

Connectivity of diagonal products of Baire one functions

by

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Abstract. We characterize those Baire one functions f for which the diagonal product $x \mapsto (f(x), g(x))$ has a connected graph whenever g is approximately continuous or is a derivative.

I. Introduction. It is well known for a long time that the product of two derivatives need not be a derivative. Moreover, the characteristic function of every closed set can be written as the product of two bounded derivatives [2]. So the graph of the product of two derivatives need not be connected, whence the graph of the diagonal product of two derivatives need not be connected. On the other hand, it is well known (and easy to prove) that the product of a bounded approximately continuous function with a bounded derivative is a derivative again, so its graph is connected. However, the graph of the diagonal product of a bounded approximately continuous function with a bounded derivative is not necessarily connected. In 1963 Neugebauer [7] constructed a bounded approximately continuous function f and a bounded derivative g such that $f(0) = g(0) = 0$ and $[f(x)]^2 + [g(x)]^2 \geq 1/4$ for $x \neq 0$, so the graph of $f \Delta g$ is not connected. In this paper our aim is to characterize those Baire one functions f for which the diagonal product $f \Delta g$ has a connected graph whenever g is approximately continuous or is a derivative.

II. Preliminaries. The real line $(-\infty, \infty)$ is denoted by \mathbb{R} and the set of positive integers by \mathbb{N} . The only measure used is Lebesgue measure in \mathbb{R} and all integrals are Lebesgue integrals. For each set $A \subset \mathbb{R}$, $\text{int } A$ denotes its (Euclidean) interior, $\text{cl } A$ its closure, $\text{fr } A$ its boundary and $|A|$ its outer measure.

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The word *function* means a mapping from \mathbb{R} into \mathbb{R} unless otherwise explicitly stated. The graph of a function $f : X \rightarrow Y$ will be denoted by $\Gamma(f)$. We denote by \mathcal{D} the family of all derivatives. If X and Y are metric spaces, then the family of all *Baire one functions* from X into Y (i.e., pointwise limits of sequences of continuous functions) will be denoted by $\mathcal{B}_1(X, Y)$. We write \mathcal{B}_1 for $\mathcal{B}_1(\mathbb{R}, \mathbb{R})$.

The terms *d-closed*, *d-interior* (*d-int*) etc. will refer to the Denjoy topology (density topology) on \mathbb{R} . (See, e.g., [3], [9], [5].) We say that a function f is *approximately continuous* if it is continuous relative to the Denjoy topology. The family of all approximately continuous functions will be denoted by \mathcal{C}_{ap} . Recall that each element of \mathcal{C}_{ap} is a Baire one function and each bounded element of \mathcal{C}_{ap} is a derivative (see, e.g., [1]).

We denote by \mathfrak{b} the family of all bounded functions. We drop the “ \cap ” sign between classes of functions, e.g., $\mathfrak{b}\mathcal{C}_{\text{ap}}$ denotes the family of all bounded approximately continuous functions.

We denote by \mathcal{C} the family of all continuous functions, by $\mathcal{C}_{\text{a.e.}}$ the family of all functions which are continuous almost everywhere and by \mathcal{C}_0 the family of all functions which are continuous except possibly at one point. It is well known that elements of $\mathcal{C}_{\text{ap}}\mathcal{C}_{\text{a.e.}}$ are exactly those functions which are continuous with respect to the so-called a.e.-topology [8]. Recall that a set $A \subset \mathbb{R}$ is a.e.-open if and only if it is *d*-open and $|A| = |\text{int } A|$.

For each set $T \subset \mathbb{R}$ and $a, b \in \mathbb{R}$, $a < b$, we denote by $\varphi_T(a, b)$ the measure of the greatest interval J contained in $(a, b) \setminus T$ if any such interval exists, and 0 otherwise. We say that a set $T \subset \mathbb{R}$ is *porous* at $x \in \mathbb{R}$ from the left if

$$\limsup_{\eta \rightarrow 0^+} \frac{\varphi_T(x - \eta, x)}{\eta} > 0.$$

Being porous from the right is defined similarly.

We say that a set $T \subset \mathbb{R}$ is *non-degenerate* at $x \in \mathbb{R}$ from the left if

$$\limsup_{\eta \rightarrow 0^+} \frac{|T \cap (x - \eta, x)|}{\eta} > 0.$$

Non-degeneracy from the right is defined similarly.

Let Y be a topological space and let $f : \mathbb{R} \rightarrow Y$. We say that

- f is a *Darboux function* if the f -image of each interval is connected,
- f is *peripherally continuous* if

$$f^{-1}(U) \cap (x - \eta, x) \neq \emptyset \neq f^{-1}(U) \cap (x, x + \eta)$$

for each $x \in \mathbb{R}$, each neighborhood $U \subset Y$ of $f(x)$ and each $\eta > 0$.

Given two functions $f, g : \mathbb{R} \rightarrow \mathbb{R}$, we define their *diagonal product* $f \Delta g : \mathbb{R} \rightarrow \mathbb{R}^2$ by $(f \Delta g)(x) = (f(x), g(x))$.

Let $\mathcal{A} \subset \mathcal{B}_1$. The *maximal diagonal class* of \mathcal{A} with respect to connectivity is the family $\mathcal{M}_\Delta(\mathcal{A})$ of those Baire one functions whose diagonal product with each function from \mathcal{A} has a connected graph.

III. Auxiliary lemmas. There are many conditions which are equivalent to the connectivity of the graph of a function $f \in \mathcal{B}_1$ (cf. [1, Theorem 1.1, p. 9]). For our purpose we will generalize two of them.

LEMMA 1. *Assume that Y is a T_1 -space and $f : \mathbb{R} \rightarrow Y$ is a Darboux function. Then f is peripherally continuous.*

PROOF. Suppose that f is not peripherally continuous from the left at some $x \in \mathbb{R}$. Let a neighborhood $U \subset Y$ of $f(x)$ and $\eta > 0$ be such that $f^{-1}(U) \cap (x - \eta, x) = \emptyset$. Set $A = f((x - \eta, x])$ and $V = Y \setminus \{f(x)\}$. Since Y is a T_1 -space, V is open. Hence $A_1 = A \cap U = \{f(x)\}$ and $A_2 = A \cap V$ are open in A , disjoint and non-empty, and $A_1 \cup A_2 = A$. So A is not connected and f is not a Darboux function, which completes the proof. ■

LEMMA 2. *Assume that Y is a metric space and $f \in \mathcal{B}_1(\mathbb{R}, Y)$ is peripherally continuous. Then the graph of f is connected.*

PROOF. Suppose that $\Gamma(f)$ is not connected. Then there exist disjoint non-empty sets $E_1, E_2 \subset \Gamma(f)$, open in $\Gamma(f)$, such that $E_1 \cup E_2 = \Gamma(f)$. For $j \in \{1, 2\}$ let D_j be the preimage of E_j under the map $x \mapsto (x, f(x))$. Then D_1, D_2 are disjoint, non-empty and $D_1 \cup D_2 = \mathbb{R}$, so $\text{fr } D_1 \neq \emptyset$. By [4] (§31.X.5, p. 397), the set of points of continuity of $f|_{\text{fr } D_1}$ is non-empty. Let x be one. By symmetry, we may assume that $x \in D_1$ and that $(x - \tau, x) \cap D_2 \neq \emptyset$ for each $\tau > 0$. Since E_1 and E_2 are disjoint and open in $\Gamma(f)$, there exists a neighborhood $U \subset Y$ of $f(x)$ and a $\tau_1 > 0$ such that $f(t) \notin U$ for $t \in (x - \tau_1, x + \tau_1) \cap D_2$. Since $f|_{\text{fr } D_1}$ is continuous at x , there exists a $\tau_2 \in (0, \tau_1)$ such that $f(t) \in U$ for each $t \in (x - \tau_2, x] \cap \text{fr } D_1$. So $D_2 \cap (x - \tau_2, x) \cap \text{fr } D_1 = \emptyset$, whence $D_2 \cap (x - \tau_2, x) \neq \emptyset$ is open. Let (a, b) be a component of $D_2 \cap (x - \tau_2, x)$. Then $b \in (x - \tau_2, x] \cap \text{fr } D_1$ and $f(b) \in U$. But $f(t) \notin U$ for $t \in (a, b)$. So f is not peripherally continuous at b from the left, contrary to the assumption. ■

The next lemma follows easily from the definitions.

LEMMA 3. *Assume that Y is an arbitrary topological space and that the graph of $f : \mathbb{R} \rightarrow Y$ is connected. Then f is a Darboux function.* ■

From the above three lemmas we get the following theorem.

THEOREM 4. *Let Y be a metric space and $f \in \mathcal{B}_1(\mathbb{R}, Y)$. Then the following conditions are equivalent:*

(A) *f is a Darboux function,*

- (B) f is peripherally continuous,
- (C) f has a connected graph.

We will also need the following lemma.

LEMMA 5 [6, Lemma 9]. *Assume that $\{J_n : n \in \mathbb{N}\}$ is a family of non-overlapping intervals and (r_n) is a sequence of non-negative numbers with $\sum_{n=1}^{\infty} r_n < \infty$. Then there exists a sequence (t_n) such that $|t_n| = 1$ for each $n \in \mathbb{N}$ and for every interval $I \subset \mathbb{R}$,*

$$\left| \sum_{J_n \subset I} t_n r_n \right| \leq 2 \sup \{r_n : J_n \subset I\}.$$

IV. Main results. First we will deal with the family \mathcal{C}_{ap} .

PROPOSITION 6. *Let $f \in \mathcal{B}_1$. Then the following conditions are equivalent:*

- (i) $f \in \mathcal{M}_{\Delta}(\mathcal{C}_{\text{ap}})$,
- (ii) $f \in \mathcal{M}_{\Delta}(\text{b}\mathcal{C}_{\text{ap}})$,
- (iii) for each $x \in \mathbb{R}$ and each $\varepsilon > 0$ the set $f^{-1}([f(x) - \varepsilon, f(x) + \varepsilon])$ is bilaterally non-degenerate at x .

PROOF. The implication (i) \Rightarrow (ii) is obvious.

(ii) \Rightarrow (iii). Suppose that for some $x \in \mathbb{R}$ and some $\varepsilon > 0$, C is not non-degenerate at x from the left, where $C = f^{-1}([f(x) - \varepsilon, f(x) + \varepsilon])$. For each $n \in \mathbb{N}$ set $x_n = x - 1/n$, $I_n = [x_n, x_{n+1}]$ and $A_n = I_n \setminus C$, find a closed set $B_n \subset d\text{-int } A_n$ such that $|A_n \setminus B_n| \leq |A_n|/n$, and use Lemma 12 of [10] to find an approximately continuous function g_n such that $0 \leq g_n \leq 1$ on \mathbb{R} , $g_n = 1$ off A_n and $g_n = 0$ on B_n . Define

$$g(t) = \begin{cases} 0 & \text{if } t \leq x_1 \text{ or } t \geq x, \\ \varepsilon g_n(t) & \text{if } t \in I_n, n \in \mathbb{N}. \end{cases}$$

Then clearly g is bounded and approximately continuous except possibly at x , and it is continuous at x from the right. Moreover, for each $t \in [x_n, x_{n+1}]$,

$$\begin{aligned} \frac{|\{z \in [t, x] : g(z) - g(x) = 0\}|}{x - t} &\geq \frac{1}{x - x_n} \left| \bigcup_{k=n+1}^{\infty} B_k \right| \\ &\geq \frac{n}{n+1} \left(1 - \frac{1}{n+1} \right) \frac{|[x_{n+1}, x] \setminus C|}{x - x_{n+1}}. \end{aligned}$$

So g is approximately continuous at x from the left. But for each $t \in [x_1, x)$ either $t \notin \bigcup_{n=1}^{\infty} A_n$, so that $g(t) = g(t) - g(x) = \varepsilon$, or $t \in \bigcup_{n=1}^{\infty} A_n$, and then $t \notin C$ and $|f(t) - f(x)| > \varepsilon$. Hence $f\Delta g$ is not peripherally continuous at x from the left and by Theorem 4, $f \notin \mathcal{M}_{\Delta}(\text{b}\mathcal{C}_{\text{ap}})$.

Similarly we proceed if C is not non-degenerate at x from the right.

(iii) \Rightarrow (i). Let $g \in \mathcal{C}_{\text{ap}}$. Fix an $x \in \mathbb{R}$ and an $\varepsilon > 0$. Let C be as above and $D = g^{-1}([g(x) - \varepsilon, g(x) + \varepsilon])$. Since C is non-degenerate at x from the left, we can find a sequence $a_n \nearrow x$ such that

$$\lim_{n \rightarrow \infty} \frac{|[a_n, x] \cap C|}{x - a_n} = 2c > 0.$$

Let $\eta \in (0, \varepsilon)$ be such that

$$\frac{|[a_n, x] \cap C|}{x - a_n} > c \quad \text{if } a_n > x - \eta, n \in \mathbb{N},$$

and

$$\frac{|[t, x] \cap D|}{x - t} > 1 - c \quad \text{if } t \in (x - \eta, x).$$

Then $|[a_n, x] \cap C \cap D| > 0$ for each $n \in \mathbb{N}$ with $a_n > x - \eta$. Hence $f \Delta g$ is peripherally continuous at x from the left.

Similarly we can prove that $f \Delta g$ is peripherally continuous at x from the right. This implies that the graph of $f \Delta g$ is connected, so $f \in \mathcal{M}_{\Delta}(\mathcal{C}_{\text{ap}})$. ■

Now we turn to the family \mathcal{D} .

PROPOSITION 7. *Let $f \in \mathcal{B}_1$. Then the following conditions are equivalent:*

- (i) $f \in \mathcal{M}_{\Delta}(\mathcal{D})$,
- (ii) $f \in \mathcal{M}_{\Delta}(\text{bDC}_0)$,
- (iii) for each $x \in \mathbb{R}$ and each $\varepsilon > 0$ the complement of $f^{-1}([f(x) - \varepsilon, f(x) + \varepsilon])$ is bilaterally porous at x .

Proof. The implication (i) \Rightarrow (ii) is obvious.

(ii) \Rightarrow (iii). Suppose that for some $x \in \mathbb{R}$ and some $\varepsilon \in (0, 1)$, $\mathbb{R} \setminus C$ is not porous at x from the left, where $C = f^{-1}([f(x) - \varepsilon, f(x) + \varepsilon])$. Then there exists a sequence $x_n \nearrow x$ such that $x_n \notin C$ for each $n \in \mathbb{N}$ and

$$(1) \quad \lim_{n \rightarrow \infty} \frac{x_{n+1} - x_n}{x - x_n} = 0.$$

For each $n \in \mathbb{N}$ set $A_n = [x_n, x_{n+1}] \setminus C$. We consider two cases:

• Assume that $\text{int } A_n \neq \emptyset$ for each $n \in \mathbb{N}$. Let $y_n \in (a_n, b_n) \subset \text{int } A_n$. For each $n \in \mathbb{N}$ put $J_n = [y_n, y_{n+1}]$, define a continuous function g_n by

$$g_n(t) = \begin{cases} 0 & \text{if } t \leq y_n \text{ or } t \geq y_{n+1}, \\ 1 & \text{if } t \in [b_n, a_{n+1}], \\ \text{linear} & \text{in } [y_n, b_n] \text{ and } [a_{n+1}, y_{n+1}], \end{cases}$$

and set $r_n = \int_{J_n} g_n$. Choose a sequence (t_n) according to Lemma 5. Define

$$g(t) = \begin{cases} 0 & \text{if } t \leq a_1 \text{ or } t \geq x, \\ t_n \varepsilon g_n(t) & \text{if } t \in J_n, n \in \mathbb{N}. \end{cases}$$

Then clearly g is bounded and continuous except possibly at x , and it is continuous at x from the right. Moreover, for each $t \in J_n$,

$$\begin{aligned} \frac{1}{x-t} \left| \int_t^x g \right| &\leq \frac{1}{x-t} \left(\left| \sum_{k=n+1}^{\infty} \int_{J_k} g \right| + \int_{J_n} |g| \right) \\ &\leq \frac{3\varepsilon \sup\{|J_k| : k \geq n\}}{x-t} \\ &\leq 3\varepsilon \sup \left\{ \frac{x_{k+2} - x_k}{x - x_{n+2}} : k \geq n \right\} \\ &\leq 3\varepsilon \sup \left\{ \frac{x_{k+2} - x_k}{x - x_k} \cdot \left(1 - \frac{x_{k+2} - x_k}{x - x_k} \right)^{-1} : k \geq n \right\}. \end{aligned}$$

So by (1), g is a derivative, whence $g \in \text{bDC}_0$. But for each $t \in [a_1, x)$ either $t \notin \bigcup_{n=1}^{\infty} (a_n, b_n)$, so that $|g(t)| = |g(t) - g(x)| = \varepsilon$, or $t \in \bigcup_{n=1}^{\infty} (a_n, b_n)$, in which case $t \notin C$ and $|f(t) - f(x)| > \varepsilon$. It follows that $f\Delta g$ is not peripherally continuous at x from the left and $f \notin \mathcal{M}_{\Delta}(\text{bDC}_0)$.

• Assume that $\text{int } A_n = \emptyset$ for some $n \in \mathbb{N}$. Then C is residual in $[x_n, x_{n+1}]$. Define $\varepsilon' = (|f(x_n) - f(x)| - \varepsilon)/2$ and $C' = f^{-1}([f(x_n) - \varepsilon', f(x_n) + \varepsilon'])$. Then C' is a G_{δ} set and $C' \cap C = \emptyset$, so C' is nowhere dense in $[x_n, x_{n+1}]$. Hence the complement of C' is not porous at x_n from the right and for every interval J , $\text{int}(J \setminus C') \neq \emptyset$. Now proceed as in the previous case, using x_n instead of x and ε' instead of ε .

Similarly we proceed if $\mathbb{R} \setminus C$ is not porous at x from the right.

(iii) \Rightarrow (i). Let $g \in \mathcal{D}$ and let G be its primitive. Fix an $x \in \mathbb{R}$ and an $\varepsilon > 0$. Define C as above. Since $\mathbb{R} \setminus C$ is porous at x from the left, we can find a sequence $a_n \nearrow x$ such that $[a_{2n-1}, a_{2n}] \subset C$ and

$$\lim_{n \rightarrow \infty} \frac{a_{2n} - a_{2n-1}}{x - a_{2n-1}} = 2c > 0.$$

Let $\eta \in (0, \varepsilon)$ be such that for each $t \in (x - \eta, x)$,

$$\frac{a_{2n} - a_{2n-1}}{x - a_{2n-1}} > c \quad \text{if } a_{2n-1} > x - \eta, \quad n \in \mathbb{N},$$

and

$$\left| \frac{G(t) - G(x)}{t - x} - g(x) \right| \leq \frac{c\varepsilon}{2} \quad \text{if } t \in (x - \eta, x).$$

Then for each $n \in \mathbb{N}$ with $a_{2n-1} > x - \eta$, we get

$$\left| \frac{G(a_{2n}) - G(a_{2n-1})}{a_{2n} - a_{2n-1}} - g(x) \right| \leq \frac{2x - a_{2n} - a_{2n-1}}{a_{2n} - a_{2n-1}} \cdot \frac{c\varepsilon}{2} < \varepsilon.$$

Hence there exists a $t \in [a_{2n-1}, a_{2n}]$ such that $|g(t) - g(x)| \leq \varepsilon$. But since $[a_{2n-1}, a_{2n}] \subset C$, also $|f(t) - f(x)| \leq \varepsilon$. It follows that $f\Delta g$ is peripherally continuous at x from the left.

Similarly we can prove that $f\Delta g$ is peripherally continuous at x from the right. This implies that the graph of $f\Delta g$ is connected, so $f \in \mathcal{M}_\Delta(\mathcal{D})$. ■

Remark. By Proposition 7, we get $\mathcal{M}_\Delta(\mathcal{DC}_{a.e.}) = \mathcal{M}_\Delta(\mathcal{D})$. It turns out that the analogous result does not hold for the family \mathcal{C}_{ap} .

PROPOSITION 8. *Let $f \in \mathcal{B}_1$. Then the following conditions are equivalent:*

- (i) $f \in \mathcal{M}_\Delta(\mathcal{C}_{ap}\mathcal{C}_{a.e.})$,
- (ii) $f \in \mathcal{M}_\Delta(\mathbf{b}\mathcal{C}_{ap}\mathcal{C}_0)$,
- (iii) *for each $x \in \mathbb{R}$ and each $\varepsilon > 0$ the closure of $f^{-1}([f(x) - \varepsilon, f(x) + \varepsilon])$ is bilaterally non-degenerate at x .*

Proof. The implication (i) \Rightarrow (ii) is obvious.

(ii) \Rightarrow (iii). Suppose that for some $x \in \mathbb{R}$ and some $\varepsilon > 0$, $\text{cl } C$ is not non-degenerate at x from the left, where $C = f^{-1}([f(x) - \varepsilon, f(x) + \varepsilon])$. For each $n \in \mathbb{N}$ set $x_n = x - 1/n$, $I_n = [x_n, x_{n+1}]$ and $A_n = \text{int } I_n \setminus \text{cl } C$, find a closed set $B_n \subset A_n$ such that $|A_n \setminus B_n| \leq |A_n|/n$, and find a continuous function g_n such that $0 \leq g_n \leq 1$ on \mathbb{R} , $g_n = 1$ off A_n and $g_n = 0$ on B_n . Define

$$g(t) = \begin{cases} 0 & \text{if } t \leq x_1 \text{ or } t \geq x, \\ \varepsilon g_n(t) & \text{if } t \in I_n, n \in \mathbb{N}. \end{cases}$$

Then clearly $g \in \mathbf{b}\mathcal{C}_0$. Repeating the argument of Proposition 6 one can see that $g \in \mathcal{C}_{ap}$ and $f\Delta g$ is not peripherally continuous at x from the left. Hence $f \notin \mathcal{M}_\Delta(\mathbf{b}\mathcal{C}_{ap}\mathcal{C}_{a.e.})$.

Similarly we proceed if $\text{cl } C$ is not non-degenerate at x from the right.

(iii) \Rightarrow (i). Let $g \in \mathcal{C}_{ap}\mathcal{C}_{a.e.}$. Fix an $x \in \mathbb{R}$ and an $\varepsilon > 0$. Let C be as above and $D = g^{-1}((g(x) - \varepsilon, g(x) + \varepsilon))$. Since $\text{cl } C$ is bilaterally non-degenerate at x , we can find a sequence $a_n \nearrow x$ such that

$$\lim_{n \rightarrow \infty} \frac{|[a_n, x] \cap \text{cl } C|}{x - a_n} = 2c > 0.$$

Let $\eta \in (0, \varepsilon)$ be such that

$$\frac{|[a_n, x] \cap \text{cl } C|}{x - a_n} > c \quad \text{if } a_n > x - \eta, n \in \mathbb{N},$$

and

$$\frac{|[t, x] \cap D|}{x - t} > 1 - c \quad \text{if } t \in (x - \eta, x).$$

Then for each $n \in \mathbb{N}$ with $a_n > x - \eta$, we get

$$|(a_n, x) \cap \text{cl } C \cap D| = |(a_n, x) \cap \text{cl } C \cap \text{int } D| > 0.$$

(We used the fact that D is a.e.-open.) Hence $(a_n, x) \cap C \cap D \neq \emptyset$, so $f\Delta g$ is peripherally continuous at x from the left.

Similarly we prove that $f \Delta g$ is peripherally continuous at x from the right. This implies that the graph of $f \Delta g$ is connected and that $f \in \mathcal{M}_\Delta(\mathcal{C}_{\text{ap}}\mathcal{C}_{\text{a.e.}})$. ■

PROPOSITION 9. $\mathcal{M}_\Delta(\mathcal{C}_{\text{ap}}\mathcal{C}_{\text{a.e.}}) \setminus \mathcal{M}_\Delta(\mathcal{C}_{\text{ap}}) \neq \emptyset$.

PROOF. For each $n \in \mathbb{N}$ set $a_n = 1$ and find a nowhere dense perfect set F_n such that $\inf F_n = a_{n+1}$, $\sup F_n = a_n$ and $|F_n| = (1-1/n)(a_n - a_{n+1})$. Observe that

$$(2) \quad 0 \in d\text{-int} \left((-\infty, 0] \cup \bigcup_{n=1}^{\infty} F_n \right).$$

Fix an $n \in \mathbb{N}$. Let $\{E_{n,k} : k \in \mathbb{N}\}$ be the family of all components of $(a_{n+1}, a_n) \setminus F_n$. For all $k \in \mathbb{N}$ if $E_{n,k} = (b_{n,k} - c_{n,k}, b_{n,k} + c_{n,k})$, then let $f_{n,k}$ be a continuous function such that $f_{n,k}(b_{n,k}) = 1$ and $f_{n,k}(x) = 0$ if $|x - b_{n,k}| \geq \max\{c_{n,k}, c_{n,k}^2\}$. Define

$$f(x) = \begin{cases} f_{n,k}(x) & \text{if } x \in E_{n,k}, \ n, k \in \mathbb{N}, \\ 1 & \text{if } x \leq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Then evidently f is approximately continuous on $\mathbb{R} \setminus \{0\}$ and it is continuous at 0 from the left, so at these points condition (iii) of Proposition 8 is satisfied. Fix an $\varepsilon \in (0, 1)$ and set $C = f^{-1}([1-\varepsilon, 1+\varepsilon])$. Then $C \cap \bigcup_{n=1}^{\infty} F_n = \emptyset$ and since $C \cap E_{n,k} \neq \emptyset$ for all $n, k \in \mathbb{N}$, we have $\text{cl } C \supset \bigcup_{n=1}^{\infty} F_n$. Hence by (2), f does not satisfy condition (iii) of Proposition 6 and it does satisfy condition (iii) of Proposition 8, which completes the proof. ■

REMARK. It is easy to see that the function constructed in the proof of Proposition 9 is discontinuous on a set of positive measure. This leads to conjecturing that for functions from $\mathcal{C}_{\text{a.e.}}$ the proposition analogous to Proposition 7 holds. This is indeed true.

PROPOSITION 10. $\mathcal{M}_\Delta(\mathcal{C}_{\text{ap}}\mathcal{C}_{\text{a.e.}}) \cap \mathcal{C}_{\text{a.e.}} = \mathcal{M}_\Delta(\mathcal{C}_{\text{ap}}) \cap \mathcal{C}_{\text{a.e.}}$.

PROOF. The inclusion “ \supset ” is obviously satisfied.

Let $f \in \mathcal{M}_\Delta(\mathcal{C}_{\text{ap}}\mathcal{C}_{\text{a.e.}}) \cap \mathcal{C}_{\text{a.e.}}$, $x \in \mathbb{R}$ and $\varepsilon > 0$. Set $C = f^{-1}([f(x) - \varepsilon, f(x) + \varepsilon])$. By Proposition 8, $\text{cl } C$ is bilaterally non-degenerate at x . Since $t \in C$ implies $|f(t) - f(x)| \leq \varepsilon$, it follows that for every $t \in \text{cl } C$, if t is a point of continuity of f , then $|f(t) - f(x)| \leq \varepsilon$. Hence $|\text{cl } C \setminus C| = 0$ and C is bilaterally non-degenerate at x , i.e., condition (iii) of Proposition 6 is satisfied. ■

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