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The space of ANR's in $\mathbb{R}^n$

by

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Abstract. The hyperspaces $\mathrm{ANR}(\mathbb{R}^n)$ and $\mathrm{AR}(\mathbb{R}^n)$ in $2^{\mathbb{R}^n}$ $(n \geq 3)$ consisting respectively of all compact absolute neighborhood retracts and all compact absolute retracts are studied. It is shown that both have the Borel type of absolute $G_{\delta\sigma\delta}$-spaces and that, indeed, they are not $F_{\sigma\delta\sigma}$-spaces. The main result is that $\mathrm{ANR}(\mathbb{R}^n)$ is an absorber for the class of all absolute $G_{\delta\sigma\delta}$-spaces and is therefore homeomorphic to the standard model space Ω_3 of this class.

1. Introduction. For a metric space Z, by 2^Z we denote the space of all nonempty compacta with the topology determined by the Hausdorff distance. We shall refer to any subspace of 2^Z as a hyperspace. The topological classification of such hyperspaces has been an object of study for many years [vM]. One of the fundamental results in the area is that 2^Z is homeomorphic to the Hilbert cube $Q = [-1, 1]^\infty$ if and only if Z is a Peano continuum. If we consider $C(Z) \subset 2^Z$, the subspace consisting of all the continua in Z, then one has $C(Z) \cong Q$ if and only if Z is a Peano continuum which does not contain a free arc.

Recently, much attention has been paid to hyperspaces which are not necessarily complete-metrizable (see [C1], [C2], [DvMM], [DR], [GvM], [CDGvM], and others). The central theme of these papers has been to isolate, for a given Z, a certain subspace of 2^Z determined by some characteristic topological property and then to topologically identify this hyperspace as a classical model of infinite-dimensional topology. Some examples are: the space of arcs or pseudoarcs in $\mathbb{R}^2$, the hyperspace of compacta in Q of a fixed dimension n, the hyperspace of infinite-dimensional compacta in Q.

Continuing in this direction, we will deal in this paper with such hyperspaces as $\mathrm{ANR}(Z)$ and $\mathrm{AR}(Z)$ consisting respectively of the (compact) absolute neighborhood retracts and absolute retracts in Z. For a space Z

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with nice local structure, the space $\mathrm{ANR}(Z)$ is a dense absolute neighborhood retract of 2^Z, which itself is an absolute neighborhood retract space. This is the case if $Z = \mathbb{R}^n$ for $n \geq 1$.

In the project of topologically identifying $\mathrm{ANR}(\mathbb{R}^n)$ and $\mathrm{AR}(\mathbb{R}^n)$, especially if one expects to put forward the techniques of infinite-dimensional topology, one first encounters a question concerning the Borel structure of these spaces. We presume that such questions were asked in the thirties by the so-called Polish school. Later, the problem of finding a reasonable complete metric on $\mathrm{ANR}(Z)$ was successfully investigated by Borsuk [Bor] and Kuratowski [Kur1]. Borsuk [Bor] discovered that for a finite-dimensional compactum Z, there exists such a metric that is stronger than the Hausdorff metric (i.e., convergence in Borsuk's metric implies convergence in the Hausdorff metric). Hence, at least for a finite-dimensional compactum Z, $\mathrm{ANR}(Z)$ is the continuous injective image of a complete space, so it is Borel by an application of Suslin's result (see [Kur2]).

In this paper, we shall deal with $\mathrm{ANR}(\mathbb{R}^n)$ and $\mathrm{AR}(\mathbb{R}^n)$ for $n \geq 3$. The cases $\mathrm{ANR}(\mathbb{R}^2)$ and $\mathrm{AR}(\mathbb{R}^2)$ are handled in [CDGvM]. It turns out that $\mathrm{ANR}(\mathbb{R}^2)$ is a difference of two absolute $F_{\sigma\delta}$-sets; indeed, the exact Borel class of $\mathrm{ANR}(\mathbb{R}^2)$ is the so-called first small Borel class of the second ambiguous Borel class (see [Kur2] for this terminology). By contrast, $\mathrm{AR}(\mathbb{R}^2)$ is an absolute $F_{\sigma\delta}$-set which is not a $G_{\delta\sigma}$-set. It is evident that $\mathrm{AR}(\mathbb{R})$ is homeomorphic to $\{(x, y) \in \mathbb{R}^2 \mid x \geq y\}$. On the other hand, one can show that $\mathrm{ANR}(\mathbb{R}) \cong Q_{\mathrm{f}}$, where $Q_{\mathrm{f}} = \{(x_i) \in Q \mid x_i = 0 \text{ a.e.}\}$—see the Appendix.

Our first result on the topological identification of $\mathrm{ANR}(\mathbb{R}^n)$, $n \geq 3$, states that both $\mathrm{ANR}(\mathbb{R}^n)$ and $\mathrm{AR}(\mathbb{R}^n)$ are absolute $G_{\delta\sigma\delta}$-sets; moreover, they are not $F_{\sigma\delta\sigma}$-sets. Hence they are of the exact third multiplicative Borel class.

Results related to the topological classification are obtained with use of the absorbing set method (see Section 3). We use a variation of this method by [BM] (and then developed and applied by many authors, see, e.g., [DvMM]). In [BM] it is shown that, for each of the absolute Borel classes $\mathcal{A}_\alpha$ and $\mathcal{M}_\alpha$ with $\alpha \geq 2$, there exist spaces (one calls them models) Λ_α and Ω_α, respectively, which are absorbing sets for $\mathcal{A}_\alpha$ and $\mathcal{M}_\alpha$. The spaces Λ_α and Ω_α are of the exact additive and multiplicative class, topologically contain all elements of $\mathcal{A}_\alpha$, $\mathcal{M}_\alpha$ as closed subsets, and are "minimal" with respect to these two properties. One can find copies of them nicely embedded as dense subsets of Q (see [BM]).

We usually will identify $\mathbb{R}^n$ with the open cube $(-1, 1)^n \subset [-1, 1]^n = D_n$, and then will treat $\mathrm{ANR}(\mathbb{R}^n)$ as a subspace of $2^{D_n} \cong Q$. Our main result states that $\mathrm{ANR}(\mathbb{R}^n)$ with $n \geq 3$ is an $\mathcal{M}_3$-absorber. By the Uniqueness Theorem on absorbers, $\mathrm{ANR}(\mathbb{R}^n)$ is homeomorphic to Ω_3. We do not settle the case of $\mathrm{AR}(\mathbb{R}^n)$ for $n \geq 3$. There are some indications that, for large n, the space $\mathrm{AR}(\mathbb{R}^n)$ might not even be an absolute retract; see the Introduction of [CDGvM]).

To complete the picture, let us mention that in [CDGvM] it was shown that $\mathrm{ANR}(\mathbb{R}^2)$ is a $\varGamma$-absorber, where $\varGamma$ is the first small Borel class of the second ambiguous Borel class. Moreover, it was proved that $\mathrm{ANR}(\mathbb{R}^2) \cong \varOmega_2 \times (Q \setminus \varOmega_2)$, a subset of $Q \times Q$, which is another $\varGamma$-absorber. In [CDGvM], it was also shown that $\mathrm{AR}(\mathbb{R}^2) \cong \varOmega_2$.

The facts that $\mathrm{ANR}(\mathbb{R}^n)$ and $\mathrm{AR}(\mathbb{R}^n)$ are absolute $G_{\delta\sigma\delta}$-sets are obtained in Section 2. They are consequences of our characterization of (compact) absolute neighborhood retracts and absolute retracts among compacta in the Hilbert cube Q.

That the spaces $\mathrm{ANR}(\mathbb{R}^n)$ and $\mathrm{AR}(\mathbb{R}^n)$ for $n \geq 3$ are absolute $G_{\delta\sigma\delta}$-sets which are not $F_{\sigma\delta\sigma}$-sets is derived in Section 4. This is obtained by using a "cylindrical" variation of the approach used in [CDGvM].

The main difficulty of showing that $\mathrm{ANR}(\mathbb{R}^n)$ for $n \geq 3$ is an $\mathcal{M}_3$-absorbing set is in verification of the so-called strong $\mathcal{M}_3$-universality (see Section 3). Here our method differs from the ad hoc arguments developed before. The details are in Section 6.

In our case, the proof comes down to this. We will be given a map $f : U \to V \cap 2^{\mathbb{R}^n}$, where U is an open subset of Q, and V is an open subset of 2^{D_n}. Then for an arbitrary $G_{\delta\sigma\delta}$-subset C of U we must be able to closely approximate f by an injective map $g : U \to V \cap 2^{\mathbb{R}^n}$ so that $g^{-1}(\mathrm{ANR}(\mathbb{R}^n)) = C$.

First we employ an approximate factorization $\alpha : U \to P$, $\beta : P \to V \cap 2^{\mathbb{R}^n}$ of f through a polyhedron P. We design a certain map $\varphi : U \to 2^{I^3}$ so that $\varphi(q) \in \mathrm{ANR}(I^3)$ for $q \in C$ and $\varphi(q) \in 2^{I^3} \setminus \mathrm{ANR}(I^3)$ for $q \in U \setminus C$. We then obtain a map $\widetilde{\beta} : P \times 2^{I^3} \to V \cap 2^{\mathbb{R}^n}$ so that for a fixed $K \in 2^{I^3}$, $\widetilde{\beta} : P = P \times \{K\} \to V \cap 2^{\mathbb{R}^n}$ closely approximates β. Moreover, this $\widetilde{\beta}$ is defined in such a way that if $q \in C$, then $\widetilde{\beta}(p, \varphi(q)) \in \mathrm{ANR}(\mathbb{R}^n)$ and if $q \in U \setminus C$, then $\widetilde{\beta}(p, \varphi(q)) \in 2^{\mathbb{R}^n} \setminus \mathrm{ANR}(\mathbb{R}^n)$ for each $p \in P$. Finally, our map $g : U \to V \cap 2^{\mathbb{R}^n}$ is given as the composition $(\alpha, \varphi) : U \to P \times 2^{I^3}$ followed by $\widetilde{\beta} : P \times 2^{I^3} \to V \cap 2^{\mathbb{R}^n}$. This g will closely approximate f, and by a procedure involving certain coding given by the map φ and retained by $\widetilde{\beta}$, we shall simultaneously deduce that g is injective.

For the reader familiar with the techniques in [CN], we mention that our construction of the map $\widetilde{\beta}$ bears some resemblance to a method used there. Our definition of the map $\widetilde{\beta}$ was in part inspired by their work.

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2. Borel type of $\mathrm{ANR}(\mathbb{R}^n)$**.** In this section we apply the following characterization of compact absolute neighborhood retracts and compact absolute retracts in the Hilbert cube Q to determine the Borel types of $\mathrm{ANR}(\mathbb{R}^n)$ and $\mathrm{AR}(\mathbb{R}^n)$.

2.1. PROPOSITION. *Let X be a compact subset of the Hilbert cube Q. Then X is an absolute neighborhood retract* (*resp., an absolute retract*) *if and only if for every $\varepsilon > 0$ there exists a neighborhood U of X such that*

(∗) *for every $\delta > 0$ there exists a map $f : \overline{U} \to B(X,\delta)$* (*resp., $f : Q \to B(X,\delta)$*), *where*

(i) *$d(x, f(x)) < \varepsilon$ for $x \in \overline{U}$,*

(ii) *f is the identity map on some neighborhood of X in U.*

Proof. We will assume that Q is a convex subset of a Hilbert space and d is the metric induced by the norm.

Necessity. There is a neighborhood V of X and a retraction $r : V \to X$ (resp., $r : Q \to X$). Choose a neighborhood U of X, $U \subset V$, such that

$$d(x, r(x)) < \varepsilon, \qquad x \in \overline{U}. \tag{1}$$

Given $\delta > 0$, choose $\beta > 0$, $\beta < \delta$, so that $B(X,\beta) \subset U$ and

$$d(x, r(x)) < \delta \quad \text{whenever } x \in B(X,\beta). \tag{2}$$

Find a map $\lambda : Q \to I$ so that $\lambda(Q \backslash B(X,\beta)) \subset \{1\}$ and $\lambda(B(X,\beta/2)) \subset \{0\}$. Define $f : U \to Q$ (resp., $f : Q \to Q$) by

$$f(x) = \lambda(x)r(x) + (1-\lambda(x))x.$$

We claim that $f(\overline{U}) \subset B(X,\delta)$ (resp., $f(Q) \subset B(X,\delta)$). Since $f(x) = r(x)$ off $B(X,\beta)$ and since $r(x) \in X$, we only need to check $f(x)$ for $x \in B(X,\beta)$. In this case, however, we have

$$\begin{aligned} d(r(x), f(x)) &= d(\lambda(x)r(x) + (1-\lambda(x))r(x), \lambda(x)r(x) + (1-\lambda(x))x) \\ &\le d(r(x), x) < \delta \end{aligned}$$

(see (2)). Since $r(x) \in X$ it follows that $f(x) \in B(X,\delta)$.

To check (i) notice that, according to (1), $d(x, r(x)) < \varepsilon$ for $x \in \overline{U}$; therefore we only need to check $f(x)$ for $x \in B(X,\beta)$. As previously, we have $d(x, f(x)) = d(\lambda(x)x + (1-\lambda(x))x, \lambda(x)r(x) + (1-\lambda(x))x) \le d(r, r(x)) < \varepsilon$ (see (1)).

The condition (ii) follows from the fact that λ carries $B(X,\beta/2)$ to 0, and hence f is the identity map on $B(X,\beta/2) \subset U$.

Sufficiency. Let X be a compactum in Q. For each $\varepsilon = 1/2^n$ we select a neighborhood U_n of X so that condition (∗) is satisfied. Pick $k_n \in \mathbb{N}$ so that

$$B(X, 1/k_n) \subset U_{n+1},$$

and $k_1 < k_2 < \ldots$ Apply $(*)$ to $\delta_n = 1/k_n$ to obtain a map $f_n : \overline{U}_n \to B(X, 1/k_n) \subset U_{n+1}$ (resp., $f_n : Q \to B(X, 1/k_n) \subset U_{n+1}$) such that

$(\mathrm{i})_n$ $d(x, f_n(x)) < 1/2^n$ for $x \in \overline{U}_n$,
$(\mathrm{ii})_n$ f_n is the identity map on X.

Let $g_1 = f_1 : \overline{U}_1 \to B(X, 1/k_1)$ (resp., $g_1 = f_1 : Q \to B(X, 1/k_1)$). Assume $g_1, \ldots, g_n$, $n \geq 1$, have been defined so that $g_i : \overline{U}_1 \to B(X, 1/k_i)$ (resp., $g_i : Q \to B(X, 1/k_i)$) and g_i is the identity map on X for $1 \leq i \leq n$. We let $g_{n+1} = f_{n+1} \circ g_n$. By $(\mathrm{i})_n$, we get

$$d(g_{n+1}(x), g_n(x)) = d(f_{n+1}(g_n(x)), g_n(x)) < 1/2^{n+1}$$

for all $x \in \overline{U}_1$ (resp., $x \in Q$). Hence, the sequence $\{g_n\}_{n=1}^{\infty}$ converges uniformly on $\overline{U}_1$ (resp., on Q). Let $g : \overline{U}_1 \to Q$ (resp., $g : Q \to Q$) be the limit of $\{g_n\}_{n=1}^{\infty}$.

Since g_n is the identity on X for each n, so is g. Clearly, $g_n(\overline{U}_1) \subset f_n(U_n) \subset B(X, 1/k_n)$ (resp., $g_n(Q) \subset f_n(Q) \subset B(X, 1/k_n)$). We have that the sequence $\{k_n\}_{n=1}^{\infty}$ increases and X is compact, so $\operatorname{im} g \subset \bigcap_{n=1}^{\infty} B(X, 1/k_n) = X$. This shows that g is a retraction of $\overline{U}_1$ (resp., of Q) onto X. ■

For a space Z we define the following subspaces of the hyperspace 2^Z:

(1) $\mathrm{ANR}(Z) = \{X \in 2^Z \mid X$ is an absolute neighborhood retract$\}$,
(2) $\mathrm{AR}(Z) = \{X \in 2^Z \mid X$ is an absolute retract$\}$,
(3) $\mathrm{AR}_{\mathrm{f}}(Z) = \{X \in \mathrm{ANR}(Z) \mid$ each component of X is an absolute retract$\}$,
(4) $\mathrm{ANR}_{\mathrm{c}}(Z) = \{X \in \mathrm{ANR}(Z) \mid X$ is connected$\}$.

The spaces $\mathrm{AR}(Z)$ and $\mathrm{ANR}_{\mathrm{c}}(Z)$ will be treated as subspaces of $C(Z)$, the hyperspace of all continua of Z, rather than as subspaces of 2^Z.

2.2. THEOREM. *The spaces* $\mathrm{ANR}(Q)$, $\mathrm{AR}(Q)$, $\mathrm{AR}_{\mathrm{f}}(Q)$ *and* $\mathrm{ANR}_{\mathrm{c}}(Q)$ *are absolute* $G_{\delta\sigma\delta}$*-sets.*

Proof. We first establish the cases of $\mathrm{ANR}(Q)$ and $\mathrm{AR}(Q)$ by showing that these spaces are $G_{\delta\sigma\delta}$-subsets of the compactum 2^Q.

Let $\mathcal{U}$ be a countable base for Q which is closed under finite unions. It follows that for every compactum $X \subset Q$ and every neighborhood G of X there exists $U \in \mathcal{U}$ such that $X \subset U \subseteq G$. Let $F(n, U, m)$ be the subset of 2^Q consisting of X such that $X \subset U$, and there exists a map $f : \overline{U} \to B(X, 1/m)$ with $d(f(x), x) < 1/n$ for $x \in \overline{U}$, and f is the identity on a neighborhood of X in U; here $n, m \in \mathbb{N}$ and $U \in \mathcal{U}$. According to 2.1, we have

$$\mathrm{ANR}(Q) = \bigcap_{n=1}^{\infty} \bigcup_{U \in \mathcal{U}} \bigcap_{m=1}^{\infty} F(n, U, m).$$

Since each $F(n,U,m)$ is open and $\mathcal{U}$ is countable, $\mathrm{ANR}(Q)$ is a $G_{\delta\sigma\delta}$-subset of 2^Q. Changing the domain of f in the definition of $F(n,U,m)$ from $\overline{U}$ into Q and applying the absolute retract part of 2.1, we deduce that $\mathrm{AR}(Q)$ is also a $G_{\delta\sigma\delta}$-subset of 2^Q.

Since $\mathrm{ANR}_{\mathrm{c}}(Q) = \mathrm{ANR}(Q) \cap C(Q)$ and $C(Q)$ is a closed subset of 2^Q, $\mathrm{ANR}_{\mathrm{c}}(Q)$ is a $G_{\delta\sigma\delta}$-subset of 2^Q.

Now we will establish the case of $\mathrm{AR}_{\mathrm{f}}(Q)$. Let $C_n = \{X \in 2^Q \mid X$ has $\le n$ components$\}$, $n \ge 1$. Then C_n is a closed subset of 2^Q. In what follows, we show that $\mathrm{AR}_{\mathrm{f}}(Q) \cap C_n$ is a $G_{\delta\sigma\delta}$-subset of 2^Q. Consequently, $\mathrm{AR}_{\mathrm{f}}(Q)$, being a countable union $\bigcup_{n=1}^{\infty}(\mathrm{AR}_{\mathrm{f}}(Q) \cap C_n)$ of *closed* $G_{\delta\sigma\delta}$-subsets, is itself an absolute $G_{\delta\sigma\delta}$-set (see Proposition 3.8 of [DM]).

Consider the map $u : C(Q)^n \to 2^Q$ given by $u(A_1, \ldots, A_n) = A_1 \cup \ldots \ldots \cup A_n$. Write $D_1 = C_1$, and $D_n = C_n \setminus \bigcup_{i=1}^{n-1} C_i$ for $n \ge 2$. We have

$$u^{-1}(\mathrm{AR}_{\mathrm{f}}(Q) \cap D_n) = \mathrm{AR}(Q)^n \cap Y_n,$$

where $Y_n = \{(A_1, \ldots, A_n) \in C(Q)^n \mid A_i \cap A_j = \emptyset \text{ when } i \ne j\}$. Since Y_n is open and the product $\mathrm{AR}(Q)^n$ is a $G_{\delta\sigma\delta}$-subset of $C(Q)^n$, $u^{-1}(\mathrm{AR}_{\mathrm{f}}(Q) \cap D_n)$ is also a $G_{\delta\sigma\delta}$-subset of $C(Q)^n$. Applying a result of [SR], $\mathrm{AR}_{\mathrm{f}}(Q) \cap D_n$ is an absolute $G_{\delta\sigma\delta}$-set. Finally, $\mathrm{AR}_{\mathrm{f}}(Q) \cap C_n$, being a finite union of absolute $G_{\delta\sigma\delta}$-sets, $\bigcup_{i=1}^{n} \mathrm{AR}_{\mathrm{f}}(Q) \cap D_n$, is also an absolute $G_{\delta\sigma\delta}$-set. ∎

2.3. Corollary. *For every separable*, *complete-metrizable space* Z, *the spaces* $\mathrm{ANR}(Z)$, $\mathrm{AR}(Z)$, $\mathrm{AR}_{\mathrm{f}}(Z)$ *and* $\mathrm{ANR}_{\mathrm{c}}(Z)$ *are absolute* $G_{\delta\sigma\delta}$*-sets.*

Proof. Embed Z in the Hilbert cube Q. Then Z is a G_δ-subset of Q. It is then clear that $2^Z = \{X \in 2^Q \mid X \subset Z\}$ is a G_δ-subset of 2^Q. Since $\mathrm{ANR}(Z) = \mathrm{ANR}(Q) \cap 2^Z$, $\mathrm{AR}(Z) = \mathrm{AR}(Q) \cap 2^Z$, $\mathrm{AR}_{\mathrm{f}}(Z) = \mathrm{AR}_{\mathrm{f}}(Q) \cap 2^Z$ and $\mathrm{ANR}_{\mathrm{c}}(Z) = \mathrm{ANR}_{\mathrm{c}}(Q) \cap 2^Z$, the assertion follows from 2.2. ∎

3. Main result. Let us recall that a closed subset A of an absolute neighborhood retract M is a *Z-set* if every map of the n-dimensional cube I^n, $n \ge 1$, can be approximated by a map whose image misses A. An arbitrary set A (i.e., not necessarily closed) with the above property is called a *locally homotopy negligible set*. A countable union of Z-sets is called a *σZ-set*. An embedding whose image is a Z-set is called a *Z-embedding*.

Here, the Hilbert cube Q will be represented as $[-1,1]^\infty$, the countable product of the interval $[-1,1]$.

Let $\mathcal{C}$ be a class of spaces which is topological (i.e., every topological copy of an element of $\mathcal{C}$ belongs to $\mathcal{C}$). Let M be a Hilbert cube manifold. A set $X \subseteq M$ is called *strongly* $\mathcal{C}$*-universal* if for every $C \in \mathcal{C}$ with $C \subset Q$, every map $f : Q \to M$ that restricts to a Z-embedding on some compact subset K of Q can be arbitrarily closely approximated by a Z-embedding $g : Q \to M$ such that $g|K = f|K$ and $g^{-1}(X) \setminus K = C \setminus K$. (This notion is closely related to so-called strong $(\mathcal{K},\mathcal{C})$-universality, where $\mathcal{K}$ is the class of all compacta; see [C3].)

A subset $X \subseteq M$ is called a $\mathcal{C}$-*absorber* in M if:

(1) $X \in \mathcal{C}$,
(2) $X \subseteq S$ for some σZ-set S in M,
(3) X is strongly $\mathcal{C}$-universal in M.

The uniqueness theorem [DvMM, Theorem 2.1] states that two $\mathcal{C}$-absorbers X and Y in respective copies M and M' of Q are homeomorphic, i.e., there exists a homeomorphism h of M onto M' so that $h(X) = Y$. This result can be extended to Hilbert cube manifolds as well [BGvM].

The pseudoboundary of Q, i.e., the set

$$B = \{(x_i) \in Q \mid \exists (i \in \mathbb{N}) \, (|x_i| = 1)\},$$

is an example of an absorber for the class of all σ-compact spaces. The space B^∞ in Q^∞ is an absorber for the Borel class of absolute $F_{\sigma\delta}$-sets. Let

$$P = \{(x_i) \in (Q^\infty)^\infty \mid [\forall (i \in \mathbb{N}) \, (x_i \in Q^\infty \backslash B^\infty)] \ \& \ x_i = 0 \text{ a.e.}\}.$$

The space P^∞ in $((Q^\infty)^\infty)^\infty$ is an absorber for the Borel class of absolute $G_{\delta\sigma\delta}$-sets (and P is an absorber in $(Q^\infty)^\infty$ for the Borel class of absolute $G_{\delta\sigma}$-sets). These absorbers (together with all absorbers related to the Borel hierarchy) were described in [BM] and denoted by $B = \Lambda_1, B^\infty = \Omega_2$, $P = \Lambda_2$ and $P^\infty = \Omega_3$.

For $n \geq 3$, we let $D_n = [-1,1]^n \subset \mathbb{R}^n$. We will consider the hyperspaces 2^{D_n} and $C(D_n)$, copies of the Hilbert cube [vM]. We will identify the hyperspaces $2^{\mathbb{R}^n}$ and $C(\mathbb{R}^n)$ with $2^{(-1,1)^n}$ and $C((-1,1)^n)$ in 2^{D_n} and $C(D_n)$, respectively.

Here is our main result.

3.1. Theorem. *For every* $n \geq 3$, $\mathrm{ANR}(\mathbb{R}^n)$ *and* $\mathrm{AR_f}(\mathbb{R}^n)$ *are* $G_{\delta\sigma\delta}$-*absorbers in* 2^{D_n}, *and* $\mathrm{ANR_c}(\mathbb{R}^n)$ *is a* $G_{\delta\sigma\delta}$-*absorber in* $C(D_n)$.

Proof. We need to check the conditions (1)–(3) of the above definition of an absorber with $\mathcal{C} = \mathcal{G}_{\delta\sigma\delta}$, the Borel class of absolute $G_{\delta\sigma\delta}$-sets, in the respective copy $M = 2^{D_n}$ or $M = C(D_n)$ of the Hilbert cube Q.

By 2.3, $\mathrm{ANR}(\mathbb{R}^n)$, $\mathrm{AR_f}(\mathbb{R}^n)$ and $\mathrm{ANR_c}(\mathbb{R}^n)$ are absolute $G_{\delta\sigma\delta}$-sets; hence (1) is satisfied.

Since every element of $\mathrm{ANR}(\mathbb{R}^n)$ is either a finite set or is of dimension ≥ 1, we have

$$\mathrm{ANR}(\mathbb{R}^n) \subset \mathcal{F}(\mathbb{R}^n) \cup \dim^{\geq 1}(\mathbb{R}^n),$$

where $\mathcal{F}(\mathbb{R}^n)$ is the hyperspace of finite subsets of $\mathbb{R}^n$, and $\dim^{\geq 1}(\mathbb{R}^n)$ is the hyperspace of compacta in $\mathbb{R}^n$ with dimension ≥ 1. Both $\mathcal{F}(\mathbb{R}^n)$ and $\dim^{\geq 1}(\mathbb{R}^n)$ are σZ-sets in 2^{D_n} (see [Cu1], [DvMM], [DR]). Consequently, the spaces $\mathrm{ANR}(\mathbb{R}^n)$ and $\mathrm{AR_f}(\mathbb{R}^n) \subset \mathrm{ANR}(\mathbb{R}^n)$ are contained in a σZ-set in 2^{D_n}.

Consider the hyperspace $L(\mathbb{R}^n)$ consisting of locally connected continua in $\mathbb{R}^n$. We have $\mathrm{ANR_c}(\mathbb{R}^n) \subset L(\mathbb{R}^n)$. By a result of [GvM], the space $L(\mathbb{R}^n)$ for $n \geq 3$ is contained in a σZ-set in $C(D_n)$; hence $\mathrm{ANR_c}(\mathbb{R}^n)$ also is contained in a σZ-set in $C(D_n)$. The verification of (2) is complete.

We will verify condition (3) in Section 6; more exactly, condition (3) follows from the condition (abs) (formulated in Proposition 3.2) which will be proved in Section 6. ■

3.2. Proposition. *Let M be a copy of the Hilbert cube and $\mathcal{C}$ be a topological class which is hereditary with respect to open sets. A set $X \subset M$ is strongly $\mathcal{C}$-universal provided the following holds*:

(abs) *given open sets $U \subseteq Q$ and $V \subseteq M$, an element $C \in \mathcal{C}$ with $C \subseteq U$ and a map $\varepsilon : V \to (0,1)$, for every map $\widetilde{f} : U \to V$ there exists an injective map $g : U \to M$ such that $d(g(x), \widetilde{f}(x)) < \varepsilon(\widetilde{f}(x))$ for $x \in U$, $g(U)$ is locally homotopy negligible in M and $g^{-1}(X) = C$.*

Proof. Suppose we are given an element $C \in \mathcal{C}$, $C \subset Q$, and a map $f : Q \to M$ which restricts to a Z-embedding on a compact set $K \subset Q$. We may assume that $K \neq \emptyset$. Since $f(K)$ is a Z-set, we can approximate f by $\widetilde{f}$ such that $\widetilde{f}(Q\backslash K) \cap \widetilde{f}(K) = \emptyset$ and $\widetilde{f}|K = f|K$. For given $\varepsilon > 0$, set $\widetilde{\varepsilon}(m) = \varepsilon d(m, f(K))$ for $m \in M$. We may assume that $\varepsilon < 1$ and d is bounded by 1. Now apply the condition (abs) with $U = M\backslash K$, $V = M \setminus f(K)$, C replaced by $C\backslash K$, and $\widetilde{f}|U$ to find an injective map $g : U \to M$ such that

$$d(g(x), \widetilde{f}(x)) < \widetilde{\varepsilon}(\widetilde{f}(x)) = \varepsilon d(\widetilde{f}(x), \widetilde{f}(K)), \qquad x \in U,$$

$g(U)$ is locally homotopy negligible in M, and $g^{-1}(X) = C\backslash K$. Since $\varepsilon < 1$, actually g maps U into $V = M\backslash f(K)$. Extending g by f over K we see that $g : Q \to M$ is injective and continuous, and $g^{-1}(X)\backslash K = C\backslash K$. Since $g(Q) = f(K) \cup g(U)$ is the union of a Z-set and a locally homotopy negligible set, it is a Z-set. The assumption that d is bounded by 1 yields that $d(g(x), \widetilde{f}(x)) \leq \varepsilon$; the proof is complete. ■

Let $P_0 = (((Q^\infty)^\infty)^\infty, P^\infty) \setminus \{0\}$, where $0 \in P^\infty$ is the point all of whose coordinates are 0. The symbol (Q, Ω_3) will stand for $(((Q^\infty)^\infty)^\infty, P^\infty)$.

3.3. Proposition. (a) *A pair (M, X) is homeomorphic to the pair P_0 if and only if*

(i) *M is homeomorphic to $Q \setminus \{pt\}$;*
(ii) *$X \in \mathcal{G}_{\delta\sigma\delta}$;*
(iii) *X is contained in some σZ-set of M;*
(iv) *X is strongly $\mathcal{G}_{\delta\sigma\delta}$-universal in M.*

(b) *For every* $q \in \Omega_3$, *the pairs* $(Q, \Omega_3) \setminus \{q\}$ *and* P_0 *are homeomorphic.*

Proof. It is standard that P_0 satisfies (i)–(iv). Conversely, suppose that a pair (M, X) satisfies (i)–(iv). Let $\widetilde{M} = M \cup \{\infty\}$ be the one-point compactification of M. Obviously, $\widetilde{M}$ is a topological copy of Q. Write $\widetilde{X} = X \cup \{\infty\}$. We deduce that $\widetilde{X}$ is contained in some σZ-set of $\widetilde{M}$ and that $\widetilde{X} \in G_{\delta\sigma\delta}$. Moreover, using (iv), it can easily be checked that $\widetilde{X}$ is strongly $G_{\delta\sigma\delta}$-universal in $\widetilde{M}$. By [CDGvM, Theorem 2.5], there exists a homeomorphism $h : \widetilde{M} \to ((Q^\infty)^\infty)^\infty$ such that $h(\widetilde{X}) = P^\infty$. Further, one can choose h in such a way that $h(\infty) = 0 \in P^\infty$. As a result, $h(X) = P^\infty \setminus \{0\}$, and the proof of (a) is complete.

Part (b) follows from the fact that $(Q, \Omega_3) \setminus \{q\}$ satisfies (i)–(iv) of (a). ∎

3.4. Corollary. *For every* $n \geq 3$, *the spaces* $\mathrm{ANR}(\mathbb{R}^n)$, $\mathrm{AR}_{\mathrm{f}}(\mathbb{R}^n)$ *and* $\mathrm{ANR}_{\mathrm{c}}(\mathbb{R}^n)$ *are homeomorphic to* P^∞. *More precisely*, *the pairs* $(2^{\mathbb{R}^n}, \mathrm{ANR}(\mathbb{R}^n))$, $(2^{\mathbb{R}^n}, \mathrm{AR}_{\mathrm{f}}(\mathbb{R}^n))$, *and* $(C(\mathbb{R}^n), \mathrm{ANR}_{\mathrm{c}}(\mathbb{R}^n))$ *are homeomorphic to* $(Q, \Omega_3) \setminus \{q\}$ *independently of the choice of* $q \in Q$.

Proof. We will apply 3.2(a) and (b). By results of [Cu2], the spaces $M = 2^{\mathbb{R}^n}$ and $M = C(\mathbb{R}^n)$ are homeomorphic to $Q \setminus \{pt\}$. By 2.3, $X = \mathrm{ANR}(\mathbb{R}^n)$, $X = \mathrm{AR}_{\mathrm{f}}(\mathbb{R}^n)$, and $X = \mathrm{ANR}_{\mathrm{c}}(\mathbb{R}^n)$ are absolute $G_{\delta\sigma\delta}$-sets. An argument of the proof of 3.1 shows that each X is contained in a σZ-set in the respective M. The strong $G_{\delta\sigma\delta}$-universality of X in 2^{D_n} (resp., in $C(D_n)$) implies the strong universality of X in the suitable M. ∎

We close this section with a remark concerning one property of $\mathcal{C}$ which is usually required in the definition of a $\mathcal{C}$-absorber. Namely, the uniqueness theorem on absorbers was stated and proved in [DvMM] for a class $\mathcal{C}$ which is closed hereditary, i.e., every closed subset of an element of $\mathcal{C}$ also belongs to $\mathcal{C}$. It is obvious that the class $G_{\delta\sigma\delta}$ is closed hereditary and the version of the uniqueness theorem from [DvMM] applies to conclude 3.4. Let us, however, observe that the closed hereditary assumption is superfluous.

3.5. Lemma. *Let* $\mathcal{C}$ *be a topological class. If a subset* X *of a copy* M *of the Hilbert cube is strongly* $\mathcal{C}$-*universal then* X *is also* $\widetilde{\mathcal{C}}$-*universal*, *where* $\widetilde{\mathcal{C}} = \{C_0 \mid \exists (C \in \mathcal{C})\, (C_0$ *is a closed subset of* $C)\}$.

Proof. Let $C \in \mathcal{C}$, C_0 be a closed subset of C and suppose $C_0 \subset Q$. Choose a homeomorphism $h : C \to C'$ such that $h(C_0) = C_0$ and $C' \cap Q = C_0$. There exists a copy Q' of Q such that $Q \cup C' \subset Q'$.

Given a map $f : Q \to M$ which restricts to a Z-embedding on some compactum $K \subset Q$, extend f to a map $\overline{f} : Q' \to M$. Approximate $\overline{f}$ by $\overline{g}$ such that $\overline{g}|K = \overline{f}|K$, $\overline{g}^{-1}(X) \setminus K = C' \setminus K$, and $\overline{g}$ is a Z-embedding. Set $g = \overline{g}|Q$, and notice that $g^{-1}(X) \setminus K = C_0 \setminus K$. ∎

4. Allowable sequences and universality of $\mathrm{ANR}(\mathbb{R}^n)$**.** In this section we make the first step towards verification of the strong universality of $\mathrm{ANR}(\mathbb{R}^n)$ in the version contained in 3.2(abs). We shall show that the space $\mathrm{ANR}(\mathbb{R}^n)$ (resp., $\mathrm{AR}(\mathbb{R}^n)$) for $n \geq 3$ is universal for the class $\mathcal{G}_{\delta\sigma\delta}$, that is, every $B \in \mathcal{G}_{\delta\sigma\delta}$ admits a *closed* embedding in $\mathrm{ANR}(\mathbb{R}^n)$ (resp., $\mathrm{AR}(\mathbb{R}^n)$). This will be achieved by using the notion of allowable sequence and the operation $X \to Z(X)$ which to every allowable sequence X assigns an element $Z(X) \in 2_0^{I^3} = \{Y \in 2^{I^3} \mid 0 \in Y\}$. Placing B in the Hilbert cube Q, we will describe an embedding of Q in $C(I^3) \subset 2^{I^3}$ which precisely sends B into $\mathrm{AR}(I^3)$ and $Q \setminus B$ into $C(I^3) \setminus \mathrm{ANR}(I^3)$. Write $Q \setminus B = \bigcup_{n=1}^\infty A_n$, where each A_n is an $F_{\sigma\delta}$-subset of Q. Roughly speaking, to each $x \in Q$ we assign an allowable sequence $X = \{X_i\}_{i=1}^\infty$ with $X_i \in 2^{I^2}$ so that the local structure of $\bigcup_{i=1}^\infty X_i$ is determined by the fact that $x \in A_n$ or $x \in Q \setminus A_n$. (Here we employ in an essential way a construction appearing in a preliminary version of [CDGvM].) Finally, the use of the operation $X \to Z(X)$ makes it possible to code the fact that whenever $x \in B$ then $Z(X) \in \mathrm{AR}(I^3)$, and whenever $x \in Q \setminus B$, then $Z(X) \in C(I^3) \setminus \mathrm{ANR}(I^3)$.

In this section we mostly deal with $\mathbb{R}^3$ and $I^3 \subset \mathbb{R}^3$. When appealing to $\mathbb{R}$, $\mathbb{R}^2$, I and I^2 we mean these objects are subsets of $\mathbb{R}^3$ in the way that $\mathbb{R} = \mathbb{R} \times \{0\} \times \{0\} \subset \mathbb{R}^2 = \mathbb{R} \times \mathbb{R} \times \{0\} \subset \mathbb{R}^3$ and $I = I \times \{0\} \times \{0\} \subset I^2 = I \times I \times \{0\} \subset I \times I \times I \subset \mathbb{R}^3$.

Choose a sequence $\{I_i\}_{i=1}^\infty$ of closed intervals $I_i = [a_i, b_i]$ so that $0 < b_{i+1} < a_i < b_i < 1$ and $b_{i+1} - a_{i+1} < b_i - a_i$ for $i \geq 1$, and $\lim a_i = 0$. Let $D_i = [a_i, b_i] \times [0, a_i^2] \subset I^2$. We have

(i) $I_i \cap I_j = \emptyset$ when $i \neq j$,

(ii) the sequence $\{I_i\}_{i=1}^\infty$ converges to $\{0\}$ in 2^I,

(iii) for each line $L \subset \mathbb{R}^2$ containing the origin, either $L \subset \mathbb{R} \times \{0\}$ or L intersects D_i for at most finitely many i.

We shall now describe three classes of one-dimensional continua Λ_i^1, Λ_i^2, Λ_i^3 ($i \in \mathbb{N}$). Write c_i for the midpoint of I_i. Then $\Lambda_i^1 \subset 2^{D_i}$ consists of those elements of the form $\{c\} \times [0, a_i^2]$ where $c \in I_i$. We define $\Lambda_i^2 \subset 2^{D_i}$ to be those elements $(\{a, d\} \times [0, \varepsilon]) \cup ([a, b] \times \{\varepsilon\}) \cup ([c, d] \times \{\varepsilon\})$ where $a_i \leq a \leq b < c \leq d \leq b_i$ and $0 \leq \varepsilon \leq a_i^2$. Finally, $\Lambda_i^3 \subset 2^{D_i}$ will be all those elements $([a, b] \times \{\varepsilon\}) \cup (\{a, b\} \times [a, \varepsilon])$ with $a_i \leq a < b \leq b_i$, $0 < \varepsilon \leq a_i^2$. We let

$$\Lambda_i = \Lambda_i^1 \cup \Lambda_i^2 \cup \Lambda_i^3 \subset 2^{D_i}, \quad i \geq 1.$$

Fix a decomposition

$$\mathbb{N} = \bigcup_{j=0}^\infty N_j \tag{iv}$$

of $\mathbb{N}$ into infinite sets N_j, with $N_i \cap N_j = \emptyset$ when $i \neq j$. We retain this decomposition throughout the paper.

4.1. Definition. An element X of $\prod_{i=1}^{\infty} \Lambda_i \subseteq \prod_{i=1}^{\infty} 2^{D_i}$ will be treated as a sequence $\{X_i\}_{i=1}^{\infty}$ and will be called an *allowable sequence* if $X_i \in \Lambda_i^1$ whenever $i \in N_0$. We assign to such an X a 2-dimensional continuum $Z(X) \in 2_0^{I^3}$ in the following manner. Write $D_{xz} = I \times \{0\} \times I$ and $D_{xy} = I \times I \times \{0\}$. Let

$$Z_j(X) = D_{xz} \cup D_{xy} \cup \bigcup \{X_i \times [0, 1/(j+1)] \mid i \in N_0 \cup \ldots \cup N_j\},$$

and

$$Z(X) = \bigcup_{j=0}^{\infty} Z_j(X).$$

Geometrically, $Z(X)$ is the union of two planar disks D_{xz} and D_{xy} along with the cylinders over each X_i of height $1/(j+1)$, $i \in N_j$.

The lemma below collects certain properties of $Z(X)$.

4.2. Lemma. *Let* $X = \{X_i\}_{i=1}^{\infty}$ *and* $Y = \{Y_i\}_{i=1}^{\infty}$ *be allowable sequences. Then*

(a) D_{xz} *is the only square in* $Z(X)$ *containing* $\{0\} \times \{0\} \times I$ *as one of its edges*;
(b) *if* $i \in N_0$ *and* $a_i \le x \le b_i$, *then* $Z(X)$ *is locally a* 2*-manifold at* $(x, 0, 0)$ *if and only if* $(x, 0) \notin X_i$;
(c) *relative to the rectilinear PL structure on* $\mathbb{R}^3$, $Z(X)$ *is locally polyhedral at the points not in* $\{0\} \times \{0\} \times I$ *and is not locally polyhedral at all points of* $\{0\} \times \{0\} \times I$;
(d) $Z(X)$ *is contractible*;
(e) $Z(X)$ *is locally contractible at* $0 \in Z(X)$;
(f) *if for each* $j \ge 0$, X_i *belongs to* Λ_i^3 *for at most finitely many* $i \in N_j$, *then* $Z(X)$ *is locally contractible at each* $p = (0, 0, t)$ *with* $0 < t \le 1$;
(g) *if for some* $j \ge 0$, X_i *belongs to* Λ_i^3 *for infinitely many* $i \in N_j$, *then* $Z(X)$ *is not locally simply connected at* $p = (0, 0, t)$ *whenever* $1/(j+2) < t < 1/(j+1)$;
(h) *for each line segment* $[a, b]$ *in* $\mathbb{R}^3$, $[a, b] \cap Z(X)$ *is an absolute neighborhood retract*;
(i) *the local topological structure of* $Z(X)$ *at* $(0, 0, 0)$ *is different from that of* $Z(Y)$ *at* $(0, 0, 1)$ *and from that of* $(I \times [-1, 0] \times \{0\}) \cup Z(Y)$ *at* $(0, 0, 0)$;
(j) *if* $i \in N_0$, $a_i \le x \le b_i$, *and* $(x, 0) \in X_i \cap Y_i$, *then* $X_i = Y_i$.

Proof. The statements (a), (b), (c), (j) are evident. The formula $(x_1, x_2, x_3) \to (x_1, x_2, (1 - \lambda)x_3)$, $\lambda \in I$, establishes a strong deformation retraction of $Z(X)$ onto the disk D_{xy}. The same formula restricted to a cubical neighborhood U of 0 witnesses a strong deformation retraction of U onto a disk contained in $U \cap I^2$. This concludes (d) and (e).

To get (f), (g), use the fact that for $1/(j+2) < t \leq 1/(j+1)$, $(0,0,t)$ has a neighborhood in $Z(X)$ which admits a strong deformation retraction onto $Z(X) \cap (I^2 \times \{t\})$, the latter being a copy of $I \cup (\bigcup\{X_i \mid i \in N_0 \cup \ldots \cup N_j\})$. In the case of (f), the latter is topologically a continuum consisting of I, a finite union of copies of S^1, each intersecting I in exactly one arc, and a null sequence of nonintersecting arcs each with precisely one point on I, and which converges to $\{0\}$. In the case of (g), infinitely many of the arcs become copies of S^1 each intersecting I in exactly one arc.

Turning to (h), it is sufficient to prove that for each line segment $[a,b]$ in $\mathbb{R}^3$ such that $a \in Z(X)$, there exists c between a and b such that either $[a,c] \subseteq Z(X)$ or $[a,c] \cap Z(X) = \{a\}$. If a is not in $\{0\} \times \{0\} \times I$, then (c) shows that $Z(X)$ is locally polyhedral at a, and so (h) is certainly true. Hence assume that $a \in \{0\} \times \{0\} \times I$. If $[a,b]$ lies in the xz-plane then (h) is true. If not, then the projection $[a_0, b_0]$ of $[a,b]$ to the xy-plane is a line segment with $a_0 = 0$ and b not in $\mathbb{R}$. Using (iii), find c_0 between a_0 and b_0 so that $[a_0, c_0]$ intersects no D_i. Let $c \in [a,b]$ be the point that projects to c_0. Then either $[a,c] \cap D_{xy}$ is nontrivial or $[a,c] \cap Z(X) = \{a\}$.

Lastly, to see why (i) is true, let W be a neighborhood of $(0,0,0)$ in $Z(X)$. Then there exists a 2-cell $D \subseteq W$ with $(0,0,0) \in \partial D$ and such that $D \backslash \partial D$ is an open subset of $Z(X)$. On the other hand, if W is a neighborhood of $(0,0,1)$ in $Z(Y)$, then there is no such 2-cell $D \subseteq W$. We leave the proof of the other part to the reader. ∎

4.3. Proposition. (a) *For an allowable sequence* $X = \{X_i\}_{i=1}^{\infty}$, $Z(X)$ *is an absolute retract if and only if for each* $j \geq 0$, $X_i \in \Lambda_i^3$ *for at most finitely many* $i \in N_j$; *in case* $Z(X)$ *is not an absolute retract*, *it is also not an absolute neighborhood retract.*

(b) *The assignment* $X \to Z(X)$ *is continuous and injective from* $\prod_{i=1}^{\infty} \Lambda_i$ *into* $2_0^{I^3}$.

Proof. (a) Suppose that for each $j \geq 0$, $X_i \in \Lambda_i^3$ for at most finitely many $i \in N_j$. Then from 4.2(d)–(f), $Z(X)$ is locally contractible at all points. Since $Z(X)$ is finite-dimensional (actually 2-dimensional) and contractible (4.2(d)), it is an absolute retract. Conversely, if there is $j \geq 0$ such that $X_i \in \Lambda_i^3$ for infinitely many $i \in N_j$, then by 4.2(g), $Z(X)$ is not an absolute retract. The same argument shows that if $Z(X)$ is not an absolute retract, then it is not an absolute neighborhood retract.

(b) The injectivity is obvious. Let $\{X^n\}_{n=1}^{\infty}$ be a sequence in $\prod_{i=1}^{\infty} \Lambda_i$ converging to $X^0 \in \prod_{i=1}^{\infty} \Lambda_i$. Then, for each $j \geq 1$ the sequence $\{Z(X^n) \cap (I \times I \times [1/j, 1])\}_{n=1}^{\infty}$ converges to $Z(X^0) \cap (I \times I \times [1/j, 1])$. Moreover, for each $\varepsilon > 0$ there exists $j \in \mathbb{N}$ such that for all n,

$$D_{xy} \subset Z(X^n) \cap (I \times I \times [0, 1/j]) \subset B(D_{xy}, \varepsilon). \quad \blacksquare \tag{1}$$

Let $\alpha^0 : N_0 \to \mathbb{N}$ be a bijection.

4.4. PROPOSITION. *Let A be an $F_{\sigma\delta\sigma}$-subset of the Hilbert cube Q. There exists an embedding $\Theta = (\Theta_i) : Q \to \prod_{i=1}^{\infty} \Lambda_i$ such that*

(a) *if $q \in A$, then for some $j \geq 0$ there are infinitely many $i \in N_j$ such that $\Theta_i(q) \in \Lambda_i^3$,*

(b) *if $q \in Q \setminus A$, then for each $j \geq 0$ there are at most finitely many $i \in N_j$ such that $\Theta(q) \in \Lambda_i^3$,*

(c) *for each $q \in Q$ and $i \in N_0$, $\Theta_i(q) = \{2^{-1}(b_i + a_i) + 2^{-1}q_j(b_i - a_i)\} \times [0, a_i^2] \in \Lambda_i^1$, where $j = \alpha^0(i)$.*

Let us postpone the proof of Proposition 4.4 and formulate our main result of this section which is a direct consequence of 4.3, 4.4 and 4.2.

4.5. THEOREM. *Let A be an $F_{\sigma\delta\sigma}$-subset of Q and Θ be an embedding as in Proposition* 4.4. *The map $\varphi = Z \circ \Theta : Q \to 2_0^{I^3}$ is an embedding such that*

(a) *$\varphi(q)$ is contractible for $q \in Q$,*

(b) *if $q \in A$, then $\varphi(q)$ is not an absolute neighborhood retract,*

(c) *if $q \in Q \setminus A$, then $\varphi(q)$ is an absolute retract.* $\blacksquare$

A standard immediate consequence of 4.5 and 2.3 is

4.6. COROLLARY. *For every $n \geq 3$, the spaces* $\mathrm{ANR}(\mathbb{R}^n)$, $\mathrm{AR}(\mathbb{R}^n)$, $\mathrm{AR}_{\mathrm{f}}(\mathbb{R}^n)$ *and* $\mathrm{ANR}_{\mathrm{c}}(\mathbb{R}^n)$ *are essential absolute $G_{\delta\sigma\delta}$-sets.*

Proof. Let $Y \subset 2^{D_n}$ be one of these spaces. From 2.3, Y is a $G_{\delta\sigma\delta}$-subset of the Hilbert cube 2^{D_n}. We need to show that Y is essential, i.e., that Y is not an $F_{\sigma\delta\sigma}$-subset.

There exists a $G_{\delta\sigma\delta}$-subset B of Q such that B is not an $F_{\sigma\delta\sigma}$-subset. Let $A = Q \setminus B$, and apply Theorem 4.5 to obtain a map $\varphi : Q \to 2_0^{I^3}$ such that $\varphi^{-1}(Y_0) = \varphi^{-1}(\varphi(Q) \cap Y_0) = B$, where $Y_0 = Y \cap 2_0^{I^3}$. If Y were an $F_{\sigma\delta\sigma}$-subset of 2^{D_n}, then $\varphi(Q) \cap Y_0$ would be an $F_{\sigma\delta\sigma}$-subset of $2_0^{I^3}$, which in turn would imply that B is an $F_{\sigma\delta\sigma}$-subset, a contradiction. $\blacksquare$

The next lemma will help us prove Proposition 4.4. It exists in a preliminary version of [CDGvM]; we include its proof because [CDGvM] is only a preprint now.

4.7. LEMMA. *Suppose A is an $F_{\sigma\delta}$-subset of Q. There exists a map $\psi = (\psi_i) : Q \to \prod_{i=1}^{\infty} \Lambda_i$ such that*

(a) *if $q \in A$, then $\psi_i(q) \in \Lambda_i^3$ for infinitely many i,*

(b) *if $q \in Q \setminus A$, then $\psi_i(q) \in \Lambda_i^1 \cup \Lambda_i^2$ for almost all i.*

Proof. Write $A = \bigcap_{n=1}^{\infty} \bigcup_{m=1}^{\infty} A_m^n$, where each A_m^n is compact and

(1) $A_1^n \subseteq A_2^n \subseteq \ldots$ for every $n \geq 1$, and

(2) $\bigcup_{m=1}^{\infty} A_m^1 \supseteq \bigcup_{m=1}^{\infty} A_m^2 \supseteq \ldots$

Let c_i be the midpoint of $[a_i, b_i]$ and $d_i = 2^{-1} a_i^2 (b_i - a_i)$. Choose a bijection $\gamma : \mathbb{N} \times (\mathbb{N} \cup \{0\}) \to \mathbb{N}$ such that $\gamma(n,m) < \gamma(n, m+1)$ for every (n,m). Define maps $\alpha_m^n : Q \to I$ by

$$\alpha_m^n(q) = \begin{cases} d_{\gamma(n,m)} \operatorname{dist}_d(q, A_m^n) & \text{if } n, m \geq 1, \\ d_{\gamma(n,m)} & \text{if } m = 0. \end{cases}$$

Here d is a metric for Q which is bounded by 1. Note that since $b_{i+1} - a_{i+1} < b_i - a_i$ and $a_{i+1} < a_i$, we have $d_j < d_i$ whenever $i < j$; consequently, $d_{\gamma(n,m+1)} \leq d_{\gamma(n,m)}$. By (1) ($A_m^n \subset A_{m+1}^n$), we now have $0 \leq \alpha_{m+1}^n(q) \leq \alpha_m^n(q)$ for all q.

Fix n, m and q. Write $\gamma = \gamma(n,m)$ and $\alpha_m^n = \alpha_m^n(q)$ and let $T_m^n = T_m^n(q)$ be $(\{c_\gamma - \alpha_m^n, c_\gamma + \alpha_m^n\} \times [0, a_\gamma^2]) \cup ([c_\gamma - \alpha_m^n, c_\gamma - \alpha_{m+1}^n] \times \{a_\gamma^2\}) \cup ([c_\gamma + \alpha_{m+1}^n, c_\gamma + \alpha_m^n] \times \{a_\gamma^2\})$.

We see that $T_m^n(q) \in \Lambda_{\gamma(n,m)}$ for $q \in Q$. The map T_m^n has the following properties:

(3) If $q \in Q$ and $n \geq 1$, then the sequence $\{T_m^n(q)\}_{m=1}^{\infty}$ consists entirely of elements $T_m^n(q) \in \Lambda_{\gamma(n,m)}^2$ if and only if $\alpha_m^n(q) > 0$ for $m \geq 0$ if and only if $q \notin \bigcup_{m=1}^{\infty} A_m^n$.

(4) If $q \in Q$ and $n \geq 1$, then for exactly one $j \geq 0$, $T_j^n(q) \in \Lambda_{\gamma(n,j)}^3$, whereas for all other m, $T_m^n(q) \in \Lambda_{\gamma(n,m)}^1 \cup \Lambda_{\gamma(n,m)}^2$; this is true if and only if there exists $M \geq 1$ such that for all $m \geq M$, $\alpha_m^n(q) = 0$, or, equivalently, if and only if $q \in \bigcup_{m=1}^{\infty} A_m^n$.

Now we put $\psi_i(q) = T_m^n(q)$, where $i = \gamma^{-1}(n,m)$. The requirements (a) and (b) are readily checked using (3) and (4). ∎

Proof of 4.4. For $i \in N_0$, define Θ_i as in (c). Now write $A = \bigcup_{k=1}^{\infty} A_k$ where each A_k is an $F_{\sigma\delta}$-subset of Q. For each $k \geq 1$, with the help of 4.7, find $(\Theta_i) : Q \to \prod_{i \in N_k} \Lambda_i$ satisfying 4.7(a), (b).

Our map Θ is $(\Theta_i)_{i \in \bigcup_{k=0}^{\infty} N_k} = (\Theta_i)_{i \in \mathbb{N}}$. Items (a) and (b) follow from the conditions 4.7(a), (b): if $q \in A$ then $q \in A_k$ for at least one k; if $q \in Q \setminus A$ then $q \in Q \setminus A_k$ for all k. ∎

5. Finite unions of $Z(X)$. This section contains some auxiliary facts relating to $Z(X)$ which will be used in Section 6.

For a set $K \subset \mathbb{R}^3$, $u \in \mathbb{R}$, and positive numbers λ, δ, we define

$$K_{(\lambda,u,\delta)} = \{(\delta x_1, \lambda x_2 + u, \delta x_3) \mid (x_1, x_2, x_3) \in K\}.$$

For an allowable sequence X we define

$$\mathcal{U}(X) = \{Z(X)_{(\lambda_1,u_1,\delta)} \cup \ldots \cup Z(X)_{(\lambda_k,u_k,\delta)} \mid u_1 < \ldots < u_k,\ k \in \mathbb{N}\}.$$

Our objective here is to lay out the important properties of the 2-dimensional compacta which are elements of $\mathcal{U}(X)$.

5.1. Lemma. *Let* $X = \{X_i\}_{i=1}^{\infty}$ *and* $Y = \{Y_i\}_{i=1}^{\infty}$ *be allowable sequences*, $F \in \mathcal{U}(X)$ *and* $G \in \mathcal{U}(Y)$. *Then*

(a) *if* C *is a component of* F, *then* C *is contractible*, *everywhere locally* 2-*dimensional*, *not a square*, *and* $C \in \mathcal{U}(X)$;

(b) F *is an absolute neighborhood retract if and only if* $Z(X)$ *is*;

(c) *if* F *and* G *are connected and there exists an affine isometry* $h : \mathbb{R}^3 \to \mathbb{R}^3$ *carrying* F *onto* G, *then* $X_i = Y_i$ *for each* $i \in N_0 \subseteq \mathbb{N}$.

Proof. Since the formula of the proof of 4.2 yields a strong deformation retraction of C onto a 2-cell in $\mathbb{R}^2$, C is contractible. The other statements of part (a) easily follow from the definition of $\mathcal{U}(X)$.

Let $F = \bigcup_{k=1}^{r} Z_{(\lambda_k,u_k,\delta)}$ and $G = \bigcup_{l=1}^{s} Z_{(\mu_l,v_l,\varrho)}$, where $u_1 < \ldots < u_r$ and $v_1 < \ldots < v_s$, $I_k^X = \{0\} \times \{u_k\} \times [0,\delta]$, and $I_l^Y = \{0\} \times \{v_l\} \times [0,\varrho]$. We see that, relative to the triangle PL structure of $\mathbb{R}^3$, F (resp., G) is not locally polyhedral precisely at the points of $S_F = \bigcup_{k=1}^{r} I_k^X$ (resp., $S_G = \bigcup_{l=1}^{s} I_l^Y$). Let $\pi_k : \mathbb{R}^3 \to \mathbb{R}^3$ be the affine isomorphism given by $\pi_k(x_1,x_2,x_3) = (\delta x_1, \lambda_k x_2 + u_k, \delta x_3)$. For each $1 \le k \le r$, there exists $\varepsilon_k \in \{0,1\}$ such that π_k carries some neighborhood of $\{0\} \times \{0\} \times [0,1]$ in $Z(X) \cup ([0,1] \times [-\varepsilon_k, 0] \times \{0\})$ onto a neighborhood of I_k^X in F; we always have $\varepsilon_1 = 0$ and if F is connected, then $\varepsilon_k = 1$ for each $k > 1$.

Now we are in a position to show (b). Assume that $Z(X)$ is an absolute neighborhood retract. It follows that $Z(X)$ and $Z(X) \cup ([0,1] \times [-\varepsilon_k, 0] \times \{0\})$ are locally contractible. Consequently, F is locally contractible at the points of S_F. Since, in addition, at every point $p \in F \backslash S_F$, F is locally polyhedral, F is locally contractible everywhere; hence F is an absolute neighborhood retract. Conversely, suppose F is not an absolute neighborhood retract. Then, according to 4.2(e)–(g), $Z(X)$ is not locally simply connected at $(0,u_1,t)$ for $0 < t \le \delta$; the proof of (b) is complete.

Now suppose that F, G, and $h : \mathbb{R}^3 \to \mathbb{R}^3$ are as in (c). We have $h(S_F) = S_G$. This easily shows that $\delta = \varrho$ and $r = s$. Moreover, there exists $1 \le l_0 \le r$ such that $h(I_1^X) = I_{l_0}^Y$. Suppose $l_0 > 1$. The local topological structure of F at $(0,u_1,0) \in I_1^X$ is that of $Z(X)$ at $(0,0,0)$, while the local topological structure of G at $(0,v_{l_0},0) \in I_{l_0}^Y$ and at $(0,v_{l_0},\delta) \in I_{i_0}^Y$ is that of $Z(Y) \cup ([0,1] \times [-1,0] \times \{0\})$ at $(0,0,0)$ and $(0,0,1)$ (use the fact that G is connected). Since the local topological structure of $Z(Y) \cup ([0,1] \times [-1,0] \times \{0\})$ at $(0,0,1)$ coincides with that of $Z(Y)$ at $(0,0,1)$, an application of 4.2(i) leads to a contradiction. Consequently, we have $h(I_1^X) = I_1^Y$; the above argument also shows that $h(0,u_1,0) = (0,v_1,0)$.

Applying 4.2(a) we see that $[0,\delta]\times\{u_1\}\times[0,\delta]$ (resp., $[0,\delta]\times\{v_1\}\times[0,\delta]$) is the only square in F (resp., in G) with I_1^X (resp., I_1^Y) as one of its edges. We conclude that $h([0,\delta]\times\{u_1\}\times[0,\delta]) = [0,\delta]\times\{v_1\}\times[0,\delta]$; and since $h(0,u_1,0) = (0,v_1,0)$, we have $h(x,u_1,0) = (x,v_1,0)$ for $0 \le x \le \delta$. Now, if $i \in N_0$ and $a_i \le x \le b_i$ then, applying 4.2(b), we see that F is not locally a 2-manifold at $(\delta x, u_1, 0)$ if and only if G is not locally a 2-manifold at $(\delta x, v_1, 0)$ if and only if $(x,0) \in X_i \cap Y_i$. From 4.2(j), we get $X_i = Y_i$. ∎

5.2. Corollary. *Let X be an allowable sequence, $F_i \in \mathcal{U}(X)$ and $h_i : \mathbb{R}^3 \to \mathbb{R}^n$ be affine isometric embeddings with $n \ge 3$, $1 \le i \le k$, and $h_i(F_i) \cap h_j(F_j) = \emptyset$ when $i \ne j$. Suppose that G is a finite one-dimensional polyhedron in $\mathbb{R}^n$. Then $M = h_1(F_1) \cup \ldots \cup h_k(F_k) \cup G$ is an absolute neighborhood retract if and only if $Z(X)$ is.*

Proof. Suppose $Z(X)$ is an absolute neighborhood retract. By 5.1(b), each $h_i(F_i)$ is an absolute neighborhood retract, hence so is $\bigcup_{i=1}^k h_i(F_i)$. By an application of 4.2(h), $G \cap \bigcup_{i=1}^k h_i(F_i)$ is an absolute neighborhood retract. A standard fact yields that so is M.

Suppose $Z(X)$ is not an absolute neighborhood retract. Then from 5.1(b), neither is $h_1(F_1)$. From 4.2(h) it follows that there is an open neighborhood of $h_1(F_1)$ in M which retracts onto $h_1(F_1)$. If M were an absolute neighborhood retract, this would imply that so is $h_1(F_1)$. ∎

6. Verification of strong universality. This section is entirely devoted to verification of the condition (abs) of 3.2 for the spaces $\mathrm{ANR}(\mathbb{R}^n)$, $\mathrm{AR}_{\mathrm{f}}(\mathbb{R}^n)$ and $\mathrm{ANR}_{\mathrm{c}}(\mathbb{R}^n)$ with $n \ge 3$. This will complete the proof of 3.1.

Let us recall that we identify $\mathbb{R}^n$ with $(-1,1)^n \subset [-1,1]^n = D_n$. We are supposed to prove the following fact.

6.1. Theorem. *Let $U \subseteq Q$ and $V \subseteq 2^{D_n}$, with $n \ge 3$, be open sets, and let C be a $G_{\delta\sigma\delta}$-subset of U. Given maps $f : U \to V$ (resp., $f : U \to V \cap C(D_n)$) and $\varepsilon : V \to (0,1)$, there exists an injective map $g : U \to 2^{D_n}$ (resp., $g : U \to C(D_n)$) such that $\widetilde{H}(f(q), g(q)) < \varepsilon(f(q))$ for $q \in U$, $g^{-1}(\mathrm{ANR}(\mathbb{R}^n)) = C$ and $g^{-1}(\mathrm{AR}_{\mathrm{f}}(\mathbb{R}^n)) = C$ (resp., $g^{-1}(\mathrm{ANR}_{\mathrm{c}}(\mathbb{R}^n)) = C$), and $g(U)$ is locally homotopy negligible in 2^{D_n} (resp., in $C(D_n)$).*

Here by $\widetilde{H}$ we mean the Hausdorff metric generated by a fixed metric on D_n.

A standard way of proving a statement like 6.1 is to "replace" U by a polyhedron P and then verify the assertion for f defined on P. We will slightly modify this strategy.

Let P be a locally finite countable polyhedron and let $\alpha : U \to P$ and $\beta : P \to V$ (resp., $\beta : P \to V \cap C(D_n)$) be maps so that $\beta \circ \alpha$ is as close to f as we wish (see, e.g., [MS, p. 316]). Since $2^{D_n} \setminus 2^{\mathbb{R}^n}$ (resp., $C(D_n) \setminus C(\mathbb{R}^n)$) is a Z-set in 2^{D_n} (resp., in $C(D_n)$) we can additionally require that $\beta : P \to V \cap 2^{\mathbb{R}^n}$ (resp., $\beta : P \to V \cap C(\mathbb{R}^n)$) (see [Tor]).

Here is our major technical step.

6.2. Proposition. *Given a map* $\delta : V \to (0,1)$, *there exists a map* $\widetilde{\beta} : P \times 2_0^{I^3} \to 2^{\mathbb{R}^n}$ *such that*

(a) $H(\beta(x), \widetilde{\beta}(x,K)) < \delta(\beta(x))$ *for* $(x,K) \in P \times 2_0^{I^3}$,

(b) $\widetilde{\beta}(x,\{0\}) \in \mathcal{F}(\mathbb{R}^n)$ *for* $x \in P$,

(c) *if* $K \in 2_0^{I^3}$ *is a continuum and* $x \in P$, *then* $\widetilde{\beta}(x,K)$ *has only finitely many components each of which intersects* $\widetilde{\beta}(x,\{0\})$.

Moreover, *for* $x \in P$ *and an allowable sequence* X, *set* $A = \widetilde{\beta}(x, Z(X))$. *Then*

(d) *every nontrivial component of* A *is either a square or an affine isometric copy of an element of* $\mathcal{U}(X)$,

(e) *there exists a component of* A *which is an affine isometric copy of an element of* $\mathcal{U}(X)$.

Here H is the Hausdorff metric on $2^{\mathbb{R}^n}$ induced by the Euclidean metric d on $\mathbb{R}^n$. Assuming 6.2, the proof of 6.1 goes as follows.

Proof of 6.1. Since $2^{D_n} \setminus 2^{\mathbb{R}^n}$ (resp., $C(D_n) \setminus C(\mathbb{R}^n)$) is a Z-set, we can assume that $f : U \to V \cap 2^{\mathbb{R}^n}$ (resp., $f : U \to V \cap C(\mathbb{R}^n)$).

There exists $\tilde{\varepsilon} : V \cap 2^{\mathbb{R}^n} \to (0,1)$ so that whenever $h : U \to V \cap 2^{\mathbb{R}^n}$ and $H(f(x), h(x)) < \tilde{\varepsilon}(f(x))$ for $x \in U$, then $\widetilde{H}(f(x), h(x)) < \varepsilon(f(x))$ for $x \in U$. In view of this, it is enough to restrict our attention to maps into $2^{\mathbb{R}^n}$ and the metric H.

Let $\varphi = Z \circ \Theta$ be the map of 4.5. Define

$$g(q) = \widetilde{\beta}(\alpha(q), \varphi(q)), \qquad q \in U.$$

Using 6.2(a), one sees that $H(f(q), g(q)) < \widetilde{\varepsilon}(f(q))$ for $q \in U$ provided δ is sufficiently small and $\beta \circ \alpha$ is suitably close to f.

To show that g is injective, fix $p, q \in U$ with $p \neq q$. Write $X = \Theta(p)$, $Y = \Theta(q)$, $A = \widetilde{\beta}(\alpha(p), Z(X))$, and $B = \widetilde{\beta}(\alpha(q), Z(Y))$. By Proposition 4.4, Θ is an embedding, so $X \neq Y$. To show that $g(p) = A$ and $g(q) = B$ are different sets, it is enough to argue that they have different sets of components. Using 6.2(e), select A', a component of A which is an affine isometric copy of an element F of $\mathcal{U}(X)$. Let B' be any component of B and suppose that $B' = A'$; we may assume that B' is nontrivial. Using 5.1(a), we see that B' is not a square. By 6.2(d), we find that B' is an affine isomorphic copy of an element G of $\mathcal{U}(Y)$. Using 5.1(c), one concludes that $X_i = Y_i$ for all $i \in N_0$. But $X_i = \Theta_i(p)$ and $Y_i = \Theta_i(q)$, so (c) of 4.4 shows that $p_j = q_j$ for all $j \in \mathbb{N}$, i.e., $p = q$.

Applying the fact that $\varphi^{-1}(\mathrm{ANR}(\mathbb{R}^n)) = C$ (use 4.5 with $A = Q \setminus C$) together with 6.2(d), and a special case of 5.2 (when $G = \emptyset$), we infer that $g^{-1}(\mathrm{ANR}(\mathbb{R}^n)) = C$. Moreover, if $q \in C$ then, by 6.2(d) and 5.1(a), every component of $g(q)$ is contractible; hence $g(q) \in \mathrm{AR_f}(\mathbb{R}^n)$. This shows $g^{-1}(\mathrm{AR_f}(\mathbb{R}^n)) = C$.

Finally, note that $g(q)$ is locally two-dimensional at every point $q \in U$ (see 5.1(a)). Since the hyperspace of at most two-dimensional compacta in $2^{\mathbb{R}^n}$ is homotopy negligible in 2^{D_n} for $n \geq 3$, $g(U)$ is locally homotopy negligible in 2^{D_n}.

The case of $\mathrm{ANR_c}(\mathbb{R}^n)$ needs certain adjustments. First, in this case, we have $f : U \to V \cap C(\mathbb{R}^n)$ and $\beta : P \to V \cap C(\mathbb{R}^n)$. For every $x \in P$ and $v, w \in \beta_0(x) = \widetilde{\beta}(x, \{0\})$, we let

$$A_{v,w}(x) = [v, w] \cap (\overline{B}_d(v, \delta(\beta(x))) \cup \overline{B}_d(w, \delta(\beta(x)))),$$

where $[v, w]$ is the segment joining v and w. We let $\beta'(x) = \bigcup\{A_{v,w}(x) \mid v, w \in \beta_0(x)\}$ (cf. proof of 5.4 of [DR]). Since $H(\beta_0(x), \beta(x)) < \delta(\beta(x))$ (see 6.2(a)) and $\beta(x)$ is a continuum, so is $\beta'(x)$. We see that β' maps P into $C(\mathbb{R}^n)$, and for $x \in P$ we have $\beta_0(x) \subseteq \beta'(x)$ and $H(\beta'(x), \beta(x)) < 2\delta(\beta(x))$. Let $g'(q) = \beta'(\alpha(q)) \cup g(q) = \beta'(\alpha(q)) \cup \widetilde{\beta}(\alpha(q), \varphi(q))$ for $q \in U$. Since $\varphi(q)$ is a continuum (see 4.5(a)), an application of 6.2(c) yields that so is $g'(q)$; hence $g' : U \to C(\mathbb{R}^n)$.

In showing that g' is injective it is sufficient to note that the set of points of $g'(q)$ at which $g'(q)$ is locally two-dimensional is the same as that for $g(q)$.

The fact that $(g')^{-1}(\mathrm{ANR_c}(\mathbb{R}^n)) = C$ and the local homotopy negligibility of $g(U)$ follow as previously (in this case, however, we apply the full strength of 5.2).

We have

$$H(\beta(q), g'(q)) < 2\delta(\beta(\alpha(q))), \qquad q \in U.$$

Hence, if δ is sufficiently small and if $\alpha \circ \beta$ is close to f, then g' is an injective map as desired in 6.1. ■

The rest of this section is devoted to proving Proposition 6.2. We start with the following auxiliary lemma.

6.3. Lemma. *There exists a map* $\Phi : I \times 2_0^{I^3} \to 2^{I^3}$ *such that*

(a) $\Phi(t, \{0\}) = \{(0, t, 0)\} \subset \Phi(t, K)$ *for* $(t, K) \in I \times 2_0^{I^3}$,

(b) $\Phi(t, K) = \{(0, t, 0)\}$ *for* $t \in [0, 1/16] \cup [15/16, 1]$ *and arbitrary* K,

(c) *if* K *is a continuum*, *then so is* $\Phi(t, K)$ *for all* $t \in I$,

(d) *for every $K', K'' \in 2_0^{I^3}$, we have $\Phi(t, K') \cap \Phi(s, K'') = \emptyset$ provided $t \neq s$ and $s \in I \setminus (1/8, 7/8)$.*

Moreover, for an allowable sequence X and a finite subset F of I, set $A = \bigcup\{\Phi(t, Z(X)) \mid t \in F\}$. Then

(e) *every nontrivial component of A either belongs to $\mathcal{U}(X)$ or is a translation of a certain cD_{xz}, $0 < c \leq 1$, in the y-direction; if $F \cap (1/8, 7/8) \neq \emptyset$ then there exists a component of A that belongs to $\mathcal{U}(X)$.*

Proof. Fix a map $\varphi : I \to [0, 1/16]$ such that $\varphi([1/4, 3/4]) = \{1/16\}$, $\varphi^{-1}(\{0\}) = I \setminus (1/8, 7/8)$, and $t + \varphi(t) < 7/8$ whenever $t \in [3/4, 7/8)$. Fix also a map $\psi : I \to I$ so that $\psi([1/8, 7/8]) = \{1\}$ and $\psi(I \setminus (1/16, 15/16)) = \{0\}$. Define

$$\Phi(t, K) = \{(\psi(t)x_1, t + \varphi(t)x_2, \psi(t)x_3) \mid (x_1, x_2, x_3) \in K\}$$

for $(t, K) \in I \times 2_0^{I^3}$. (Note that, according to the notation of Sec. 5, if $t \in (1/8, 7/8)$ then $\Phi(t, K) = K_{(\varphi(t), t, \psi(t))}$.) The properties (a)–(d) and the first part of (e) easily follow from the definition of Φ. For the second part of (e), use the fact that $K_{(\varphi(t), t, \psi(t))} \in \mathcal{U}(X)$ provided $\varphi(t) > 0$ and $\psi(t) > 0$, and the last part of 5.1(a). ∎

Our next lemma contains the construction of "carriers"—straight line segments in $\mathbb{R}^n$—that will be used to describe $\widetilde{\beta}$ restricted to the 1-skeleton of P. Let us make some basic assumptions on the polyhedron P that will be kept throughout the text (one sees that generality will not be lost):

(i) P is infinite,

(ii) the set of vertices $P^{(0)} = \{v_i \mid i \in \mathbb{N}\}$ of P is enumerated in the way that $v_i \neq v_j$ when $i \neq j$, and each $v_i \in P^{(0)}$ is adjacent to some $v_j \in P^{(0)}$,

(iii) P is subdivided so finely that $\operatorname{diam}_H(\beta(\operatorname{st}(\tau))) < \frac{1}{32} \min\{\delta(\beta(x)) \mid x \in \tau\}$ for every simplex τ of P,

(iv) whenever we write $\langle v_i, v_j \rangle$ for a 1-simplex (i.e., $\langle v_i, v_j \rangle \in P^{(1)}$), we mean that $i < j$, and we call (i, j) *admissible*.

The last condition fixes an orientation of $\langle v_i, v_j \rangle$ and we assign the parameterization $t \to tv_j + (1-t)v_i$, $t \in I$, to $\langle v_i, v_j \rangle$.

6.4. Lemma. *One can assign to every $v \in P^{(0)}$ a set $\varphi(v) \in \mathcal{F}(\mathbb{R}^n)$, and to every $\langle v_i, v_j \rangle \in P^{(1)}$ a finite collection of straight line segments (= segments) $S_{(i,j)}$ in $\mathbb{R}^n$ so that*

(a) *the collection $\{\varphi(v) \mid v \in P^{(0)}\}$ is pairwise disjoint,*

(b) *every segment $S \in S_{(i,j)}$ has one endpoint in $\varphi(v_i)$ and the other in $\varphi(v_j)$; every $z \in \varphi(v_i) \cup \varphi(v_j)$ is an endpoint of at least one $S \in S_{(i,j)}$,*

(c) *for every $S' \in S_{(i,j)}$ and $S'' \in S_{(k,m)}$, S' and S'' intersect in at most one common endpoint when $(i, j) \neq (k, m)$,*

(d) *there exists* $\Sigma_{(i,j)} \in S_{(i,j)}$ *such that when* $(i,j) \neq (k,m)$ *then* $\Sigma_{(i,j)} \cap \Sigma_{(k,m)} = \emptyset$,

(e) *for every simplex* τ *of* P, *every* $x \in \tau$, *every* $\langle v_i, v_j\rangle \in (\overline{\mathrm{st}}(\tau))^{(1)}$ *and every* $S \in S_{(i,j)}$, *we have* $\mathrm{diam}_d(S) < \frac{1}{5}\delta(\beta(x))$ *and* $H(\varphi(v_i), \beta(v_i)) < \frac{1}{16}\delta(\beta(x))$.

Proof. For each $v \in P^{(0)}$ let $\delta(v) = \frac{1}{16}\min\{\delta(\beta(x)) \mid x \in \overline{\mathrm{st}}^2(\mathrm{v})\}$. Suppose inductively that for $1 \le i \le r$, $\varphi(v_i) = \varphi_i \in \mathcal{F}(\mathbb{R}^n)$ have been chosen so that:

(1) the cardinality of φ_i is greater than the number of vertices of P that are adjacent to v_i,

(2) $H(\beta(v_i), \varphi_i) < \delta(v_i)$,

(3) $\varphi_i \cap \varphi_j = \emptyset$ whenever $1 \le i < j \le r$.

Suppose also that for all admissible pairs (i,j) with $1 \le i < j \le r$, a set $S_{(i,j)}$ has been assigned to satisfy (b)–(d) and

(4) $\mathrm{diam}_d(S) \le \frac{3}{2}H(\beta(v_i), \beta(v_j)) + \delta(v_i) + \delta(v_j)\,, \quad S \in S_{(i,j)}$.

Choose $\varphi(v_{r+1}) = \varphi_{r+1} \in \mathcal{F}(\mathbb{R}^n)$ so that (1)–(3) hold for $r+1$. We will adjust this choice if necessary. Suppose that for all admissible $(k, r+1)$, $S_{(k,r+1)}$ has been defined to satisfy (b)–(d) and (4) for all k with $1 \le k \le i-1$. Let $(i, r+1)$ be admissible and $x \in \varphi_i$ be arbitrary. From (2) there exists $x_1 \in \beta(v_i)$ so that $d(x, x_1) < \delta(v_i)$. Pick $x_2 \in \beta(v_{r+1})$ with $d(x_1, x_2) \le \frac{3}{2}H(\beta(v_i), \beta(v_{r+1}))$. Applying (2) again, find $y \in \varphi_{r+1}$ with $d(x_2, y) < \delta(v_{r+1})$. It is now clear that the segment S joining x and y satisfies (4) with $j = r+1$. Reverse this procedure starting from arbitrary $y \in \varphi_j$ and ending with some $x \in \varphi_i$ so that the collection $S_{(i,r+1)}$ will satisfy (b) and (4).

To get (c), just slightly revise the choice in φ_{r+1} if necessary. To obtain (d), use (1) to find $x \in \varphi_i$ so that x lies in no previously defined $\Sigma_{(i,j)}$ and look at a segment S joining $x \in \varphi_i$ with some $y \in \varphi_{r+1}$. In case y is already an endpoint of some $\Sigma_{(k,r+1)}$ with $k < i$, add y^* to φ_{r+1} near y to serve as the endpoint for $S = \Sigma_{(i,r+1)}$. Find y^* so that the previously indicated procedures with new data ensure that (a)–(c) and (4) are still true. The inductive construction is complete.

The second inequality of (e) is a direct consequence of (2) and the definition of $\delta(v)$ for $v = v_i$. Using (4), (iii) and the definition of δ, we have

$$\begin{aligned}\mathrm{diam}_d(S) &\le \tfrac{3}{2}H(\beta(v_i), \beta(v_j)) + \delta(v_i) + \delta(v_j)\\ &\le \tfrac{3}{2}\cdot\tfrac{1}{32}\delta(\beta(x)) + \tfrac{1}{16}\delta(\beta(x)) + \tfrac{1}{16}\delta(\beta(x)) < \tfrac{1}{5}\delta(\beta(x))\end{aligned}$$

for every $S \in S_{(i,j)}$ with $\langle v_i, v_j\rangle \in (\overline{\mathrm{st}}^2(\tau))^{(1)}$ and $x \in \tau$. This shows the rest of (e). ■

For each (i,j) and $S \in S_{(i,j)}$, let us orient S by the following parameterization. Let $x \in \varphi(v_i)$ and $y \in \varphi(v_j)$ be the endpoints of S; we set

(v) $\kappa_S(t) = ty + (1-t)x$ for $t \in I$.

The next lemma uses the "carriers" $\Sigma = \Sigma_{(i,j)}$, and κ_Σ to continuously assign to each $(t,K) \in I \times 2_0^{I^3}$ a compressed copy $\widetilde{\kappa}_\Sigma(t,K)$ of K which "emanates" from $p = \kappa_\Sigma(t) \in \Sigma$. Moreover, $\widetilde{\kappa}_\Sigma(t,K)$ will be an affine injective copy of K when $t \in (1/8, 7/8)$, a planar compactum when $t \in (1/16, 1/8] \cup [7/8, 15/16)$, and $\{p\}$ when $t \in [0, 1/16] \cup [15/16, 1]$.

Let us introduce a notation. For $l \in \mathbb{N}$, we let

(vi) $\mathrm{Sig}_l = \{\Sigma_{(k,m)} \mid \langle v_k, v_m\rangle \in (\overline{\mathrm{st}}^2(v_l))^{(1)}\}$.

6.5. Lemma. *To every* $\Sigma = \Sigma_{(i,j)}$, *one can assign a map* $\widetilde{\kappa}_\Sigma : I \times 2_0^{I^3} \to 2^{\mathbb{R}^n}$ *such that*

(a) $\widetilde{\kappa}_\Sigma(t, \{0\}) = \{\kappa_\Sigma(t)\} \subset \widetilde{\kappa}_\Sigma(t,K)$ *for* $(t,K) \in I \times 2_0^{I^3}$,

(b) $\widetilde{\kappa}_\Sigma(t,K) = \{\kappa_\Sigma(t)\}$ *for* $t \in [0,1/16] \cup [15/16,1]$ *and arbitrary* K,

(c) *if* K *is a continuum, then so is* $\widetilde{\kappa}_\Sigma(t,K)$ *for all* $t \in I$,

(d) *for any simplex* τ *of* P, $x \in \tau$ *and* $\langle v_k, v_m\rangle \in (\overline{\mathrm{st}}(\tau))^{(1)}$, *we have* $\mathrm{diam}_d\, \widetilde{\kappa}_{\Sigma_{(k,m)}}(t,K) < \frac{1}{4}\delta(\beta(x))$ *for* $(t,K) \in I \times 2_0^{I^3}$.

Moreover, *for an allowable sequence* X, $l \in \mathbb{N}$, *and a finite set* $F \subset \mathrm{Sig}_l \times I$, *set* $A = \bigcup\{\widetilde{\kappa}_\Sigma(t, Z(X)) \mid (\Sigma,t) \in F\}$. *Then*

(e) A *satisfies* 6.2(d), *and*

(f) *if* $(\Sigma,t) \in F$ *for some* $\Sigma \in \mathrm{Sig}_l$ *and* $1/8 < t < 7/8$, *then* A *satisfies* 6.2(e).

Proof. Using 6.4(d), for each $l \in \mathbb{N}$ we choose δ_l with $0 < \delta_l < 1$ so that

(1) $$B(\Sigma', 2\delta_l) \cap B(\Sigma'', 2\delta_l) = \emptyset, \quad \Sigma', \Sigma'' \in \mathrm{Sig}_l,\ \Sigma' \neq \Sigma''.$$

We may assume that

(2) $$\delta_l < \tfrac{1}{16} \min\{\delta(\beta(x)) \mid x \in \overline{\mathrm{st}}^2(v_l)\}.$$

Fix $\Sigma = \Sigma_{(i,j)}$ and let

$$x_\Sigma = \kappa_\Sigma(0) \in \varphi(v_i) \quad \text{and} \quad y_\Sigma = \kappa_\Sigma(1) \in \varphi(v_j)$$

be the endpoints of Σ. Write $t_\Sigma = d(x_\Sigma, y_\Sigma)$. Define a map $c_\Sigma : \mathbb{R}^3 \to \mathbb{R}^3$ by

(3) $$c_\Sigma(x) = (\delta_i x_1, t_\Sigma x_2, \delta_i x_3), \quad x = (x_1, x_2, x_3) \in \mathbb{R}^3.$$

We see that for any allowable sequence X and $A \in \mathcal{U}(X)$, $c_\Sigma(A)$ also lies in $\mathcal{U}(X)$. Fix an affine isometric injection $\tau_\Sigma : \mathbb{R}^3 \to \mathbb{R}^n$ sending $0 \in \mathbb{R}^3$ onto x_Σ and $(0, t_\Sigma, 0) \in \mathbb{R}^3$ onto y_Σ. We let $a_\Sigma = \tau_\Sigma \circ c_\Sigma : \mathbb{R}^3 \to \mathbb{R}^n$ and

(4) $$\widetilde{\kappa}_\Sigma(t,K) = a_\Sigma(\Phi(t,K)), \quad (t,K) \in I \times 2_0^{I^3},$$

where Φ is that of 6.3.

The conditions (a)–(c) are consequences of 6.3(a)–(c). To get (d), let τ be a simplex of P, $x \in \tau$ and $\langle v_k, v_m\rangle \in (\overline{\mathrm{st}}(\tau))^{(1)}$, and write $\Sigma = \Sigma_{(k,m)}$. By (3), $c_\Sigma(I^3)$ lies in $[0,\delta_k]\times[0,t_\Sigma]\times[0,\delta_k]$, so $\operatorname{diam}_d c_\Sigma(I^3) \le \sqrt{2\delta_k^2 + t_\Sigma^2}$. By (2), $\delta_k < \frac{1}{16}\delta(\beta(x))$ and by 6.4(e), $t_\Sigma < \frac{1}{5}\delta(\beta(x))$. Hence $\operatorname{diam}_d c_\Sigma(I^3) < \frac{1}{4}\delta(\beta(x))$. Since τ_Σ preserves distance and $\widetilde{\kappa}_\Sigma(t,K) = \tau_\Sigma c_\Sigma(\Phi(t,K)) \subset \tau_\Sigma c_\Sigma(I^3)$, we get (d).

The map τ_Σ is defined in such a manner that the segment $\{(0,t,0) \mid 0 \le t \le t_\Sigma\} \subset \mathbb{R}^3$ is transformed affinely onto Σ by τ_Σ. Hence any point in $[0,\delta_l]\times[0,t_\Sigma]\times[0,\delta_l]$ is mapped by τ_Σ into the $2\delta_l$-neighborhood of Σ. We conclude that

$$(5)\qquad \widetilde{\kappa}_\Sigma(t,K) \subset B(\Sigma, 2\delta_l) \quad \text{for any } \Sigma \in \mathrm{Sig}_l \text{ and any } (t,K) \in I \times 2_0^{I^3}.$$

By (1) and (5), we get

$$(6)\qquad \text{if } \Sigma', \Sigma'' \in \mathrm{Sig}_l \text{ and } \Sigma' \ne \Sigma'', \text{ then } \widetilde{\kappa}_{\Sigma'}(t',K') \cap \widetilde{\kappa}_{\Sigma''}(t'',K'') = \emptyset$$

for any choices of (t',K') and (t'',K'') in $I \times 2_0^{I^3}$.

Let A be as in the statement of the lemma and let us examine the structure of A. For each $\Sigma \in \mathrm{Sig}_l$, let A^Σ be the subset of A consisting of $\bigcup\{\widetilde{\kappa}_\Sigma(t, Z(X)) \mid \exists (t \in I)\,((\Sigma,t) \in F)\}$. As a result of (6), one sees that $A^{\Sigma'} \cap A^{\Sigma''} = \emptyset$ whenever $\Sigma' \ne \Sigma''$ and that $\{A^\Sigma \mid \Sigma \in \mathrm{Sig}_l\}$ is a partition of A into closed sets (note that possibly $A^\Sigma = \emptyset$ for a given $\Sigma \in \mathrm{Sig}_l$). Moreover, for each Σ with $A^\Sigma \ne \emptyset$, there exists a finite set $0 \le t_1 < \ldots < t_r \le 1$ such that

$$(7)\qquad a_\Sigma^{-1}(A^\Sigma) = \bigcup\{\Phi(t_q, Z(X)) \mid 1 \le q \le r\}.$$

From this discussion, one concludes that if A' is a component of A, then there exists $\Sigma \in \mathrm{Sig}_l$ such that A' is a component of A^Σ. Hence there exists a component A^* of $T = \bigcup\{\Phi(t_q, Z(X)) \mid 1 \le q \le r\}$ such that $A' = a_\Sigma(A^*)$. By the first part of 6.3(e), $A^* \in \mathcal{U}(X)$ or A^* is a square, which yields (e) of this lemma. If $1/8 < t_q < 7/8$ for some q, then the second part of 6.3(e) shows that there is a component A^* of T with $A^* \in \mathcal{U}(X)$. Then the component $A' = a_\Sigma(A^*)$ satisfies (e) of 6.2, and this proves (f) of the current lemma. ∎

We are now ready to prove Proposition 6.2. An essential step in our proof will be the following fact, which is a consequence of [CN, Lemma 3.3].

6.6. Lemma. *There exists a map $h : P \to \mathcal{F}(P^{(1)})$ such that*

(vii) $h(x) \in \mathcal{F}(\tau^{(1)})$ *for every simplex τ of P and $x \in \tau$*; $h(x) = \{x\}$ *whenever $x \in P^{(1)}$.* ∎

This lemma suggests a way of defining $\widetilde{\beta} : P \times 2_0^{I^3} \to 2^{\mathbb{R}^n}$. Assume for a moment that a certain map $\widetilde{\chi} : P^{(1)} \times 2_0^{I^3} \to 2^{\mathbb{R}^n}$ has been defined. Then we would obtain an extension $\widetilde{\beta} : P \times 2_0^{I^3} \to 2^{\mathbb{R}^3}$ by using Lemma 6.6 as follows. For $(x,K)\in P\times 2_0^{I^3}$, consider the finite subset $h(x)\subset P^{(1)}$. For each $p\in h(x)$, we have $(p,K)\in P^{(1)}\times 2_0^{I^3}$, and thus $\widetilde{\chi}(p,K)\in 2^{\mathbb{R}^n}$. We shall define $\widetilde{\beta}(x,K)=\bigcup\{\widetilde{\chi}(p,K)\mid p\in h(x)\}$.

In order to produce $\widetilde{\beta}$ as required by 6.2, it will be necessary to make a suitable choice of the map $\widetilde{\chi}$ indicated above. Let us first indicate how we could construct a certain map $\widetilde{\kappa}:P^{(1)}\times 2_0^{I^3}\to 2^{\mathbb{R}^n}$ which would extend, by the method indicated above using Lemma 6.6, to a map $\beta^*:P\times 2_0^{I^3}\to 2^{\mathbb{R}^n}$. This map β^* will satisfy (a)–(d) of 6.2 but not (e). We shall then show how to adjust $\widetilde{\kappa}$ to produce $\widetilde{\chi}$ in such a manner that the map $\widetilde{\beta}$ induced by $\widetilde{\chi}$ will still satisfy (a)–(d) of 6.2, but now also (e).

To obtain $\widetilde{\kappa}$, consider an admissible (i,j), and for $(t,K)\in I\times 2_0^{I^3}$ let

$$\text{(viii)}\qquad \widetilde{\kappa}_{(i,j)}(t,K)=\widetilde{\kappa}_{\Sigma_{(i,j)}}(t,K)\cup\{\kappa_S(t)\mid S\in S_{(i,j)}\setminus\{\Sigma_{(i,j)}\}\}.$$

The maps $\widetilde{\kappa}_{(i,j)}$ induce a map $\widetilde{\kappa}:P^{(1)}\times 2_0^{I^3}\to 2^{\mathbb{R}^n}$ via the formula

$$\widetilde{\kappa}(p,K)=\widetilde{\kappa}_{(i,j)}(t,K),\qquad p=tv_j+(1-t)v_i,\ p\in\langle v_i,v_j\rangle.$$

Using 6.5(b), one can check that, indeed, $\widetilde{\kappa}$ is well defined.

Let us indicate why β^* will not satisfy (e) of 6.2, and hence why we shall need to introduce adjusted versions χ_S of the maps κ_S in order to produce the needed $\widetilde{\chi}$ and in turn its extension $\widetilde{\beta}$.

Suppose p is a vertex of P, say $p=v_i\in\langle v_i,v_j\rangle$. For any $K\in 2_0^{I^3}$, $\widetilde{\kappa}(p,K)=\widetilde{\kappa}_{(i,j)}(0,K)=\widetilde{\kappa}_{\Sigma_{(i,j)}}(0,K)\cup\bigcup\{\kappa_S(0)\mid S\in S_{(i,j)}\setminus\{\Sigma_{(i,j)}\}\}$. By 6.5(b), $\kappa_{\Sigma_{(i,j)}}(0,K)$ is a singleton and of course $\{\kappa_S(0)\mid S\in S_{(i,j)}\setminus\{\Sigma_{(i,j)}\}\}$ is a finite set. Hence $\beta^*(p,K)=\widetilde{\kappa}(p,K)$ is a finite set which cannot satisfy 6.2(e) because $\dim Z(X)=2$ for every allowable sequence X.

Proof of Proposition 6.2. First, for every admissible (i,j) and every $S\in S_{(i,j)}$ we prescribe a parameterization $\chi_S:I\to\mathbb{R}^n$ so that $\operatorname{im}\chi_S\subset S\cup\bigcup\{\Sigma_{(k,m)}\mid\Sigma_{(k,m)}\cap S\neq\emptyset\}$. For $\Sigma=\Sigma_{(i,j)}$, we let

$$\text{(1)}\qquad \chi_\Sigma(s)=s\kappa_\Sigma(3/4)+(1-s)\kappa_\Sigma(1/4),\qquad s\in I.$$

Observe that $\operatorname{im}\chi_\Sigma\subset\kappa_\Sigma([1/4,3/4])\subset\Sigma$, and further that χ_Σ preserves orientation. A description of χ_S for $S\in S_{(i,j)}\setminus\{\Sigma_{(i,j)}\}$ requires some auxiliary preparation.

Let $x=\kappa_S(0)\in\varphi(v_i)$ and $y=\kappa_S(1)\in\varphi(v_j)$ be the endpoints of S. We fix $t_x\in\{0,1/4\}$ and $t_y\in\{3/4,1\}$ so that $t_x=1/4$ if x is an endpoint of some segment $\Sigma_{(k,m)}$ and $t_x=0$ otherwise; and $t_y=3/4$ if y is an endpoint of some $\Sigma_{(k,m)}$ and $t_y=1$ otherwise. Next we select $\overline{x}$ and $\overline{y}$ as follows. We take $\overline{x}=x$ if $t_x=0$ and $\overline{y}=y$ if $t_y=1$. If $\Sigma'=\Sigma_{(k,m)}$ has x as an endpoint, we let $\overline{x}=\kappa_{\Sigma'}(3/4)$ whenever $i=m$, and $\overline{x}=\kappa_{\Sigma'}(1/4)$ whenever $i=k$. Similarly, if $\Sigma''=\Sigma_{(q,r)}$ has y as an endpoint, we let $\overline{y}=\kappa_{\Sigma''}(1/4)$ whenever $j=q$, and $\overline{y}=\kappa_{\Sigma''}(3/4)$ whenever $j=r$. Finally, χ_S will be the unique piecewise affine map that transforms affinely, in an orientation preserving fashion, $[0,t_x)$, $[t_x,t_y]$, and $(t_y,1]$ onto $[\overline{x},x)$, $[x,y]$, and $(y,\overline{y}]$, respectively. Hence, we have

$$\chi_S(T)=\begin{cases}[\overline{x},x)\subset\Sigma'=\Sigma_{(k,m)} & \text{if } T=[0,t_x),\\ [x,y]=S & \text{if } T=[t_x,t_y],\\ (y,\overline{y}]\subset\Sigma''=\Sigma_{(q,r)} & \text{if } T=(t_y,1].\end{cases}\tag{2}$$

Note that

$$\chi_S=\kappa_S \quad \text{if and only if} \quad t_x=0 \text{ and } t_y=1.\tag{3}$$

Moreover, for the unique orientation preserving affine map $t_\Sigma : I\to[1/4,3/4]$ we have

$$\chi_\Sigma(s)=\kappa_\Sigma(t_\Sigma(s)),\qquad s\in I.\tag{4}$$

For $S\in S_{(i,j)}\setminus\{\Sigma_{(i,j)}\}$ there are unique affine mappings $t_{\Sigma'}$, of $[0,t_x)$ *into* I, t_S of $[t_x,t_y]$ *onto* I, and $t_{\Sigma''}$ of $(t_y,1]$ *into* I so that

$$\chi_S(s)=\begin{cases}\kappa_{\Sigma'}(t_{\Sigma'}(s)) & \text{if } s\in[0,t_x),\\ \kappa_S(t_S(s)) & \text{if } s\in[t_x,t_y],\\ \kappa_{\Sigma''}(t_{\Sigma''}(s)) & \text{if } s\in(t_y,1].\end{cases}\tag{5}$$

We see that t_S preserves orientation; also $t_{\Sigma'}$ and $t_{\Sigma''}$ preserve orientation provided $i=m$ and $j=q$; otherwise $t_{\Sigma'}$ and $t_{\Sigma''}$ reverse orientation.

Given $(s,K)\in I\times 2_0^{I^3}$ and $S\in S_{(i,j)}$ we define

$$\widetilde{\chi}_S(s,K)=\widetilde{\kappa}_{\Sigma_{(i,j)}}(t_{\Sigma_{(i,j)}}(s),K)\quad \text{if } S=\Sigma_{(i,j)},$$

and if $S\in S_{(i,j)}\setminus\{\Sigma_{(i,j)}\}$,

$$\widetilde{\chi}_S(s,K)=\begin{cases}\widetilde{\kappa}_{\Sigma'}(t_{\Sigma'}(s),K) & \text{if } s\in[0,t_x),\\ \{\kappa_S(t_S(s))\} & \text{if } s\in[t_x,t_y],\\ \widetilde{\kappa}_{\Sigma''}(t_{\Sigma''}(s),K) & \text{if } s\in(t_y,1].\end{cases}\tag{6}$$

By an application of 6.5(a), (5) and (6), one has

$$\widetilde{\chi}_S(s,\{0\})=\{\chi_S(s)\}\subset\widetilde{\chi}_S(s,K)\quad \text{ for } (s,K)\in I\times 2_0^{I^3}.\tag{7}$$

Applying 6.5(c) and (6) we infer that

(8) $\widetilde{\chi}_S(s,K)$ is a continuum if $(s,K)\in I\times 2_0^{I^3}$ and K is a continuum.

We see that $\widetilde{\chi}_S$ is continuous for each $S\in S_{(i,j)}$ as follows. First, if $S=\Sigma_{(i,j)}$ then the fact (Lemma 6.5) that $\widetilde{\kappa}_S$ is continuous and the formulation of $\widetilde{\chi}_S$ show the needed continuity. If $S\neq\Sigma_{(i,j)}$, then continuity hinges on whether the three formulas describing χ_S agree in the limit independently of K at the places t_x and t_y. For example, by an application of 6.5(b), $\lim_{s\to t_x^-}\widetilde{\chi}_S(s,K)=\lim_{s\to t_x^-}\widetilde{\kappa}_{\Sigma'}(t_{\Sigma'}(s),K)=\{x\}$ independently of K, whereas $\widetilde{\chi}_S(t_x,K)=\kappa_S(t_S(t_x))=\kappa_S(0)=\{x\}$. One can make a similar argument at t_y.

Define

$$\widetilde{\chi}_{(i,j)}(s,K)=\bigcup\{\widetilde{\chi}_S(s,K)\mid S\in S_{(i,j)}\}. \tag{9}$$

These maps $\widetilde{\chi}_{(i,j)}: I\times 2_0^{I^3}\to 2^{\mathbb{R}^n}$ induce a map $\widetilde{\chi}: P^{(1)}\times 2_0^{I^3}\to 2^{\mathbb{R}^n}$ by

$$\widetilde{\chi}(p,K)=\widetilde{\chi}_{(i,j)}(s,K)\quad \text{ where } p=sv_j+(1-s)v_i\in\langle v_i,v_j\rangle. \tag{10}$$

We must show that $\widetilde{\chi}$ is well defined, and this requires checking only at the vertices of $P^{(1)}$. Suppose v_i is a vertex, say of $\langle v_i,v_j\rangle$, and of another simplex, say $\langle v_k,v_i\rangle$. Then for a given K, we have to show that $\widetilde{\chi}_{(i,j)}(0,K)=\widetilde{\chi}_{(k,i)}(1,K)$. Now, $\widetilde{\chi}_{(i,j)}(0,K)=\bigcup\{\widetilde{\chi}_S(0,K)\mid S\in S_{(i,j)}\}$. When $S=\Sigma_{(i,j)}$, $\widetilde{\chi}_S(0,K)=\widetilde{\kappa}_{\Sigma_{(i,j)}}(1/4,K)$, since $t_{\Sigma_{(i,j)}}(0)=1/4$. Hence $\widetilde{\chi}_{(i,j)}(0,K)=\widetilde{\kappa}_{\Sigma_{(i,j)}}(1/4,K)\cup\bigcup\{\widetilde{\chi}_S(0,K)\mid S\in S_{(i,j)}\setminus\{\Sigma_{(i,j)}\}\}$.

There is an $S_1\in S_{(i,j)}\setminus\{\Sigma_{(i,j)}\}$ such that $\Sigma_{(k,i)}$ has an endpoint in common with S_1 (see 6.4(b)). For any such S_1, $\widetilde{\chi}_{S_1}(0,K)=\widetilde{\kappa}_{\Sigma_{(i,j)}}(3/4,K)$. We therefore have $\widetilde{\chi}_{(i,j)}(0,K)=\widetilde{\kappa}_{\Sigma_{(i,j)}}(1/4,K)\cup\widetilde{\kappa}_{\Sigma_{(k,i)}}(3/4,K)\cup\bigcup\{\widetilde{\chi}_S(0,K)\mid S\in S_{(i,j)}$ and $S\cap(\Sigma_{(i,j)}\cup\Sigma_{(k,i)})=\emptyset\}$. A similar argument shows that $\widetilde{\chi}_{(k,i)}(1,K)=\widetilde{\kappa}_{\Sigma_{(k,i)}}(3/4,K)\cup\widetilde{\kappa}_{\Sigma_{(i,j)}}(1/4,K)\cup\bigcup\{\widetilde{\chi}_S(1,K)\mid S\in S_{(k,i)}$ and $S\cap(\Sigma_{(i,j)}\cup\Sigma_{(k,i)})=\emptyset\}$.

Using 6.4(b) again, we note that if $S_0\in S_{(k,i)}$ and $S_0\cap(\Sigma_{(i,j)}\cup\Sigma_{(k,i)})=\emptyset$, then there is at least one $S_1\in S_{(i,j)}$ with $S_1\cap(\Sigma_{(i,j)}\cup\Sigma_{(k,i)})=\emptyset$ and S_1 having an endpoint in common with S_0. The construction yields $\widetilde{\chi}_{S_0}(1,K)=\widetilde{\chi}_{S_1}(0,K)$. The reverse is also true: for such S_1 there is S_0 with the same equality. Therefore, finally, we have $\widetilde{\chi}_{(i,j)}(0,K)=\widetilde{\chi}_{(k,i)}(1,K)$. The other case, that is, the second simplex is $\langle v_i,v_k\rangle$, yields the same equality.

The continuity of $\widetilde{\chi}$ follows because $P^{(1)}$ is locally finite and each $\widetilde{\chi}_{(i,j)}$ is continuous. We define the required $\widetilde{\beta}: P\times 2_0^{I^3}\to 2^{\mathbb{R}^n}$ by

$$\widetilde{\beta}(x,K)=\bigcup\{\widetilde{\chi}(p,K)\mid p\in h(x)\}, \tag{11}$$

where h is the map of Lemma 6.6. The continuity of h and of $\widetilde{\chi}$ yield that of $\widetilde{\beta}$.

Throughout the process of verification of (a)–(e) we will maintain the following notation: $x\in\tau$, τ is a simplex of P ($\dim\tau\ge 1$), $p\in h(x)\cap\langle v_i,v_j\rangle$ where $\langle v_i,v_j\rangle\in\tau^{(1)}$ (h is from Lemma 6.6) and $p=sv_j+(1-s)v_i$, $0\le s\le 1$. We start with verification of (a).

Note that $\widetilde{\kappa}_{(i,j)}(s,\{0\})=\{\kappa_S(s)\mid S\in S_{(i,j)}\}$ and $\widetilde{\chi}_{(i,j)}(s,\{0\})=\bigcup\{\widetilde{\chi}_S(s,\{0\})\mid S\in S_{(i,j)}\}=\{\chi_S(s)\mid S\in S_{(i,j)}\}$. Hence $H(\widetilde{\kappa}_{(i,j)}(s,\{0\}),\widetilde{\chi}_{(i,j)}(s,\{0\}))\le\max\{d(\kappa_S(s),\chi_S(s))\mid S\in S_{(i,j)}\}$. We see that $\kappa_S(s)\in S$ while $\chi_S(s)$ lies in, say, S' where $S'\cap S\ne\emptyset$, $S'\in S_{(k,m)}$ and $\langle v_k,v_m\rangle\cap\langle v_i,v_j\rangle\ne\emptyset$. Using 6.4(e), we get $\operatorname{diam}_d(S)<\frac{1}{5}\delta(\beta(x))$ and $\operatorname{diam}_d(S')<$

$\frac{1}{5}\delta(\beta(x))$. Hence $d(\kappa_S(s),\chi_S(s)) < \frac{2}{5}\delta(\beta(x))$ and we conclude that

$$H(\widetilde{\kappa}_{(i,j)}(s,\{0\}),\widetilde{\chi}_{(i,j)}(s,\{0\})) < \tfrac{2}{5}\delta(\beta(x)). \tag{12}$$

We now want to show that

$$H(\widetilde{\chi}_S(s,\{0\}),\widetilde{\chi}_S(s,K)) \le \tfrac{1}{4}\delta(\beta(x)), \qquad S\in S_{(i,j)}. \tag{13}$$

Since $\widetilde{\chi}_S(s,\{0\})\subset\widetilde{\chi}_S(s,K)$, it is sufficient to prove that $\operatorname{diam}_d\widetilde{\chi}_S(s,K)\le \frac{1}{4}\delta(\beta(x))$. Referring to (6), one sees that $\widetilde{\chi}_S(s,K)$ is either a singleton $\{\kappa_S(t_S(s))\}$, in which case its diameter is 0, or is of the form $\widetilde{\kappa}_{\Sigma}(t_{\Sigma}(s),K)$. In the latter instance, we apply 6.5(d).

From (13) we get

$$H(\widetilde{\chi}_{(i,j)}(s,\{0\}),\widetilde{\chi}_{(i,j)}(s,K)) \le \tfrac{1}{4}\delta(\beta(x)). \tag{14}$$

Now

$$\begin{aligned} H(\beta(x),\widetilde{\chi}(p,K)) &= H(\beta(x),\widetilde{\chi}_{(i,j)}(s,K)) \\ &\le H(\beta(x),\beta(v_i)) + H(\beta(v_i),\varphi(v_i)) \\ &\quad + H(\varphi(v_i),\widetilde{\kappa}_{(i,j)}(s,\{0\})) \\ &\quad + H(\widetilde{\kappa}_{(i,j)}(s,\{0\}),\widetilde{\chi}_{(i,j)}(s,\{0\})) \\ &\quad + H(\widetilde{\chi}_{(i,j)}(s,\{0\}),\widetilde{\chi}_{(i,j)}(s,K)) \\ &< \big(\tfrac{1}{32}+\tfrac{1}{16}+\tfrac{1}{5}+\tfrac{2}{5}+\tfrac{1}{4}\big)\delta(\beta(x)) < \delta(\beta(x)) \end{aligned}$$

by respectively applying (iii), the second part of 6.4(e), the first part of 6.4(e), (12) and (14). Since $H(\beta(x),\widetilde{\beta}(x,K))\le\max\{H(\beta(x),\widetilde{\chi}(p,K))\mid p\in h(x)\}$, the estimate (a) follows.

To obtain (b), note that $\widetilde{\beta}(x,\{0\}) = \bigcup\{\widetilde{\chi}(p,\{0\})\mid p\in h(x)\} = \bigcup\{\chi_{(i,j)}(s,\{0\})\mid p\in h(x)\} = \bigcup\{\widetilde{\chi}_S(s,\{0\})\mid S\in S_{(i,j)},\ p\in h(x)\}$. By (7) each $\widetilde{\chi}_S(s,\{0\})$ is the singleton $\{\chi_S(s)\}$. Since $S_{(i,j)}$ and $h(x)$ are finite, so is $\widetilde{\beta}(x,\{0\})$; hence we have (b).

If we replace $\{0\}$ in the above argument by a continuum $K\in 2_0^{I^3}$, then using (7) and (8) instead of just (7), we get (c).

Before we check (d), (e) let us make some observations on the structure of $\widetilde{\beta}(x,K)$. From (6), (9), (10), $\widetilde{\chi}(p,K)$ is a union of a finite subcollection of $\{\widetilde{\kappa}_{\Sigma_{(k,m)}}(t,K)\mid t\in I \text{ and } \langle v_k,v_m\rangle\cap\langle v_i,v_j\rangle\ne\emptyset\}$ and a finite set. Hence $\widetilde{\chi}(p,K)=\Omega(p,K)\cup\bigcup\mathcal{L}(p,K)$ where $\Omega(p,K)$ is a finite set and $\mathcal{L}(p,K)$ is a finite subset of $\{\widetilde{\kappa}_{\Sigma_{(k,m)}}(t,K)\mid(\Sigma_{(k,m)},t)\in\mathrm{Sig}_i\times I \text{ and } \langle v_k,v_m\rangle\cap\tau\ne\emptyset\}$. Indeed, if we choose any vertex v_r of τ, then $\mathcal{L}(p,K)\subset\{\widetilde{\kappa}_{\Sigma_{(k,m)}}(t,K)\mid (\Sigma_{(k,m)},t)\in\mathrm{Sig}_r\times I\}$ because of the definition of Sig_r (see (vi)). This, together with (11), shows that

$$\widetilde{\beta}(x,K)=\Omega_0(x,K)\cup\bigcup\mathcal{L}_0(x,K) \tag{15}$$

where $\Omega_0(x,K)$ is a finite set and

(16) $\mathcal{L}_0(x, K)$ is a finite subset of $\{\widetilde{\kappa}_{\Sigma_{(k,m)}}(t, K) \mid (\Sigma_{(k,m)}, t) \in \mathrm{Sig}_i \times I\}$ where v_i is a vertex of τ and τ is any simplex of P containing x.

From (15), (16) and 6.5(e), one obtains (d). To prove (e), it is sufficient to show that for $\Sigma = \Sigma_{(i,j)}$, we have $\widetilde{\kappa}_\Sigma(t_\Sigma(s), Z(X)) \subset \widetilde{\beta}(x, Z(X))$, because $t_\Sigma(s) \in [1/4, 3/4] \subset (1/8, 7/8)$ and we can apply 6.5(f). By (6), $\widetilde{\chi}_\Sigma(s, Z(X)) = \widetilde{\kappa}_\Sigma(t_\Sigma(s), Z(X))$. Then one applies (9), (10) and (11) to get $\widetilde{\chi}_\Sigma(s, Z(X)) \subset \widetilde{\chi}_{(i,j)}(s, Z(X)) \subset \widetilde{\chi}(p, Z(X)) \subset \widetilde{\beta}(x, Z(X))$. ∎

Appendix. Let $D = [-1, 1]$ and identify $\mathbb{R}$ with $(-1, 1) \subset D$. Denote by $\mathcal{K}_{\mathrm{fd}}$ the class of spaces that are countable unions of finite-dimensional compacta.

We have:

THEOREM. *The space* $\mathrm{ANR}(\mathbb{R})$ *is a* $\mathcal{K}_{\mathrm{fd}}$*-absorber in* 2^D.

Since $Q_{\mathrm{f}} = \{(x_i) \in Q \mid x_i = 0 \text{ a.e.}\}$ is another $\mathcal{K}_{\mathrm{fd}}$-absorber in Q, the uniqueness theorem on absorbers yields

COROLLARY. *The space* $\mathrm{ANR}(\mathbb{R})$ *is homeomorphic to* Q_{f}. ∎

Proof of Theorem. According to a result of Curtis [Cu1], the space $\mathcal{F}(\mathbb{R})$ is a $\mathcal{K}_{\mathrm{fd}}$-absorber. In order to show that $\mathrm{ANR}(\mathbb{R}) \supset \mathcal{F}(\mathbb{R})$ is also a $\mathcal{K}_{\mathrm{fd}}$-absorber, we use the maximality theorem of Toruńczyk (see [BP, Theorem 4.2, p. 131]). We only need to check that (1) $\mathrm{ANR}(\mathbb{R}) \in \mathcal{K}_{\mathrm{fd}}$, and (2) $\mathrm{ANR}(\mathbb{R})$ is a σZ-set in 2^D.

The subset of $\mathrm{ANR}(\mathbb{R})$ consisting of exactly n components can obviously be identified with $\{(a_1, b_1, a_2, b_2, \ldots, a_n, b_n) \in \mathbb{R}^{2n} \mid a_1 \le b_1 < a_2 \le b_2 < \ldots < a_n \le b_n\}$. Since the latter set is σ-compact, it follows that $\mathrm{ANR}(\mathbb{R})$ is a countable union of finite-dimensional compacta; this shows (1).

We will show that every compactum $K \subset \mathrm{ANR}(\mathbb{R})$ is a Z-set in 2^D (then (2) follows from (1)). Let $f : I^n \to 2^D$ be a map of the n-dimensional cube I^n and $0 < \varepsilon < 1$ be given. We let $f_\varepsilon(q) = [(1-\varepsilon/2)f(q)] \cup [\sup(1-\varepsilon/2)f(q) + (\varepsilon/2)C]$ for $q \in I^n$, where C is the standard Cantor set in $[0, 1]$. We see that $d(f, f_\varepsilon) < \varepsilon$ and $f_\varepsilon(q) \notin \mathrm{ANR}(\mathbb{R})$ for all $q \in I^n$. Hence, $f_\varepsilon(I^n) \cap K = \emptyset$, and K is a Z-set. ∎

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