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Classical-type characterizations of non-metrizable ANE(n)-spaces

by

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Abstract. The Kuratowski–Dugundji theorem that a metrizable space is an absolute (neighborhood) extensor in dimension n iff it is $LC^{n-1}\& C^{n-1}$ (resp., LC^{n-1}) is extended to a class of non-metrizable absolute (neighborhood) extensors in dimension n. On this base, several facts concerning metrizable extensors are established for non-metrizable ones.

1. Introduction. Throughout this paper n is always an integer ≥ 0; a space means a completely regular space; $\mathcal{T}(X)$ denotes the topology of a space X; a map between two spaces is a continuous map, and dim stands for the dimension defined by finite functionally open covers.

To define a concept of a space being an absolute (neighborhood) extensor in dimension n, we adopt Chigogidze's approach [1]. Suppose Y is a subspace of Z and $g : Y \to X$ is a map. Then g is Z-*normal* if, for every real-valued map φ on X, the map $\varphi \circ g$ is continuously extendable to the whole of Z. We say that X is an *absolute neighborhood extensor* for n-dimensional spaces (briefly, ANE(n)) if, for every space Z with $\dim(Z) \leq n$ and $Y \subset Z$, every Z-normal map $g : Y \to X$ is continuously extendable to a map $f : U \to X$ for some functionally open neighborhood U of Y in Z; if one can take $U = Z$, then X is said to be an AE(n) [1; Definition 1.1]. For X Lindelöf and Čech-complete, these definitions coincide with the classical ones.

The starting point of the present paper is the familiar Kuratowski–Dugundji theorem [7, 3] that a metrizable space X is an ANE(n) (resp., AE(n)) for metrizable spaces if and only if X is LC^{n-1} (resp., LC^{n-1} & C^{n-1}). Unfortunately, in this form, the above result fails when X is non-metrizable. A compact AE($n-1$)-space which is LC^{n-1} & C^{n-1} but not

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an AE(n) is found in [2]. Almost all characterizations of non-metrizable AE(n)'s are in terms of inverse systems with n-soft projections [6, 10, 2, 1]. These characterizations imply, in particular, that each non-metrizable AE(n) is certainly LC^{n-1} & C^{n-1}. In this way, the following question arises: Can non-metrizable ANE(n)'s (resp., AE(n)'s) be characterized by LC^{n-1} (resp., LC^{n-1} & C^{n-1}) type properties?

Only one such attempt to characterize the compact AE(n)-spaces is known to the authors [11]. To recall it briefly, the property of a space X to be LC^{n-1} & C^{n-1} is equivalent to the existence of a special-type "base for X". Under "base for X" we mean every map $\gamma : \mathcal{T}(X) \to \exp(\mathcal{T}(X))$ with $U = \bigcup \gamma(U)$ for every $U \in \mathcal{T}(X)$. Then X is LC^{n-1} & C^{n-1} if and only if it admits a base $\gamma : \mathcal{T}(X) \to \exp(\mathcal{T}(X))$ such that $X \in \gamma(X)$ and, for every $U \in \mathcal{T}(X)$, every continuous map of a k-sphere ($k < n$) in some $W \in \gamma(U)$ is contractible in U. These "LC^{n-1} & C^{n-1}" bases satisfying, in addition, an assumption of regularity appear (in [11]) in a characterization of the compact AE(n)'s among the compact AE(0)'s. Since every metrizable space is an AE(0) for metrizable spaces, this result agrees with the classical one. To extend in such a manner the Kuratowski–Dugundji theorem to all ANE(n)'s (resp., AE(n)'s) and to show how the resulting theorems can be applied, is the purpose of this paper.

After these preliminaries, we now turn to the central concepts of the paper. If $\mathcal{U}(\alpha), \alpha \in \mathcal{A}$, are families of subsets of a space X, then $\bigwedge\{\mathcal{U}(\alpha) : \alpha \in \mathcal{A}\}$ will denote the family of all intersections $\bigcap\{U_\alpha : \alpha \in \mathcal{A}\}$, where $U_\alpha \in \mathcal{U}(\alpha)$. A base $\gamma : \mathcal{T}(X) \to \exp(\mathcal{T}(X))$ for X is said to be *n-regular* [11] if, for every $U, V \in \mathcal{T}(X)$, the following conditions hold:

1) $\gamma(U) \wedge \gamma(V) = \gamma(U \cap V)$ (*regularity condition*);

2) Every continuous map of a k-sphere ($k < n$) in some $W \in \gamma(U)$ is contractible in U.

Suppose $X \subset Y$. A map $e : \mathcal{T}(X) \to \mathcal{T}(Y)$ is called an *extension operator* if $e(\emptyset) = \emptyset$ and $e(U) \cap X = U$ for every $U \in \mathcal{T}(X)$. Following Shirokov [11], an extension operator $e : \mathcal{T}(X) \to \mathcal{T}(Y)$ is said to be *n-regular* if, for every $U, V \in \mathcal{T}(X)$, the following conditions hold:

1) $e(U) \cap e(V) = e(U \cap V)$ (*regularity condition*);

2) Every continuous map of a k-sphere ($k < n$) in U is contractible in U provided it is contractible in $e(U)$.

Our first result is the following theorem.

THEOREM 1.1. *For a Čech-complete* AE(0)*-space X and $n > 0$ the following conditions are equivalent*:

(i) *X is an* ANE(n);

(ii) *For every C-embedding of X in a space Y there is an n-regular extension operator $e : \mathcal{T}(X) \to \mathcal{T}(Y)$;*

(iii) *X has an n-regular base.*

As will be shown in Section 4 (Corollary 4.6), a space X is an AE(n) if and only if $X \in \mathrm{ANE}(n) \cap C^{n-1}$. In view of that, Theorem 1.1 reduces to [11; Theorem 4] when X is compact and C^{n-1}.

We do not know whether Theorem 1.1 remains valid if "Čech-complete" is omitted. But, under slight modifications of the concepts of an n-regular extension operator and n-regular base, the following characterizations of arbitrary ANE(n)'s hold.

THEOREM 1.2. *For an* AE(0)*-space X and $n > 0$ the following conditions are equivalent*:

(i) *X is an* ANE(n);

(ii) *For every C-embedding of X in a space Y there is an $(n, \aleph_1)$-regular extension operator $e : \mathcal{T}(X) \to \mathcal{T}(Y)$;*

(iii) *X has an $(n, \aleph_1)$-regular base.*

The precise definitions of an $(n, \aleph_1)$-regular extension operator and base are given in Section 2.

Despite its complication, the characterization of ANE(n)'s in terms of $(n, \aleph_1)$-regular bases is very successful in proving that a space X is an ANE(n) provided X admits a countable functionally open cover of ANE(n)'s (Theorem 7.1). On the base of Theorems 1.1 and 1.2, we also establish some homotopic and realization properties of ANE(n)'s (Section 6).

The proofs of Theorems 1.1 and 1.2 take up most of the paper and will be finally accomplished in Sections 5 and 4, respectively.

2. Some properties of ANE(n)**-spaces.** Throughout this section τ is an infinite cardinal. By $\mathcal{T}_{\mathrm{fo}}(X)$ we denote the functionally open subsets of X. For two subsets V and U of X we write $V \overset{n}{\hookrightarrow} U$ if every continuous map of a k-sphere $(k < n)$ in V is contractible in U. If $\mathcal{V}$ and $\mathcal{U}$ are two families of subsets of X, then $\mathcal{V} \overset{n}{\hookrightarrow} \mathcal{U}$ means that every continuous map of a k-sphere $(k < n)$ in some element of $\mathcal{V}$ is contractible in an element of $\mathcal{U}$.

Suppose $X \subset Y$. We say that an extension operator $e : \mathcal{T}(X) \to \mathcal{T}(Y)$ is (n, τ)-*regular* if

$1)_e$ $e(U) \in \mathcal{T}_{\mathrm{fo}}(Y)$ provided $U \in \mathcal{T}_{\mathrm{fo}}(X)$;

$2)_e$ Whenever $\mathcal{U} \subset \mathcal{T}(X)$ with $\mathrm{card}(\mathcal{U}) < \tau$, every continuous map of a k-sphere $(k < n)$ in $\bigcap \mathcal{U}$ is contractible in $\bigcap \mathcal{U}$ provided it is contractible in $\bigcap\{e(U) : U \in \mathcal{U}\}$.

We say that a base $\gamma : \mathcal{T}(X) \to \exp(\mathcal{T}(X))$ for X is (n, τ)-*regular* if

$1)_\gamma$ $\gamma(U) \subset \mathcal{T}_{\rm fo}(X)$ is countable if $U \in \mathcal{T}_{\rm fo}(X)$;

$2)_\gamma$ $\bigwedge\{\gamma(U) : U \in \mathcal{U}\} \overset{n}{\hookrightarrow} \bigcap \mathcal{U}$ for every $\mathcal{U} \subset \mathcal{T}(X)$ with $\operatorname{card}(\mathcal{U}) < \tau$.

LEMMA 2.1. *Suppose X is Lindelöf and $X \subset Y$. Then*:

(i) *The existence of an n-regular extension operator $e : \mathcal{T}(X) \to \mathcal{T}(Y)$ implies the existence of an $(n, \aleph_0)$-regular one*;

(ii) *X has an n-regular base if and only if X has an $(n, \aleph_0)$-regular base.*

Proof. (i) In fact, it suffices to construct an extension operator $e_0 : \mathcal{T}(X) \to \mathcal{T}(Y)$ such that, for every $U \in \mathcal{T}(X)$, $e_0(U) \subset e(U)$ and $e_0(U) \in \mathcal{T}_{\rm fo}(Y)$ provided $U \in \mathcal{T}_{\rm fo}(X)$. For $U \in \mathcal{T}_{\rm fo}(X)$, take a countable $v(U) \subset \mathcal{T}_{\rm fo}(Y)$ such that $U \subset \bigcup v(U) \subset e(U)$. This is possible because each functionally open subset of X is Lindelöf. Next, merely set $e_0(U) = \bigcup v(U)$ if $U \in \mathcal{T}_{\rm fo}(X)$ and $e_0(U) = e(U)$ otherwise.

(ii) Suppose $\gamma : \mathcal{T}(X) \to \exp(\mathcal{T}(X))$ is an n-regular base for X. Let $\gamma_0(U) \subset \mathcal{T}_{\rm fo}(X)$ be a countable cover of U refining $\gamma(U)$ if $U \in \mathcal{T}_{\rm fo}(X)$ and $\gamma_0(U) = \gamma(U)$ otherwise. It is easily seen that this defines an $(n, \aleph_0)$-regular base γ_0 for X. Suppose now γ_0 is an $(n, \aleph_0)$-regular base for X. Define $\gamma : \mathcal{T}(X) \to \exp(\mathcal{T}(X))$ by

$$\gamma(U) = \bigcup \Big\{ \bigwedge\{\gamma_0(V) : V \in \mathcal{V}\} : \mathcal{V} \subset \mathcal{T}_{\rm fo}(X) \text{ is finite and } \bigcap \mathcal{V} \subset U \Big\}.$$

Clearly, $\gamma(U)$ covers U and $\gamma(U) \wedge \gamma(V) = \gamma(U \cap V)$ for every $U, V \in \mathcal{T}(X)$. It follows from the definition of γ that, for each $W \in \gamma(U)$, there is a finite $\mathcal{V} \subset \mathcal{T}_{\rm fo}(X)$ such that $\bigcap \mathcal{V} \subset U$ and $W \in \bigwedge\{\gamma_0(V) : V \in \mathcal{V}\}$. Then, by the $(n, \aleph_0)$-regularity of γ_0, $W \overset{n}{\hookrightarrow} U$. So, γ is n-regular. ∎

LEMMA 2.2. *Suppose an* ANE(n)*-space X is C-embedded in $\mathbb{R}^T$. Then there exists an extension operator $e : \mathcal{T}(X) \to \mathcal{T}(\mathbb{R}^T)$ with the following properties*:

(i) *e is both n-regular and (n, τ)-regular*;

(ii) *For every space Z, a closed subset F of Z with* $\dim(Z - F) \le n$, *and every map $u : Z \to e(X)$ with $u(F) \subset X$, there is a map $h : Z \to X$ such that $h|u^{-1}(X) = u|u^{-1}(X)$ and $h(u^{-1}(e(U))) \subset U$ for every $U \in \mathcal{T}(X)$.*

Proof. By [1; Lemma 3.1], there is an n-dimensional space P and a perfect functionally closed n-invertible map f from P onto $\mathbb{R}^T$. The last means that, whenever Z is a space with $\dim(Z) \le n$ and $u : Z \to \mathbb{R}^T$, there is a map $q : Z \to P$ such that $f \circ q = u$. Set $A = f^{-1}(X)$. Since X is C-embedded in $\mathbb{R}^T$, $f|A$ is P-normal. Therefore, there exists a functionally open (in P) neighborhood W_1 of A and a continuous extension $g_1 : W_1 \to X$ of $f|A$. Note that $V = \mathbb{R}^T - f(P - W_1)$ contains X and it is functionally open in $\mathbb{R}^T$ because f is functionally closed. Then $W = f^{-1}(V)$ is functionally open in P and $W \subset W_1$.

Define $e : \mathcal{T}(X) \to \mathcal{T}(\mathbb{R}^T)$ by $e(U) = V - f(P - g^{-1}(U))$ for every $U \in \mathcal{T}(X)$, where $g = g_1|W$. It is easily seen that e is an extension operator such that, for every $U, V \in \mathcal{T}(X)$, $e(U) \cap e(V) = e(U \cap V)$ and $e(U) \in \mathcal{T}_{\mathrm{fo}}(\mathbb{R}^T)$ provided $U \in \mathcal{T}_{\mathrm{fo}}(X)$. The n-regularity and the (n,τ)-regularity of e follow from (ii). So, we only have to check (ii).

Take Z, F and u as in (ii). Since $e(X) = V$ and f is n-invertible, there is a $q : Z - F \to W$ with $f \circ q = u|Z - F$. Since $F \subset Z$ is closed and $f^{-1}(e(U)) \subset g^{-1}(U)$ for every $U \in \mathcal{T}(X)$, setting $h|Z - F = g \circ q$ and $h|F = u|F$, we get a map h from Z into X such that $h(u^{-1}(e(U))) \subset U$, $U \in \mathcal{T}(X)$ (h is continuous because $h^{-1}(U) = (h|Z - F)^{-1}(U) \cup u^{-1}(e(U))$ for every $U \in \mathcal{T}(X)$). Finally,

$$g \circ q|(u^{-1}(X) - F) = f \circ q|(u^{-1}(X) - F) = u|(u^{-1}(X) - F)$$

implies $h|u^{-1}(X) = u|u^{-1}(X)$. ∎

LEMMA 2.3. *For any space X, we have* (i)⇒(ii)⇒(iii), *where*

(i) *X is an* ANE(n);

(ii) *For every C-embedding of X in a space Y there is an (n,τ)-regular extension operator $e : \mathcal{T}(X) \to \mathcal{T}(Y)$*;

(iii) *X has an (n,τ)-base.*

P r o o f. (i)⇒(ii). Suppose X is C-embedded in Y. Consider Y as a C-embedded subset of $\mathbb{R}^T$ for some T. By Lemma 2.2, there is an (n,τ)-regular extension operator $e' : \mathcal{T}(X) \to \mathcal{T}(\mathbb{R}^T)$. Then $e : \mathcal{T}(X) \to \mathcal{T}(Y)$ defined by $e(U) = e'(U) \cap Y$ is as required.

(ii)⇒(iii). Consider X as a C-embedded subset of $\mathbb{R}^T$ for some T. Let $e : \mathcal{T}(X) \to \mathcal{T}(\mathbb{R}^T)$ be an (n,τ)-regular extension operator and let $\mathcal{B} \subset \mathcal{T}(\mathbb{R}^T)$ be a topological base of $\mathbb{R}^T$ consisting of standard convex sets. Suppose $U \in \mathcal{T}_{\mathrm{fo}}(X)$. Since $e(U) \in \mathcal{T}_{\mathrm{fo}}(\mathbb{R}^T)$, by [9], there is a countable $D \subset T$ such that $\pi^{-1}(\pi(e(U))) = e(U)$, where $\pi : \mathbb{R}^T \to \mathbb{R}^D$ is the projection. Hence, there exists a countable $\mathcal{B}(U) \subset \mathcal{B}$ such that $e(U) = \bigcup \mathcal{B}(U)$. Define a base $\gamma : \mathcal{T}(X) \to \exp(\mathcal{T}(X))$ for X by

$$\gamma(U) = \begin{cases} \{H \cap X : H \in \mathcal{B}(U)\} & \text{if } U \in \mathcal{T}_{\mathrm{fo}}(X), \\ \{H \cap X : H \in \mathcal{B} \text{ and } H \subset e(U)\} & \text{otherwise.} \end{cases}$$

To show that γ is (n,τ)-regular, we only have to check $2)_\gamma$. Suppose $\mathcal{U} \subset \mathcal{T}(X)$ with $\mathrm{card}(\mathcal{U}) < \tau$. Take $g : \mathbb{S}^k \to \bigcap\{G_U : U \in \mathcal{U}\}$, where $G_U \in \gamma(U)$ for every $U \in \mathcal{U}$ and $k < n$. According to the definition of γ, $G_U = H_U \cap X$ for some $H_U \in \mathcal{B}$ with $H_U \subset e(U)$. By the convexity of $\bigcap\{H_U : U \in \mathcal{U}\}$, there is an extension $h : \mathbb{B}^{k+1} \to \bigcap\{e(U) : U \in \mathcal{U}\}$ of g. Since, finally, e is (n,τ)-regular, g is certainly extendable to a map $u : \mathbb{B}^{k+1} \to \bigcap \mathcal{U}$. ∎

3. γ-admissible sets. In this section, X is a closed C-embedded AE(0)-subspace of a product $Y = \prod\{Y_t : t \in T\}$ of Polish spaces, where $\mathrm{card}(T) = w(X)$, and $\gamma : \mathcal{T}(X) \to \exp(\mathcal{T}(X))$ is an (n,τ)-regular base. For every $A \subset T$, let $Y(A) = \prod\{Y_t : t \in A\}$. If $B \subset D \subset T$, we write $\pi_B : Y \to Y(B)$ and $\pi_B^D : Y(D) \to Y(B)$ for the projections, $X(B) = \pi_B(X)$, $p_B = \pi_B|X$, and $p_B^D = \pi_B^D|X(D)$. Take an upper semicontinuous compact-valued (briefly, usco) retraction $r : Y \to X$ and a family $\mathcal{A}(T)$ of subsets of T with the following properties:

(3.1) $\bigcup \mathcal{M} \in \mathcal{A}(T)$ for every subfamily $\mathcal{M} \subset \mathcal{A}(T)$;

(3.2) Whenever $D \in \mathcal{A}(T)$ and $B \subset D$ is countable, there is a countable $A \in \mathcal{A}(T)$ with $B \subset A \subset D$;

(3.3) For any $A \in \mathcal{A}(T)$, $p_A : X \to X(A)$ is a functionally open map and $X(A)$ is a closed C-embedded AE(0)-subspace of $Y(A)$;

(3.4) For any $A \in \mathcal{A}(T)$, there exists a usco retraction $r_A : Y(A) \to X(A)$ such that $p_A \circ r = r_A \circ \pi_A$.

For the existence of r and $\mathcal{A}(T)$, see [13; pp. 201–203], where the elements of $\mathcal{A}(T)$ are called r-admissible subsets of T. In the sequel, for $D \in \mathcal{A}(T)$, we sometimes write $\mathcal{A}(D)$ for $\{B \in \mathcal{A}(T) : B \subset D\}$.

For every $t \in T$ fix a countable topological base $\mathcal{B}_t \subset \mathcal{T}(Y_t)$ of Y_t. Denote by $\mathcal{P}$ the subbase for $\mathcal{T}(Y)$ consisting of all $U \in \mathcal{T}(Y)$ for which there are $t(U) \in T$ and $V \in \mathcal{B}_{t(U)}$ such that $U = \pi_{t(U)}^{-1}(V)$. Next, for every $A \subset T$, set

$$\mathcal{P}(A) = \{U \cap X : U \in \mathcal{P} \text{ and } t(U) \in A\}.$$

Clearly, $\mathcal{P}(A) \subset \mathcal{T}_{\mathrm{fo}}(X)$ and $\mathrm{card}(\mathcal{P}(A)) = \mathrm{card}(A) \cdot \aleph_0$.

We now turn to the central concept of this section. A set $A \in \mathcal{A}(T)$ is called γ-*admissible* if

$$p_A^{-1}(p_A(W)) = W \quad \text{for every } W \in \gamma(U) \text{ and } U \in \mathcal{P}(A). \tag{3.5}$$

Lemma 3.6. *Let $D \subset T$ be γ-admissible. Then for every countable set $B \subset D$ there is a countable γ-admissible $A \in \mathcal{A}(D)$ containing B.*

Proof. The proof follows some arguments of the proof of [11; Theorem 4]. First, observe that for every functionally open $W \subset X$ with $p_D^{-1}(p_D(W)) = W$ there is a countable $A(W) \in \mathcal{A}(D)$ such that

$$p_{A(W)}^{-1}(p_{A(W)}(W)) = W. \tag{a}$$

Indeed, since (by (3.3)) $p_D(W)$ is functionally open in $X(D)$ and $X(D)$ is C-embedded in $Y(D)$, there is a functionally open $W' \subset Y(D)$ with $W' \cap X(D) = p_D(W)$. By [9], there exists a countable $S \subset D$ such that $(\pi_S^D)^{-1}(\pi_S^D(W')) = W'$. Then, by (3.2), S is a subset of a countable $A(W) \in \mathcal{A}(D)$. This $A(W)$ satisfies (a).

By induction on i, we now construct an increasing sequence $\{A(i) : i = 1, 2, \ldots\}$ of countable sets $A(i) \in \mathcal{A}(D)$ such that, for every $U \in \mathcal{P}(A(i))$ and $W \in \gamma(U)$,

$$\text{(b)} \qquad p^{-1}_{A(i+1)}(p_{A(i+1)}(W)) = W.$$

By (3.2), there is a countable $A(1) \in \mathcal{A}(D)$ containing B. Suppose $A(i) \in \mathcal{A}(D)$ is countable. Since the families $\mathcal{P}(A(i))$ and $\gamma(U), U \in \mathcal{P}(A(i))$, are countable, the set

$$A(i+1) = A(i) \cup \bigcup\{A(W) : W \in \gamma(U) \text{ and } U \in \mathcal{P}(A(i))\}$$

is also countable and, by (a), $A(i+1)$ satisfies (b). According to (3.1), $A(i+1) \in \mathcal{A}(D)$. This completes the inductive construction.

We finally check that $A = \bigcup\{A(i) : i = 1, 2, \ldots\}$ is the required set. It is obvious that A is countable and $B \subset A$. By (3.1), $A \in \mathcal{A}(D)$. In order to check (3.5), take a $W \in \gamma(U)$ for some $U \in \mathcal{P}(A)$. Since $U \in \mathcal{P}(A(i))$ for some i, by (b), $p^{-1}_{A(i+1)}(p_{A(i+1)}(W)) = W$. Therefore, $p_A^{-1}(p_A(W)) = W$ because $A(i+1) \subset A$. ∎

LEMMA 3.7. *Let $\mathcal{M} \subset \mathcal{A}(T)$ be a family of γ-admissible sets. Then $\bigcup \mathcal{M}$ is also γ-admissible.*

Proof. This follows from the definition of γ-admissibility. ∎

LEMMA 3.8. *Suppose A and B are γ-admissible subsets of T and B is countable. Then there exists a countable γ-admissible subset $C \subset A$ containing $A \cap B$ such that $X(A \cup B)$ is homeomorphic to the fibered product*

$$Z = \{(x_1, x_2) \in X(B \cup C) \times X(A) : p^{B\cup C}_C(x_1) = p^A_C(x_2)\}.$$

Proof. Since $A \cap B$ is countable, by Lemma 3.6, it is a subset of some countable γ-admissible $C \subset A$. The usco retractions $r_{A\cup B} : Y(A \cup B) \to X(A \cup B)$, $r_A : Y(A) \to X(A)$ and $r_{B\cup C} : Y(B \cup C) \to X(B \cup C)$, existing by (3.4), form the following commutative diagram:

$$\begin{tikzcd}
Y(A\cup B) \arrow[rr, "\pi^{A\cup B}_A"] \arrow[dr, "r_{A\cup B}"] \arrow[dd, "\pi^{A\cup B}_{B\cup C}"'] & & Y(A) \arrow[dr, "r_A"] & \\
& X(A\cup B) \arrow[rr, "p^{A\cup B}_A"] \arrow[dd, "p^{A\cup B}_{B\cup C}"] & & X(A) \arrow[dd, "p^A_C"] \\
Y(B\cup C) \arrow[dr, "r_{B\cup C}"'] & & & \\
& X(B\cup C) \arrow[rr, "p^{B\cup C}_C"] & & X(C)
\end{tikzcd}$$

We now show that $h : X(A \cup B) \to Z$ defined by $h(x) = (p^{A\cup B}_{B\cup C}(x), p^{A\cup B}_{A}(x))$ is a homeomorphism. Take $z = (x_1, x_2) \in Z$. Since $(B \cup C - C) \cap (A - C) = \emptyset$, there is exactly one $x \in Y(A \cup B)$ for which $\pi^{A\cup B}_{B\cup C}(x) = x_1$ and $\pi^{A\cup B}_{A}(x) = x_2$. Note that $p^{A\cup B}_{B\cup C}(r_{A\cup B}(x)) = x_1$, $p^{A\cup B}_{A}(r_{A\cup B}(x)) = x_2$ and $r_{A\cup B}(x) \subset X(A \cup B)$. Therefore, $r_{A\cup B}(x) = x$ is the unique point of $X(A \cup B)$ with $h(x) = z$. That is, h is a one-to-one map. Since, finally, h is open, it is certainly a homeomorphism. ∎

4. Proof of Theorem 1.2. Below we need the following definitions: A surjection $f : X \to Y$ between realcompact spaces is *n-soft* [1] if for any realcompact space Z with $\dim(Z) \le n$, $Z_0 \subset Z$ closed and any two maps $g : Z_0 \to X$ and $h : Z \to Y$ such that g is Z-normal and $f \circ g = h|Z_0$, there exists a lifting $u : Z \to X$ of h (i.e., $f \circ u = h$) such that $u|Z_0 = g$. A map $f : X \to Y$ is said to have a *Polish kernel* [1] if there is a Polish space P such that X is C-embedded in $Y \times P$ and $f = \pi|X$, where $\pi : Y \times P \to Y$ is the projection. A subset A of a space Y is C^k provided $A \overset{k+1}{\hookrightarrow} A$. A family $\mathcal{A}$ of subsets of a space Y is *equi-LC^k* in Y [8] if, for any $y \in Y$ and any neighborhood V of y in Y, there is a neighborhood W of y such that $W \cap A \overset{k+1}{\hookrightarrow} V \cap A$ for every $A \in \mathcal{A}$.

Now we turn to the proof of Theorem 1.2. Suppose X is an AE(0)-space. By Lemma 2.3, we only have to prove (iii)⇒(i). Assume $\gamma : \mathcal{T}(X) \to \exp(\mathcal{T}(X))$ is an $(n, \aleph_1)$-regular base for X, and X is a closed C-embedded subset of $\mathbb{R}^{w(X)}$. Set $T = \{\lambda : \lambda < w(X)\}$. Below we freely use all notation of Section 3 with Y replaced by $\mathbb{R}^T$. For every $\lambda \in T$, by Lemma 3.6, fix a countable γ-admissible $B(\lambda) \subset T$ containing λ. Next, put $A(\lambda) = \bigcup\{B(\mu) : \mu < \lambda\}$ if λ is a limit ordinal and $A(\lambda) = \bigcup\{B(\mu) : \mu \le \lambda\}$ otherwise. By Lemma 3.7, these $A(\lambda)$ are γ-admissible. For convenience, we set

$$X_\lambda = X(A(\lambda)),\ p_\lambda = \pi_{A(\lambda)}|X : X \to X_\lambda,\ p^\mu_\lambda = \pi^{A(\mu)}_{A(\lambda)}|X_\mu \text{ for } \lambda < \mu.$$

Since $A(\lambda) \in \mathcal{A}(T)$, by (3.3), X_λ is a closed C-embedded AE(0)-subspace of $\mathbb{R}^{A(\lambda)}$. Thus, we get a continuous inverse system $S = \{X_\lambda, p^\mu_\lambda : \lambda < \mu < w(X)\}$ with $X = \varprojlim S$.

We now prove that $X \in \mathrm{ANE}(n)$ by showing X_1 is LC^{n-1} and each $p^{\lambda+1}_\lambda$ is n-soft. First we show by induction on k ($k \le n$) that $p^{\lambda+1}_\lambda$ is k-soft. By (3.3), p_λ is functionally open. Hence, for every $\lambda < w(X)$, $p^{\lambda+1}_\lambda$ is functionally open with a Polish kernel. So, all $p^{\lambda+1}_\lambda$ are 0-soft [1; Corollary 1.20]. Assuming all $p^{\lambda+1}_\lambda$ are k-soft for some $k < n$, we prove that they are $(k+1)$-soft. For a particular $\lambda < w(X)$, by Lemma 3.8, there exists a countable γ-admissible subset $C(\lambda) \subset A(\lambda)$ containing $A(\lambda) \cap (B(\lambda) \cup B(\lambda+1))$ such that the diagram

$$\begin{tikzcd} X_{\lambda+1} \arrow[r, "p_\lambda^{\lambda+1}"] \arrow[d, "p_{D(\lambda)}^{A(\lambda+1)}"'] & X_\lambda \arrow[d, "p_{C(\lambda)}^{A(\lambda)}"] \\ X(D(\lambda)) \arrow[r, "p_{C(\lambda)}"] & X(C(\lambda)) \end{tikzcd} \tag{4.1}$$

is a Cartesian square (i.e., $X_{\lambda+1}$ is a fibered product, where $D(\lambda) = B(\lambda) \cup B(\lambda+1) \cup C(\lambda)$). To prove now that $p_\lambda^{\lambda+1}$ is $(k+1)$-soft, it suffices to show that $q = p_{C(\lambda)}^{D(\lambda)}$ is $(k+1)$-soft. Notice $X(D(\lambda))$ and $X(C(\lambda))$ are Polish spaces and q is open (see (3.3)). Then, by [8; Theorem 1.2], we only have to check that each fiber $q^{-1}(z)$ is C^k and the family $\{q^{-1}(z) : z \in X(C(\lambda))\}$ is equi-LC^k in $X(D(\lambda))$. To this end, suppose $x \in X(D(\lambda))$ and V is a neighborhood of x. First, take a finite $\mathcal{U}(x) \subset \mathcal{P}(D(\lambda))$ such that

$$x \in p_{D(\lambda)}\Big(\bigcap \mathcal{U}(x)\Big) \subset V. \tag{4.2}$$

Next, for every $U \in \mathcal{U}(x)$, pick a $W_U \in \gamma(U)$ with $W_U \cap p_{D(\lambda)}^{-1}(x) \neq \emptyset$. Note that

$$p_{D(\lambda)}^{-1}(p_{D(\lambda)}(W_U)) = W_U \quad \text{for every } U \in \mathcal{U}(x), \tag{4.3}$$

because $D(\lambda)$ is γ-admissible (see (3.5)). Setting now

$$W = p_{D(\lambda)}\Big(\bigcap\{W_U : U \in \mathcal{U}(x)\}\Big),$$

we get a neighborhood W of x because $p_{D(\lambda)}$ is open. Let us check that this W is as required. So, let $u : \mathbb{S}^m \to q^{-1}(z) \cap W$ be a map, where $m \le k$ and $z \in X(C(\lambda))$. Take a countable $\mathcal{U}(z) \subset \mathcal{P}(C(\lambda))$ with

$$p_{C(\lambda)}\Big(\bigcap \mathcal{U}(z)\Big) = z. \tag{4.4}$$

For every $U \in \mathcal{U}(z)$ we fix a $W_U \in \gamma(U)$ for which $W_U \cap p_{C(\lambda)}^{-1}(z) \neq \emptyset$. Since $C(\lambda)$ is γ-admissible,

$$p_{C(\lambda)}^{-1}(p_{C(\lambda)}(W_U)) = W_U \quad \text{for every } U \in \mathcal{U}(z). \tag{4.5}$$

Take $z' \in X_\lambda$ such that $p_{C(\lambda)}^{A(\lambda)}(z') = z$, and then define $g_{\lambda+1} : \mathbb{S}^m \to X_{\lambda+1}$ by $g_{\lambda+1}(s) = (u(s), z')$, $s \in \mathbb{S}^m$. Since the diagram (4.1) is a Cartesian square, $g_{\lambda+1}$ is correctly defined. By the k-softness of all adjacent projections in the system S, there is a lifting $g : \mathbb{S}^m \to X$ of $g_{\lambda+1}$. It follows from (4.3) and (4.5) that

$$g(\mathbb{S}^m) \subset \bigcap\{W_U : U \in \mathcal{U}\} \in \bigwedge\{\gamma(U) : U \in \mathcal{U}\},$$

where $\mathcal{U} = \mathcal{U}(x) \cup \mathcal{U}(z)$. Since γ is $(n, \aleph_1)$-regular and $\mathcal{U} \subset \mathcal{T}(X)$ is countable, there exists an extension $h : \mathbb{B}^{m+1} \to \bigcap \mathcal{U}$ of g. Finally, by (4.2) and (4.4), $p_{D(\lambda)} \circ h$ is a map from $\mathbb{B}^{m+1}$ into $q^{-1}(z) \cap V$ extending u.

To prove that $q^{-1}(z)$ is C^k, we repeat the previous proof with W replaced by $X(D(\lambda))$ and with $\mathcal{U}$ replaced by $\mathcal{U}(z)$. This completes the induction. Thus, all $p_\lambda^{\lambda+1}$ are n-soft.

Finally, we show that X_1 is LC^{n-1}. Fix an $x \in X_1$ and its neighborhood V. Take a point $x_0 \in p_1^{-1}(x)$ and a finite family $\mathcal{U}(x) \subset \mathcal{P}(A(1))$ of its neighborhoods such that $p_1(\bigcap \mathcal{U}(x)) \subset V$. Set $W = p_1(\bigcap\{W_U : U \in \mathcal{U}(x)\})$, where $W_U \in \gamma(U)$ is a neighborhood of x_0. Since p_1 is n-soft (all $p_\lambda^{\lambda+1}$ are n-soft and S is continuous) and

$$p_1^{-1}(W) \subset \bigcap\{W_U : U \in \mathcal{U}(x)\} \in \bigwedge\{\gamma(U) : U \in \mathcal{U}(x)\} \overset{n}{\hookrightarrow} \bigcap \mathcal{U}(x),$$

we conclude that $W \overset{n}{\hookrightarrow} V$. ■

Corollary 4.6. *A space $X \in \mathrm{AE}(n)$ if and only if $X \in \mathrm{ANE}(n) \cap C^{n-1}$.*

Proof. We only have to prove the sufficiency. Suppose $X \in \mathrm{ANE}(n) \cap C^{n-1}$. First, let us show that X is an AE(0). Suppose Z is a zero-dimensional space, $A \subset Z$ and $g : A \to X$ is a Z-normal map. Let X be a C-embedded subset of $\mathbb{R}^T$ (for some T) and let $e : \mathcal{T}(X) \to \mathcal{T}(\mathbb{R}^T)$ be as in Lemma 2.2. By [1], $e(X)$ is an AE(0) because it is functionally open in $\mathbb{R}^T$. Hence, there is a map $u : Z \to e(X)$ with $u|A = g$. Then, by Lemma 2.2(ii) (with $F = \emptyset$), there exists a map $h : Z \to X$ such that $h|A = g$. That is, X is an AE(0). It now follows from the proof of Theorem 1.2 ((iii)⇒(i)) that X is the limit space of a continuous inverse system $S = \{X_\lambda, p_\lambda^\mu : \lambda < \mu < w(X)\}$, where all projections are n-soft and X_1 is an LC^{n-1} Polish space. In particular, p_1 is n-soft and therefore $X_1 \in C^{n-1}$. Then $X_1 \in \mathrm{AE}(n)$ (see [1; Proposition 1.11]). Hence, $X \in \mathrm{AE}(n)$ as an n-soft preimage of an AE(n)-space. ■

5. Proof of Theorem 1.1. Following [2] we introduce weak versions of C^k and equi-LC^k. We say that a subset A of a space Y is *weakly* C^k (briefly, wC^k) in Y provided $A \overset{k+1}{\hookrightarrow} G$ for every neighborhood G of A in Y. A family $\mathcal{A}$ of subsets of a space Y is *equi-weakly* LC^k (equi-wLC^k) in Y if, for any $y \in Y$ and any neighborhood V of y in Y, there is a neighborhood W of y such that $W \cap A \overset{k+1}{\hookrightarrow} V \cap G$ for every $A \in \mathcal{A}$ and every neighborhood G of A in Y.

Proposition 5.1. *Let Y be a completely metrizable space and $\mathcal{A}$ be a family of closed subsets of Y such that $\mathcal{A}$ is equi-wLC^k in Y and every $A \in \mathcal{A}$ is wC^k in Y. Then $\mathcal{A}$ is equi-LC^k in Y and every $A \in \mathcal{A}$ is C^k.*

Proof. Following the arguments of [8; Proposition 2.1] we find a complete metric ϱ on Y (agreeing with the topology of Y) with the following property: for every $\varepsilon > 0$ there is an $\eta(\varepsilon) > 0$ such that $B_{\eta(\varepsilon)}(y) \cap A \overset{k+1}{\hookrightarrow}$

$B_\varepsilon(y) \cap B_\xi(A)$ for any $y \in Y$, $A \in \mathcal{A}$ and $\xi > 0$. Here, $B_\xi(A)$ denotes the ξ-neighborhood of A in (Y, ϱ); similarly for $B_\varepsilon(y)$, etc. Note, in addition, that $A \overset{k+1}{\hookrightarrow} B_\xi(A)$ for any $\xi > 0$. Thus, by [5; Theorem 3.1], $\mathcal{A}$ is equi-LC^k in Y and each $A \in \mathcal{A}$ is C^k. ∎

We now proceed to the proof of Theorem 1.1. Suppose X is a Čech-complete AE(0)-space. By [13; Theorem 3], X is Lindelöf. Then, according to Lemmas 2.1, 2.2 and 2.3, we only have to prove the implication (iii)⇒(i). Assume X has an n-regular base. Then, by Lemma 2.1(ii), there is an $(n, \aleph_0)$-regular base γ for X. Let $T = \{\lambda : \lambda \in w(X)\}$. Embed X as a closed subset of $Y = \prod\{Y_\lambda : \lambda \in T\}$, where Y_1 is a Polish space and $Y_\lambda = [0,1]$ for $\lambda > 1$ (this can be done because X is Lindelöf and Čech-complete). Using Lemma 3.6, for every $\lambda \in T$, fix a countable γ-admissible subset $B(\lambda) \subset T$ containing both 1 and λ. Following the notation of the proof of Theorem 1.2(iii)⇒(i), we construct a continuous inverse system $S = \{X_\lambda, p^\mu_\lambda : \lambda < \mu < w(X)\}$ such that $X = \varprojlim S$ and each $p^{\lambda+1}_\lambda$ is functionally open with a Polish kernel.

Our proof now continues just as that of Theorem 1.2. Everything goes through as before, except that a bit more care is necessary to check that the map $q = p^{D(\lambda)}_{C(\lambda)}$ (see diagram (4.1)) is $(k+1)$-soft provided all adjacent projections of S are k-soft ($k < n$). In that proof, the hypothesis that γ is $(n, \aleph_1)$-regular was needed because, for every $z \in X(C(\lambda))$, we took a countable family $\mathcal{U}(z) \subset \mathcal{P}(C(\lambda))$ such that $p_{C(\lambda)}(\bigcap \mathcal{U}(z)) = z$ (see (4.4)). In our present situation γ is only $(n, \aleph_0)$-regular but, in addition, q is now perfect. Indeed, $\pi^{D(\lambda)}_{C(\lambda)} : Y(D(\lambda)) \to Y(C(\lambda))$ is perfect because $1 \in A(\lambda) \cap (B(\lambda) \cup B(\lambda+1)) \subset C(\lambda)$, and therefore so is $q = \pi^{D(\lambda)}_{C(\lambda)} | X(D(\lambda))$. Whenever G is a neighborhood of $q^{-1}(z)$, this allows one to find a neighborhood G_0 of z such that $q^{-1}(G_0) \subset G$. Take then a finite $\mathcal{U}(z, G) \subset \mathcal{P}(C(\lambda))$ with

$$z \in p_{C(\lambda)}\Big(\bigcap \mathcal{U}(z, G)\Big) \subset G_0. \tag{4.4*}$$

Since $\mathcal{U}(x) \cup \mathcal{U}(z, G) \subset \mathcal{T}(X)$ is finite and γ is $(n, \aleph_0)$-regular, repeating precisely the previous proof with (4.4) replaced by (4.4*), we see that each fiber $q^{-1}(z)$ is wC^k in $X(D(\lambda))$ and the family $\{q^{-1}(z) : z \in X(C(\lambda))\}$ is equi-wLC^k in $X(D(\lambda))$. Then Proposition 5.1 completes the proof. ∎

6. Homotopic and realization properties of ANE(n)**-spaces.** The characterizations of ANE(n)'s in terms of regular extension operators and regular bases allow one to obtain some other characterizations. In this section we show that some classical descriptions of metrizable ANE(n)'s admit natural extensions to all ANE(n)'s.

Let $f, g : Z \to X$ and let ν be a family of subsets of X. Then f and g are said to be *ν-close* if, for every $z \in Z$, there is an $H \in \nu$ such that

$f(z), g(z) \in H$; and f and g are *ν-homotopic* if there is a homotopy $h : Z \times I \to X$ between f and g such that $\{h(\{z\} \times I) : z \in Z\}$ refines ν, where $I = [0,1]$. We say that a map $\delta : \mathrm{Cov}(X) \to \mathrm{Cov}(X)$ is *regular* if $\delta(\omega) \wedge \delta(\nu) = \delta(\omega \wedge \nu)$ for every $\omega, \nu \in \mathrm{Cov}(X)$. Here, $\mathrm{Cov}(X)$ denotes the family of all open covers of X.

Since there is no such thing as a (-1)-sphere, every extension operator e (or base γ) satisfying the regularity condition is 0-regular (see Section 1). For convenience, each 0-regular extension operator (resp., base) will be called regular.

THEOREM 6.1. *For a Čech-complete* AE(0)*-space X and $n > 0$ the following conditions are equivalent*:

(i) *X is an* ANE(n);

(ii) *There is a regular map $\delta : \mathrm{Cov}(X) \to \mathrm{Cov}(X)$ such that, whenever $\omega \in \mathrm{Cov}(X)$ and Z is a space with* $\dim(Z) < n$, *any two $\delta(\omega)$-close maps $f, g : Z \to X$ are ω-homotopic.*

Proof. (i)⇒(ii). Consider X as a C-embedded subset of $\mathbb{R}^T$ for some T. Let $e : \mathcal{T}(X) \to \mathcal{T}(\mathbb{R}^T)$ be as in Lemma 2.2, and let $\mathcal{B} \subset \mathcal{T}(\mathbb{R}^T)$ be a topological base of $\mathbb{R}^T$ consisting of standard convex sets. Set $\beta(U) = \{H \in \mathcal{B} : H \subset U\}$ for every $U \in \mathcal{T}(\mathbb{R}^T)$. Thus, we obtain a regular base β for $\mathbb{R}^T$. Define $\delta : \mathrm{Cov}(X) \to \mathrm{Cov}(X)$ by

$$\delta(\omega) = \{H \cap X : H \in \beta(e(U)),\ U \in \omega\} \quad \text{for } \omega \in \mathrm{Cov}(X).$$

Since β and e are regular, so is δ. Let $\omega \in \mathrm{Cov}(X)$, Z be a space with $\dim(Z) < n$, and let $f, g : Z \to X$ be two $\delta(\omega)$-close maps. Let $u : Z \times I \to \mathbb{R}^T$ be the linear homotopy between f and g. Since f and g are $\delta(\omega)$-close, for every $z \in Z$ there are $U(z) \in \omega$ and $H(z) \in \beta(e(U(z)))$ such that $f(z), g(z) \in H(z)$. Therefore, $u(\{z\} \times I) \subset H(z) \subset e(U(z))$. By [4; Theorem 1.5], $\dim(Z \times I) \le n$. So, by Lemma 2.2(ii) (with $F = \emptyset$), there is a map $h : Z \times I \to X$ such that $h|Z \times \{0,1\} = u|Z \times \{0,1\}$ and $h(\{z\} \times I) \subset U(z)$ for every $z \in Z$. Hence, f and g are ω-homotopic.

(ii)⇒(i). Suppose $\delta : \mathrm{Cov}(X) \to \mathrm{Cov}(X)$ is as in (ii). Define a regular base η for X by $\eta(U) = \{V \in \mathcal{T}(X) : \mathrm{cl}(V) \subset U\}$ for every $U \in \mathcal{T}(X)$. Next, define a regular base $\gamma : \mathcal{T}(X) \to \exp(\mathcal{T}(X))$ for X by

$$\gamma(U) = \{W \cap V : V \in \eta(U),\ W \in \delta(\{U, X - \mathrm{cl}(V)\})\} \quad \text{for } U \in \mathcal{T}(X).$$

It follows from the construction of γ and from the properties of δ that $\gamma(U) \overset{n}{\hookrightarrow} U$ for $U \in \mathcal{T}(X)$. So, γ is an n-regular base for X and, by Theorem 1.1(iii), X is an ANE(n). ■

Our next result characterizes ANE(n)'s by realization of simplicial complexes. We only consider locally finite simplicial complexes. For such a complex M we use $|M|$ to denote the polytope on M, and $M^{(k)}$ for the k-skeleton of M. A subcomplex L of M is said to be *full* if $M^{(0)} \subset L$. Suppose ν is a cover of X and L is a subcomplex of M. A map $g : |L| \to X$ is a *partial* ν*-realization* of M provided $\{g(|\sigma \cap L|) : \sigma \in M\}$ refines ν. If $L = M$, then g is said to be a *full* ν*-realization* of M.

THEOREM 6.2. *For a Čech-complete* AE(0)*-space* X *and* $n > 0$ *the following conditions are equivalent*:

(i) X *is an* ANE(n);

(ii) *There is a regular map* $\delta : \mathrm{Cov}(X) \to \mathrm{Cov}(X)$ *such that*, *whenever* $\omega \in \mathrm{Cov}(X)$, M *is a simplicial complex and* L *is a full subcomplex of* M *with* $\dim(M - L) \le n$, *every partial* $\delta(\omega)$*-realization* $g : |L| \to X$ *of* M *can be extended to a full* ω*-realization of* M.

P r o o f. The proof follows the pattern of that of Theorem 6.1. For (ii)⇒(i), we repeat precisely the corresponding implication of that theorem. As for (i)⇒(ii), let us check that the map δ constructed in (i)⇒(ii) of Theorem 6.1 works in our present situation. Suppose M is a simplicial complex, L is a full subcomplex of M with $\dim(M-L) \le n$ and $g : |L| \to X$ is a partial $\delta(\omega)$-realization of M for some $\omega \in \mathrm{Cov}(X)$. For every $\sigma \in M$ pick a $U(\sigma) \in \omega$ and $H(\sigma) \in \beta(e(U(\sigma)))$ such that $g(|\sigma \cap L|) \subset H(\sigma)$. By inductively climbing up the complexes $L_k = L \cup M^{(k)}$ ($k \le n$), we now extend g to a map $u : |L_n| = |M| \to \mathbb{R}^T$ such that $u(|\sigma|) \subset \mathrm{conv}(g(|\sigma \cap L|))$ for every $\sigma \in M$. Since $H(\sigma)$ is convex, $u(|\sigma|) \subset H(\sigma) \subset e(U(\sigma))$. Then, by Lemma 2.2(ii) (with $F = |L|$), there is an extension $h : |M| \to X$ of g such that, for every $\sigma \in M$, $h(|\sigma|) \subset U(\sigma)$. This h is a full ω-realization of M. ∎

Similar results remain valid for arbitrary ANE(n)-spaces:

THEOREM 6.3. *For an* AE(0)*-space* X *and* $n > 0$ *the following conditions are equivalent*:

(i) X *is an* ANE(n);

(ii) *There is a map* $\delta : \mathrm{Cov}_{\mathrm{cfo}}(X) \to \mathrm{Cov}_{\mathrm{cfo}}(X)$ *such that*, *whenever* $\Omega \subset \mathrm{Cov}_{\mathrm{cfo}}(X)$ *is countable and* Z *is a space with* $\dim(Z) < n$, *any two* $\bigwedge\{\delta(\omega) : \omega \in \Omega\}$*-close maps* $f, g : Z \to X$ *are* $\bigwedge \Omega$*-homotopic*;

(iii) *There is a map* $\delta : \mathrm{Cov}_{\mathrm{cfo}}(X) \to \mathrm{Cov}_{\mathrm{cfo}}(X)$ *such that*, *whenever* $\Omega \subset \mathrm{Cov}_{\mathrm{cfo}}(X)$ *is countable*, M *is a simplicial complex and* L *is a full subcomplex of* M *with* $\dim(M - L) \le n$, *every partial* $\bigwedge\{\delta(\omega) : \omega \in \Omega\}$*-realization* $g : |L| \to X$ *of* M *can be extended to a full* $\bigwedge \Omega$*-realization of* M.

Here, $\mathrm{Cov}_{\mathrm{cfo}}(X)$ denotes the family of all countable functionally open covers of X. The proof of Theorem 6.3 closely follows the proofs of Theorems 6.1 and 6.2.

Our last result in this section is a characterization of Čech-complete ANE(n)-spaces by extension of maps with metric domains.

Theorem 6.4. *For a Čech-complete* AE(0)*-space X and $n > 0$ the following conditions are equivalent*:

(i) *X is an* ANE(n);

(ii) *There is a regular map $\delta : \mathrm{Cov}(X) \to \mathrm{Cov}(X)$ with the following property*: *If Z is a metric space*, *$A \subset Z$ is closed with* $\dim(Z - A) \le n$, *$f : Z \to X$ and $\omega \in \mathrm{Cov}(X)$*, *then every map $g : A \to X$ which is $\delta(\omega)$-close to $f|A$ can be extended to a map which is ω-close to f.*

Proof. (i)⇒(ii). Let δ be as in the proof of Theorem 6.1(i)⇒(ii). Suppose Z is a metric space, $A \subset Z$ is closed with $\dim(Z - A) \le n$, $f : Z \to X$ and $\omega \in \mathrm{Cov}(X)$. Take $g : A \to X$ such that g and $f|A$ are $\delta(\omega)$-close. Let $q : Z \to \mathbb{R}^T$ be an extension of g. Consider $\nu = \{H \in \beta(e(U)) : U \in \omega\}$ and the open set $W = \bigcup\{f^{-1}(H) \cap q^{-1}(H) : H \in \nu\}$ in Z. Since $f|A$ and $q|A = g$ are $\delta(\omega)$-close, W contains A. Take $\varphi : Z \to I$ such that $\varphi|A \equiv 1$ and $\varphi|(Z - W) \equiv 0$. Next define $u : Z \to \mathbb{R}^T$ by $u(z) = (1 - \varphi(z))f(z) + \varphi(z)q(z)$, $z \in Z$. Observe that $u|A = g$, $u|(Z - W) = f|(Z - W)$ and f is ν-close to u. So, $u(Z) \subset \bigcup \nu \subset e(X)$. Since $\dim(Z - A) \le n$, by Lemma 2.2(ii) (with $F = A$), there is an extension $h : Z \to X$ of g such that $h(u^{-1}(e(U))) \subset U$ for every $U \in \mathcal{T}(U)$. This implies that h is ω-close to f.

(ii)⇒(i). By Theorem 1.1(iii), it suffices to show that X has an n-regular base. This can be done by following closely the proof of Theorem 6.1(ii)⇒(i). ■

7. Open subspaces of ANE(n)'s.

In this section we give some applications of the main theorems.

Theorem 7.1. *Suppose X is a space having a countable functionally open cover consisting of* ANE(n)*'s. Then X is an* ANE(n).

To prove Theorem 7.1 we need three auxiliary propositions.

Proposition 7.2. *Under the assumptions of Theorem* 7.1, *X is an* AE(0)*-space.*

Proof. Let ω be a countable functionally open cover of X consisting of ANE(0)'s. By Corollary 4.6, it suffices to show that $X \in$ ANE(0). To this end, let Z be a zero-dimensional space, $A \subset Z$ and $g : A \to X$ be a Z-normal map. Since ω is countable functionally open, there is a map p from X onto a separable metric space Y such that $p^{-1}(p(U)) = U$ and $p(U)$ is open in Y, for all $U \in \omega$. Using a countable open cover of Y that is a closure refinement of $p(\omega)$, we obtain an open cover $\{W_U : U \in \omega\}$ of X such that $\mathrm{cl}(W_U) \subset U$. Note that, for every $U \in \omega$,

(a) $\mathrm{cl}(W_U)$ is C-embedded in X.

Indeed, let $\varphi : \mathrm{cl}(W_U) \to \mathbb{R}$. Since, by Corollary 4.6, $U \in \mathrm{AE}(0)$, it follows from [1] that there is a map h from U onto a Polish space M with $h(\mathrm{cl}(W_U)) \subset M$ closed, and a function $\psi : h(\mathrm{cl}(W_U)) \to \mathbb{R}$ such that $\psi \circ (h|\,\mathrm{cl}(W_U)) = \varphi$. Take a neighborhood G of $h(\mathrm{cl}(W_U))$ in M and an extension $\Psi : M \to \mathbb{R}$ of ψ with $\Psi|(M - G) = 0$. Define, finally, the required extension $\Phi : X \to \mathbb{R}$ of φ by $\Phi|U = \Psi \circ h$ and $\Phi|(X - U) = 0$.

Since g is Z-normal, there is a functionally open (in Z) family $\nu = \{V_U : U \in \omega\}$ with $V_U \cap A = g^{-1}(W_U)$. Next, there exists a disjoint open cover μ of $V = \bigcup \nu$ refining ν because V is zero-dimensional (being a functionally open subset of Z). Our final step is to get an extension $f : V \to X$ of g. Suppose $P \in \mu$. If $P \cap A = \emptyset$, then merely take $f|P$ to be a constant map. If $P \cap A \neq \emptyset$, let $U(P) \in \omega$ be such that $g(P \cap A) \subset W_{U(P)}$. By (a), $\mathrm{cl}(W_{U(P)})$ is C-embedded in X. This implies that $g|P\cap A : P \cap A \to U(P)$ is P-normal. Hence, there is an extension $f|P : P \to U(P)$ of $g|P\cap A$ because $\dim(P) = 0$ and $U(P) \in \mathrm{AE}(0)$. ∎

Proposition 7.3. *Let $X = G_1 \cup G_2$, where both G_1 and G_2 are functionally open subsets of X. If each G_i has an $(n, \aleph_1)$-regular base, then the same is true for X.*

Proof. Let $\gamma_i : \mathcal{T}(G_i) \to \exp(\mathcal{T}(G_i))$ be an $(n, \aleph_1)$-regular base for G_i, $i = 1, 2$. Since each G_i is functionally open in X, there is a map $h : X \to I^2$ such that $h^{-1}(h(G_i)) = G_i$ and $h(G_i) \subset I^2$ is open. So, every G_i contains a functionally closed (in X) subset F_i such that $F_1 \cup F_2 = X$. Set $L_1 = X - F_2$, $L_2 = X - F_1$ and $L_0 = G_1 \cap G_2$. Define $\gamma : \mathcal{T}(X) \to \exp(\mathcal{T}(X))$ by

$$\gamma(U) = \gamma_1(U \cap L_1) \cup \gamma_2(U \cap L_2) \cup (\gamma_1(U \cap L_0) \wedge \gamma_2(U \cap L_0)), \qquad U \in \mathcal{T}(X).$$

Note $U \cap L_j \in \mathcal{T}_{\mathrm{fo}}(X)$, $j = 0, 1, 2$, provided $U \in \mathcal{T}_{\mathrm{fo}}(X)$. Hence, for $U \in \mathcal{T}_{\mathrm{fo}}(X)$, $\gamma(U) \subset \mathcal{T}_{\mathrm{fo}}(X)$ is countable because each family $\gamma_i(U \cap L_i), \gamma_i(U \cap L_0) \subset \mathcal{T}_{\mathrm{fo}}(X)$, $i = 1, 2$, is countable. So, it only remains to check $2)_\gamma$ of the definition of an $(n, \aleph_1)$-regular base (see Section 2). Take a countable family $\mathcal{U} \subset \mathcal{T}(X)$ and a non-empty $W \in \bigwedge\{\gamma(U) : U \in \mathcal{U}\}$. Then for every $U \in \mathcal{U}$ there is a $W_U \in \gamma(U)$ such that $W = \bigcap\{W_U : U \in \mathcal{U}\}$. Since $L_1 \cap L_2 = \emptyset$, there exists an $i(W) \in \{1, 2\}$ for which

$$W_U \in \gamma_{i(W)}(U \cap L_{i(W)}) \cup (\gamma_1(U \cap L_0) \wedge \gamma_2(U \cap L_0)), \qquad U \in \mathcal{U}.$$

Therefore, for every $U \in \mathcal{U}$ we can choose a $V_U \in \gamma_{i(W)}(U \cap L_{i(W)}) \cup \gamma_{i(W)}(U \cap L_0)$ containing W_U. Next, for any $U \in \mathcal{U}$ pick a $j(U) \in \{i(W), 0\}$ such that $V_U \in \gamma_{i(W)}(U \cap L_{j(U)})$. Finally, note that

$$W \subset \bigcap\{V_U : U \in \mathcal{U}\} \in \bigwedge\{\gamma_{i(W)}(U \cap L_{j(U)}) : U \in \mathcal{U}\} \overset{n}{\hookrightarrow} \bigcap \mathcal{U}.$$

That is, $\bigwedge\{\gamma(U) : U \in \mathcal{U}\} \overset{n}{\hookrightarrow} \bigcap \mathcal{U}$. ∎

PROPOSITION 7.4. *Suppose ν is a countable disjoint functionally open cover of a space X. If every $V \in \nu$ has an $(n, \aleph_1)$-regular base, then the same is true for X.*

Proof. Merely set $\gamma(U) = \bigcup\{\gamma_V(U \cap V) : V \in \nu\}$ $(U \in \mathcal{T}(X))$, where γ_V is an $(n, \aleph_1)$-regular base for V. ■

Proof of Theorem 7.1. The proof follows as closely as possible the proof of the classical result that every metric space that is a countable union of open ANR's is an ANR.

Suppose ω is a functionally open cover of X consisting of ANE(n)'s. By Theorem 1.2(iii) and Proposition 7.2, it suffices to show that X admits an $(n, \aleph_1)$-regular base. First, we note that X is the union of an increasing sequence $\{U_k : k = 1, 2, \ldots\}$ of functionally open subsets having $(n, \aleph_1)$-regular bases. Indeed, by Theorem 1.2(iii), any $U \in \omega$ has an $(n, \aleph_1)$-regular base. So, by Proposition 7.3, the same is true for the union of every finite subfamily of ω. Next, take a countable functionally open cover $\nu_1 \cup \nu_2$ of X such that each ν_i is a pairwise disjoint refinement of $\{U_k : k = 1, 2, \ldots\}$. Such a cover $\nu_1 \cup \nu_2$ certainly exists because there is a map g from X onto a separable metric space Y such that, for every k, $g^{-1}(g(U_k)) = U_k$ and $g(U_k) \subset Y$ is open. Since every element of ν_i (being a functionally open subset of some U_k) admits an $(n, \aleph_1)$-regular base, by Proposition 7.4, $G_i = \bigcup \nu_i$ $(i = 1, 2)$ also has an $(n, \aleph_1)$-regular base. Finally, by Proposition 7.3, the same is true for $X = G_1 \cup G_2$. ■

COROLLARY 7.5. *Suppose U and V are functionally open subsets of a space X such that $U, V, U \cap V \in$ AE(n). Then $X \in$ AE(n).*

Proof. By Theorem 7.1 and Corollary 4.6, it suffices to show X is C^{n-1}. Let $g : \mathbb{S}^m \to X$ $(m < n)$. As in the proof of Proposition 7.3, we take two closed subsets F_1 and F_2 of X such that $F_1 \cup F_2 = X$, $F_1 \subset U$ and $F_2 \subset V$. Set $A_i = g^{-1}(F_i)$ and pick two closed $B_1, B_2 \subset \mathbb{B}^{m+1}$ such that $B_i \cap \mathbb{S}^m = A_i$ and $B_1 \cup B_2 = \mathbb{B}^{m+1}$. Using the hypotheses that $U \cap V$, U and V are AE(n)'s we can easily obtain the required extension of g by combining extensions on $B_1 \cap B_2$, B_1 and B_2. ■

In conclusion, we prove an analogue of a result of Toruńczyk [12] concerning metrizable ANR's.

THEOREM 7.6. *Suppose X is a Čech-complete* ANE(1)*-space which has a topological base $\mathcal{B} \subset \mathcal{T}(X)$ such that, for every finite subfamily $\mathcal{V} \subset \mathcal{B}$, every component of $\bigcap \mathcal{V}$ is C^{n-1}. Then X is an* ANE(n).

Proof. By Theorem 1.1(iii), it suffices to show that X admits an n-regular base. Suppose $\eta : \mathcal{T}(X) \to \exp(\mathcal{T}(X))$ is a 1-regular base for X (Theorem 1.1). Define $\gamma : \mathcal{T}(X) \to \exp(\mathcal{T}(X))$ by setting

$$\gamma(U)=\Big\{\bigcap\mathcal{V}:\mathcal{V}\subset\mathcal{B}\text{ is finite and }\bigcap\mathcal{V}\subset W\in\eta(U)\Big\}.$$

Suppose $U_1, U_2 \in \mathcal{T}(X)$. Since $\eta(U_1)\wedge\eta(U_2)=\eta(U_1\cap U_2)$, the above definition guarantees that $\gamma(U_1)\wedge\gamma(U_2)=\gamma(U_1\cap U_2)$. So, it only remains to check that $\gamma(U)\overset{n}{\hookrightarrow}U$ for every $U\in\mathcal{T}(X)$. To this end, take $g:\mathbb{S}^m\to H$ with $H\in\gamma(U)$ and $m<n$. If $m=0$, then $g(\mathbb{S}^0)$ is contractible in U because $\gamma(U)$ refines $\eta(U)$ and $\eta(U)\overset{1}{\hookrightarrow}U$. If $m>0$, then $g(\mathbb{S}^m)$ is connected. From the definition of γ, there is a finite $\mathcal{V}\subset\mathcal{B}$ such $g(\mathbb{S}^m)\subset\bigcap\mathcal{V}\subset U$. Hence, $g(\mathbb{S}^m)$ lies in a component of $\bigcap\mathcal{V}$ and thus, it is contractible in U. ∎

References

[1] A. Chigogidze, *Noncompact absolute extensors in dimension n, n-soft mappings, and their applications*, Izv. Akad. Nauk SSSR 50 (1) (1986), 156–180 (in Russian); English transl.: Math. USSR-Izv. 28 (1987), 151–174.

[2] A. Dranishnikov, *Absolute extensors in dimension n and dimension-raising n-soft maps*, Uspekhi Mat. Nauk 39 (5) (1984), 55–95 (in Russian); English transl.: Russian Math. Surveys 39 (1984).

[3] J. Dugundji, *Absolute neighborhood retracts and local connectedness in arbitrary metric spaces*, Compositio Math. 13 (1958), 229–246.

[4] V. Filippov, *On the dimension of products of topological spaces*, Fund. Math. 106 (1980), 181–212 (in Russian).

[5] V. Gutev, *Selections for quasi-l.s.c. mappings with uniformly equi-LC^n range*, Set-Valued Anal. 1 (1993), 319–328.

[6] R. Haydon, *On a problem of Pełczyński*: *Milutin spaces*, *Dugundji spaces and* AE(0-dim), Studia Math. 52 (1974), 23–31.

[7] K. Kuratowski, *Sur les espaces localement connexes et péaniens en dimension n*, Fund. Math. 24 (1935), 269–287.

[8] E. Michael, *Continuous selections II*, Ann. of Math. 64 (1956), 562–580.

[9] R. Pol and E. Puzio-Pol, *Remarks on Cartesian products*, Fund. Math. 93 (1976), 57–69.

[10] E. V. Ščepin [E. V. Shchepin], *Functors and uncountable powers of compacta*, Uspekhi Mat. Nauk 36 (3) (1981), 3–62 (in Russian); English transl.: Russian Math. Surveys 36 (1981).

[11] L. Shirokov, *On* AE(n)*-compacta and n-soft mappings*, Sibirsk. Mat. Zh. 33 (1992), 151–156 (in Russian).

[12] H. Toruńczyk, *Concerning locally homotopy negligible sets and characterizations of l_2-manifolds*, Fund. Math. 101 (1978), 93–110.

[13] V. Valov, *Another characterization of* AE(0)*-spaces*, Pacific J. Math. 127 (1987), 199–208.

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