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On the open-open game

by

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Abstract. We modify a game due to Berner and Juhász to get what we call "the open-open game (of length ω)": a round consists of player I choosing a nonempty open subset of a space X and II choosing a nonempty open subset of I's choice; I wins if the union of II's open sets is dense in X, otherwise II wins. This game is of interest for ccc spaces. It can be translated into a game on partial orders (trees and Boolean algebras, for example). We present basic results and various conditions under which I or II does or does not have a winning strategy. We investigate the games on trees and Boolean algebras in detail, completely characterizing the game for ω_1-trees. An undetermined game is also defined. (In contrast, it is still open whether there is an undetermined game using the definition due to Berner and Juhász.) Finally, we show that various variations on the game yield equivalent games.

0. Introduction. The following game is due to Berner and Juhász [BJ]: two players take turns playing; a round consists of player I choosing a nonempty open set $U \subseteq X$ and player II choosing a point $p \in U$; a round is played for each ordinal less than ω (more generally, for each ordinal less than some given ordinal α); I wins the game if the set of points II plays is dense; otherwise, II wins. Denote the game by $G^{\mathrm{o}}_{\mathrm{p}}(X)$.

Consider the following modification, which we call the *open-open game* of length ω: a round consists of I choosing a nonempty open set $U \subseteq X$ and II choosing a nonempty open $V \subseteq U$; I wins if the union of II's open sets is dense in X, otherwise II wins. Denote this game by $G(X)$. (In comparison with $G^{\mathrm{o}}_{\mathrm{p}}(X)$, $G^{\mathrm{o}}_{0}(X)$ might be a better notation.)

$G^{\mathrm{o}}_{\mathrm{p}}(X)$ is only interesting if X is separable, as otherwise II has a trivial winning strategy. $G(X)$ is interesting for a wider class of spaces, the ccc spaces (if X is not ccc, it is easy to see that II has a winning strategy). Furthermore, the open-open game can be translated into an interesting game

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on partial ordered sets (posets), for example on trees or Boolean algebras. In this case, II's move will be smaller than I's move. We denote the games by $G(T)$ or $G(B)$ when T and B are trees or Boolean algebras respectively. In these cases, of course, the moves of both players are elements of T or B and I wins if the set of all II's moves is predense.

§1 and §2 contain the basic results and various conditions under which player I or player II has or does not have a winning strategy. In §3, we investigate the game on trees and Boolean algebras in detail. We show that if T is an ω_1-tree, then I has a winning strategy if and only if T is the union of countably many branches, and II has a winning strategy if and only if T has an uncountable antichain. Further discussion on the game for Boolean algebras leads to a weak version of the von Neumann problem: is every ccc weakly $\aleph_0$-distributive algebra a measure algebra? Finally, an undetermined game is defined. In contrast, it is still open whether there is a real example of a space X on which $G^{\mathrm{o}}_{\mathrm{p}}(X)$ is undetermined. In §4 we give more games and show they are all equivalent to the open-open game.

All spaces are assumed to be regular.

1. When the game favors player I or player II. Let X be a space. A *π-base* for X is a collection $\mathcal{U}$ of open sets such that for each open subset U there exists $V \in \mathcal{U}$ with $V \subseteq U$. Further, $\pi(X)$ is the minimum cardinality of a π-base, and is called the *π-weight* (the corresponding notion for Boolean algebras is the density). One of the main results for the game $G^{\mathrm{o}}_{\mathrm{p}}(X)$ in [BJ] is that I has a winning strategy iff $\pi(X) = \omega$. We will show that is not true for the open-open game G. First, it is easy to show the following is true for G.

THEOREM 1.1. (i) *If* $\pi(X) = \omega$, *then I has a winning strategy.*
(ii) *If* X *is not ccc*, *then II has a winning strategy.*

Proof. (i) The strategy for I is to keep choosing every member of a countable π-base $\mathcal{U}$.

(ii) The strategy for II is to choose at each stage a nonempty subset of a member of a fixed uncountable maximal disjoint collection of open subsets. ∎

Another example for which II has a winning strategy is any uncountable measure algebra. Player II just chooses an element of measure less than $2^{-(n+2)}$ in the nth round.

Not every ccc space for which II has a winning strategy has a strictly positive measure. The most interesting example is the Pixley–Roy space $\mathrm{PR}(\mathbb{R})$ of the reals $\mathbb{R}$ (see Comfort and Negrepontis [CN]).

Let us recall some definitions. A space X has the *C'' property* (resp. *weak C'' property*) if for any countable sequence of open coverings $\langle \mathcal{U}_n \rangle$ of X, there is a sequence $\langle F_n \rangle$ with $F_n \in \mathcal{U}_n$ such that $\bigcup_n F_n = X$ (resp.

$\overline{\bigcup_n F_n} = X$). A family $D \subseteq {}^\omega\omega$ is *strongly dominating* if for any $f \in {}^\omega\omega$, there is a $g \in D$ such that for each n, $f(n) < g(n)$. A game G_{p} played on X is called the *point-open game* if I plays a point x_n and II plays an open neighborhood U_n of x_n in each round, and I wins iff $\bigcup_n U_n = X$ (see [DG]).

THEOREM 1.2. *In the following cases*, *II has a winning strategy in the games* $G(\mathrm{PR}(X))$, *where* $X \subseteq \mathbb{R}$:

(i) X *is not a null set.*

(ii) *Player II has a winning strategy in the point-open game* $G_{\mathrm{p}}(X)$.

COROLLARY. *If* X *is not* C'', *then II has a winning strategy in* $G(\mathrm{PR}(X))$.

Proof of Theorem 1.2. (i) The basic open sets of $\mathrm{PR}(X)$ have the form $[F,U] = \{E \in [X]^{<\omega} : F \subseteq E \subseteq U\}$, where $F \in [X]^{<\omega}$ and U is an open set containing F. The strategy for II is to play a set of the form $[E_n, V_n]$, where V_n is an open set of small measure. Since the outer measure of X is positive, $\bigcup_n V_n$ cannot cover X. If $x \notin \bigcup_n V_n$, then $[\{x\}, X]$ does not meet any $[E_n, V_n]$.

The proof of (ii) is obvious since $G_{\mathrm{p}}(X)$ is equivalent to the similarly defined game where I plays a finite number of points at each stage. ∎

The corollary follows since if X is not C'' it is easy to see that II has a winning strategy in $G_{\mathrm{p}}(X)$. It is also easy to see that if $\mathrm{PR}(X)$ is not weakly C'', II has a winning strategy in $G(\mathrm{PR}(X))$. If X is not C'' then $\mathrm{PR}(X)$ is not weakly C'', but there is a Lusin (hence C'') set X such that $\mathrm{PR}(X)$ is not weakly C'' (see [DG]). We do not know whether the fact that II has a winning strategy in the game $G(\mathrm{PR}(X))$ implies II has a winning strategy in the point-open game $G_{\mathrm{p}}(X)$. It is not difficult to show, however, that for a second countable space X, I has a winning strategy in $G(\mathrm{PR}(X))$ if and only if X is countable. But Telgársky has shown that for a metric space X, I has a winning strategy in $G_{\mathrm{p}}(X)$ if and only if X is countable.

Dominating families are used in Theorem 2.3. At this point, let us note that such a family D is not C'', and so II has a winning strategy in $G(\mathrm{PR}(D))$.

At first, the authors guessed the open-open games for separable spaces might not favor II. This is not the case and the example we present is essentially due to Szymański [S]. First, let us state the following:

FACT 1.3. *Let* $\mathrm{RO}(X)$ *denote the regular open algebra of a space* X *and let* $E(X)$ *be the Stone space of* X. *Then the game* $G(E(X))$ *is equivalent to* $G(\mathrm{RO}(X))$, *which is equivalent to the game* $G(X)$.

COROLLARY 1.4. *If* $\mathrm{RO}(X)$ *is isomorphic to* $\mathrm{RO}(Y)$ *for spaces* X *and* Y, *then* $G(X)$ *and* $G(Y)$ *are equivalent. Thus*, *if* Y *is a dense subspace* *of X, or Y is a closed irreducible image of X, then $G(X)$ and $G(Y)$ are equivalent.*

Example 1.5. There is a countable space X such that II has a winning strategy. By Theorem 1.2, we know II has a winning strategy in the game $G(\mathrm{PR}(\mathbb{R}))$, hence II has a winning strategy in the game $G(Z)$ where Z is $E(\mathrm{PR}(\mathbb{R}))$ by Fact 1.3. Note that Z is separable ($\mathrm{PR}(\mathbb{R})$ has a σ-centered base), therefore if X is a countable dense subset of Z, then II has a winning strategy in $G(X)$ by Fact 1.3 again.

Returning to results for player I to have a winning strategy, an equivalent condition involving the club filter on the countable subsets of a set is useful.

Definition. A collection $\mathcal{C} \subset [X]^{\le\omega}$ is *closed* provided that if $C_1 \subset C_2 \subset \ldots$ is an increasing ω-chain from $\mathcal{C}$, then $\bigcup_{n\in\omega} C_n \in \mathcal{C}$. The collection $\mathcal{C}$ is *unbounded* if for each $B \in [X]^{\le\omega}$ there is a $C \in \mathcal{C}$ such that $B \subset C$.

Definition. If $\mathbb{Q}$ is a partial order and $i : \mathbb{P} \to \mathbb{Q}$ then i is a *complete embedding* of $\mathbb{P}$ into $\mathbb{Q}$ if and only if

(1) $\forall p, p' \in \mathbb{P}\,(p \le p' \Rightarrow i(p) \le i(p'))$,
(2) $\forall p, q \in \mathbb{P}\,(p \perp_{\mathbb{P}} q \leftrightarrow i(p) \perp_{\mathbb{Q}} i(q))$, and
(3) $\forall W \subset \mathbb{P}\,((W \text{ is predense in } \mathbb{P}) \leftrightarrow (i(W) \text{ is predense in } \mathbb{Q}))$.

(3) may be replaced by the following (see [K]):

(3′) $\forall q \in \mathbb{Q}\, \exists p \in \mathbb{P}\, \forall p' \in \mathbb{P}\,(p' \le p \Rightarrow i(p') \text{ and } q \text{ are compatible})$.

Definition. If $\mathbb{P} \subset \mathbb{Q}$, then $\mathbb{P}$ is *completely embedded* in $\mathbb{Q}$, written $\mathbb{P} \subset_{\mathrm{c}} \mathbb{Q}$, if and only if the identity map is a complete embedding of $\mathbb{P}$ into $\mathbb{Q}$, where $<_{\mathbb{P}} = <_{\mathbb{Q}} \cap (\mathbb{P} \times \mathbb{P})$.

Note that playing G on a space X is equivalent to playing G on the partial order of the open sets of X (or of any open π-base).

Theorem 1.6. *I has a winning strategy in $G(\mathbb{Q})$ if and only if $\{\mathbb{P} \in [\mathbb{Q}]^{\le\omega} : \mathbb{P} \subset_{\mathrm{c}} \mathbb{Q}\}$ contains a club (closed and unbounded subcollection).*

Proof. Suppose $\{\mathbb{P} \in [\mathbb{Q}]^{\le\omega} : \mathbb{P} \subset_{\mathrm{c}} \mathbb{Q}\}$ contains a club $\mathcal{C}$. We define a winning strategy σ for I. Let $\sigma\emptyset \in \mathbb{Q}$. Suppose $q_0 \le \sigma\emptyset$. Let $\mathbb{P}_0 \in \mathcal{C}$ be such that $\{\sigma\emptyset, q_0\} \subset \mathbb{P}_0$. Let $\{A_n : n \in \omega\}$ be a partition of ω with $A_n \subset \omega \setminus n$. Let $f_0 : A_0 \overset{\text{1-1}}{\twoheadrightarrow} \mathbb{P}_0$. Note $0 \in A_0$. Let $\sigma\langle q_0\rangle = f_0(0)$. Suppose $q_1 \le \sigma\langle q_0\rangle$. Let $\mathbb{P}_1 \in \mathcal{C}$, $\mathbb{P}_1 \supset \mathbb{P}_0 \cup \{q_1\}$. Let $f_1 : A_1 \overset{\text{1-1}}{\twoheadrightarrow} \mathbb{P}_1$. Let $\alpha_1 \in \{0, 1\}$ be such that $1 \in A_{\alpha_1}$, and let $\sigma\langle q_0, q_1\rangle = f_{\alpha_1}(1)$. Continue in this way, so that given $\sigma\langle q_0, \ldots, q_n\rangle = f_{\alpha_n}(n)$ and $q_{n+1} \le \sigma\langle q_0, \ldots, q_n\rangle$, we choose $\mathbb{P}_{n+1} \in \mathcal{C}$, $\mathbb{P}_{n+1} \supset \mathbb{P}_n \cup \{q_{n+1}\}$, and $f_n : A_n \overset{\text{1-1}}{\twoheadrightarrow} \mathbb{P}_n$; then letting $\alpha_{n+1} \in n+2$ be such that $n+1 \in A_{\alpha_{n+1}}$, we let $\sigma\langle q_0, \ldots, q_{n+1}\rangle = f_{\alpha_{n+1}}(n+1)$. I thus wins this play of the game, since every element of $\bigcup_{n\in\omega} \mathbb{P}_n$ is played, $\{q_n : n \in \omega\}$ is (pre-)dense in $\bigcup_{n\in\omega}\mathbb{P}_n$, $\bigcup_{n\in\omega}\mathbb{P}_n \in \mathcal{C}$, and so $\bigcup_{n\in\omega}\mathbb{P}_n \subset_{\mathrm{c}} \mathbb{Q}$, which means $\{q_n : n\in\omega\}$ is predense in $\mathbb{Q}$. So σ is a winning strategy for I.

Now suppose that I has a winning strategy σ in $G(\mathbb{Q})$. We show that $\{\mathbb{P}\in[\mathbb{Q}]^{\le\omega} : \mathbb{P}\subset_{\mathrm{c}}\mathbb{Q}\}$ contains a club. Let $\mathcal{C} = \{\mathbb{P}\in[\mathbb{Q}]^{\le\omega}$: (a) $\forall p,q\in\mathbb{P}\,(p\perp_{\mathbb{P}} q \Leftrightarrow p\perp_{\mathbb{Q}} q)$, and (b) $\sigma''[\mathbb{P}]^{<\omega}\subset\mathbb{P}\}$.

It is easy to check that $\mathcal{C}$ is a club.

To show that for each $\mathbb{P}\in\mathcal{C}$, $\mathbb{P}\subset_{\mathrm{c}}\mathbb{Q}$, we need only check that if $W\subset\mathbb{P}$ is predense in $\mathbb{P}$, then W is predense in $\mathbb{Q}$. So suppose $\mathbb{P}\in\mathcal{C}$ and $W\subset\mathbb{P}$ is predense in $\mathbb{P}$, but not in $\mathbb{Q}$. Let $r\in\mathbb{Q}$ be such that $\forall w\in W\,(w\perp_{\mathbb{Q}} r)$. We have $\sigma\emptyset\in\mathbb{P}$. Let $w_0\in W$ be such that $\sigma\emptyset$ and w_0 are compatible in $\mathbb{P}$, and let $q_0\in\mathbb{P}$ be such that $q_0\le\sigma\emptyset, w_0$. Then $\sigma\langle q_0\rangle\in\mathbb{P}$. Let $w_1\in W$ be such that $\sigma\langle q_0\rangle$ and w_1 are compatible in $\mathbb{P}$, and let $q_1\in\mathbb{P}$ be such that $q_1\le\sigma\langle q_0\rangle, w_1$. Continue in this way. Since σ is a winning strategy for I, $\{q_n : n\in\omega\}$ must be predense in $\mathbb{Q}$. Let $n\in\omega$ be such that q_n and r are compatible. But then w_n and r are compatible, a contradiction. So W is predense in $\mathbb{Q}$. Thus $\mathcal{C}$ is the desired club. ∎

For brevity, let us call $G(X)$ a *I-favorable* (or *II-favorable) game* if I (or II) has a winning strategy. We may also say X is I-favorable (or II-favorable).

COROLLARY 1.7. *The finite support product of I-favorable partial orders is I-favorable.* (*For spaces*, *the product of I-favorable spaces is I-favorable.*)

Proof. Suppose $\mathbb{Q} = \{p\in\prod_{\alpha<\kappa}\mathbb{Q}_\alpha : |\mathrm{supt}(p)|<\omega\}$, where each $\mathbb{Q}_\alpha$ is I-favorable. For each $\alpha<\kappa$, let $\mathcal{C}_\alpha$ be a club subset of $\{\mathbb{P}\in[\mathbb{Q}_\alpha]^{\le\omega} : \mathbb{P}\subset_{\mathrm{c}}\mathbb{Q}_\alpha\}$. Let $\mathcal{C} = \{\{q\in\mathbb{Q} : \mathrm{supt}(q)\subset C$ and for each $\alpha\in\mathrm{supt}(q),\ q(\alpha)\in C_\alpha\} : C\in[\kappa]^{\le\omega}$ and for each $\alpha\in C, 1_\alpha\in C_\alpha\in\mathcal{C}_\alpha\}$. Note $\mathcal{C}\subset[\mathbb{Q}]^{\le\omega}$. Also note that given $C,D\in[\kappa]^{\le\omega}$, in order to have $\{q\in\mathbb{Q} : \mathrm{supt}(q)\subset C$ and for each $\alpha\in\mathrm{supt}(q),\ q(\alpha)\in C_\alpha\}\subset\{q\in\mathbb{Q} : \mathrm{supt}(q)\subset D$ and for each $\alpha\in\mathrm{supt}(q)$, $q(\alpha)\in D_\alpha\}$, we must have $C\subset D$ and for each $\alpha\in C$, $C_\alpha\subset D_\alpha$. Using this fact and the fact that each $\mathcal{C}_\alpha$ is a club, it is not difficult to check that $\mathcal{C}$ is a club. Using the fact that each element of a $\mathcal{C}_\alpha$ is completely embedded in $\mathbb{Q}_\alpha$, it is not difficult to check that each element of $\mathcal{C}$ is completely embedded in $\mathbb{Q}$. By Theorem 1.6, $\mathbb{Q}$ is I-favorable. ∎

Adam Krawczyk also independently discovered a characterization for I-favorable topological spaces which is essentially equivalent to ours (for the partial order use the poset of all open sets (or a base) ordered by inclusion), from which he also obtained the result of Corollary 1.7.

COROLLARY 1.8. *If X_α has countable π-weight for each $\alpha<\kappa$, then $X=\prod_{\alpha<\kappa}X_\alpha$ is I-favorable.*

Corollary 1.8 shows the difference between the game $G(X)$ on the one hand, and the games $G_{\mathrm{p}}(X)$ and $G^{\mathrm{o}}_{\mathrm{p}}(X)$ on the other: $G_{\mathrm{p}}(X)$ is not I-favorable if X is uncountable, and $G^{\mathrm{o}}_{\mathrm{p}}(X)$ is not favorable if $\pi(X)>\omega$.

We give two more results involving embeddings.

THEOREM 1.9. *If* $i : \mathbb{P} \to \mathbb{Q}$ *is a complete embedding and* $\mathbb{Q}$ *is I-favorable*, *then* $\mathbb{P}$ *is I-favorable. Also*, *if* $\mathbb{P}$ *is II-favorable*, *then* $\mathbb{Q}$ *is II-favorable.*

Proof. Assuming the hypothesis and given a winning strategy for I in $G(\mathbb{Q})$, one uses property $(3')$ of complete embeddings to define a strategy for I in $G(\mathbb{P})$; property (2) is used in showing the strategy is a winning one. A similar technique shows the second statement is true. ∎

The trivial counter-example to the converse statements is $i : \mathbb{P} = \{1\} \to \mathbb{Q}$, where $\mathbb{Q}$ is II-favorable.

DEFINITION. $i : \mathbb{P} \to \mathbb{Q}$ is a *dense embedding* if and only if

(1) $\forall p, p' \in \mathbb{P}\, (p' \le p \Rightarrow i(p') \le i(p))$,
(2) $\forall p, p' \in \mathbb{P}\, (p \perp p' \Rightarrow i(p) \perp i(p'))$,
(3) $i''\mathbb{P}$ is dense in $\mathbb{Q}$.

THEOREM 1.10. *If* $i : \mathbb{P} \to \mathbb{Q}$ *is a dense embedding*, *then* $\mathbb{Q}$ *is I-favorable iff* $\mathbb{P}$ *is I-favorable.* (See Corollary 1.4.)

We have seen that the game on a continuous, closed, irreducible image Y of a space X is equivalent to the game on X (Corollary 1.4). For certain compact spaces, we can remove the requirement of irreducibility.

THEOREM 1.11. *Every dyadic space is I-favorable.*

This theorem can be proved directly, but the details are rather messy. We give a slicker proof that involves showing that a property stronger than I-favorability, possessed by dyadic spaces, is preserved by continuous maps whose domain is compact.

DEFINITION. Let $\mathrm{op}(X)$ be the partial order of all open subsets of X. If $\mathbb{Q} \subset \mathrm{op}(X)$, define $\mathbb{Q} \subset_! \mathrm{op}(X)$ by requiring that conditions (1) and (2) in the definition of $\subset_{\mathrm{c}}$ hold, together with the following strengthening of $(3')$:

(3!) whenever $S \subset \mathbb{Q}$ and $x \notin \overline{\bigcup S}$, there is a $W \in \mathbb{Q}$ such that $x \in W$ and $W \cap \bigcup S = \emptyset$.

$\mathbb{Q} \subset_! \mathrm{op}(X)$ implies $\mathbb{Q} \subset_{\mathrm{c}} \mathrm{op}(X)$.

DEFINITION. X is *very I-favorable* provided that there is a club $\mathcal{C} \subset [\mathrm{op}(X)]^{\le\omega}$ such that for each $C \in \mathcal{C}$, $C \subset_! \mathrm{op}(X)$.

We need to use such a club in the domain space to build one in the range space. To accomplish this, the following lemma proves useful.

LEMMA 1.12. *Suppose* $\mathcal{C} \subset [I]^{\le\omega}$ *is a club and* $J \subset I$. *Then there is a club* $\mathcal{D} \subset [J]^{\le\omega}$ *such that for each* $D \in \mathcal{D}$ *there is a* $C \in \mathcal{C}$ *such that* $D = C \cap J$.

P r o o f. Note that $\{C \cap J : C \in \mathcal{C}\}$ will not necessarily work: it may not be closed. For each $s \in [J]^{<\omega}$, choose $f(s) \in \mathcal{C}$ such that $s \subset f(s)$, and such that $t \subset s$ implies $f(t) \subset f(s)$. Then $\mathcal{D} = \{D \in [J]^{\le\omega} : \forall s \in [D]^{<\omega}\, (f(s) \cap J \subset D)\}$ works. ■

Any 2^κ is very I-favorable: let $\mathcal{C}$ be the collection of $C \in [\mathrm{op}(X)]^{\le\omega}$ for which there is an $I \in [\kappa]^{\le\omega}$ such that

(1) $\{[\sigma] : \sigma \in \mathrm{Fn}(I,2)\} \subset C$;
(2) for each $U \in C, \{[\sigma] : \sigma \in \mathrm{Fn}(I,2)$ and $[\sigma] \subset U\}$ is dense in U; and
(3) C is closed under finite intersections.

Then $\mathcal{C}$ witnesses that 2^κ is very I-favorable.

With the above fact, to prove Theorem 1.11 we need to prove that *if* X *is compact and very I-favorable, and* $f : X \twoheadrightarrow Y$ *is continuous, then* Y *is very I-favorable.*

P r o o f. Suppose X is compact and very I-favorable, and $f : X \twoheadrightarrow Y$ is continuous. Let $\mathcal{C} \subset [\mathrm{op}(X)]^{\le\omega}$ witness the fact that X is very I-favorable. By passing to a subcollection, we may assume without loss of generality that for each $C \in \mathcal{C}$, C is closed under finite unions and intersections, and that $f^{-1}(\{y \in Y : f^{-1}(y) \subset U\}) \in C$ whenever $U \in C$. (Since f is a closed map, for an open set U, $\{y \in Y : f^{-1}(y) \subset U\}$ is open.)

Let $\mathcal{J} = \{f^{-1}(U) : U^{\mathrm{open}} \subset Y\} \subset \mathrm{op}(X)$. By Lemma 1.12, let $\mathcal{D}$ be a club, $\mathcal{D} \subset [\mathcal{J}]^{\le\omega}$, such that for each $D \in \mathcal{D}$ there is a $C \in \mathcal{C}$ such that $D = C \cap \mathcal{J}$. For $D \in \mathcal{D}$, let $D_Y = \{U \in \mathrm{op}(Y) : f^{-1}(U) \in D\}$. It is not difficult to check that $\mathcal{D}_Y = \{D_Y : D \in \mathcal{D}\}$ is a club in $[\mathrm{op}(Y)]^{\le\omega}$, and, using the fact that f is perfect (continuous, closed, and inverse images of points are compact), that $\mathcal{D}_Y$ witnesses that Y is very I-favorable. ■

QUESTION 1.13. If X is compact and is I-favorable, is X co-absolute with a dyadic space?

If the weight of X is $\le \aleph_1$, then X *is* co-absolute with a dyadic space.

Corollary 1.7 takes care of Cohen forcing. Another partial order in which I has a winning strategy is the order for adding an increasing κ-sequence of functions from ω into the rationals.

EXAMPLE 1.14. Let κ be uncountable and let $\mathbb{P}_\kappa$ be the partial order where elements are of the form $p = \langle a_p, n_p, f_p\rangle$, where $a_p \in [\kappa]^{<\omega}$, $n_p \in \omega$, and $f_p : a_p \times n_p \to$ (the rationals). Define $p \le q$ iff $a_p \supset a_q$, $n_p \ge n_q$, $f_p \supset f_q$, and for each $\alpha, \beta \in a_q$ and $i \in n_p \setminus n_q$, if $\alpha < \beta$ then $f_p(\alpha, i) < f_p(\beta, i)$.

We claim that $\mathcal{C} = \{\mathbb{P}_A : A \in [\kappa]^{\le\omega}\}$ is a club subset of $\{\mathbb{Q} \in [\mathbb{P}_\kappa]^{\le\omega} : \mathbb{Q} \subset_{\mathrm{c}} \mathbb{P}_\kappa\}$. Suppose $A \in [\kappa]^{\le\omega}$. We show $\mathbb{P}_A \subset_{\mathrm{c}} \mathbb{P}_\kappa$. Conditions (1) and (2) of the definition of $\subset_{\mathrm{c}}$ are easy to check. We show $(3')$ holds. Suppose $q \in \mathbb{P}_\kappa$. Let $a_q = \{\beta_0, \ldots, \beta_l\}$ be listed in increasing order. Let $p = \langle a_q \cap$

$A, n_q, f_q|(a_q \cap A) \times n_q\rangle$. Note $p = \langle a_p, n_p, f_p\rangle \in \mathbb{P}_A$. Suppose $p' \in \mathbb{P}_A$ and $p' \le p$. We need to show that p' and q are compatible. Let $a_t = a_{p'} \cup a_q$ and $n_t = n_{p'}$. Now if $(\beta, i) \in \operatorname{dom} f_{p'} \cap \operatorname{dom} f_q$, then $(\beta, i) \in (a_q \cap A) \times n_q$, so $f_{p'}(\beta, i) = f_p(\beta, i) = f_q(\beta, i)$. We need to define $f_t : a_t \times n_t \to$ (the rationals) to extend $f_{p'} \cup f_q$ in such a way that $t \le p', q$.

For $\beta \in a_q \setminus A$, let β' be the greatest element of $a_q \cap A$ less than β, if possible, and let β'' be the least element of $a_q \cap A$ greater than β, if possible. Let $f_{p'}(\beta', i) < f_t(\beta, i) < f_{p'}(\beta'', i)$. This completes the definition of f_t, since if $(\beta, i) \in \operatorname{dom} f_t \setminus (\operatorname{dom} f_{p'} \cup f_q)$, then $\beta \in a_q$ and $i \in n_t \setminus n_q = n_{p'} \setminus n_q$; $\beta \notin a_{p'}$, so $\beta \notin A$, and by the above, we have defined $f_t(\beta, i)$. Now, t trivially extends p' since there is no $i \in n_t \setminus n_{p'}$. To see that t extends q, suppose $\varrho, \psi \in a_q$, $i \in n_t \setminus n_q = n_{p'} \setminus n_q$, and $\varrho < \psi$. If $\varrho, \psi \in A$, then $\varrho, \psi \in a_p$, and since p' extends p, $(f_t(\varrho, i) =) f_{p'}(\varrho, i) < f_{p'}(\psi, i) (= f_t(\psi, i))$. If $\varrho \in A$ and $\psi \notin A$, then $\varrho < \psi'$, and $(f_t(\varrho, i) =) f_{p'}(\varrho, i) \le f_{p'}(\psi', i) \le f_t(\psi, i)$. The remaining cases are similar.

$\mathcal{C}$ is closed since if $\mathbb{P}_{A_0} \subset \mathbb{P}_{A_1} \subset \ldots$, then $A_0 \subset A_1 \subset \ldots$, and $\bigcup_n \mathbb{P}_{A_n} = \mathbb{P}_{\bigcup_n A_n}$. Clearly $\mathcal{C}$ is unbounded.

In the above example, if $\kappa \ge \mathfrak{c}^+$, then Cohen forcing does not add a κ-sequence in ω^ω which is increasing in $<^*$. So $\mathbb{P}_\kappa$ is not a complete suborder of Cohen forcing.

2. When the players do not have a winning strategy. Since the open-open game on any completely regular space has the same behavior as the game on its compactifications, we just consider the games played on compact spaces.

THEOREM 2.1. *If in a compact* (*or even Baire*) *space X every dense set is separable*, *then II does not have a winning strategy.*

Proof. Suppose σ is a winning strategy for II. In the open family $\{\sigma\langle C\rangle : C$ is open in $X\}$, take a maximal disjoint subfamily $\mathcal{C}_\emptyset$. Then $\bigcup \mathcal{C}_\emptyset$ is dense in X, hence separable, and so $\mathcal{C}_\emptyset$ is countable.

Let $\mathcal{C}_\emptyset = \{\sigma\langle C_n\rangle : n \in \omega\}$. Similarly, for each $n \in \omega$ a maximal disjoint subfamily of $\{\sigma\langle C_n, C\rangle : C$ is open in $X\}$ has dense union in X, and hence is countable: let $\mathcal{C}_{\langle n\rangle} = \{\sigma\langle C_n, C_{nm}\rangle : m \in \omega\}$ be such a subfamily. Continue in this way, so that for each $n \in \omega$ and $s \in {}^n\omega$, $\mathcal{C}_s = \{\sigma\langle C_s|_1, C_s|_2, \ldots, C_s, C_{s^\frown\langle m\rangle}\rangle : m \in \omega\}$ is a maximal subfamily of $\{\sigma\langle C_s|_1, C_s|_2, \ldots, C_s, C\rangle : C$ is open in $X\}$ whose union is dense in X. Since X is Baire, $\bigcap_{s \in {}^{<\omega}\omega} \bigcup \mathcal{C}_s$ is dense in X, hence separable; let $\{d_n : n \in \omega\}$ be a countable dense subset of $\bigcap_{s \in {}^{<\omega}\omega} \bigcup \mathcal{C}_s$. Let $n_0 \in \omega$ be such that $d_0 \in \sigma\langle C_{n_0}\rangle$; in general, let $n_m \in \omega$ be such that $d_m \in \sigma\langle C_{n_0}, C_{n_0 n_1}, \ldots, C_{n_0 n_1 \ldots n_m}\rangle$. But this defines a play of the game according to the strategy σ that II does not win. Hence II has no winning strategy. ∎

We have the following implications for any compact space X:

$$
\begin{tikzcd}
 & & \text{every dense subset is separable} \arrow[dr, shift right] \arrow[d, shift right, "(?)"'] & \\
\text{hereditarily separable} \arrow[r] & \text{countable } \pi\text{-weight} \arrow[ur] \arrow[dr, shift right] & & \text{II has no w.s.} \arrow[ul, shift right, "/" marking] \arrow[dl, shift right, "/" marking] \arrow[d, shift right] \\
 & & \text{I has a w.s.} \arrow[ul, shift right, "/" marking] \arrow[u, shift right, "/" marking] \arrow[ur, shift right] & \text{ccc} \arrow[u, shift right, "/" marking]
\end{tikzcd}
$$

Assuming MA, we have the following:

Theorem 2.2. *(MA) If (X,τ) is ccc and $\pi(X) < \mathfrak{c}$, then II does not have a winning strategy.*

Proof. Like the last proof, we build countably many maximal disjoint families $\mathcal{C}_t$. Let $\mathcal{B}$ be a π-base with $|\mathcal{B}| < \mathfrak{c}$. For each function f such that $\operatorname{dom} f \in \omega$ and $\operatorname{rng} f \subset \tau$, let $\mathcal{C}_{f|i}$ be a countable subset of τ such that $\{\sigma\langle f(0),\ldots,f(i-1),C\rangle : C \in \mathcal{C}_{f|i}\}$ has dense union in X. Now define a poset $P = \{f : \operatorname{dom} f \in \omega$, and for each $i \in \operatorname{dom} f, f(i) \in \mathcal{C}_{f|i}\}$. The order is reverse inclusion. For each $B \in \mathcal{B}$, the set $D_B = \{f :$ there exists $i \in \operatorname{dom} f$ such that $B \cap \sigma\langle f(0),\ldots,f(i)\rangle \neq \emptyset\}$ is a dense subset of P. Since $|\mathcal{B}| < \mathfrak{c}$, there is a generic function h with dom $h = \omega$ and for any $B \in \mathcal{B}$, there is an n such that $\sigma\langle h(0),\ldots,h(n)\rangle \cap B \neq \emptyset$. But then player I wins the play $h(0), \sigma\langle h(0)\rangle, h(1), \sigma\langle h(0),h(1)\rangle, \ldots,$ since $\bigcup_{n<\omega} h(n)$ is dense in X, contradicting the supposition that σ is a winning strategy for II. ∎

Since the poset used in the above proof is countable, in fact the assertion in Theorem 2.2 follows from the statement $B(\mathfrak{c})$: in the reals $\mathbb{R}$, any union of less than $\mathfrak{c}$ many meager sets is meager. Recall that $U(\mathfrak{m})$ is the statement that any set of size $< \mathfrak{c}$ has measure zero; and $\mathfrak{d} = \mathfrak{c}$ is the statement that any dominating family in ${}^{\omega}\omega$ must have size $\mathfrak{c}$. If we denote the statement "every ccc space X with $\pi(X) < \mathfrak{c}$ is not II-favorable" by $NF(\mathfrak{c})$, the following is true.

Theorem 2.3.

$$
\begin{tikzcd}
B(\mathfrak{c}) \arrow[r] & NF(\mathfrak{c}) \arrow[r, shift right, "/" marking] \arrow[dr, shift right] & U(\mathfrak{m}) \arrow[l, shift right] \\
 & & \mathfrak{d} = \mathfrak{c} \arrow[ul, shift right, "/" marking]
\end{tikzcd}
$$

Proof. The implications follow from Theorems 2.2 and 1.2. (See the Corollary of 1.2 and the comment on dominating families following it for $NF(\mathfrak{c}) \Rightarrow \mathfrak{d} = \mathfrak{c}$.) The example where $U(\mathfrak{m})$ is true but $NF(\mathfrak{c})$ is false is the model obtained by adding ω_2 infinitely equal reals to a model of CH (see Miller [M]). $NF(\mathfrak{c})$ is false because $\mathfrak{d} = \omega_1$. The second example is the model obtained by adding ω_1 random reals to a model of $\mathfrak{c} = \omega_2$, where $\mathfrak{d} = \mathfrak{c}$ is true but $U(\mathfrak{m})$ is false. ∎

QUESTION 2.4. Does $NF(\mathfrak{c})$ imply $B(\mathfrak{c})$?

As we know, there are not many techniques to show certain spaces are not II-favorable. The following might be interesting.

LEMMA 2.5. *Suppose $\mathbb{P}$ is a notion of forcing with precaliber ω_1 (i.e., for any uncountable subset $Q \subset \mathbb{P}$, there is an uncountable centered $Q' \subset Q$), and $\mathbb{P}$ adds a Cohen real. Then for any ccc space X in the ground model M and any winning strategy σ of player II in M, σ is no longer a winning strategy in $M^{\mathbb{P}}$.*

The argument is essentially the same as the one presented in Theorem 2.2. Since $\mathbb{P}$ has precaliber ω_1, X remains ccc in the extension. The Cohen real gives the play for I which defeats II's strategy σ.

3. Games on trees. In the game on trees, in each round player II chooses a node $\geq$ I's choice, and I wins if and only if every node of the tree is comparable to some choice of II's.

THEOREM 3.1. *If T is an ω_1-tree, then I has a winning strategy iff T is the union of countably many branches.*

Proof. It suffices to prove the "only if" part. Suppose I has a winning strategy but T is not the union of countably many branches.

Let σ be a winning strategy of I. I plays $\sigma\emptyset$ to start the game. If there is only one branch through $\sigma\emptyset$, let $R_0 = \{r : r \text{ is comparable to } \sigma\emptyset\}$ and let $S_0 = \{\sigma\emptyset\}$. Otherwise, let $\alpha_0 = \min\{\alpha : |\{r : r \text{ is a successor of } \sigma\emptyset \text{ and } \mathrm{ht}(r) = \alpha\}| > 1\}$, let $R_0 = \{r : r \text{ is a successor of } \sigma\emptyset \text{ and } \mathrm{ht}(r) = \alpha_0\}$. Let $S_0 = T_{a_0+1}$, where T_{α_0+1} is all $r \in T$ with $\mathrm{ht}(r) \leq \alpha_0$. In both cases, S_0 is countable.

For $n \in \omega$, and $r_m \in S_m$ for each $m \leq n$, we proceed similarly. If there is only one branch through $\sigma\langle r_m : m \leq n\rangle$, let $R_{\langle r_m : m\leq n\rangle} = \{s : s \text{ is comparable to } \sigma\langle r_m : m \leq n\rangle\} = S_{\langle r_m : m \leq n\rangle}$. Otherwise, let $\alpha_{\langle r_m : m \leq n\rangle} = \min\{\alpha : |\{s : s \text{ is a successor of } \sigma\langle r_m : m \leq n\rangle \text{ and } \mathrm{ht}(s) = \alpha\}| > 1\}$, let $R_{\langle r_m : m \leq n\rangle} = \{s : s \text{ is a successor of } \sigma\langle r_m : m \leq n\rangle \text{ and } \mathrm{ht}(s) = \alpha_{\langle r_m : m \leq n\rangle}\}$, and let $S_{\langle r_m : m \leq n\rangle} = T_{\alpha_{\langle r_m : m \leq n\rangle}+1}$. Let $R_{n+1} = \bigcup\{R_{\langle r_m ; m \leq n\rangle}\text{: for each } m \leq n,\ r_m \in R_m\}$ and $S_{n+1} = \bigcup\{S_{\langle r_m : m \leq n\rangle}\text{: for } m \leq n, r_m \in R_m\}$. Then S_{n+1} is countable.

Since $\bigcup_{n<\omega} S_n$ is countable and T is not the union of countably many branches, we may let $t \in T - (\bigcup_{n<\omega} S_n \cup \bigcup_{n<\omega} R_n)$. Consider the following play of the game. I starts by playing $\sigma\emptyset$. If there is only one branch through $\sigma\emptyset$, II plays $t_0 = \sigma\emptyset \in S_0$; since $t \notin R_0$, t is not comparable to t_0. Otherwise, since $t \notin T_{\alpha_0+1}$, there is a $t_0 \in R_0 \subset S_0$ that is not comparable to t; let II play such a t_0. In both cases $t_0 \in S_0$. I must now play $\sigma\langle t_0\rangle$. If there is only one branch through $\sigma\langle t_0\rangle$, II plays $t_1 = \sigma\langle t_0\rangle \in S_1$; since $t \notin R_1$, t

is not comparable to t_1. Otherwise, II plays a $t_1 \in R_{\langle t_0 \rangle} \subseteq S_1$ that is not comparable to t. Clearly, II can continue to play the game by choosing a node that is not comparable to t, thus II defeats I's strategy σ. ∎

One can construct a Suslin tree from a ccc, nonseparable, linearly ordered topological space in such a way that I wins in the space if and only if I wins in the tree, and hence if X is a linearly ordered space, I has a winning strategy in $G(X)$ if and only if X is separable.

From Theorem 3.1 we can see that the game on any ever-branching $\aleph_1$-tree or on any $\aleph_1$-Aronszajn tree is not I-favorable. The games on Suslin trees are not II-favorable. Here we present a proof which leads to more information concerning the game on trees. Let us call σ a *stationary winning strategy* if σ depends only upon the last move of the opponent.

THEOREM 3.2. *X is ccc iff II has no stationary winning strategy.*

Proof. $\Rightarrow$ Suppose X is ccc and σ is a stationary winning strategy for II. Let $\{\sigma(U_n) : n < \omega\}$ be a maximal pairwise disjoint subcollection of $\{\sigma(U) : U \text{ is open in } X\}$. Then II does not win the play $U_0, \sigma(U_0), U_1, \sigma(U_1), \ldots$, and so σ is not a winning strategy.

$\Leftarrow$ See Theorem 1.1(ii). ∎

The following lemma developed from Theorem 3.2, and is useful in unifying some proofs of results that follow.

LEMMA 3.3. *Suppose $\mathbb{P}$ is a ccc poset. Suppose that for each countable collection $\{C_n : n \in \omega\}$ of maximal antichains in $\mathbb{P}$ there is a countable $Q \subset \mathbb{P}$ with the following property: for each $p \in \mathbb{P}$ there is a $q \in Q$ such that for each $r \in \bigcup_n C_n$, if r is compatible with q, then r is compatible with p. Then II has no winning strategy.*

Proof. Suppose $\mathbb{P}$ is as above, and σ is a winning strategy for II. Let $A_\emptyset = \{\sigma(p_n) : n \in \omega\}$ be a maximal antichain of $\{\sigma(p) : p \in \mathbb{P}\}$; note $\{\sigma(p_n) : n \in \omega\}$ is a maximal antichain of $\mathbb{P}$. Similarly, for each $n \in \omega$, $\{\sigma(p_n, p) : p \in \mathbb{P}\}$ contains a countable maximal antichain, say $A_{\langle n \rangle} = \{\sigma(p_n, p_{n,m}) : m \in \omega\}$. Continue in this way, so that given $s \in {}^{<\omega}\omega$, $A_s = \{\sigma(p_{s|1}, p_{s|2}, \ldots, p_s, p_{s^\frown\langle m \rangle}) : m \in \omega\}$ is a maximal antichain. Let $\mathbb{Q}$ be as in the hypothesis for $\{A_s : s \in {}^{<\omega}\omega\} = \mathcal{A}$. Let $\mathbb{Q} = \{q_n : n \in \omega\}$. Let $n_0 \in \omega$ be such that $\sigma(p_{n_0})$ is compatible with q_0; let $s_0 = \langle n_0 \rangle$. In general, given $s_m = \langle n_0, \ldots, n_m \rangle$, let $n_{m+1} \in \omega$ be such that $\sigma(p_{s_0}, \ldots, p_{s_m}, p_{s_m{}^\frown\langle n_{m+1} \rangle})$ is compatible with q_{m+1}. Since II wins the play $p_{s_0}, \sigma(p_{s_0}), p_{s_1}, \sigma(p_{s_0}, p_{s_1}), \ldots$, let $p \in \mathbb{P}$ be incompatible with each $\sigma(p_{s_0}, \ldots, p_{s_m})$. We may let $q \in \mathbb{Q}$ be such that for each $r \in \bigcup \mathcal{A}$, if r is compatible with q, then r is compatible with p. As there is an m with $\sigma(p_{s_0}, \ldots, p_{s_m}) \in \bigcup \mathcal{A}$ compatible with q, it is also compatible with p, a contradiction. ∎

Theorem 3.4. *If T is an ω_1-tree, then II has a winning strategy in $G(T)$ iff T has an uncountable antichain.*

Proof. Suppose T is an ω_1-tree without an uncountable antichain. So $(T, \geq) = \mathbb{P}$ is ccc. We show $\mathbb{P}$ satisfies the hypothesis of Lemma 3.3. Suppose $\mathcal{A} = \{\mathcal{C}_n : n \in \omega\}$ is a collection of maximal antichains in $\mathbb{P}$. Let $\alpha < \omega_1$ be such that $\bigcup \mathcal{A} \subset T_\alpha$. Let $Q = T_{\alpha+1}$. Suppose $p \in \mathbb{P}$. If $p \in Q$, let $q = p$. If $p \notin Q$, let q be the predecessor of p of height α. Clearly if $r \in \bigcup \mathcal{A} \subset T_\alpha$ is compatible with q, then r is compatible with p. From the lemma, II has no winning strategy. ∎

Corollary 3.5. *If T is an ever-branching ω_1-tree, i.e., for all $t \in T$, $\{s \in T : s > t\}$ is not totally ordered, then II has no winning strategy iff T is Suslin.*

By extending the ideas of the above proof, we can construct an undetermined game, i.e., a game in which neither I nor II has a winning strategy.

Example 3.6. We simply state the example as a game played on a poset, since it is easy to translate it into games on a Boolean algebra or a topological space. Let T be an *Aronszajn tree*, i.e., an ω_1-tree in which all chains (branches) are countable. Let $\mathbb{P}(T) = \{A \subseteq T : A \text{ is a finite antichain}\}$, ordered by $\supseteq$. Then $\mathbb{P}(T)$ is the partial order for "specializing" T and is well known to be ccc (see [T]).

We show $\mathbb{P}(T)$ satisfies the hypothesis of Lemma 3.3, and so II has no winning strategy. Suppose $\mathcal{A} = \{\mathcal{C}_n : n \in \omega\}$ is a collection of maximal antichains in $\mathbb{P}(T)$. Let $\alpha < \omega_1$ be such that $\bigcup \mathcal{A} \subset \mathbb{P}(T_\alpha)$. Let $Q = \mathbb{P}(T_{\alpha+1})$. Suppose $A \in \mathbb{P}(T)$. If $A \in Q$, then A itself has the desired property. If $A \notin Q$, proceed as follows. For each $p \in A \setminus T_{\alpha+1}$, let t_p be the predecessor of p of height α. Let $B = [A \cap T_{\alpha+1}] \cup \{t_p : p \in A \setminus T_{\alpha+1}\}$. Then $B \in \mathbb{P}(T_{\alpha+1}) = Q$, and for $R \in \bigcup \mathcal{A}$, if R is compatible with B, then since $R \cup B$ is an antichain and each element of R has height $< \alpha$, it follows that $R \cup A$ is an antichain, and hence R is compatible with A.

Now, suppose I has a winning strategy σ. Let $\alpha_0 < \omega_1$ be such that $\sigma\emptyset \subset T_{\alpha_0}$, and $S_0 = \{\sigma(B) : B \in \mathbb{P}(T_{\alpha_0+1})\}$. Let $\alpha_1 < \omega_1$ be such that $\alpha_0 < \alpha_1$ and $\bigcup S_0 \subset T_{\alpha_1}$, and let $S_1 = \{\sigma(B_0, B_1) : B_0 \in \mathbb{P}(T_{a_0+1}), B_1 \in \mathbb{P}(T_{a_1+1})\}$. In general, let $\alpha_n < \alpha_{n+1} < \omega_1$ be such that $\bigcup S_n \subset T_{\alpha_{n+1}}$ and $S_{n+1} = \{\sigma(B_0, B_1, \ldots, B_{n+1})$: for each $m \leq n+1$, $B_m \in \mathbb{P}(T_{\alpha_m+1})\}$. Let $\alpha = \sup_n \alpha_n$, and let $t \in T - T_\alpha$. We now show that there is a play of the game in which $\{t\}$ is incompatible with each choice of II, which means σ is not a winning strategy. Note that for $\{t\}$ to be incompatible with $A \in P(T)$ means that t is comparable with, but not equal to, some element of A.

I must first play $\sigma\emptyset$. If t is comparable to an element of $\sigma\emptyset$, II plays $B_0 = \sigma\emptyset$. Otherwise, let t_0 be the predecessor of t with $\operatorname{ht}(t_0) = \alpha_0$; let II play $B_0 = \sigma\emptyset \cup \{t_0\}$, an antichain in T_{α_0+1}. So $\{t\}$ is compatible with B_0. I must now play $\sigma(B_0) \subset T_{\alpha_1}$. If t is comparable to an element of $\sigma(B_0)$, II plays $B_1 = \sigma(B_0)$. Otherwise let t_1 be the predecessor of t with $\mathrm{ht}(t_1) = \alpha_1$; let II play $B_1 = \sigma(B_0) \cup \{t_1\}$, an antichain in T_{α_1+1}. So $\{t\}$ is incompatible with B_1. I must now play $\sigma(B_0, B_1) \subset T_{\alpha_2}$. Clearly, II can continue in such a way as to defeat I's strategy σ. ■

By extending some of the techniques presented so far, we may obtain results for (ccc) Boolean algebras.

A Boolean algebra B is called $\aleph_0$-*distributive* if for any sequence of partitions (i.e., maximal antichains) W_i $(i < \omega)$ there is a common refinement W, i.e., a maximal antichain W such that for each $i \in \omega$ and each $a \in W$ there is a $b \in W_i$ such that $a \le b$. For complete Boolean algebras, the above is just one of its equivalent definitions. An algebra B is called *weakly* $\aleph_0$-*distributive* iff every $f : \omega \to \omega$ in $\mathcal{M}[G]$ is majorized by some $g : \omega \to \omega$ that is in $\mathcal{M}$ (i.e., $f(n) < g(n)$ for all $n < \omega$). It is easy to see that every uncountable, separative, σ-closed algebra is II-favorable. But for $\aleph_0$-distributive algebras, we have:

THEOREM 3.7. *For any ccc* $\aleph_0$-*distributive algebra* B, *II has no winning strategy.*

Proof. Such an algebra satisfies the hypothesis of Lemma 3.3. For a $b \in B$, we take $q \in W$ such that b and q are compatible. Suppose $r \in \bigcup_n W_n$ is compatible with q. Let n be such that $r \in W_n$. Since W is a refinement of W_n, we must have $q \le r$. But then clearly r and b are compatible. ■

LEMMA 3.8. *Let* $[b]$ *denote the algebra generated by* $b \in B$ (*i.e.*, $[b] = \{u \in B : u \le b\}, 1_{[b]} = b, 0_{[b]} = 0$, *and for each* $c \in [b], -c = b - c$). *Then* $G(B)$ *is I-favorable iff* $G([b])$ *is I-favorable for all* $b \in B - \{0\}$.

Proof. $\Leftarrow$ Trivial: let $b = 1$.

$\Rightarrow$ Suppose $G(B)$ is I-favorable and let σ be a w.s. for I in $G(B)$. Suppose $b \in B - \{0\}$. Let $B - \{0\} = \{b_\alpha : \alpha < \kappa\}$. A winning strategy σ' for I in $G([b])$ is obtained as follows. Let $\sigma'\emptyset = \sigma\emptyset \wedge b$ if $\sigma\emptyset \wedge b \ne 0$. Otherwise, let $\alpha_0 = \min\{\alpha : b_\alpha \le \sigma\emptyset\}$, and let $\sigma'\emptyset = \sigma\langle b_{\alpha_0}\rangle \wedge b$ if $\sigma\langle b_{\alpha_0}\rangle \wedge b \ne 0$. Otherwise, let $\alpha_1 = \min\{\alpha : b_\alpha \le \sigma\langle b_{\alpha_0}\rangle\}$, and let $\sigma'\emptyset = \sigma\langle b_{\alpha_0}, b_{\alpha_1}\rangle \wedge b$ if $\sigma\langle b_{\alpha_0}, b_{\alpha_1}\rangle \wedge b \ne \emptyset$.

Otherwise define α_2 similarly. Since σ is a w.s., this procedure is finite, i.e., there is an n_0 such that $\sigma\langle b_{\alpha_0}, \ldots, b_{\alpha_{n_0}}\rangle \wedge b \ne \emptyset$, and so $\sigma'\emptyset = \sigma\langle b_{\alpha_0}, \ldots, b_{\alpha_{n_0}}\rangle \wedge b$. Now suppose II's first move in $G([b])$ is c_0. Let $c_0 = b_{\alpha_{n_0+1}}$. Let $\sigma'\langle c_0\rangle = \sigma\langle b_{\alpha_0}, \ldots, b_{\alpha_{n_0+1}}\rangle \wedge b$ if $\sigma\langle b_{\alpha_0}, \ldots, b_{\alpha_{n_0+1}}\rangle \wedge b \ne \emptyset$. Otherwise let $\alpha_{n_0+2} = \min\{\alpha : b_\alpha \le \sigma\langle b_{\alpha_0}, \ldots, b_{\alpha_{n_0+1}}\rangle\}$ and proceed as above. Continuing in this way, the σ' so defined is a w.s. for I in $G([b])$.

THEOREM 3.9. *For any ccc weakly $\aleph_0$-distributive B, I has a winning strategy iff the atoms are dense. Thus, every atomless weakly $\aleph_0$-distributive algebra B is not I-favorable.*

Proof. $\Leftarrow$ is obvious, since the atoms then form a maximal (countable) antichain. So we only need to show that if there is an $a \in B$ such that no atom b is smaller than a, then I has no winning strategy. By Lemma 3.8, it suffices to show I has no winning strategy in the game $G([a])$. Without loss of generality, assume B has no atoms. Suppose B is I-favorable. Then $\{\mathbb{P} \in [B]^{\leq\omega} : \mathbb{P} \subset_{\mathrm{c}} B\}$ contains a club $\mathcal{C}$. We show $\mathcal{C}$ contains a nonatomic p.o. Let $\mathbb{P}_0 \in \mathcal{C}$. Since B is atomless, for each $p \in B$, let $a_p \in B$ be such that $0 < a_p < p$. Let $b_p = p - a_p \neq 0$. Let $\mathbb{P}_1 \in \mathcal{C}$ be such that $\mathbb{P}_0 \cup \{a_p : p \in \mathbb{P}_0\} \cup \{b_p : p \in \mathbb{P}_0\} \subset \mathbb{P}_1$. Given $\mathbb{P}_n$, let $\mathbb{P}_{n+1} \in \mathcal{C}$ be such that $\mathbb{P}_n \cup \{a_p : p \in \mathbb{P}_n\} \cup \{b_p : p \in \mathbb{P}_n\} \subset \mathbb{P}_{n+1}$. Let $\mathbb{P} = \bigcup_{n\in\omega} \mathbb{P}_n \in \mathcal{C}$. Then $\mathbb{P}$ is a nonatomic p.o., that is, for each $p \in \mathbb{P}$ there are $q, r \in \mathbb{P}$ (namely a_p and b_p) such that $q \leq p$, $r \leq p$, and $q \perp r$. As $\mathbb{P}$ is countable and nonatomic, $\mathbb{P}$ yields the same generic extension as $\mathrm{Fn}(\omega, 2)$, the partial order for adding a Cohen real (see [K], exercise VII C4). A standard density argument shows the added real is not majorized by any ground model real. Since $\mathbb{P}$ is completely embedded in B, B also adds this nonmajorized real, so B cannot be weakly $\aleph_{\mathrm{o}}$-distributive, a contradiction. So B is not I-favorable.

Von Neumann asked [Ma] whether every ccc weakly $\aleph_0$-distributive complete algebra is a measure algebra. Since every measure algebra is II-favorable, we have the following "weak" question.

QUESTION 3.10. Is it consistent that every ccc weakly $\aleph_0$-distributive algebra is II-favorable?

A *Suslin algebra* is a ccc, $\aleph_0$-distributive, atomless complete Boolean algebra. By Theorems 3.7 and 3.9, a Suslin algebra is neither I-favorable nor II-favorable. In other words, the game played on a Suslin algebra is undetermined.

4. More games. Consider the following game G_1 on a Boolean algebra B. Player I must first choose $0 \in B$, then II chooses a partition (maximal antichain) C_0. As a response, I chooses a member a_1 in C_0, then the next move of II is a partition C_1 again. Every time, I chooses a member $a_{n+1} \in C_n$. I wins iff $\sum_n a_n = 1$. Let G_2 and G_3 be similarly defined games, where in G_2 player II always chooses a dense (cofinal) set and in G_3 player II always chooses a predense set and I still chooses a member of II's last choice.

THEOREM 4.1. *G is equivalent to G_i $(i = 1, 2, 3)$.*

Proof. First we show G is equivalent to G_1. The proofs for G_2, G_3 are similar. Let σ_1 be a winning strategy for II_1, i.e., the second player of the game G_1. Now define a winning strategy σ for II. For I's first move a_0 in the game G, in the partition C_0, which is decided by σ_1, take a member $c_0 \in C_0$ with $a_0 \wedge c_0 = b_0 \neq 0$. Let $C_1 = \sigma_1(0, C_0, c_0)$. In the next round, as a response to I's move a_1, II chooses a member $c_1 \in C_1$ with $c_1 \wedge a_1 = b_1 \neq 0$. Clearly, since $\sum_n c_n \neq 1$, $\sum_n b_n$ cannot be 1. On the other hand, let σ be a winning strategy of II; we are going to define a winning strategy σ_1 for II_1, as follows. Let $B_0 = \{b : \text{there is an } a \text{ such that } b = \sigma(a)\}$. Since B_0 is dense in B, take a partition $C_0 \subset B_0$. Then C_0 will be $\sigma_1(0)$, the first move of II_1. For the next move $a_1 \in C_0$ of player I, in the game G_1, let the first move of I in the game G be a_0', where $a_1 = \sigma(a_0')$ and define $b_0' = a_1$. Consider the dense set $B_1 = \{b \in B : \text{there is an } a \text{ such that } b = \sigma(a_0', b_0', a)\}$. Take a partition $C_1 \subset B_1$. Continue the play as above. Finally, if $\sum_n b_n' \neq 1$, then $\sum_n a_n \neq 1$ since $b_n' = a_{n+1}$.

Now suppose τ_1 is a winning strategy of I_1, the first player in the game G_1. We claim that there is an $a_0 \in B$ such that for all $b \leq a_0$, there is a partition W such that $b = \tau_1(0, W)$. Otherwise, $B_0 = \{b : \text{there is no partition with } W \text{ such that } b = \tau_1(0, W)\}$ is dense. Take a partition $C_0 \subset B_0$. Then $\tau_1(0, C_0) \in B_0$ is a contradiction. Let such an a_0 be the first move of I in the game G. For any choice $b_0 \leq a_0$ of II, let W_0 be the corresponding partition with $b_0 = \tau_1(0, W_0)$. Now, by the same argument, there is an $a_1 \in B$ such that for any $b \leq a_1$, there is a partition W with $b = \tau_1(0, W_0, b_0, W)$. Let a_1 be the next move of I. Clearly, $\sum b_n = 1$ by the fact that τ_1 is a winning strategy of I_1.

Now, assume τ is a winning strategy for I. For any partition W_0, let $a_0' \in W_0$ such that $a_0' \wedge a_0 = b_0 \neq 0$ where $a_0 = \tau(\emptyset)$ is I's first move. Continue to choose such a_n' for each partition W_n so that $a_n' \wedge a_n = b_n \neq 0$ where $a_n = \tau(a_0, b_0, \ldots, a_{n-1}, b_{n-1})$. Obviously, $\sum a_n' = 1$ since $\sum b_n = 1$. ∎

Let G_4 be the following game on B: in each round, I plays a finite subset $A_n \subset B - \{0\}$ and II plays a finite subset $B_n \subset B - \{0\}$ of the same size as A_n such that for any $a \in A_n$, there exists $b \in B_n$ with $b \leq a$. I wins if $\sum_n \sum B_n = 1$. If we replace finite subsets by countable subsets, call the new game G_5.

It is easy to show that G is equivalent to G_4, but we do not know the following.

QUESTION 4.2. Is G equivalent to G_5?

For another equivalence, consider the following "forcing version" of G. Let G_6 be the following game. It starts by I choosing 0 and II choosing a name $\dot{f}(0)$ for an element of ω. Then I plays a number $n_1 \in \omega$, and II plays a name $\dot{f}(1)$. I wins if $\Vdash$ "there exists $\dot{n}$ such that $\dot{f}(\dot{n}) = g(\dot{n})$" where $g \in {}^{\omega}\omega$ and $g(i) = n_{i+1}$. Clearly, II wins iff there is a condition q such that $q \Vdash$ "for each $\dot{n}$, $\dot{f}(\dot{n}) \neq \dot{g}(\dot{n})$". The game G_6 is equivalent to G. Define the game G_7

by modifying G_6 as follows: II wins iff there exists q such that $q \Vdash$ "for each $\dot{n}, \dot{f}(\dot{n}) > g(\dot{n})$". The following question is open.

QUESTION 4.3. Is G equivalent to G_7?

The game G_7 is equivalent to the game where in the game G_1 we let player I choose finitely many members of the partition played by II_1. It also should be noted that the name $\dot{f}$ is not provided by player II in one play. So it is different from saying that a new function can dominate all old functions.

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