Minor cycles for interval maps

by

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Abstract. For continuous maps of an interval into itself we consider cycles (periodic orbits) that are non-reducible in the sense that there is no non-trivial partition into blocks of consecutive points permuted by the map. Among them we identify the *miror* ones. They are those whose existence does not imply existence of other non-reducible cycles of the same period. Moreover, we find minor patterns of a given period with minimal entropy.

1. Introduction and basic definitions. The aim of this paper is to find the simplest types of non-reducible cycles for continuous interval maps. Let us try to state this in a more rigorous way. The terminology used here is mainly the same as in [ALM]. Perhaps also [ALM] gives a good background for understanding this paper; there is also an extensive list of relevant publications there.

We consider the space $\mathcal{I}$ of all continuous maps $f : I \to I$, where I is a closed interval. A *cycle* of f is a periodic orbit of f, and by its period we mean the smallest period. If $f, g \in \mathcal{I}$ and P, Q are cycles of f, g respectively, then the pairs (P, f) and (Q, g) are *equivalent* if there exists a homeomorphism φ of the smallest interval containing P onto the smallest interval containing Q such that $\varphi(P) = Q$ and $\varphi \circ f \circ \varphi^{-1}|_Q = g|_Q$ (in other words, the permutations on both orbits are the same, up to the change of orientation). The equivalence classes of this relation are called *patterns*. If P (more precisely, (P, f), but we shall usually suppress f) has pattern A then we shall write $A = [P]$ and we shall call P a *representative of A in f*. We shall also say in this case that *f exhibits A*.

A pattern A *forces* a pattern B if all maps in $\mathcal{I}$ exhibiting A also exhibit B. A cycle P of f has a *non-trivial block structure* if $P = \{x_1, \ldots, x_k\}$ with $x_1 < \ldots < x_k$ and it can be divided into blocks $P_1 = \{x_1, \ldots, x_m\}$, $P_2 = \{x_{m+1}, \ldots, x_{2m}\}$, $\ldots$, $P_s = \{x_{sm-m+1}, \ldots, x_{sm}\}$ such that $m, s > 1$, $sm = k$ and f maps each block P_i onto some block $P_{j(i)}$. A cycle without

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a non-trivial block structure will be called *non-reducible*. Clearly, if two cycles are equivalent and one of them is non-reducible, so is the other one. Therefore we may speak about *non-reducible patterns*.

Now we can state our aim in a more precise way. Namely, we are going to find all minimal non-reducible patterns of a given period k ("minimal" means not forcing any other pattern in this class). We shall call them *minor patterns*. As the reader may have guessed, *minor* stands for "*mi*nimal *non-r*educible"; this explains also why the term *non-reducible* is used rather than *irreducible*. For their representatives we shall use the term *minor cycles*. We shall also find among them the ones with minimal entropy (the *entropy* of a pattern is the infimum of the topological entropies of maps exhibiting this pattern). Entropy measures to some extent how complicated a pattern is, so minimizing it is a natural idea.

The way the problem solved in this paper arose is the following. Recently a similar theory for the orientation preserving homeomorphisms of the disk has been developed (see e.g. [FM]). There non-reducible patterns are either twist (exhibited by rotations) or pseudo-Anosov (after removing the cycle, the map is isotopic to a pseudo-Anosov homeomorphism). The class of pseudo-Anosov patterns seems to be an important one, and one of the natural questions is to find the ones with minimal entropy among those of a given period. This generates immediately another question: how is it for interval maps? Once this question is posed, the next one arises: for interval maps, can we find not only entropy-minimal, but also all forcing-minimal patterns among the non-reducible patterns of a given period? These two questions are answered in this paper.

We shall use some standard notions and techniques of combinatorial dynamics. In particular, we shall assume that the reader is familiar with the following notions and their elementary use: f-covering, P-basic intervals, a P-graph (= a Markov graph), a loop, a P-monotone map, a P-linear map, topological entropy, a rome, a horseshoe.

The author is grateful to Jarosław Wróblewski, who wrote the computer programs which started this research. Those programs were based on the ideas of [BCMM] and made it possible to find experimentally minor patterns of low periods.

2. Preliminary results. We are looking for the minor patterns of period k. If $k = 1$ or 2 then there is only one pattern of period k, so it is minor. If $k \geq 3$ is odd then there is a unique primary pattern of period k, namely the Štefan pattern (see e.g. [S], [ALM]). Since there are no other primary patterns of period k, it is forced by every pattern of this period. This pattern is non-reducible, and therefore it is a unique minor pattern of period k.

Thus, we are left with the case $k \geq 4$ even. To avoid unnecessary fractions, we shall usually write $k = 2n$.

There are two cases of a block structure of a cycle P, particularly interesting from our point of view. The first one is when there are two blocks. Then we will say that *there is a division for P*. Otherwise, we will say that *there is no division for P* (see [LMPY]). The other interesting case is when each block consists of two points. In this case we will say that P is a *2-extension* (see [ALM]). In fact, these are the only consequences of non-reducibility of P we will use: there is no division for P and P is not a 2-extension. For the patterns we shall also use those terms: division, no division, 2-extension.

Very often we will use (without referring to it) the following basic fact (see [ALM]):

If f is P-monotone and exhibits a pattern B then $[P]$ forces B.

We will also use once an existence of a P-adjusted map, that is, a P-monotone map which has only one representative of $[P]$ (namely P itself) (see [ALM$_Y$]).

The situation when there is no division for P has been investigated thoroughly in [LMPY]. We will use some consequences of the results of [LMPY].

LEMMA 2.1. *Let A be a pattern of period $2n$, $n \geq 2$, with no division. Then A forces the Štefan pattern of period n if n is odd, and $n+1$ if n is even.*

P r o o f. Let $P = \{x_0, x_1, \ldots, x_{2n-1}\}$ be a representative of A in a P-linear map f and let $f(x_i) = x_{i+1}$ for $i = 0, 1, \ldots, 2n-2$, and $f(x_{2n-1}) = x_0$. In the notation of [LMPY], $x_{2n} = x_0$, so by Proposition 2.4 of [LMPY], f has a cycle of period n if n is odd, and $n+1$ if n is even. Thus, A forces some pattern of this period, so it forces the Štefan pattern of this period. ∎

To state the next lemma, we introduce the following property:

($\star$)   if $x < z$ then $f(x) > x$ and if $x > z$ then $f(x) < x$.

LEMMA 2.2. *Let P be a cycle of f with no division. Then either there is a fixed point z of f such that ($\star$) holds for all $x \in P$, or there exist closed intervals K, M with disjoint interiors and a common endpoint belonging to P, such that each of them f-covers $K \cup M$, and $P \setminus (K \cup M) \neq \emptyset$.*

P r o o f. This is basically Lemma 3.1 and Corollary 3.2 of [LMPY]. The statement is slightly different, but the proof of Lemma 3.1 of [LMPY] shows the existence of such intervals. The only thing we have to add is an argument showing that $P \setminus (K \cup M) \neq \emptyset$. We have $\inf(K \cup M) = x_n \geq x_{n+1}$ (in the notation of Lemma 3.1 of [LMPY]) and if $x_{n+1} = x_n$ then the period of P is 1, so $x_n = x_0$, a contradiction. Therefore $x_{n+1} \in P \setminus (K \cup M)$. ∎

We use the above lemma, which is rather technical, to get a nice property of minor cycles.

LEMMA 2.3. *Let P be a minor cycle of f of period $k > 2$. Then f has a fixed point z such that $(\star)$ holds for every $x \in P$.*

Proof. Since P is minor, there is no division for it. Therefore Lemma 2.2 holds. Suppose first that f is P-adjusted and intervals K, M with the properties described in Lemma 2.2 exist. Then f has a periodic point x such that $f^k(x) = x$, with $x, f(x) \in K$ and $f^i(x) \in M$ for $i = 2, 3, \ldots, k-1$. Let Q be the cycle of f to which x belongs. It is contained in $K \cup M$, and $P \setminus (K \cup M) \neq \emptyset$, so it is disjoint from P. Therefore no point of Q is the common point of K and M. The period q of Q divides k. Since $f(x) \in K$, we have $f^{q+1}(x) = f(x) \in K$, and since $1 \leq q \leq k$, we get $q = k-1$ or $q = k$. However, $k > 2$, so $k - 1$ does not divide k. Therefore $q = k$. If Q has a non-trivial block structure then the two leftmost points of Q belong to the same block. Those points are x and $f(x)$ and this contradicts the property that the image of a block is disjoint from the block itself. Thus, Q is non-reducible, that is, $[Q]$ is a non-reducible pattern of period k. Since f is P-adjusted and $Q \neq P$, we get that $[P]$ forces $[Q]$ and $[Q] \neq [P]$, so $[P]$ is not a minor pattern, a contradiction. Hence, f has a fixed point z such that $(\star)$ holds for all $x \in P$.

Now, if f is not necessarily P-adjusted, we take a P-adjusted map g and get a fixed point t of g such that $(\star)$ holds for all $x \in P$ and g, t replacing f, z. The point t belongs to some P-basic interval $[a, b]$. Then $f(a) = g(a) \geq b$ and $f(b) = g(b) \leq a$. Therefore f has a fixed point z in (a, b). Since $f|_P = g|_P$, $(\star)$ is satisfied for all $x \in P$. ∎

Remark 2.4. It is easy to see that if, in Lemma 2.3, f is P-monotone then z is the unique fixed point of f and $(\star)$ holds for all $x \in I$.

3. Green patterns. Assume that P is a cycle of a P-linear map $f \in \mathcal{I}$, z is a fixed point of f such that $(\star)$ holds for all $x \in P$, P has period $2n$, $n \geq 2$, and there is no division for P. Notice that here z plays a similar role to 0 in $[\text{ALM}_Y]$ for maps of the triod Y. Therefore we shall use here a similar terminology. Namely, if $x \in P$ then the pair $A = (x, f(x))$ is an *arrow in P* with the *beginning* $x = b(A)$ and *end* $f(x) = e(A)$. An arrow is *green* if its beginning and end are on the same side of z, and *black* otherwise. We shall write P_- for $\{x \in P : x < z\}$ and P_+ for $\{x \in P : x > z\}$.

LEMMA 3.1. *The number of green arrows in P is even. Moreover, $\operatorname{Card}(P_-)$ minus the number of green arrows in P_- is equal to $\operatorname{Card}(P_+)$ minus the number of green arrows in P_+.*

P r o o f. Let g_- (respectively g_+) be the number of green arrows in P_- (respectively P_+), and let b_- (respectively b_+) be the number of black arrows beginning in P_- (respectively P_+). When we look at the arrows beginning in P_- and P_+, then we get $\mathrm{Card}(P_-) = b_- + g_-$ and $\mathrm{Card}(P_+) = b_+ + g_+$. When we look at the arrows ending in P_- and P_+, then we get $\mathrm{Card}(P_-) = b_+ + g_-$ and $\mathrm{Card}(P_+) = b_- + g_+$. Therefore $b_- = b_+$, so $2n = \mathrm{Card}(P_-) + \mathrm{Card}(P_+) = 2b_- + (g_- + g_+)$. Hence the number $g_- + g_+$ of all green arrows in P is even. Moreover, $\mathrm{Card}(P_-) - g_- = b_- = b_+ = \mathrm{Card}(P_+) - g_+$. ∎

We shall call a cycle P of period $2n$, $n \geq 2$, for which there exists a P-monotone map f with a fixed point z such that $(\star)$ holds for all $x \in P$, there are exactly two green arrows in P and f restricted to the set of the beginnings of black arrows in P is decreasing, a *green cycle*. Clearly, if P is a green cycle then all P-monotone maps have those properties. Moreover, if P is green then all cycles with the same pattern as P are green, so we may speak about a *green pattern* $[P]$.

It is easy to classify and describe green patterns of a given period. We start with a simple lemma. When we have a green cycle P then we shall use f and z as in the above definition.

LEMMA 3.2. *Let P be a green cycle. Then there is a green arrow in P which begins at $\min P$ or $\max P$ and there is a green arrow in P which ends at $\max P_-$ or $\min P_+$.*

P r o o f. Suppose that $\min P$ and $\max P$ are the beginnings of black arrows A_1 and A_2 respectively. Since no green arrow can end at $\min P$ or $\max P$ and f restricted to the set of the beginnings of black arrows in P is decreasing, we have $\max P = e(A_1)$ and $\min P = e(A_2)$. Thus, the period of P is 2, a contradiction.

Suppose now that $\max P_-$ and $\min P_+$ are the ends of black arrows A_1 and A_2 respectively. Since no green arrow can begin at $\max P_-$ or $\min P_+$ and f restricted to the set of the beginnings of black arrows in P is decreasing, we have $\min P_+ = b(A_1)$ and $\max P_- = b(A_2)$. As before, we get a contradiction. ∎

Let P be a green cycle of period $2n$. Suppose first that both green arrows in P are on the same side of z. Without loss of generality we may assume that they are to the left of z. By Lemma 3.1, there are $n+1$ points in P_- and $n-1$ points in P_+. We shall denote those points by $x_1, \ldots, x_{n+1}$ and $y_1, \ldots, y_{n-1}$ in such a way that $x_{n+1} < x_n < \ldots < x_1 < z < y_1 < y_2 < \ldots < y_{n-1}$. By Lemma 3.2, there is a green arrow beginning at x_{n+1} and there is a green arrow ending at x_1. Thus, there are $s, l \in \{2, \ldots, n\}$ such that the green arrows begin at x_{n+1} and x_l and end at x_s and x_1 (although we do not know yet which arrow ends where).

Once s and l are fixed, we know all the black arrows (since f restricted to the set of the beginnings of black arrows in P is decreasing). Namely, the black arrows beginning at x_i with $i < l$ end at y_i; the ones beginning at x_i with $l < i \le n$ end at y_{i-1}; the ones beginning at y_i with $i < s - 1$ end at x_{i+1} and the ones beginning at y_i with $i \ge s - 1$ end at x_{i+2}.

If $l < s$ then $f(x_l) = x_1$, and then $f^2(x_i) = x_{i+1}$ for $i = 1, \dots, l - 1$, so the trajectory of x_l never passes through x_{n+1}. Since P is a cycle, we get a contradiction. Therefore we have $s \le l$. If $s \le i < l$ then we get $f^2(x_i) = x_{i+2}$. Thus, again since P is a cycle, we get the following rule:

(i) if $l - s$ is even then the green arrows go from x_{n+1} to x_s and from x_l to x_1;

(ii) if $l - s$ is odd then the green arrows go from x_{n+1} to x_1 and from x_l to x_s.

Hence, we see that specifying s and l defines a green pattern of period $2n$ completely. We shall denote such a pattern by $B_n(s, l)$.

By tracing arrows, we also get the following lemma, which will be useful later.

LEMMA 3.3. *If $[P] = B_n(s, l)$ then $f^2(x_i) = x_{i+1}$ for $1 \le i \le s - 2$ and for $l + 1 \le i \le n$, and $f^2(x_i) = x_{i+2}$ for $s - 1 \le i \le l - 1$. Moreover, $f(x_{l-1}) = y_{l-1}$, $f(y_{l-1}) = x_{l+1}$ and $f(x_{l+1}) = y_l$.*

Let us assume now that the green arrows are on different sides of z. By Lemma 3.1, there are n points in P_- and n points in P_+. We shall denote those points by $x_1, \dots, x_n$ and $y_1, \dots, y_n$ in such a way that $x_n < x_{n-1} < \dots < x_1 < z < y_1 < y_2 < \dots < y_n$. By Lemma 3.2, there is a green arrow beginning at x_n or y_n. Without loss of generality we may assume that this is x_n. Again by Lemma 3.2, there is a green arrow ending at y_1 or x_1. In the first case, one green arrow goes from x_n to x_s for some s with $1 \le s < n$ and the other one from y_l to y_1 for some l with $1 < l \le n$. In the second case, one green arrow goes from x_n to x_1 and the other one from y_l to y_s for some s, l with $1 \le s < l \le n$. As before, this information is enough to tell how the black arrows go, so it determines the pattern. As before, we easily see that in the first case $l > s$ and $l - s$ is odd, and in the second case $l - s$ is also odd. In the first case we shall denote the pattern by $C_n(s, l)$ and in the second case by $D_n(s, l)$. Notice also that $D_n(1, l) = C_n(1, l)$ and $D_n(s, n) = C_n(s, n)$, so we have to consider $D_n(s, l)$ only for $1 < s < l < n$.

Again by tracing arrows, we get the next lemma.

LEMMA 3.4. *If $[P] = C_n(s, l)$ then $f^2(x_i) = x_{i+2}$ for $s \le i \le l - 2$, $f^2(x_i) = x_{i+1}$ for $l \le i \le n - 1$, $f^2(y_i) = y_{i+1}$ for $1 \le i \le s - 1$, and $f^2(y_i) = y_{i+2}$ for $s \le i \le l - 1$. If $[P] = D_n(s, l)$ then $f^2(y_i) = y_{i+2}$ for*

$s \leq i \leq l - 1$. *Moreover*, $f(y_{l-1}) = x_l$, $f(x_l) = y_{l+1}$, $f(y_{l+1}) = x_{l+1}$ *and* $f(x_{n-1}) = y_n$; *if* $l = n$ *then additionally* $f(y_{n-1}) = x_n$.

The proofs of Lemmas 3.3 and 3.4 are elementary and we leave them to the reader. We conclude this section by listing the green patterns for further reference and drawing some examples of green cycles.

The green patterns are the following:

(i) $B_n(s, l)$ with $2 \leq s \leq l \leq n$ and $l - s$ even. Then the green arrows go from x_{n+1} to x_s and from x_l to x_1.

(ii) $B_n(s, l)$ with $2 \leq s \leq l \leq n$ and $l - s$ odd. Then the green arrows go from x_{n+1} to x_1 and from x_l to x_s.

(iii) $C_n(s, l)$ with $1 \leq s < l \leq n$ and $l - s$ odd. Then the green arrows go from x_n to x_s and from y_l to y_1.

(iv) $D_n(s, l)$ with $1 < s < l < n$ and $l - s$ odd. Then the green arrows go from x_n to x_1 and from y_l to y_s.
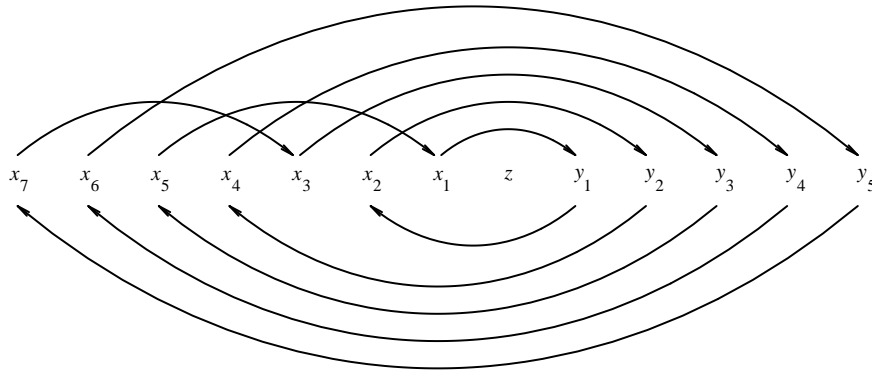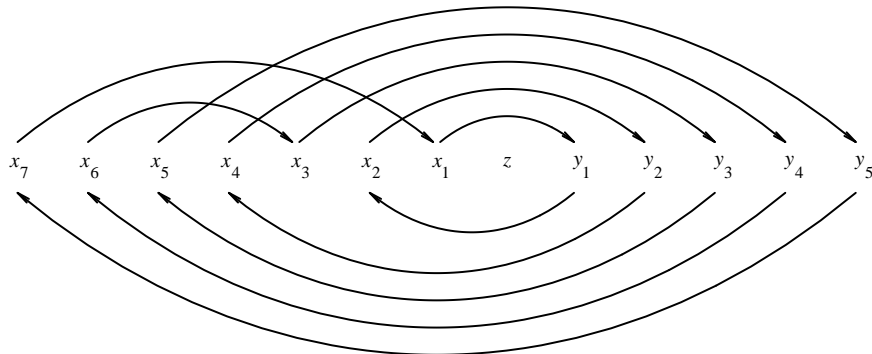

Fig. 3.1. A cycle with pattern $B_6(3, 5)$

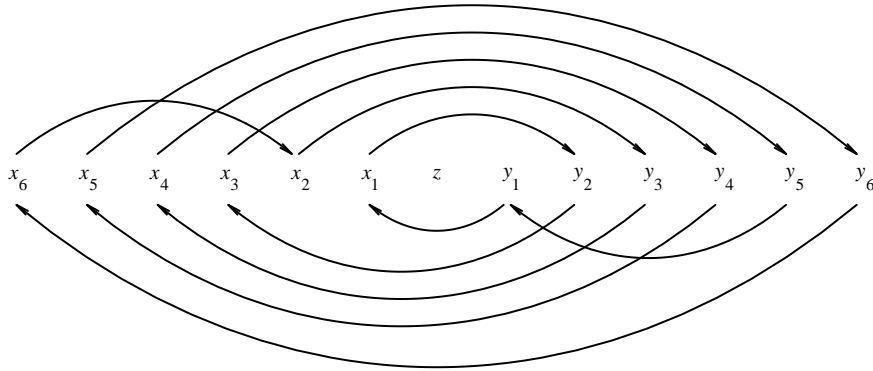Fig. 3.2. A cycle with pattern $B_6(3, 6)$
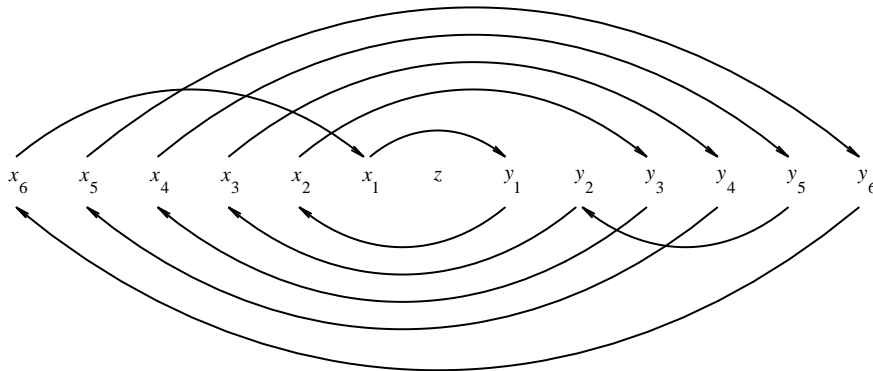
Fig. 3.3. A cycle with pattern $C_6(2,5)$

Fig. 3.4. A cycle with pattern $D_6(2,5)$

4. Minor patterns are green. In this section we assume that P is a cycle of a P-linear map $f \in \mathcal{I}$, z is a fixed point of f such that $(\star)$ holds for all $x \in P$, P has period $2n$, $n \geq 2$, and there is no division for P.

If $x, y \in I$ then $\langle x, y \rangle$ will denote the interval $[x, y]$ if $x < y$, the interval $[y, x]$ if $y < x$, and the set $\{x\}$ if $x = y$. Similarly, $\langle P \rangle$ will denote the smallest interval containing P.

For $x \in I$ we set

$$F(x) = \begin{cases} \max f([x,z]) & \text{if } x \leq z, \\ \min f([z,x]) & \text{if } x \geq z. \end{cases}$$

Clearly, F is non-decreasing and $F(x) = f(x)$ unless F is constant in a neighborhood of x.

If $g \in \mathcal{I}$ has a unique fixed point z then we shall say that a trajectory $(g^i(x))_{i=0}^m$ is *spiraling outwards* if $g^i(x) \in \langle z, g^{i+2}(x) \rangle$ for $i = 0, 1, \ldots, m-2$ and $z \in \langle g^i(x), g^{i+1}(x) \rangle$ for $i = 0, 1, \ldots, m-1$.

LEMMA 4.1. *All F-trajectories of points of $\langle P \rangle$ are spiraling outwards.*

P r o o f. We have $F(x) \leq z$ for $x \geq z$ and $F(x) \geq z$ for $x \leq z$. Therefore z is the unique fixed point of F and $z \in \langle x, F(x) \rangle$ for any $x \in I$. Moreover, since f is P-linear, we have $f(I) = \langle P \rangle$, so $F(I) = \langle P \rangle$. Hence, if some F-trajectory of a point of $\langle P \rangle$ is not spiraling outwards then there exists $y \in \langle P \rangle$ such that $F^2(y) \in \langle y, z \rangle$ and $F^2(y) \neq y$. In such a case the interval $\langle y, F(y) \rangle$ is F-invariant, so by the definition of F, it is also f-invariant. Since f is P-linear and the period of P is greater than 2, the absolute value of the slope of f on the P-basic interval containing z is larger than 1. Therefore $\langle y, F(y) \rangle$ cannot be contained in this interval, so it contains some point of P. Consequently, it contains the whole set P. Therefore $y = \min P$ or $\max P$. However, since $(\star)$ holds for all $x \in P$, we get $F(\min P) = \max P$ and $F(\max P) = \min P$, a contradiction. ∎

We set, for any $x, y \in P$,

$$m(x, y) = \min\{m \geq 0 : y \in \langle z, F^m(x) \rangle\}.$$

Since F is monotone, we have $m(x, t) \leq m(x, y) + m(y, t)$ for any $x, y, t \in P$.

We need some technical lemmas.

LEMMA 4.2. *Let $x \in P$ and $y = f^j(x)$. Then $m(x, y) \leq j$. Moreover, if $m(x, y) = j$ then $f^i(x) \in \langle z, F^i(x) \rangle$ for $i = 0, 1, \ldots, j$ and $(f^i(x))_{i=0}^j$ is spiraling outwards.*

P r o o f. There are numbers $0 \leq p_0 < p_1 < \ldots < p_s \leq j$ such that if $i \in \{0, 1, \ldots, j-1\}$ then $f^i(x)$ is the beginning of a black arrow if and only if $i \in \{p_0, p_1, \ldots, p_s\}$. We have $f^{p_0}(x) \in \langle z, x \rangle$ and if $f^{p_i}(x) \in \langle z, F^i(x) \rangle$ then $f^{p_{i+1}}(x) \in \langle z, F^{i+1}(x) \rangle$. Therefore by induction we get $f^{p_i}(x) \in \langle z, F^i(x) \rangle$ for all i, so in particular $f^{p_s}(x) \in \langle z, F^s(x) \rangle$. But then $y = f^j(x) \in \langle z, f^{p_s}(x) \rangle \subset \langle z, F^s(x) \rangle$. Since $s \leq j$, we get $m(x, y) \leq s \leq j$.

Assume now that $m(x, y) = j$. Then $s = j$, so $p_i = i$ for all i. Thus, $f^i(x) \in \langle z, F^i(x) \rangle$ for $i = 0, 1, \ldots, j$. By Lemma 4.1, $(F^i(x))_{i=0}^j$ is spiraling outwards, and thus $z \in \langle f^i(x), f^{i+1}(x) \rangle$ for $i = 0, 1, \ldots, j-1$. Suppose that $f^{i+2}(x) \in \langle z, f^i(x) \rangle$ for some $i \leq j-2$. Then $f^{i+2}(x) \in \langle z, F^i(x) \rangle$, so by induction we get $f^j(x) \in \langle z, F^{j-2}(x) \rangle$, that is, $m(x, y) \leq j - 2$, a contradiction. Therefore $f^i(x) \in \langle z, f^{i+2}(x) \rangle$ for $i = 0, 1, \ldots, j-2$, so $(f^i(x))_{i=0}^j$ is spiraling outwards. ∎

LEMMA 4.3. *Assume that n is odd, $A_1 \neq A_2$ are green arrows in P on the same side of z and all four numbers $m(e(A_i), b(A_j))$, $i, j = 1, 2$, are larger than or equal to $n - 1$. Then P is a 2-extension.*

P r o o f. We have $b(A_2) = f^{m_1}(e(A_1))$ and $b(A_1) = f^{m_2}(e(A_2))$ for some $m_1, m_2 \geq 0$ with $m_1 + m_2 + 2 = 2n$. By Lemma 4.2, $m_1, m_2 \geq n - 1$, so $m_1 = m_2 = m(e(A_1), b(A_2)) = m(e(A_2), b(A_1))$.

Suppose that $f^j(e(A_1)) \in \langle z, f^i(e(A_2)) \rangle$ for some $0 \le i < j \le n - 1$. By Lemma 4.2, we have $f^i(e(A_2)) \in \langle z, F^i(e(A_2)) \rangle$, so $f^j(e(A_1)) \in \langle z, F^i(e(A_2)) \rangle$. Consequently, $m(e(A_2), f^j(e(A_1))) \le i$. On the other hand, by Lemma 4.2, since $b(A_2) = f^{n-1}(e(A_1))$, we have $m(f^j(e(A_1)), b(A_2)) \le n - 1 - j$. Thus, $m(e(A_2), b(A_2)) \le i + n - 1 - j < n - 1$, a contradiction. Therefore if $f^j(e(A_1)) \in \langle z, f^i(e(A_2)) \rangle$ for some $i, j \in \{0, 1, \ldots, n-1\}$ then $i \ge j$. Similarly, if $f^j(e(A_2)) \in \langle z, f^i(e(A_1)) \rangle$ for some $i, j \in \{0, 1, \ldots, n-1\}$ then also $i \ge j$. Moreover, by Lemma 4.2, the trajectories $(f^i(e(A_1)))_{i=0}^{n-1}$ and $(f^i(e(A_2)))_{i=0}^{n-1}$ are spiraling outwards. Therefore, assuming that A_1 and A_2 are to the left of z (which we can do without loss of generality), the ordering of the points of P is the following. First come the points $f^{n-1}(e(A_i))$, $i = 1, 2$; then $f^{n-3}(e(A_i))$, $i = 1, 2$; then $\ldots$; then $e(A_i)$, $i = 1, 2$; then $f(e(A_i))$, $i = 1, 2$; then $f^3(e(A_i))$, $i = 1, 2$; then $\ldots$; then $f^{n-2}(e(A_i))$, $i = 1, 2$. Thus, P is a 2-extension (in fact, it is a 2-extension of a Štefan cycle of period n). ∎

LEMMA 4.4. *All green patterns are non-reducible except* $B_n(2, n)$ *for n odd, which is a 2-extension.*

P r o o f. Assume that a green cycle P has a non-trivial block structure. Then the map obtained after contracting the smallest interval containing each block into a point has still a fixed point, so z is outside those intervals. Since P has green arrows, the cycle Q obtained in such a way also has a green arrow, so the number of green arrows in P is at least as large as the cardinality of each block. Since there are only 2 green arrows in P, there are only 2 points in each block, that is, P is a 2-extension. Then the green arrows in P begin at adjacent points of P and end at adjacent points of P. The only green patterns with this property are $B_n(2, n)$. The direct checking shows that $B_n(2, n)$ is a 2-extension for n odd and is not a 2-extension for n even. ∎

The following is the key lemma of this section, if not of the whole paper.

LEMMA 4.5. *Assume that there is no division for P and that P is not a 2-extension. Then f has a cycle Q such that one of the following holds*:

   (i) *Q has period m odd, $3 \le m < n$,*
   (ii) *Q is green of period $2m$, $1 < m < n$,*
   (iii) *Q is green of period $2n$ and is not a 2-extension.*

Moreover, if there is a green arrow A in P such that $m(e(A), b(A)) < n - 1$ then f has a cycle Q such that (i) *holds.*

P r o o f. Since P is not a 2-extension, there is a green arrow in P. By Lemma 3.1, there are at least two of them. Choose two green arrows $A_1 \ne A_2$ in P and denote for $i = 1, 2$ by $c(A_i)$ the point closest to $b(A_i)$ among those

points of $P \cap \langle b(A_i), z \rangle$ which are the beginnings of black arrows. Such a point exists since the points of P_- and P_+ closest to z are the beginnings of black arrows. Consider a loop

$$\langle b(A_1), c(A_1) \rangle \to \langle z, e(A_1) \rangle \to \langle z, F(e(A_1)) \rangle \to \langle z, F^2(e(A_1)) \rangle \to \ldots \to$$
$$\langle z, F^{m_1-1}(e(A_1)) \rangle \to \langle b(A_2), c(A_2) \rangle \to \langle z, e(A_2) \rangle \to \langle z, F(e(A_2)) \rangle \to$$
$$\langle z, F^2(e(A_2)) \rangle \to \ldots \to \langle z, F^{m_1-1}(e(A_1)) \rangle \to \langle b(A_1), c(A_1) \rangle,$$

where $m_1 = m(e(A_1), b(A_2))$ and $m_2 = m(e(A_2), b(A_1))$. In this loop each interval of the form $\langle z, F^i(e(A_j)) \rangle$ F-covers the next one with orientation reversed. Moreover, $\langle b(A_j), c(A_j) \rangle$ f-covers $\langle z, e(A_j) \rangle$ with orientation preserved. When we go along the loop from $\langle b(A_j), c(A_j) \rangle$ to $\langle b(A_{3-j}), c(A_{3-j}) \rangle$, we change the side of z m_j times. When we go around the whole loop, we have to return to the same side of z, so $m_1 + m_2$ is even. Therefore when we return, we do it with orientation preserved. Thus, there is an interval $K \subset \langle b(A_1), c(A_1) \rangle$ such that $\varphi^i(K)$ is contained in the ith interval of the loop for $i = 0, 1, \ldots, m_1 + m_2 + 1$ (we consider $\langle b(A_1), c(A_1) \rangle$ to be the 0th interval), and the left (respectively right) endpoint of K is mapped by $\psi = \varphi^{m_1 + m_2 + 2}$ to the left (respectively right) endpoint of $\langle b(A_1), c(A_1) \rangle$. Here $\varphi = F$ when we consider the map on $\langle z, F^i(e(A_j)) \rangle$, and $\varphi = f$ on $\langle b(A_j), c(A_j) \rangle$.

There is a fixed point $x \in K$ of ψ such that ψ is not constant in any neighborhood of x (the graph of ψ has to intersect the diagonal from "below" to "above"). Thus, none of the points $\varphi^i(x)$ has a neighborhood on which φ is constant. Hence, $\varphi(\varphi^i(x)) = f(\varphi^i(x))$, so the φ-orbit of x is the same as the f-orbit of x. Therefore this orbit is a cycle of f. Call this cycle Q.

Let q be the period of Q. The beginnings of the green arrows in Q correspond to the intervals $\langle b(A_j), c(A_j) \rangle$ of the loop. Since f^q maps the set of the beginnings of green arrows in Q onto itself, there are only two possibilities: either $q = m_1 + m_2 + 2$ or $q = (m_1 + m_2 + 2)/2$. Moreover, in the second case $m_1 = m_2$. In both cases, f restricted to the set of the beginnings of black arrows in Q is equal to F restricted to the same set, so it is decreasing. In the first case, there are two green arrows in Q, in the second case there is only one. By Remark 2.4, $(\star)$ holds for all $x \in Q$. Therefore, in the first case Q is a green cycle.

Suppose that for one of A_j we have $m(e(A_j), b(A_j)) < n-1$. Then in the same way as Q (using the loop $\langle b(A_j), c(A_j) \rangle \to \langle z, e(A_j) \rangle \to \langle z, F(e(A_j)) \rangle \to \langle z, F^2(e(A_j)) \rangle \to \ldots \to \langle z, F^{m_j-1}(e(A_j)) \rangle \to \langle b(A_j), c(A_j) \rangle)$ we get a cycle Q' of period $m(e(A_j), b(A_j)) + 1 < n$, with exactly one green arrow. Since there is only one green arrow in Q', the period of Q' is odd. Moreover, $m(e(A_j), b(A_j)) \geq 2$, so this period is at least 3. Thus, we get a cycle of type described in (i). In particular, this proves the last statement of the lemma.

Assume now that $m(e(A_j), b(A_j)) \geq n - 1$ for $j = 1, 2$. If $q = (m_1 + m_2 + 2)/2$, then $m_1 = m_2$, so if $q = n$, we get $m_1 = m_2 = n - 1$. In this case, since Q has only one green arrow, n is odd. Since m_1 is even, A_1 and A_2 are on the same side of z. Thus, by Lemma 4.3, P is a 2-extension, a contradiction. By Lemma 4.2, $m_1 + m_2 + 2 \leq 2n$. Therefore $q < n$. Again, since Q has only one green arrow, q is odd. At least one of the numbers m_1, m_2 is larger than or equal to 2, and since they are equal, both are larger than or equal to 2. Therefore $q \geq 3$. Thus, Q is of type described in (i).

The remaining case is $q = m_1 + m_2 + 2$. If $q < 2n$ then Q is of type described in (ii) (we have $q > 2$ since for $q = 2$ there would be no green arrows). Assume that $q = 2n$. Suppose that Q is a 2-extension. By Lemma 4.4, $[Q] = B_n(2, n)$ and n is odd. Then $m_1 = m_2 = n - 1$, and by Lemma 4.3, P is a 2-extension, a contradiction. Therefore Q is not a 2-extension, so it is of type described in (iii). ∎

LEMMA 4.6. *Let $3 \leq m < n$ and let m be odd. Then any pattern of period m forces a non-reducible pattern of period $2n$.*

P r o o f. Let Q be a Štefan cycle of period m of a Q-linear map g. With the standard notation the Q-graph looks as in Figure 4.1 (see e.g. [BGMY], [ALM]), where the fixed point of g is in I_1.
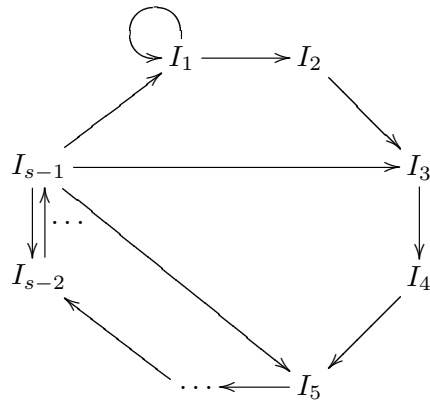

Fig. 4.1. The P-graph of a Štefan cycle P

Consider the loop $I_1 \to I_1 \to I_2 \to I_3 \to \ldots \to I_{m-1} \to I_1 \to I_1 \to \ldots \to I_1 \to I_2 \to I_3 \to \ldots \to I_{m-1} \to I_1$ of length $2n$ (that is, we go along the short loop $I_1 \to I_1$ once, then along the long loop $I_1 \to I_2 \to I_3 \to \ldots \to I_{m-1} \to I_1$ once, then along the short loop $2n - 2m + 1$ times, and again along the long loop once). This loop is simple (it is not a repetition of a shorter loop), so it is associated with a cycle R of period $2n$. The map g on $\bigcup_{i=1}^{m-2} I_i$ is decreasing, so all arrows in R beginning there are black.

There are two arrows in R beginning in I_{m-1}. They are followed by an odd number of arrows $I_1 \to I_1$ and then an arrow $I_1 \to I_2$, and I_2 is on the same side of the fixed point as I_{m-1}. Therefore those arrows are green. Thus, the cycle R is green. The distances along the loop between the occurrences of I_{m-1} are different, so R is not a 2-extension. Thus, by Lemma 4.4, R is non-reducible. Since g is Q-linear, $[Q]$ forces $[R]$.

Now, any pattern of period m forces $[Q]$, so it forces $[R]$. ∎

LEMMA 4.7. *For $n \geq 2$, any pattern $B_n(s,l)$ with $l - s$ odd which is not a 2-extension, forces a pattern of some period m odd with $3 \leq m < n$.*

P r o o f. Let P be as usual, with $[P] = B_n(s,l)$, $l - s$ odd and P not a 2-extension. If A is the green arrow from x_l to x_s, by Lemma 3.3 we have

$$m(e(A), b(A)) = 2 \cdot \frac{l - s + 1}{2} = l - s + 1.$$

If n is odd, then by Lemma 4.4, $(s,l) \neq (2,n)$, so $l - s + 1 < n - 1$. If n is even, $l - s + 1 \leq n - 1$, but $l - s + 1$ is even, whereas $n - 1$ is odd, so $l - s + 1 < n - 1$. Hence, in both cases $m(e(A), b(A)) < n - 1$. By Lemma 4.5, $[P]$ forces some pattern B of period m odd with $3 \leq m < n$. ∎

LEMMA 4.8. *Let $1 < m < n$ and let B be a green pattern of period $2m$. Then B forces a non-reducible pattern of period $2n$.*

P r o o f. If B is a 2-extension then by Lemma 4.4, m is odd, so B forces a pattern of period m odd with $3 \leq m < n$ (the one whose 2-extension B is). Therefore by Lemma 4.6 it forces a non-reducible pattern of period $2n$.

Assume that B is not a 2-extension. If $B = B_m(s,l)$ with $l - s$ odd, then by Lemma 4.7, it forces a pattern of some period r odd with $3 \leq r < m$, so again it forces a non-reducible pattern of period $2n$.

The rest of the green patterns of period $2m$ have the following property. Let Q be a representative of B in a Q-linear map g with the fixed point t. For $x \in Q$ let $J(x)$ be the $(Q \cup \{t\})$-basic interval adjacent to x and lying on the same side of t as x. Then $J(x)$ g-covers $J(g(x))$ for each $x \in Q$. Indeed, the only way this condition could be violated is that x and the other endpoint of $J(x)$ are beginnings of green arrows and g reverses orientation on $J(x)$. This can happen only if $B = B_m(s,m)$ with $m - s$ odd, but those patterns have been considered before, so we are not considering them now.

The condition described above implies that the fundamental loop of Q in the $(Q \cup \{t\})$-graph is $J(x) \to J(g(x)) \to J(g^2(x)) \to \ldots \to J(g^{2m-1}(x)) \to J(x)$. Therefore the fundamental loop of Q in the Q-graph passes twice through the Q-basic interval I_1 containing t (and once through any other Q-basic interval). Thus, the fundamental loop of Q in the Q-graph can be written as $\alpha = I_1 \to J_2 \to J_3 \to \ldots \to J_{2m} \to I_1$ (where one of J_j's is I_1). Consider the loop $\beta = I_1 \to J_2 \to J_3 \to \ldots \to J_{2m} \to I_1 \to I_1 \to \ldots \to I_1$

of length $2n$ (we go along the fundamental loop once and then along the loop $I_1 \to I_1$ $2n - 2m$ times). This loop is simple, so there is a cycle R of period $2n$ associated with it. The same arrows in α and β correspond to the green arrows in Q and R respectively, and new arrows in β correspond to black arrows (g is orientation reversing on I_1). Therefore the cycle R is green. The point corresponding to the first appearance of I_1 in the final block of I_1's in β is the end of a green arrow in R. However, the point of R adjacent to it and further from t corresponds to the third appearance of I_1 in this block, so it is the end of a black arrow. Therefore, by Lemma 4.4, $[R]$ is non-reducible. Thus, $B = [Q]$ forces the non-reducible pattern $[R]$ of period $2n$. ∎

Now we are ready to prove the result announced in the title of this section.

THEOREM 4.9. *All minor patterns of period $2n$ with $n \geq 2$ are green.*

P r o o f. Let P, f, z be as usual, and assume that P is minor. By Lemma 4.5, f has a cycle Q of one of the types described there.

If Q is as in (i) then $[P]$ forces the pattern $[Q]$ of period m odd with $3 \leq m < n$. By Lemma 4.6, $[Q]$ forces a non-reducible pattern B of period $2n$. Therefore $[P]$ forces B. Since $[P] \neq [Q]$, by the antisymmetry of the forcing relation we get $[P] \neq B$. Hence, $[P]$ is not minor, a contradiction.

If Q is as in (ii) then $[P]$ forces the green pattern $[Q]$ of period $2m$ with $1 < m < n$. By Lemma 4.8, $[Q]$ forces a non-reducible pattern C of period $2n$. Hence, as before, we get a contradiction.

Therefore Q has to be of the type described in (iii). Thus, $[P]$ forces the green pattern $[Q]$ of period $2n$ which is not a 2-extension. By Lemma 4.4, $[Q]$ is non-reducible. Since $[P]$ is minor, it follows that $[P] = [Q]$. Therefore $[P]$ is green. ∎

5. Green patterns which are minor. In this section we are going to find out which green patterns are minor.

THEOREM 5.1. *Any minor pattern of period $2n$, $n \geq 2$, is one of the following*:

(i) $B_n(s, l)$ *with* $2 \leq s \leq l \leq n$ *and* $s + l$ *equal to* $n + 1$ *or* $n + 3$ *if* n *is odd, and* $n + 2$ *if* n *is even,*

(ii) $C_n(s, l)$ *with* $1 \leq s < l \leq n$ *and* $s + l$ *equal to* n *or* $n + 2$ *if* n *is odd, and* $n + 1$ *if* n *is even.*

P r o o f. We shall use the notation of Section 3. Assume that a green cycle P is minor. One of the green arrows in P begins at $\min P$. We shall call this arrow A_1 and the other green arrow A_2. By Lemmas 4.5 and 4.6 we see that for both green arrows A_j, $j = 1, 2$, we have $m(e(A_j), b(A_j)) \geq n - 1$.

If $[P] = B_n(s, l)$ then from Lemmas 4.7 and 4.6 we deduce (in the same way as in the previous section) that $l - s$ is even. Lemma 3.3 gives us

$$m(e(A_1), b(A_1)) = 2\left(\frac{l-s}{2} + 1 + n - l\right) = 2n + 2 - l - s$$

and

$$m(e(A_2), b(A_2)) = 2\left(s - 2 + \frac{l-s}{2} + 1\right) = s + l - 2.$$

Therefore $2n+2-l-s \geq n-1$ and $s+l-2 \geq n-1$, that is, $n+1 \leq s+l \leq n+3$. Since $l - s$ is even, so is $s + l$, and we get $s + l = n + 1$ or $n + 3$ if n is odd, and $s + l = n + 2$ if n is even.

If $[P] = C_n(s, l)$ then Lemma 3.4 gives us

$$m(e(A_1), b(A_1)) = 2\left(\frac{l-1-s}{2} + 1 + n - l\right) = 2n + 1 - s - l$$

and

$$m(e(A_2), b(A_2)) = 2\left(s - 1 + \frac{l+1-s}{2}\right) = s + l - 1.$$

Therefore $2n+1-s-l \geq n-1$ and $s+l-1 \geq n-1$, that is, $n \leq s+l \leq n+2$. Since $l - s$ is odd, so is $s + l$, and we get $s + l = n$ or $n + 2$ if n is odd, and $s + l = n + 1$ if n is even.

If $[P] = D_n(s, l)$ with $2 \leq s < l \leq n - 1$ then Lemma 3.4 gives us

$$m(e(A_2), b(A_2)) = 2 \cdot \frac{l+1-s}{2} = l + 1 - s.$$

Therefore $l + 1 - s \geq n - 1$, so $n - 2 \geq n - 1$, a contradiction. ∎

Now we have good candidates for minor patterns and we have to prove that they are really minor.

THEOREM 5.2. *All patterns listed in Theorem* 5.1 *are minor.*

P r o o f. Again we shall use the notation of Section 3. Look at the $(P \cup \{z\})$-graph. Set $I_0 = [x_1, z]$, $J_0 = [z, y_1]$ and $I_i = [x_{i+1}, x_i]$, $J_i = [y_i, y_{i+1}]$ for $i > 0$.

The patterns $B_n(3, n)$ for n odd and $B_n(2, n)$ for n even are unimodal. There are no other unimodal patterns listed in Theorem 5.1, so those have to be minor. Consider the rest of the patterns listed there. If P is a representative of one of them then no P-basic interval has both endpoints which are the beginnings of green arrows. Therefore if in the $(P \cup \{z\})$-graph there is an arrow S from I_i to I_j (or from J_i to J_j) then exactly one of the endpoints of I_i (respectively J_i) is the beginning of a green arrow A. When we follow some path in the $(P \cup \{z\})$-graph and use such an arrow S then we shall say that we *move along* A (cf. [ALM$_Y$]). To get a loop associated with a green cycle we have to move along green arrows exactly twice.

In such a loop of length $2n$, if we move twice along the same green arrow A then the number of steps between those moves is at least $n-1$ (otherwise we would have $m(e(A), b(A)) < n-1$, and we would get a contradiction in the same way as we did already several times), and we do not get earlier to a $(P \cup \{z\})$-basic interval from which this move begins. Therefore in the cycle Q associated with this loop, both green arrows begin at the two leftmost (or the two rightmost) points. This means that if $[Q]$ is listed in Theorem 5.1 then $[Q] = B_n(3, n)$ if n is odd, and $[Q] = B_n(2, n)$ if n is even. However, in $B_n(i, n)$ the numbers of steps necessary to get from the end of one green arrow to the beginning of the other one are $n-i$ and $n+i-2$, whereas for Q those numbers are both $n-1$, a contradiction.

Thus, to get a loop associated with Q such that $[Q]$ is listed in Theorem 5.1, we have to move along both green arrows in P. As in the proof of Lemma 4.8, since no P-basic interval has both endpoints which are beginnings of green arrows, if a $(P \cup \{z\})$-basic interval $J \subset \langle x, z \rangle$ with x as an endpoint f-covers some $(P \cup \{z\})$-basic interval J' which is on the same side of z as $f(x)$, then $J' \subset \langle f(x), z \rangle$. Therefore if we start from any interval to which we moved along a green arrow, and we want to get to an interval adjacent to the beginning of the other green arrow, the fastest way is to follow the fundamental loop, and any other way is longer. Since the length of the loop associated with Q is $2n$, and this is also the length of the fundamental loop, we see that those two loops have to coincide. Consequently, $Q = P$. In view of Theorem 5.1, this shows that $[P]$ is minor. ∎

Just for statistics, notice that for a given $n \geq 2$, there are $2n - 2$ minor patterns of period $2n$ if n is odd, and n if n is even. In each case half of them are of the form $B_n(s, l)$ and another half of the form $C_n(s, l)$. One of them ($B_n(3, n)$ if n is odd and $B_n(2, n)$ if n is even) is unimodal, one ($C_n(2, n)$ if n is odd and $C_n(1, n)$ if n is even) is bimodal, and all the others are trimodal.
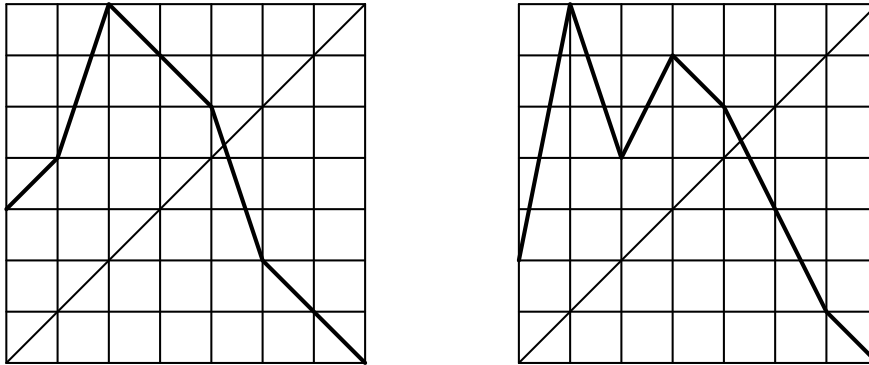

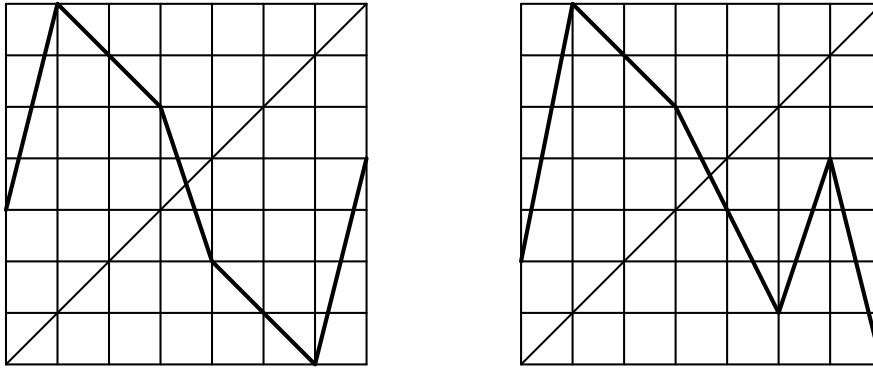Fig. 5.1. The graphs of f for $[P] = B_4(2, 4)$ (left) and $[P] = B_4(3, 3)$ (right)

Fig. 5.2. The graphs of f for $[P] = C_4(1,4)$ (left) and $[P] = C_4(2,3)$ (right)

6. Minor patterns with minimal entropy. In this section we shall find minor patterns of a given period $2n$ with minimal entropy. Clearly, they are also non-reducible patterns of this period with minimal entropy. We also compute this minimal entropy.

Among the minor patterns of period $2n$ there is one which is unimodal. Unimodal is in some sense simplest. Therefore it is a good candidate for having minimal entropy. We shall show that this is indeed so.

LEMMA 6.1. *The entropy of the pattern* $B_n(s,l)$ *with* $2 \leq s \leq l < n$, *with* $l - s$ *even and with* $n \geq 2$, *is larger than or equal to* $\log \sqrt{3}$.

Proof. Let P be a cycle with $[P] = B_n(s,l)$, and let f, z, x_i, y_i be as usual. Consider the intervals $J = [x_{l+1}, x_l]$, $K = [x_l, x_{l-1}]$, $L = [x_{l-1}, z]$ and $M = [z, y_{l-1}]$. By Lemma 3.3, we have $f(x_{l+1}) = y_l > y_{l-1}$, $f(x_{l-1}) = y_{l-1}$ and $f(y_{l-1}) = x_{l+1}$. Moreover, $f(x_l) = x_1 < z$ and $f(z) = z$. Therefore each of the intervals J, K, L f-covers M and M f-covers each of J, K, L (see Figure 6.1). Thus, there is a 3-horseshoe for f^2, so $h(f) \geq \log \sqrt{3}$. Hence, $h(P) \geq \log \sqrt{3}$. ∎

LEMMA 6.2. *The entropy of the pattern* $C_n(s,l)$ *with* $1 \leq s < l \leq n$ *even and with* $n \geq 2$ *is larger than or equal to* $\log \sqrt{3}$.

Proof. Let P be a cycle with $[P] = C_n(s,l)$, and let f, z, x_i, y_i be as usual. Assume first that $l < n$. Consider the intervals $J = [x_l, z]$, $K = [z, y_{l-1}]$, $L = [y_{l-1}, y_l]$ and $M = [y_l, y_{l+1}]$. By Lemma 3.4, we have $f(x_l) = y_{l+1}$, $f(y_{l-1}) = x_l$ and $f(y_{l+1}) = x_{l+1} < x_l$. Moreover, $f(z) = z$ and $f(y_l) = y_1 > z$. Therefore J f-covers each of the intervals K, L, M and each of K, L, M f-covers J (see Figure 6.2). Thus, as in the proof of the preceding lemma, $h(P) \geq \log \sqrt{3}$.
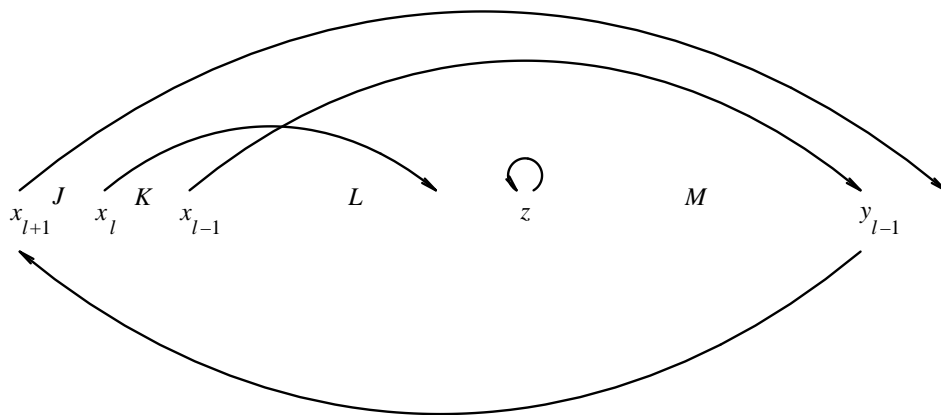
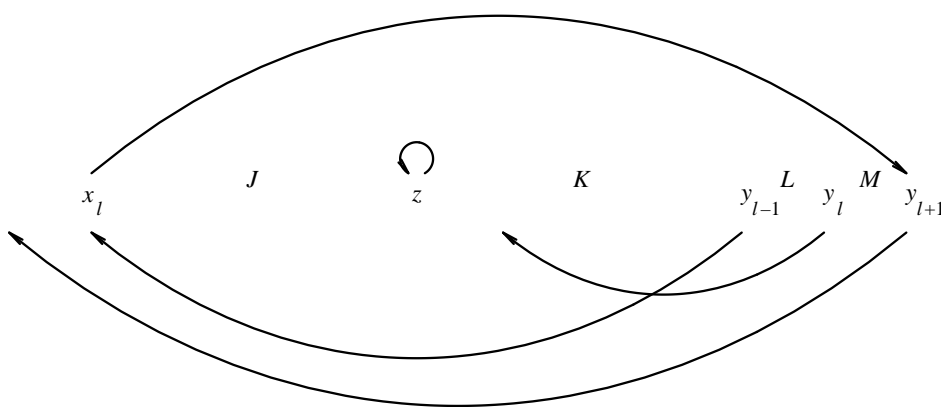Fig. 6.1. Intervals J, K, L, M for $[P] = B_n(s, l)$

Fig. 6.2. Intervals J, K, L, M for $[P] = C_n(s, l)$, $l < n$

Assume now that $l = n$. Consider the intervals $J = [x_n, x_{n-1}]$, $K = [x_{n-1}, z]$, $L = [z, y_{n-1}]$ and $M = [y_{n-1}, y_n]$. By Lemma 3.4, $f(x_{n-1}) = y_n$ and $f(y_{n-1}) = x_n$. Moreover, $f(x_n) = x_s < z$, $f(z) = z$ and $f(y_n) = y_1 > z$. Therefore each of the intervals J, K f-covers each of L, M and vice versa (see Figure 6.3). We get a 4-horseshoe for f^2, so in the same way as before we obtain $h(P) \geq \log \sqrt{4} > \log \sqrt{3}$. ∎

The only minor patterns of period $2n$ whose entropy we did not estimate are the patterns $B_n(3, n)$ for n odd and $B_n(2, n)$ for n even, that is, the unimodal ones.

Let n be odd and $[P] = B_n(3, n)$. We use our standard notation (additionally we denote by K the P-basic interval containing z).
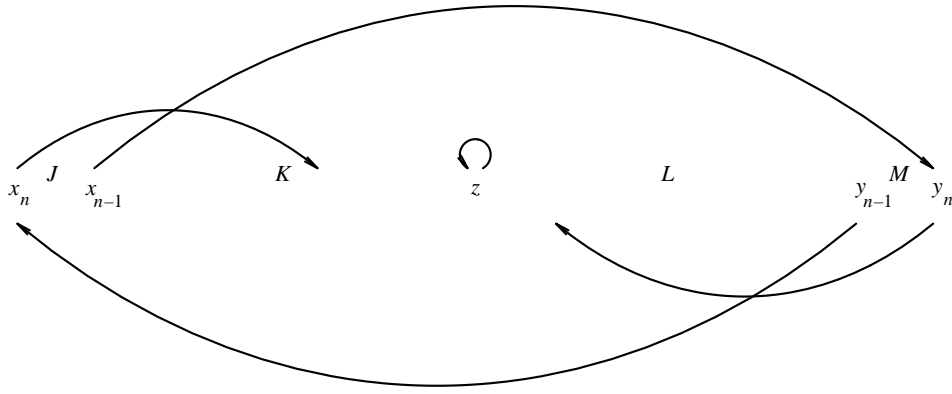
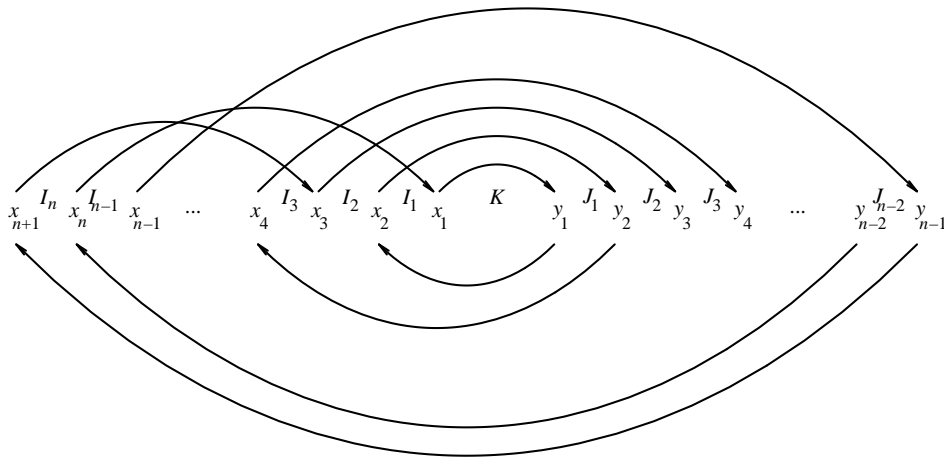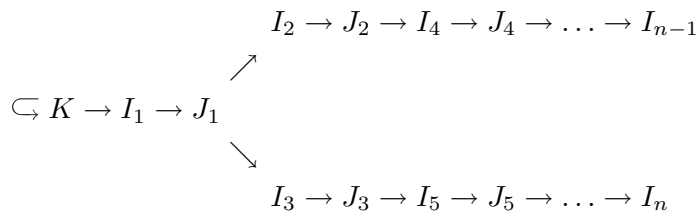Fig. 6.3. Intervals J, K, L, M for $[P] = C_n(s, n)$

Fig. 6.4. A cycle with pattern $B_n(3, n)$, n odd

The P-graph of f is the following:

$$I_2 \to J_2 \to I_4 \to J_4 \to \ldots \to I_{n-1}$$

$$\nearrow$$

$$\hookleftarrow K \to I_1 \to J_1$$

$$\searrow$$

$$I_3 \to J_3 \to I_5 \to J_5 \to \ldots \to I_n$$

and additionally there are arrows from I_n to I_1 and I_2, and from I_{n-1} to K and all J_i's. The set $\{K, I_{n-1}, I_n\}$ is a rome and there are paths of the following length from the rome to itself: $K \to K$: 1; $K \to I_{n-1}$: n; $K \to I_n$: n; $I_{n-1} \to K$: 1; $I_{n-1} \to I_{n-1}$: $n-1, n-3, \ldots, 2$; $I_{n-1} \to I_n$: $n-1, n-3, \ldots, 2$; $I_n \to K$: none; $I_n \to I_{n-1}$: $n, n-2$; $I_n \to I_n$: n. Thus, the entropy of P is

the logarithm of the largest root of the equation $d(x) = 0$, where

$$d(x) = \begin{vmatrix} x^{-1}-1 & x^{-n} & x^{-n} \\ x^{-1} & x^{-(n-1)}+x^{-(n-3)}+\ldots+x^{-2}-1 & x^{-(n-1)}+x^{-(n-3)}+\ldots+x^{-2} \\ 0 & x^{-n}+x^{-(n-2)} & x^{-n}-1 \end{vmatrix}.$$

We have

$$x^{2n+1}d(x)$$

$$= \begin{vmatrix} 1-x & 1 & 1 \\ 1 & x+x^3+\ldots+x^{n-2}-x^n & x+x^3+\ldots+x^{n-2} \\ 0 & 1+x^2 & 1-x^n \end{vmatrix}$$

$$= \begin{vmatrix} 1-x & 1 & 1 \\ 1 & \frac{x-x^n}{1-x^2}-x^n & \frac{x-x^n}{1-x^2} \\ 0 & 1+x^2 & 1-x^n \end{vmatrix} = \begin{vmatrix} 1-x & 0 & 1 \\ 1 & -x^n & \frac{x-x^n}{1-x^2} \\ 0 & x^2+x^n & 1-x^n \end{vmatrix},$$

so

$$(1-x^2)x^{2n+1}d(x) = \begin{vmatrix} 1-x & 0 & 1 \\ 1-x^2 & -x^{n-2}(1-x^2) & x-x^n \\ 0 & 1+x^{n-2} & 1-x^n \end{vmatrix}$$

$$= (1-x)\begin{vmatrix} 1 & 0 & 1 \\ 1+x & x^n-x^{n-2} & x-x^n \\ 0 & 1+x^{n-2} & 1-x^n \end{vmatrix}.$$

Hence, we get
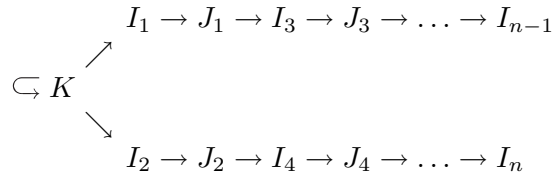
$$(1+x)x^{2n-1}d(x) = (x^n-x^{n-2})(1-x^n)+(1+x)(1-x^{n-2})$$
$$- (1+x^{n-2})(x-x^n)$$
$$= -x^{2n}+2x^{2n-2}+2x^n+1.$$

Thus, the entropy of $B_n(3,n)$ is equal to the logarithm of the largest root of the equation $b(x) = 0$, where (here n is odd)

$$b_n(x) = x^{2n}-2x^{2n-2}-2x^n-1 .$$

Let now n be even and $[P] = B_n(2,n)$. We use again the same standard notation. The P-graph of f is

$$I_1 \to J_1 \to I_3 \to J_3 \to \ldots \to I_{n-1}$$

$$\circlearrowleft K \qquad \nearrow$$

$$\searrow$$

$$I_2 \to J_2 \to I_4 \to J_4 \to \ldots \to I_n$$

and additionally there are arrows from I_n to I_1, and from I_{n-1} to K and all J_i's. As before, the set $\{K, I_{n-1}, I_n\}$ is a rome (we can take also $\{K, I_{n-1}\}$, but then counting paths is more complicated) and there are paths of the following length from the rome to itself: $K \to K$: 1; $K \to I_{n-1}$: $n-1$; $K \to I_n$: $n-1$; $I_{n-1} \to K$: 1; $I_{n-1} \to I_{n-1}$: $n-2, n-4, \ldots, 2$; $I_{n-1} \to I_n$: $n-2$, $n-4, \ldots, 2$; $I_n \to K$: none; $I_n \to I_{n-1}$: $n-1$; $I_n \to I_n$: none. Thus, the entropy of P is the logarithm of the largest root of the equation $c(x) = 0$, where

$$c(x) =$$

$$\begin{vmatrix} x^{-1}-1 & x^{-(n-1)} & x^{-(n-1)} \\ x^{-1} & x^{-(n-2)}+x^{-(n-4)}+\ldots+x^{-2}-1 & x^{-(n-2)}+x^{-(n-4)}+\ldots+x^{-2} \\ 0 & x^{-(n-1)} & -1 \end{vmatrix}.$$
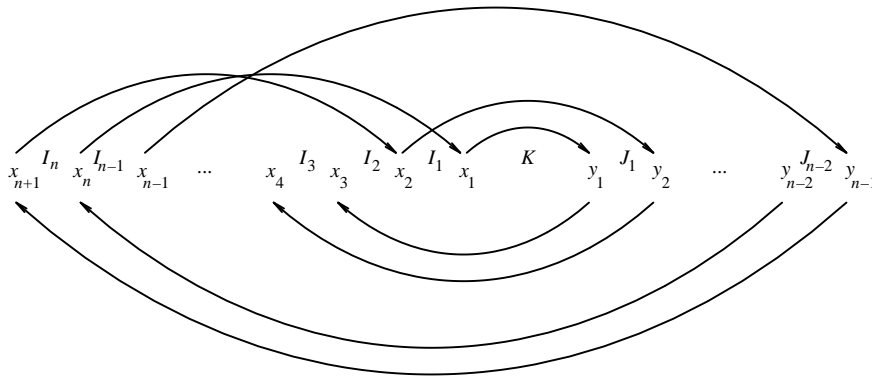

Fig. 6.5. A cycle with pattern $B_n(2, n)$, n even

We have

$$x^{2n-1}c(x)$$

$$= \begin{vmatrix} 1-x & 1 & 1 \\ 1 & x+x^3+\ldots+x^{n-3}-x^{n-1} & x+x^3+\ldots+x^{n-3} \\ 0 & 1 & -x^{n-1} \end{vmatrix}$$

$$= \begin{vmatrix} 1-x & 1 & 1 \\ 1 & \frac{x-x^{n-1}}{1-x^2}-x^{n-1} & \frac{x-x^{n-1}}{1-x^2} \\ 0 & 1 & -x^{n-1} \end{vmatrix} = \begin{vmatrix} 1-x & 0 & 1 \\ 1 & -x^{n-1} & \frac{x-x^{n-1}}{1-x^2} \\ 0 & 1+x^{n-1} & -x^{n-1} \end{vmatrix},$$

so

$$(1-x^2)x^{2n-1}c(x) = \begin{vmatrix} 1-x & 0 & 1 \\ 1-x^2 & -x^{n-1}(1-x^2) & x-x^{n-1} \\ 0 & 1+x^{n-1} & -x^{n-1} \end{vmatrix}$$

$$= (1-x)\begin{vmatrix} 1 & 0 & 1 \\ 1+x & x^{n+1}-x^{n-1} & x-x^{n-1} \\ 0 & 1+x^{n-1} & -x^{n-1} \end{vmatrix}.$$

Hence, we get

$$(1+x)x^{2n-1}c(x) = (x^{n+1}-x^{n-1})(-x^{n-1}) + (1+x)(1+x^{n-1})$$
$$- (1+x^{n-1})(x-x^{n-1})$$
$$= -x^{2n} + 2x^{2n-2} + 2x^{n-1} + 1.$$

Thus, the entropy of $B_n(2,n)$ is equal to the logarithm of the largest root of the equation $b(x) = 0$, where (here n is even)

$$b_n(x) = x^{2n} - 2x^{2n-2} - 2x^{n-1} - 1.$$

THEOREM 6.3. *For a given $n \geq 2$, among minor patterns of period $2n$ the one with the smallest entropy is $B_n(3,n)$ if n is odd and $B_n(2,n)$ if n is even. Its entropy is the logarithm of the largest root of the equation $b_n(x) = 0$.*
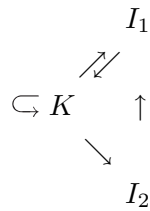
P r o o f. We know already that $h(B_n(3,n))$ for n odd and $h(B_n(2,n))$ for n even is equal to the logarithm of the largest root of the equation $b_n(x) = 0$.
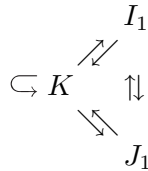Let $n \geq 4$. For $x \geq \sqrt{3}$ we have

$$b_n(x) \geq x^{2n} - 2x^{2n-2} - 2x^n - 1 = (x^2-2)x^{2n-2} - 2x^n - 1$$
$$\geq x^{2n-2} - 2x^n - 1 = x^n(x^{n-2}-2) - 1$$
$$\geq x^n(x^2-2) - 1 \geq x^n - 1 > 0.$$

Therefore $h(B_n(3,n))$ for n odd and $h(B_n(2,n))$ for n even is smaller than $\log \sqrt{3}$. By Theorem 5.1 and Lemmas 6.1 and 6.2, the entropy of any other minor pattern of period $2n$ is larger. This completes the proof in this case.
For $n = 2$ there are two patterns to consider: $B_2(2,2)$ and $C_2(1,2)$. The P-graph in the first case is

$$I_1$$
$$\nearrow\!\!\!\!\nwarrow$$
$$\circlearrowleft K \qquad \uparrow$$
$$\searrow$$
$$I_2$$

and in the second case it is

$$\begin{array}{ccc} & & I_1 \\ & \nearrow\!\!\!\nearrow & \\ \hookrightarrow K & & \uparrow\!\downarrow \\ & \searrow\!\!\!\searrow & \\ & & J_1 \end{array}$$

We see that the first graph is isomorphic to a proper subgraph of the second one and it is transitive, so $h(B_2(2,2)) < h(C_2(1,2))$.

For $n = 3$ there are four patterns to consider. Using standard techniques of [BGMY] we can easily compute the polynomials for which the logarithms of the largest roots are equal to the entropies of those patterns. They are

$$w_1(x) = x^5 - x^4 - x^3 - x^2 + x - 1 \quad \text{for } B_3(3,3),$$
$$w_2(x) = x^5 - x^4 - 3x^3 + x^2 + x - 1 \quad \text{for } B_3(2,2),$$
$$w_3(x) = x^5 - x^4 - 3x^3 + x^2 - x - 1 \quad \text{for } C_3(1,2) \text{ and } C_3(2,3).$$

We see that $w_2(x) = w_1(x) - 2x^2(x-1) < w_1(x)$ for $x > 1$ and $w_3(x) = w_2(x) - 2x < w_2(x)$ for $x > 0$. Since the coefficient of the largest power is positive, this shows that the largest zero of w_1 (it is larger than 1) is smaller than the largest zero of w_2, which is smaller than the largest zero of w_3. Hence, $h(B_3(3,3)) < h(B_3(2,2)) < h(C_3(1,2)) = h(C_3(2,3))$. ∎

Finally, notice that the equation $b_n(x) = 0$ is equivalent to $x^2 - 2 = (2x^n + 1)/x^{2n-2}$ for n odd and to $x^2 - 2 = (2x^{n-1} + 1)/x^{2n-2}$ for n even. Therefore, as $n \to \infty$, the smallest entropy of a minor pattern of period $2n$ tends to $\log\sqrt{2}$ (remaining always larger than $\log\sqrt{2}$).

This can also be deduced in a different way. Namely, the patterns $B_n(3,n)$ for n odd and $B_n(2,n)$ for n even are unimodal. Let S_n be the Štefan pattern of period $n \geq 3$ odd. By Lemma 2.1, if $n \geq 3$ is odd then $B_n(3,n)$ forces S_n. By Lemma 4.8, $B_{n-1}(2, n-1)$ forces a non-reducible pattern of period $2n$, so it forces a minor pattern of period $2n$. Since $B_{n-1}(2, n-1)$ is unimodal, this minor pattern of period $2n$ also has to be unimodal, so it is $B_n(3,n)$. Thus, $B_{n-1}(2, n-1)$ forces $B_n(3,n)$. By Lemma 4.6, S_n forces a non-reducible pattern of period $2(n+1)$, so it forces a minor pattern of that period. Since S_n is unimodal, this minor pattern of period $2(n+1)$ has to be unimodal, so it is $B_{n+1}(2, n+1)$. Thus, S_n forces $B_{n+1}(2, n+1)$. Therefore we see that $B_2(2,2)$ forces $B_3(3,3)$ forces S_3 forces $B_4(2,4)$ forces $B_5(3,5)$ forces S_5 etc. Since we know that the limit of the entropies of S_n as $n \to \infty$ is $\log\sqrt{2}$, the limits of the entropies of $B_n(3,n)$ for n odd and $B_n(2,n)$ for n even are also equal to $\log\sqrt{2}$.

7. Final remarks. There are striking similarities between the structure of minor patterns for interval maps and the structure of primary patterns for the maps of the triod Y (see [ALM$_Y$]) which fix the central point. Moreover,

the ordering of periods of unimodal minor patterns obtained at the end of
the previous section is

$$4, 6, 3, 8, 10, 5, 12, 14, 7, \ldots,$$

so it is constructed in the same way as green and red orderings in [ALM$_Y$],
except that we add 2 instead of 3. Therefore we may regard an interval as
a "diod", with the central point z fixed by the maps under consideration.

In some sense, for both I and Y the simplest patterns are the ones which
spiral out from the central point (black arrows) and come closer to it only
from time to time (colored arrows). The interval from the central point to
the end of a colored arrow cannot be stretched too soon (under the iterates
of the map) to an interval containing the beginning of this arrow. This
condition reduces the number of patterns under consideration significantly.
Among those which remain, some are extensions, but the rest are good
candidates for minor patterns. Then to find primary patterns, we have to
consider minor ones and their extensions. Here one has to take into account
the forcing among minor patterns and the divisibility of their periods.

It seems that the procedure described above can be used for looking for
primary patterns for maps of n-ods and even more complicated trees.

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