $n \geq 1$, an $\mathbb{O}_G$-module $\boldsymbol{H}_n(X,Y) : \mathbb{O}_G \to \mathbb{A}\mathrm{b}$ can be defined as follows: $\boldsymbol{H}_n(X,Y)(G/H) = H_n(X^H, Y^H)$, where H_n denotes the nth singular homology functor and X^H is the H-fixedpoint subspace of X. Similarly, with $n \geq 3$ ($n \geq 2$ if Y is the base point $*$) we define $\boldsymbol{\pi}_n(X,Y) : \mathbb{O}_G \to \mathbb{A}\mathrm{b}$ using the nth homotopy functor π_n. For

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$Y = *$, these $\mathbb{O}_G$-modules are denoted by $\widetilde{\boldsymbol{H}}_n(X)$ and $\boldsymbol{\pi}_n(X)$, respectively.

Now let $\boldsymbol{A} : \mathbb{O}_G \to \mathbb{A}\mathrm{b}$ be an $\mathbb{O}_G$-module and $n \geq 2$ an integer. Following Kahn [Ka$_1$], we define a *Moore G-space of type* $(\boldsymbol{A}, n)$ to be a G-space X such that

(1) X^H is 1-connected for all subgroups H of G,
(2) $\widetilde{\boldsymbol{H}}_n(X) \cong \boldsymbol{A}$ as $\mathbb{O}_G$-modules,
(3) $\widetilde{\boldsymbol{H}}_i(X) = 0$ for $i \neq n$.

If the H-fixedpoint sets X^H are disregarded when H is a nontrivial subgroup, then we obtain a classical Moore G-space. More precisely, if A is a G-module and $n \geq 2$, then a *classical Moore G-space of type* (A, n) is a G-space X such that

(1) X is 1-connected,
(2) $\widetilde{H}_n(X) \cong A$ as G-modules,
(3) $\widetilde{H}_i(X) = 0$ for $i \neq n$.

Moore G-spaces have been considered in several papers ([Do$_1$], [Do$_2$], [Ka$_1$], [Ka$_2$]) and shown to be important in equivariant homotopy theory (e.g., the construction of an equivariant homology decomposition [Ka$_2$]). Furthermore, classical Moore spaces have been extensively studied in connection with the Steenrod problem (e.g., [Ca], [Ka$_3$], [Sm]). Unlike the nonequivariant case, Moore G-spaces need not exist for any $\mathbb{O}_G$-module $\boldsymbol{A}$, and when they exist, they need not be unique (see Section 2 for known existence and uniqueness results). This is so even for classical Moore G-spaces.

In this paper we extend the results of [A–G] to the equivariant case and investigate the set of G-homotopy classes of comultiplications of a Moore G-space. We begin with some generalities on closed model categories $\mathbb{C}$. We show that if X is a cogroup object in $\mathrm{Ho}\,\mathbb{C}$, the associated homotopy category of $\mathbb{C}$, then the collection of comultiplications of X is in one-one correspondence with the set of morphisms $\mathrm{Ho}\,\mathbb{C}(X, F)$, where F is the fibre of the canonical morphism $X \vee X \to X \times X$. Next we introduce two closed model structures on G-$\mathbb{T}\mathrm{op}_*$, one to be used for Moore G-spaces and the other for classical Moore G-spaces. We then deduce in the next section that a Moore G-space X of type $(\boldsymbol{A}, n)$ which is a cogroup has a unique comultiplication if $\dim X < 2n - 1$. If $\dim X = 2n - 1$, we show that the set of comultiplications of X is in one-one correspondence with $\mathrm{Ext}^{n-1}(\boldsymbol{A}, \boldsymbol{A} \otimes \boldsymbol{A})$. Analogous results are established for classical Moore G-spaces. We then apply these considerations to the case $G = \mathbb{Z}_{p^k}$. This leads to an example of infinitely many G-homotopically distinct G-maps $\varphi_i : X \to Y$ such that $\varphi_i^H, \varphi_j^H : X^H \to Y^H$ are homotopic for all i, j and all subgroups $H \subseteq G$.

2. Background. The general reference here for category theory is [Qu]. Let $\mathbb{C}$ be a pointed category with finite products and coproducts. For objects X and Y of $\mathbb{C}$, morphisms are written $f : X \to Y$ or $f \in \mathbb{C}(X,Y)$. In particular, the zero morphism is $0 : X \to Y$ and the identity morphism is $1_X : X \to X$. Let $X \vee Y$ denote the coproduct of X and Y and $X \times Y$ the product of X and Y. Then for an object X, there is a canonical morphism $j : X \vee X \to X \times X$ determined by two morphisms $(1_X, 0), (0, 1_X) : X \to X \times X$. Let $\triangle = (1_X, 1_X) : X \to X \times X$ be the diagonal morphism. A morphism $\varphi : X \to X \vee X$ such that $j\varphi = \triangle$ is called a *comultiplication* of X, and X is said to have *co-structure* φ. If $(1 \vee \varphi)\varphi = (\varphi \vee 1)\varphi : X \to X \vee X \vee X$ then φ is *associative*. If there exists a morphism $\eta : X \to X$ such that $\nabla(\eta \vee 1_X)\varphi = \nabla(1_X \vee \eta)\varphi = 0 : X \to X$, where $\nabla : X \vee X \to X$ is the folding morphism, we say that η is an *inverse*. The triple (X, φ, η) is then called a *cogroup object* in $\mathbb{C}$. If (X, φ, η) is a cogroup object in $\mathbb{C}$ and Y is any object, then φ and η induce a group structure on the set $\mathbb{C}(X,Y)$ such that for every morphism $g : Y \to Y'$, the induced map $g_* : \mathbb{C}(X,Y) \to \mathbb{C}(X,Y')$ is a homomorphism.

Now let $\mathbb{C}$ be a pointed closed model category. We localize $\mathbb{C}$ with respect to the class of weak equivalences and obtain the homotopy category $\operatorname{Ho}\mathbb{C}$ [Qu]. A co-structure on an object in $\operatorname{Ho}\mathbb{C}$ is called a *co-H-structure* and a cogroup object in $\operatorname{Ho}\mathbb{C}$ is called a *co-H-group*. Quillen [Qu] has defined a suspension functor $\Sigma \operatorname{Ho}\mathbb{C} \to \operatorname{Ho}\mathbb{C}$ such that ΣX is a co-H-group. For any objects X, Y in $\operatorname{Ho}\mathbb{C}$, let us denote $\operatorname{Ho}\mathbb{C}(X,Y)$ by $[X,Y]$. Then if $f : X \to Y$, there exists an object F, called the *fibre* of f, such that for any object Z, the following sequence is exact [Qu]:

$$\ldots \to [\Sigma Z, X] \xrightarrow{f_*} [\Sigma Z, Y] \to [Z, F] \to [Z, X] \xrightarrow{f_*} [Z, Y].$$

Let X be a co-H-group, $\mathcal{C}(X) \subseteq [X, X \vee X]$ the set of co-H-structures of X and F the fibre of the canonical morphism $j : X \vee X \to X \times X$. Then the set $\mathcal{C}(X)$ is an orbit of the action of the group $[X,F]$ on $[X, X \vee X]$ by (right) translation. So there is, in general, no natural group structure on $\mathcal{C}(X)$. However, if an element of $\mathcal{C}(X)$ is chosen as a base point it is possible to offer a direct interpretation of the group structure of $\mathcal{C}(X)$.

PROPOSITION 2.1. *For any co-H-group object X in* $\operatorname{Ho}\mathbb{C}$, *there is a group isomorphism*

$$\mathcal{C}(X) \xrightarrow{\cong} [X,F].$$

The proof follows from the above long exact sequence applied to j together with the methods of [A–G].

Next let $\mathbb{T}\mathrm{op}_*$ be the category of pointed topological spaces. We give $\mathbb{T}\mathrm{op}_*$ the structure of a pointed closed model category by defining weak equivalences, fibrations and cofibrations in $\mathbb{T}\mathrm{op}_*$ in the usual way [Qu]. Let

G-$\mathbb{T}\mathrm{op}_*$ be the category with objects pointed G-spaces and morphisms G-maps. We define a closed model category structure **I** on G-$\mathbb{T}\mathrm{op}_*$ as follows:

I-1. A G-map $f : X \to Y$ is a *weak equivalence* if the maps $f^H : X^H \to Y^H$ of H-fixedpoint subspaces are weak equivalences in $\mathbb{T}\mathrm{op}_*$ for all subgroups $H \subseteq G$.

I-2. A G-map $f : E \to B$ is a *fibration* if $f^H : E^H \to B^H$ are fibrations in $\mathbb{T}\mathrm{op}_*$ for all subgroups $H \subseteq G$.

I-3. *Cofibrations* are determined by weak equivalences and fibrations by means of the lifting property [Qu, p. 5.1].

We also define a second closed model category structure **II** on G-$\mathbb{T}\mathrm{op}_*$:

II-1. A G-map $f : X \to Y$ is a *weak equivalence* if f is a weak equivalence in $\mathbb{T}\mathrm{op}_*$.

II-2. A G-map $f : E \to B$ is a *fibration* if f is a fibration in $\mathbb{T}\mathrm{op}_*$.

II-3. *Cofibrations* are determined by weak equivalences and fibrations as above.

One checks that **I** and **II** satisfy the axioms for a pointed closed model category (cf. [D–D–K]) and thus one obtains homotopy categories

$$\mathrm{Ho}^{\mathbf{I}}\, G\text{-}\,\mathbb{T}\mathrm{op}_* \quad \text{and} \quad \mathrm{Ho}^{\mathbf{II}}\, G\text{-}\,\mathbb{T}\mathrm{op}_*$$

by localizing with respect to the weak equivalences of **I** and **II**, respectively.

Finally, we summarize from [Ka_1] conditions for the existence and uniqueness of a Moore G-space X of type $(\boldsymbol{A}, n)$, where $\boldsymbol{A}$ is an $\mathbb{O}_G$-module. We are especially interested in when X is a cogroup in the appropriate category. If $\operatorname{proj\,dim} \boldsymbol{A} \le 1$, then a Moore G-space X of type $(\boldsymbol{A}, n)$ exists and any two are G-equivalent (i.e., equivalent objects in $\mathrm{Ho}^{\mathbf{I}}\, G$-$\mathbb{T}\mathrm{op}_*$). We denote X by $M(\boldsymbol{A}, n)$. Thus, for $\operatorname{proj\,dim} \boldsymbol{A} \le 1$, $\Sigma M(\boldsymbol{A}, n) \cong M(\boldsymbol{A}, n+1)$. Therefore, a Moore G-space of type $(\boldsymbol{A}, n)$ with $n \ge 3$ and $\operatorname{proj\,dim} \boldsymbol{A} \le 1$ is a co-H-group. This is also true for $n = 2$. For, following Kahn's methods [Ka_1], we can find a G-space K such that

$$\widetilde{\boldsymbol{H}}_i(K) = \begin{cases} \boldsymbol{A} & \text{for } i = 1, \\ 0 & \text{for } i \ne 1. \end{cases}$$

By uniqueness, $M(\boldsymbol{A}, 2) \cong \Sigma K$. Therefore, $M(\boldsymbol{A}, n)$ is a cogroup object in $\mathrm{Ho}^{\mathbf{I}}\, G$-$\mathbb{T}\mathrm{op}_*$ for $n \ge 2$ and $\operatorname{proj\,dim} \boldsymbol{A} \le 1$.

If A is a G-module and $\operatorname{proj\,dim} A < \infty$, then by [$\mathrm{Ka}_1$, p. 260] a classical Moore G-space of type (A, n) exists and any two are equivalent (i.e., are equivalent objects in $\mathrm{Ho}^{\mathbf{II}}\, G$-$\mathbb{T}\mathrm{op}_*$). This is seen by assigning an $\mathbb{O}_G$-module $\widetilde{\boldsymbol{A}}$ to A as follows: let $\widetilde{\boldsymbol{A}}(G/H) = 0$ for $H \ne E$ and $\widetilde{\boldsymbol{A}}(G/E) = A$, where E is the trivial subgroup of G. Then $\operatorname{proj\,dim} \widetilde{\boldsymbol{A}} \le 1$ and the existence of a classical Moore G-space follows from the previous paragraph. Uniqueness is also established and one concludes as above that a classical Moore G-space of type (A, n) with $\operatorname{proj\,dim} A < \infty$ and $n \geq 2$ is a cogroup object in $\mathrm{Ho}^{\mathbf{II}}\, G\text{-}\mathbb{T}\mathrm{op}_*$.

We next assume that $\boldsymbol{A}$ is a *rational* $\mathbb{O}_G$-module, that is, an $\mathbb{O}_G$-module such that each $\boldsymbol{A}(G/H)$ is a vector space over the field $\mathbb{Q}$ of rational numbers. Using the above results and work of [Un], we conclude that a Moore G-space of type $(\boldsymbol{A}, n)$ always exists. If, in addition, $\operatorname{proj\,dim} \boldsymbol{A} < n$, then all such Moore G-spaces are equivalent. Thus if $\boldsymbol{A}$ is a rational $\mathbb{O}_G$-module of proj dim $< n$, the Moore G-space of type $(\boldsymbol{A}, n)$ is a cogroup object in $\mathrm{Ho}^{\mathbf{I}}\, G$-$\mathbb{T}\mathrm{op}_*$, $n \geq 2$. Similar considerations apply to classical Moore G-spaces.

3. Comultiplications. In this section we use Proposition 2.1 to determine the set $\mathcal{C}(X)$ of co-H-structures of X, where X is a Moore G-space of type $(\boldsymbol{A}, n)$, a co-group and $\dim X \leq 2n-1$. In preparation for this we need some results on Bredon cohomology.

For a given $\mathbb{O}_G$-module $\boldsymbol{B}$, Bredon [Br] and Illman [Il_2] construct an equivariant cohomology theory $H^*_G(-, \boldsymbol{B})$ defined on the category of pairs of G-spaces and G-maps. This cohomology theory satisfies all the Eilenberg–Steenrod axioms for cohomology suitably interpreted for equivariant spaces and maps. The category of $\mathbb{O}_G$-modules (i.e., the category whose objects are $\mathbb{O}_G$-modules and whose morphisms are natural transformations) contains sufficiently many projectives and injectives [Br]. Thus one can define Ext^p for this category in the usual way as the right derived functor of the Hom functor.

For a pair (X, Y) of G-CW-complexes, Bredon [Br] derives a spectral sequence $\{E_r^{p,q}\}$ with

$$E_2^{p,q} = \mathrm{Ext}^p(\boldsymbol{H}_q(X, Y), \boldsymbol{B}) \Rightarrow H_G^{p+q}(X, Y; \boldsymbol{B}).$$

There is a decreasing filtration of the group $H^{p+q} = H_G^{p+q}(X, Y; \boldsymbol{B})$,

$$H^{p+q} = F^{-1}H^{p+q} \supseteq F^0 H^{p+q} \supseteq \ldots \supseteq F^{p+q}H^{p+q} = 0,$$

with

$$F^p H^{p+q}/F^{p+1}H^{p+q} = E_\infty^{p,q}.$$

Let now X be a Moore G-space of type $(\boldsymbol{A}, n)$ for an $\mathbb{O}_G$-module $\boldsymbol{A}$ and $n \geq 2$. Then the Bredon spectral sequence degenerates, i.e., $E_2^{p,q} = 0$ for $p \geq 0$ and $q \neq n$ and $E_2^{p,n} = \mathrm{Ext}^p(\boldsymbol{A}, \boldsymbol{B})$. Thus

$$0 = E_2^{p,q} = E_3^{p,q} = \ldots = E_\infty^{p,q} \quad \text{for } q \neq n \quad \text{and} \quad E_2^{p,n} = E_3^{p,n} = \ldots = E_\infty^{p,n}.$$

Hence $F^{p-q}H^p/F^{p-q+1}H^p = E_\infty^{p-q,q} = 0$ for $q \neq n$ and so (cf. [Ka_1])

$$H_G^p(X, \boldsymbol{B}) = \mathrm{Ext}^{p-n}(\boldsymbol{A}, \boldsymbol{B}). \tag{3.1}$$

For a Moore G-space X of type $(\boldsymbol{A}, n)$, let F denote the fibre of the map $j : X \vee X \to X \times X$ in the category $\mathrm{Ho}^{\mathbf{I}}\, G$-$\mathbb{T}\mathrm{op}_*$ and let X be a cogroup in $\mathrm{Ho}^{\mathbf{I}}\, G$-$\mathbb{T}\mathrm{op}_*$. We denote by $[-,-]_G$ the set of morphisms in $\mathrm{Ho}^{\mathbf{I}}\, G$-$\mathbb{T}\mathrm{op}_*$.

Theorem 3.2. *Under the above assumptions*, *if* $\dim X = d \le 2n-1$ *then the set* $\mathcal{C}(X)$ *of co-H-structures of* X *is in one-one correspondence with the group* $\operatorname{Ext}^{d-n}(\boldsymbol{A}, \boldsymbol{\pi}_d(F))$.

P r o o f. Since $\boldsymbol{H}_i(X\times X, X\vee X) = 0$ for $i < 2n$ and $\boldsymbol{H}_{2n}(X\times X, X\vee X) = \boldsymbol{H}_n(X)\otimes \boldsymbol{H}_n(X) = \boldsymbol{A}\otimes\boldsymbol{A}$, by the Hurewicz theorem, $\boldsymbol{\pi}_i(X\times X, X\vee X) = 0$ for $i<2n$ and $\boldsymbol{\pi}_{2n}(X\times X, X\vee X) = \boldsymbol{A}\otimes\boldsymbol{A}$. Thus $\boldsymbol{\pi}_i(F) = 0$ for $i < 2n-1$ and $\boldsymbol{\pi}_{2n-1}(F) = \boldsymbol{A}\otimes\boldsymbol{A}$. Let F_d denote the dth term of the Postnikov G-tower of the G-space F ([D–D–K], [Tr$_1$]) and $f_d : F \to F_d$ the canonical map. Then the morphism $\boldsymbol{\pi}_i(F) \to \boldsymbol{\pi}_i(F_d)$ induced by f_d is an isomorphism for $i \le d$ and epimorphism for $i = d+1$. Since $\dim X = d$, the equivariant Whitehead theorem ([Il$_1$], [Ma]) implies that $(f_d)_* : [X,F]_G \to [X,F_d]_G$ is a bijection. But $F_d \cong K(\boldsymbol{\pi}_d(F), d)$, the Eilenberg–MacLane space of type $(\boldsymbol{\pi}_d(F), d)$, since $d \le 2n-1$. Therefore

$$[X,F]_G \cong [X,F_d]_G \cong [X, K(\boldsymbol{\pi}_d(F), d)]_G \cong H^d_G(X, \boldsymbol{\pi}_d(F))$$

and this is $\operatorname{Ext}^{d-n}(\boldsymbol{A}, \boldsymbol{\pi}_d(F))$ by (3.1). The result now follows from Proposition 2.1. ∎

Corollary 3.3. *If* $\dim X < 2n-1$, *then* $\mathcal{C}(X)$ *has one element. If* $\dim X = 2n-1$, *then* $\mathcal{C}(X)$ *is in one-one correspondence with* $\operatorname{Ext}^{n-1}(\boldsymbol{A}, \boldsymbol{A}\otimes\boldsymbol{A})$.

Now let A be a G-module and X a classical Moore G-space of type (A,n) and a cogroup object in the category $\mathrm{Ho}^{\mathbf{II}}\, G\text{-}\mathbb{T}\mathrm{op}_*$. Then, by Proposition 2.1, $\mathcal{C}(X)$ is in one-one correspondence with the set $[X,F]_{\mathbf{II}}$ of morphisms in $\mathrm{Ho}^{\mathbf{II}}\, G\text{-}\mathbb{T}\mathrm{op}_*$ of X to F, where F is the fibre of $j : X\vee X \to X\times X$. From the Hurewicz theorem we deduce that $\pi_i(F) = 0$ for $i < 2n-1$ and $\pi_{2n-1}(F) \cong A\otimes A$ as G-modules. Suppose that $\dim X = d \le 2n-1$ and F_d is the dth term of the Postnikov G-tower of F. Then as above $[X,F]_{\mathbf{II}}$ is in one-one correspondence with $[X,F_d]_{\mathbf{II}}$ and F_d is an Eilenberg–MacLane G-space $K(\pi_d(F), d)$. Let $\widetilde{\boldsymbol{\pi}}_d(F)$ be the $\mathbb{O}_G$-module defined by $\widetilde{\boldsymbol{\pi}}_d(F)(G/H) = 0$ for $H \ne E$ and $\widetilde{\boldsymbol{\pi}}_d(F)(G/E) = \pi_d(F)$, where E is the trivial subgroup of G. Then $[X,F_d]_{\mathbf{II}} \cong [X, K(\widetilde{\boldsymbol{\pi}}_d(F), d)]$, where $K(\widetilde{\boldsymbol{\pi}}_d(F), d)$ is the Eilenberg–MacLane G-space of type $(\widetilde{\boldsymbol{\pi}}_d(F), d)$. Hence by (3.1), $[X,F]_{\mathbf{II}} \cong \operatorname{Ext}^{d-n}_G(A, \pi_d(F))$, where Ext^p_G denotes the pth Ext functor in the category of G-modules. Thus we obtain

Corollary 3.4. *Let* A *be a* G*-module and* X *a classical Moore* G*-space of type* (A,n) *and a cogroup object in* $\mathrm{Ho}^{\mathbf{II}}\, G\text{-}\mathbb{T}\mathrm{op}_*$. *If* $\dim X < 2n-1$, *then* $\mathcal{C}(X)$ *has one element. If* $\dim X = 2n-1$, *then* $\mathcal{C}(X)$ *is in one-one correspondence with* $\operatorname{Ext}^{n-1}_G(A, A\otimes A)$.

R e m a r k 3.5. Corollary 3.2 (and 3.3) can also be proved by using a spectral sequence derived from an exact couple based on dual Puppe sequences obtained from the fibrations $K(\boldsymbol{\pi}_q, q) \to F_q \to F_{q-1}$ (cf. [M–T, Chap. 14]). In addition, this method shows, under the hypothesis of Corollary 3.3, that if $\dim X > 2n-1$ and $\operatorname{Ext}^{n-1}(\boldsymbol{A}, \boldsymbol{A} \otimes \boldsymbol{A}) \neq 0$, then $\mathcal{C}(X)$ has more than one element.

4. An example. Let $\mathbb{Z}_{p^k}$ be the group of integers $\operatorname{mod} p^k$, where p is a prime, and let us denote $\mathbb{O}_{\mathbb{Z}_{p^k}}$ by $\mathbb{O}(\mathbb{Z}_{p^k})$. Any $\mathbb{O}(\mathbb{Z}_{p^k})$-module $\boldsymbol{A}$ determines a sequence

$$A_0 \xrightarrow{m_1} A_1 \xrightarrow{m_2} \ldots \xrightarrow{m_k} A_k$$

where $A_i = \boldsymbol{A}(\mathbb{Z}_{p^k}/\mathbb{Z}_{p^{k-i}})$ and $m_i = \boldsymbol{A}(\pi_i)$, where $\pi_i : \mathbb{Z}_{p^k}/\mathbb{Z}_{p^{k-i}} \to \mathbb{Z}_{p^k}/\mathbb{Z}_{p^{k-i+1}}$ are projections.

We restrict our considerations to *rational* $\mathbb{O}(\mathbb{Z}_{p^k})$-modules, where all A_i are $\mathbb{Q}$-vector spaces and all m_i are linear maps. Triantafillou [Tr] shows that for any such $\mathbb{O}(\mathbb{Z}_{p^k})$-module $\boldsymbol{A}$, $\operatorname{proj}\dim \boldsymbol{A} \leq 1$. Furthermore, $\boldsymbol{A}$ is projective if and only if all m_i are injections.

We define a rational $\mathbb{O}(\mathbb{Z}_{p^k})$-module $\boldsymbol{A}$ to be *null* if all $m_i = 0$. With such a null $\mathbb{O}(\mathbb{Z}_{p^k})$-module we associate the commutative diagram

$$\begin{array}{ccccccccc}
A_0 & \xrightarrow{0} & A_1 & \xrightarrow{0} & A_2 & \xrightarrow{0} & \ldots & \xrightarrow{0} & A_k \\
\| & & \uparrow p_1 & & \uparrow p_2 & & \ldots & & \uparrow p_k \\
A_0 & \xrightarrow{i_0} & A_0 \oplus A_1 & \xrightarrow{i_{0,1}} & A_0 \oplus A_1 \oplus A_2 & \xrightarrow{i_{0,1,2}} & \ldots & \xrightarrow{i_{0,1,\ldots,k-1}} & A_0 \oplus \ldots \oplus A_k \\
\uparrow & & \uparrow i_0 & & \uparrow i_{0,1} & & \ldots & & \uparrow i_{0,1,\ldots,k-1} \\
0 & \to & A_0 & \xrightarrow{i_0} & A_0 \oplus A_1 & \xrightarrow{i_{0,1}} & \ldots & \xrightarrow{i_{0,1,\ldots,k-2}} & A_0 \oplus \ldots \oplus A_{k-1}
\end{array}$$

where the arrows represent canonical projections and injections. Here the second horizontal line gives a rational $\mathbb{O}(\mathbb{Z}_{p^k})$-module $\boldsymbol{P}_0$ and the third horizontal line gives a rational $\mathbb{O}(\mathbb{Z}_{p^k})$-module $\boldsymbol{P}_1$ such that

$$\boldsymbol{P}_0(\mathbb{Z}_{p^k}/\mathbb{Z}_{p^{k-i}}) = A_0 \oplus \ldots \oplus A_i = \boldsymbol{P}_1(\mathbb{Z}_{p^k}/\mathbb{Z}_{p^{k-i+1}}).$$

Since all the maps in $\boldsymbol{P}_0$ and $\boldsymbol{P}_1$ are injective, $\boldsymbol{P}_0$ and $\boldsymbol{P}_1$ are projective $\mathbb{O}(\mathbb{Z}_{p^k})$-modules. Therefore, we have a projective resolution

$$0 \to \boldsymbol{P}_1 \xrightarrow{d} \boldsymbol{P}_0 \xrightarrow{\varepsilon} \boldsymbol{A} \to 0.$$

If $\boldsymbol{B}$ is another null $\mathbb{O}(\mathbb{Z}_{p^k})$-module, then the induced map

$$d^* : \operatorname{Hom}(\boldsymbol{P}_0, \boldsymbol{B}) \to \operatorname{Hom}(\boldsymbol{P}_1, \boldsymbol{B})$$

is zero. Hence

$$\operatorname{Ext}^1(\boldsymbol{A}, \boldsymbol{B}) = \operatorname{Hom}(\boldsymbol{P}_1, \boldsymbol{B}) / \operatorname{Im} d^* = \operatorname{Hom}(\boldsymbol{P}_1, \boldsymbol{B}) = \bigoplus_{i=0}^{n-1} \operatorname{Hom}(A_i, B_{i+1}).$$

Thus we have proved

Proposition 4.1. *If* $\boldsymbol{A}$ *and* $\boldsymbol{B}$ *are null* $\mathbb{O}(\mathbb{Z}_{p^k})$*-modules then*

$$\operatorname{Ext}^1(\boldsymbol{A},\boldsymbol{B})=\bigoplus_{i=0}^{k-1}\operatorname{Hom}(A_i,B_{i+1}).$$

This leads to the following example.

Example 4.2. For the group $G=\mathbb{Z}_{p^k}$, there are G-spaces X and Y and G-maps $\varphi_i: X\to Y$, $i=1,2,\ldots$, such that φ_i and φ_j are not G-homotopic for all $i\neq j$ and $\varphi_i^H,\varphi_j^H: X^H\to Y^H$ are homotopic for all i, j and all subgroups H of G.

For this example we let $\boldsymbol{A}$ be a null $\mathbb{O}(\mathbb{Z}_{p^k})$-module such that $\operatorname{Hom}(A_i, A_{i+1}\otimes A_{i+1})\neq 0$ for some $i\in\{0,1,\ldots,k-1\}$, for example, $A_i=A_{i+1}=\mathbb{Q}$. Since $\operatorname{proj}\dim\boldsymbol{A}\le 1$, there is a Moore G-space X of type $(\boldsymbol{A},2)$ which is a co-H-group (see Section 2). Kahn [Ka$_1$, p. 259] has shown how to construct X such that $\dim X=3$. By Corollary 3.3, $\mathcal{C}(X)$ is in one-one correspondence with $\operatorname{Ext}^1(\boldsymbol{A},\boldsymbol{A}\otimes\boldsymbol{A})$. By Proposition 4.1, this latter group is isomorphic to $\bigoplus_{i=1}^{n-1}\operatorname{Hom}(A_i,A_{i+1}\otimes A_{i+1})\neq 0$. Thus $\mathcal{C}(X)$ is an infinite set and so there are infinitely many co-H-structures $\varphi_i: X\to X\vee X=Y$ in $\mathrm{Ho}^{\mathbf{I}}\,G\text{-}\mathbb{T}\mathrm{op}_*$. However, for any subgroup H, X^H is the nonequivariant Moore space of type $(\boldsymbol{A}(G/H),2)$ and each φ_i^H is a comultiplication of X^H. But by [A–G] the comultiplications of X^H are in one-one correspondence with $\operatorname{Ext}(\boldsymbol{A}(G/H),\boldsymbol{A}(G/H)\otimes\boldsymbol{A}(G/H))$. This group is trivial since $\boldsymbol{A}(G/H)$ is a $\mathbb{Q}$-vector space. Thus for each subgroup H of G, φ_i^H is homotopic to φ_j^H for all $i,j=1,2,\ldots$ ∎

Finally, we close with a problem suggested by [A–G]. Given an action of a finite group G on $\mathbb{Z}_m$, the integers $\bmod m$. Suppose there is a classical Moore G-space X of type $(\mathbb{Z}_m,2)$ which is a co-H-group.

Problem 4.3. Describe the set $\mathcal{C}(X)$ of all comultiplications of X.

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