Cohomology of some graded differential algebras

by

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Abstract. We study cohomology algebras of graded differential algebras which are models for Hochschild homology of some classes of topological spaces (e.g. homogeneous spaces of compact Lie groups). Explicit formulae are obtained. Some applications to cyclic homology are given.

1. Introduction. Let K-$\mathrm{ADG}_{(c)}$ be the category of graded commutative differential algebras over a field K of zero characteristic. Let $(\mathcal{A}, d) \in K$-$\mathrm{ADG}_{(c)}$ be an algebra of the form

(1)
$$(\mathcal{A}, d) = (K[X_1, \ldots, X_n] \otimes \bigwedge(y_1, \ldots, y_n), d),$$
$$d(X_i) = 0, \quad i = 1, \ldots, n, \quad d(y_j) = f_j(X_1, \ldots, X_n), \quad j = 1, \ldots, n,$$

with polynomials $f_1, \ldots, f_n$ constituting a regular sequence. As usual, $\mathcal{A}$ is endowed with a grading by assigning to the variables X_i even degrees and to the variables y_j odd degrees, and d is supposed to be of degree $+1$. Here and in the sequel $K[X_1, \ldots, X_n]$ denotes the polynomial algebra and $\bigwedge(y_1, \ldots, y_n)$ is the exterior algebra generated by the free variables $y_1, \ldots, y_n$. We denote the degree of X by $\deg(X)$.

In the present paper we study the cohomology algebra $H^*(\mathcal{H}, \delta)$, where $(\mathcal{H}, \delta) \in K$-$\mathrm{ADG}_{(c)}$ is defined as follows:

$$(\mathcal{H}, \delta) = (\mathcal{A} \otimes \bigwedge(x_1, \ldots, x_n) \otimes K[Y_1, \ldots, Y_n], \delta),$$
$$\delta|_{\mathcal{A}} = d, \quad \delta(x_i) = 0, \quad i = 1, \ldots, n,$$

(2)
$$\delta(Y_j) = \sum_{i=1}^{n} \frac{\partial f_j}{\partial X_i} \otimes x_i, \quad j = 1, \ldots, n,$$

$$\deg(x_i) = \deg(X_i) + 1, \quad \deg(Y_i) = \deg(y_i) + 1, \quad i = 1, \ldots, n.$$

R e m a r k. The definitions of $(\mathcal{A}, d)$ and $(\mathcal{H}, \delta)$ could be rewritten in the form of Burghelea–Vigué-Poirrier [4], [19]. Let $V = \bigoplus_{i \geq 2} V_i$ be a graded

vector space over K. Let us write $\bigwedge(V) = \bigotimes_i \bigwedge(V_i)$ where $\bigwedge(V_i)$ denotes either the symmetric algebra $K[V_i]$ when i is even or the exterior algebra when i is odd. For the graded vector space $V = \bigoplus_{i \geq 2} V_i$ define $\overline{V}_i = V_{i+1}$ and $\overline{V} = \bigoplus_{i \geq 1} \overline{V}_i$. Consider $\bigwedge(V) \otimes \bigwedge(\overline{V})$. Define the derivation

$$\beta : \bigwedge(V) \otimes \bigwedge(\overline{V}) \to \bigwedge(V) \otimes \bigwedge(\overline{V})$$

of degree -1 by the equalities

$$\beta(v) = \overline{v}, \quad \beta(\overline{v}) = 0, \quad v \in V, \ \overline{v} \in \overline{V}.$$

Introduce the derivation δ by the equalities

$$\delta(v) = d(v), \quad \delta(\overline{v}) = -\beta(d(v)), \quad v \in V, \ \overline{v} \in \overline{V}.$$

Then $(\mathcal{A}, d)$ is a particular case of $(\bigwedge(V), d)$, and $(\mathcal{H}, \delta)$ is a particular case of $(\bigwedge(V) \otimes \bigwedge(\overline{V}), \delta)$. We shall use both notations, choosing the most convenient in each separate case.

By means of some spectral sequence associated with (1), we obtain a complete description of $H^*(\mathcal{H}, \delta)$ in the case (1) (Theorem 1). This description is applied to the theory of Hochschild and cyclic homology of topological spaces (in the sense of Burghelea and Goodwillie [2], [7], Theorem 2). As an application we give an alternative proof of the Burghelea–Vigué-Poirrier conjecture [19] about quasifree cyclic homology of topological spaces with cohomology algebra being a truncated polynomial algebra (Theorem 3; the original proof was obtained recently by M. Vigué-Poirrier [18]).

Then we obtain some sufficient conditions for *non-quasifreeness* of cyclic homology in terms of the conormal module of some associated polynomial ideal and show by examples that such spaces do exist (Theorems 4 and 5). Some explicit calculations of Hochschild homology for compact homogeneous spaces are given.

Let us, first, formulate the results.

THEOREM 1. *Let $(\mathcal{A}, d)$ satisfy the conditions* (1). *Then*

$$(3) \quad H^*(\mathcal{H}, \delta) = H^*(\mathcal{A}) \otimes \bigwedge(x_1, \dots, x_n) \bigg/ \bigg(\sum_{i=1}^n \frac{\partial f_1}{\partial X_i} \otimes x_i, \dots, \sum_{i=1}^n \frac{\partial f_n}{\partial X_i} \otimes x_i \bigg)$$

$$\oplus \sum_{s=1}^n \sum_{i_1 < \dots < i_s} \bigg(\mathrm{Ann}_{H^*(\mathcal{A}) \otimes \bigwedge(x_1, \dots x_n)} \bigg(\sum_{i=1}^n \frac{\partial f_{i_1}}{\partial X_i} \otimes x_i, \dots, \sum_{i=1}^n \frac{\partial f_{i_s}}{\partial X_i} \otimes x_i \bigg) \bigg/$$

$$\bigg(\sum_{i=1}^n \frac{\partial f_1}{\partial X_i} \otimes x_i, \dots, \sum_{i=1}^n \frac{\partial f_n}{\partial X_i} \otimes x_i \bigg) \bigg) \otimes K^+[Y_{i_1}, \dots, Y_{i_s}],$$

where $\bigwedge(x_1, \dots, x_n)$ is the exterior algebra generated by the free variables x_i of odd degrees, $\deg(x_i) = \deg(X_i) + 1$ and $K[Y_1, \dots, Y_n]$ is the polynomial algebra in the free variables $Y_1, \dots Y_n$ such that $\deg(Y_i) = \deg(f_i)$ for all i.

R e m a r k. In (3) we use the following notation: for any ring A the symbol $(a_1, \ldots, a_n)$ denotes the ideal generated by $a_1, \ldots, a_n$. If $I \subset A$ is an ideal, the factor ring $I/I \cap (a_1, \ldots, a_n)$ is denoted simply by $I/(a_1, \ldots, a_n)$. If $K[Y_1, \ldots, Y_n]$ is a polynomial algebra, then $K^+[Y_1, \ldots, Y_n]$ is the subalgebra generated by the monomials $Y_1^{k_1} \ldots Y_n^{k_n}$ with $k_i > 0$ for all i.

THEOREM 2. *Let X be any simply connected topological space with cohomology algebra of the form*

$$(4) \qquad H^*(X, K) = K[X_1, \ldots, X_n]/(f_1, \ldots, f_n)$$

satisfying the following assumptions:

(i) *$f_1, \ldots, f_n$ is a regular sequence in $K[X_1, \ldots, X_n]$*;
(ii) *each f_i is decomposable, that is, f_i is a polynomial only in the variables X_j satisfying $\deg(X_j) < \deg(f_i) - 1$.*

Then the following isomorphism of graded algebras is valid:

$$(5) \quad HH^*(X) \simeq H^*(X) \otimes \bigwedge(x_1, \ldots, x_n) \Big/ \Big(\sum_{i=1}^{n} \frac{\partial f_1}{\partial X_i} \otimes x_i, \ldots, \sum_{i=1}^{n} \frac{\partial f_n}{\partial X_i} \otimes x_i \Big)$$

$$\oplus \sum_{s=1}^{n} \sum_{i_1 < \ldots < i_s} \Big(\mathrm{Ann}_{H^*(X) \otimes \bigwedge(x_1, \ldots, x_n)} \Big(\sum_{i=1}^{n} \frac{\partial f_{i_1}}{\partial X_i} \otimes x_i, \ldots, \sum_{i=1}^{n} \frac{\partial f_{i_s}}{\partial X_i} \otimes x_i \Big) \Big/$$

$$\Big(\sum_{i=1}^{n} \frac{\partial f_1}{\partial X_i} \otimes x_i, \ldots, \sum_{i=1}^{n} \frac{\partial f_n}{\partial X_i} \otimes x_i \Big) \Big) \otimes K^+[Y_{i_1}, \ldots, Y_{i_s}].$$

R e m a r k. In fact, the assumption (ii) in the formulation of the theorem is not restrictive, because if $H^*(X)$ can be represented in the form (4) with the regularity condition, then it can also be represented in the form satisfying both assumptions.

THEOREM 3 (M. Vigué-Poirrier [18]). *Any simply connected topological space with cohomology algebra (4) has quasifree cyclic homology.*

Theorem 3 was proved recently in [18]; we give an alternative proof in order to illustrate the usefulness of Theorems 1 and 2.

Let R be a ring, and M a finitely generated R-module. By $\mu(M)$ we denote the least number of elements in a system of generators of M. In particular, $\mu(I)$ is defined for any ideal $I \subset R$.

THEOREM 4. *Let X be any simply connected topological space with minimal model*

$$(\mathcal{M}_X, d) = (K[X_1, \ldots, X_n] \otimes \bigwedge(y_1, \ldots, y_m), d),$$

$$d(X_i) = 0, \quad d(y_j) = f_j(X_1, \ldots, X_n), \quad i = 1, \ldots, n, \ j = 1, \ldots, m,$$

and with finite-dimensional cohomology algebra $H^*(X)$. Let $I = (f_1, \ldots, f_m)$ be the ideal in $K[X_1, \ldots, X_n]$ generated by f_j and I/I^2 be its conormal

module. If

$$\mu(I) > \mu(I/I^2),$$

then the cyclic cohomology $HC^*(X)$ is not quasifree.

THEOREM 5. *The following homogeneous spaces are topological spaces with non-quasifree cyclic homology:*

  (i) $M = \mathrm{Sp}(20)/\mathrm{SU}(6)$,
  (ii) $M = \mathrm{SU}(6)/\mathrm{SU}(3) \times \mathrm{SU}(3)$.

Now, we give some motivation for our results. Theorem 1 is the main algebraic tool for proving Theorems 2–5. Theorems 2–5 describe Hochschild and cyclic homology of a wide class of topological spaces.

The cyclic and Hochschild homology $HC_*(X)$ and $HH_*(X)$ of a topological space X have been the subject of wide interest since the papers of D. Burghelea [2], D. Burghelea and Z. Fiedorowicz [3] and T. Goodwillie [7]. Since then many papers and books on this theme have been written, e.g. [4], [5], [9], [12], [14], [17], [19]. Since $HH_*(X)$ can be identified with $H_*(X^{S^1})$ (the homology of the free loop space X^{S^1}), and $HC_*(X)$ with the homology of the associated bundle $ES^1 \times_{S^1} X^{S^1}$ ([7]), cyclic and Hochschild homology provide a powerful technique for studying free loop spaces (see e.g. [10]). The investigation of various topological invariants of X^{S^1} is very important in view of their role in mathematical physics [21].

The first explicit calculation of $HH^*(X)$ in the case (4) was done by D. Burghelea and M. Vigué-Poirrier [19], namely a formula for the Poincaré series $P_{HH^*(X)}(t)$ was obtained for any simply connected topological space X with cohomology algebra either in the form $K[X_1]/(X_1^{n+1})$, or $\bigwedge(V)$.

Theorem 2 gives an interesting formula for an *arbitrary* space whose cohomology algebra is of the form (4). We show by examples how to apply it. Our result can be applied to a wide class of topological spaces (e.g. homogeneous spaces of compact semisimple Lie groups G/H with $\mathrm{rank}(G) = \mathrm{rank}(H)$). When the multiplicative structure of $H^*(X)$ can be described explicitly by generators and relations, our formula is of particular interest (see examples below). Of course, the formula for $P_{HH^*(X)}(t)$ in [19] can be derived from ours in the case $n = 1$, $f_1 = X_1^{n+1}$. The usefulness of the result is also illustrated by another proof of the Burghelea–M. Vigué-Poirrier conjecture about quasifree cyclic homology (its validity was proved recently by M. Vigué-Poirrier [18]).

It would be interesting to find a criterion for a topological space to have quasifree cyclic homology. Therefore, to begin the investigation, one needs at least some examples of spaces with *non-quasifree* cyclic homology. Theorems 4 and 5 give such examples. Note that the condition $\mu(I) > \mu(I/I^2)$ involving the conormal module can be verified in many cases by methods of

commutative algebra and algebraic geometry, e.g. by the homological characterization of local complete intersections etc. (see [11]).

In the context of our approach we also mention the recent work of the "Buenos Aires cyclic homology group" [8]. Hochschild homology of complete intersections was also studied in [12].

2. Algebraic part (proof of Theorem 1). In what follows, whenever (C, d) is a graded differential algebra equipped with a derivation d of degree $+1$, its cohomology algebra is denoted by $H^*(C, d)$, while if d has degree -1, the notation is $H_*(C, d)$. Recall that a sequence $a_1, \ldots, a_i, \ldots$ in a ring R is called *regular* if a_i is not a zero divisor in $R/(a_1, \ldots, a_{i-1})$.

LEMMA 1. *Let (C, d) be a graded differential algebra over a field K $(\mathrm{char}(K) = 0)$ of the form*

$$(C, d) = (A \otimes K[Y_1, \ldots, Y_n], d),$$
$$d|_A = 0, \quad d(Y_i) = a_i \in A, \quad \deg(Y_i) = 2l_i, \quad i = 1, \ldots, n,$$

and with d being a derivation of degree -1. Let

$$C_{n-1} = A \otimes K[Y_1, \ldots, Y_{n-1}].$$

Then there exists a homological type spectral sequence of modules $(E_{p,q}^r, d^r)$ converging to $H_*(C, d)$ and such that

(6) $$E^2 = H_*(C_{n-1})/([a_n]) \oplus (\mathrm{Ann}_{H_*(C_{n-1})}([a_n])) \otimes K^+[Y_n].$$

P r o o f. Define an increasing filtration on C by

(7) $$F^{-1}C = \{0\} \subset F^0C = A \otimes K[Y_1, \ldots, Y_{n-1}] \subset \ldots$$
$$\subset F^pC = A \otimes K[Y_1, \ldots, Y_{n-1}] \otimes K[Y_n]^{\leq p} \subset \ldots,$$

where $K[Y_n]^{\leq p}$ denotes all polynomials of degree $\leq p$. Clearly, d respects the filtration (7). Consider the associated spectral sequence of modules $(E_{p,q}^r, d_{p,q}^r)$ (recall that it is not a spectral sequence of algebras). We use the explicit construction of spectral sequences, coming from exact pairs (see [15], with appropriate changes for homological type). We take a long exact sequence

$$\ldots \xrightarrow{\partial} H_{p+q}(F^{p-1}C) \xrightarrow{i} H_{p+q}(F^pC) \xrightarrow{j} H_{p+q}(F^pC/F^{p-1}C)$$
$$\xrightarrow{\partial} H_{p+q-1}(F^{p-1}C) \to \ldots$$

and construct the following exact pair:

$$D_{p,q}^1 = H_{p+q}(F^pC), \quad E_{p,q}^1 = H_{p+q}(F^pC/F^{p-1}C),$$
$$i_{p,q} : H_{p+q}(F^pC) \to H_{p+q}(F^{p+1}C),$$
$$j_{p,q} : H_{p+q}(F^pC) \to H_{p+q}(F^pC/F^{p-1}C),$$
$$\partial = k_{p,q} : H_{p+q}(F^pC/F^{p-1}C) \to H_{p+q-1}(F^{p-1}C).$$

The maps $i_{p,q}$ and $j_{p,q}$ are induced by i and j, ∂ is a connecting homomorphism and

$$(8) \qquad\qquad d^1_{p,q} = j_{p-1,q} \circ k_{p,q}.$$

By direct calculation,

$$E^1_{p,q} = H_{p+q}(C_{n-1} \otimes L(Y^p_n), \bar{d}) = H_{p+q}(C_{n-1}) \otimes L(Y^p_n)$$

(here $\bar{d}$ denotes the derivation induced by d, and $L(v_1, \ldots, v_s)$ is the vector space spanned by $v_1, \ldots, v_s$). By (8), $d^1_{p,q}$ is induced by d. Therefore

$$E^2 = \bigoplus_{p,q} E^2_{p,q} = H_*(H_*(C_{n-1}) \otimes K[Y_n], \widetilde{d}),$$

(9)

$$\widetilde{d}|_{H_*(C_{n-1})} = 0, \qquad \widetilde{d}(Y_n) = [a_n]_{H_*(C_{n-1})}$$

and $[a_n]_{H^*(C_{n-1})}$ denotes the cohomology class of a_n in C_{n-1}. Obviously (9) can be represented in a general form

$$E^2 = H_*(B \otimes K[Y_n], \widetilde{d}), \qquad \widetilde{d}|_B = 0, \quad \widetilde{d}(Y_n) = b \in B.$$

An easy calculation shows that for every differential algebra $(B \otimes K[Y_n], \widetilde{d})$ satisfying the conditions above,

$$H_*(B \otimes K[Y_n], \widetilde{d}) = B/(b) \oplus (\mathrm{Ann}_B(b)/(b)) \otimes K^+[Y_n].$$

Applying the above formula to (9) one obtains (6). It remains to show that $(E^r_{p,q}, d^r_{p,q})$ converges to $H_*(C,d)$. It is well known that the following conditions guarantee the convergence:

(i) $F^p C = 0$ if $p < 0$,
(ii) $E^1_{p,q} = H_{p+q}(F^p C / F^{p-1} C) = 0$ if $q < 0$,
(iii) $C = \bigcup_p F^p C$.

The conditions (i)–(iii) are verified by direct calculation. Lemma 1 is proved.

LEMMA 2. *Let (C, d) be a graded differential algebra satisfying the assumptions of Lemma 1. Then the following isomorphism of graded differential algebras is valid:*

$$(10) \qquad H_*(C,d) \simeq A/(a_1, \ldots, a_n)$$

$$\oplus \sum_{s=1}^{n} \sum_{i_1 < \ldots < i_s} (\mathrm{Ann}_A(a_{i_1}, \ldots, a_{i_s})/(a_1, \ldots, a_n)) \otimes K^+[Y_{i_1}, \ldots, Y_{i_s}].$$

P r o o f. We use induction on n. We strengthen (10) by the additional statement that the isomorphism (10) is canonical in the following sense: if φ denotes the isomorphism (10), then

$$(C) \qquad\qquad [a] \in H_*(C,d), \ a \in A \Rightarrow \varphi([a]) = \pi(a),$$

where $\pi : A \to A/(a_1, \ldots, a_n)$ is the natural projection. For $n = 1$ the condition (C) and (10) are evident. Suppose that (10) and (C) are valid for all numbers $\leq n - 1$. In particular,

$$H_*(C_{n-1}) \simeq A/(a_1, \ldots, a_{n-1})$$
$$\oplus \sum_{s=1}^{n-1} \sum_{i_1 < \ldots < i_s} (\operatorname{Ann}_A(a_{i_1}, \ldots, a_{i_s})/(a_1, \ldots, a_{n-1})) \otimes K^+[Y_{i_1}, \ldots, Y_{i_s}].$$

By Lemma 1 there is a spectral sequence $(E_{p,q}^r, d_{p,q}^r)$ converging to $H_*(C, d)$ and with E^2-term of the form (6). Obviously

$$H_*(C_{n-1})/([a_n]) = A/(a_1, \ldots, a_n)$$
$$\oplus \sum_{s=1}^{n-1} \sum_{i_1 < \ldots < i_s} (\operatorname{Ann}_A(a_{i_1}, \ldots, a_{i_s})/(a_1, \ldots, a_n)) \otimes K^+[Y_{i_1}, \ldots, Y_{i_s}].$$

Using (C) one immediately obtains

(11) $\quad \operatorname{Ann}_{H_*(C_{n-1})}([a_n]) \otimes K^+[Y_n] = \operatorname{Ann}_{R_{n-1}}(\pi(a_n)) \otimes K^+[Y_n]$

$$\oplus \sum_{s=1}^{n-1} \sum_{i_1 < \ldots < i_s} \operatorname{Ann}_{L_{i_1 \ldots i_s}}(\pi(a_n)) \otimes K^+[Y_{i_1}, \ldots, Y_{i_s}] \otimes K^+[Y_n],$$

where $R_{n-1} = A/(a_1, \ldots, a_{n-1})$, $L_{i_1 \ldots i_s} = \operatorname{Ann}_A(a_{i_1}, \ldots, a_{i_s})/(a_1, \ldots, a_{n-1})$ and π is the natural projection. Since $(E_{p,q}^r, d_{p,q}^r)$ converges to $H_*(C, d)$, cohomology classes of $H^*(C)$ are those surviving in the spectral sequence. Thus without loss of generality one can consider only elements of (11) surviving in the spectral sequence. Obviously it is enough to consider elements from $\operatorname{Ann}_{R_{n-1}}(\pi(a_n)) \otimes K^+[Y_n]$ and from each of the annihilators $\operatorname{Ann}_{L_{i_1 \ldots i_s}}(\pi(a_n)) \otimes K^+[Y_{i_1}, \ldots, Y_{i_s}] \otimes K^+[Y_n]$ separately.

In the first case the corresponding representative u of the cohomology class in $H^*(C)$ can be written in the form $\sum b_i \otimes Y_n^i$, and since it must be a cocycle, one easily obtains $b_i \in \operatorname{Ann}_A(a_n)$ and $[u] \in (\operatorname{Ann}_A(a_n)/(a_n)) \otimes K^+[Y_n]$. On the other hand, the latter algebra can be embedded in $H_*(C)$ in an obvious way.

The same argument can be applied to the second case. By (11) the corresponding representative v of the cohomology class in $H_*(C)$ can be taken in the form

$$v = \sum a_{i, i_1, \ldots, i_s} \otimes Y_{i_1}^{k_1} \ldots Y_{i_s}^{k_s} Y_n^i \quad \text{with } a_{i, i_1, \ldots, i_s} \in \operatorname{Ann}_A(a_{i_1}, \ldots, a_{i_s}).$$

Therefore $d(v) = 0$ implies $a_{i, i_1, \ldots, i_s} \in \operatorname{Ann}_A(a_{i_1}, \ldots, a_{i_s}, a_n)$ (direct computation). Finally, $[v] \in \operatorname{Ann}_A(a_{i_1}, \ldots, a_{i_s}, a_n)/(a_1, \ldots, a_n) \otimes K^+[Y_{i_1}, \ldots, Y_{i_s}] \otimes K^+[Y_n]$ in cohomology. Since the latter subalgebra can obviously be embedded in $H_*(C)$, the isomorphism (10) (of both modules and algebras) holds. This completes the proof.

COROLLARY. *Lemma 2 is also valid for the algebra $(A \otimes K[Y_1, \ldots, Y_n], d)$ with d being a derivation of degree +1.*

It is enough to replace the given grading by a new one $\deg(Y_i) + 2$, then obtain the appropriate isomorphism and return to the previous grading.

In what follows we need a fact proved by M. Vigué-Poirrier:

LEMMA 3 ([20]). *Let $(\bigwedge, d)$ be a graded differential algebra. Let ϑ be the ideal of $\bigwedge$ generated by the exterior generators, and let $A = \bigwedge/\vartheta$. If y is an exterior generator of $\bigwedge$ such that the image of dy in A is nonzero, we have $H^*(\overline{\bigwedge}, \bar{d}) = H^*(\bigwedge, d)$, where $\overline{\bigwedge} = \bigwedge/(y, dy)$ and $\bar{d}$ is the induced differential on $\overline{\bigwedge}$.*

Proof of Theorem 1. Recall that

$$(12) \quad (\mathcal{H}, \delta)$$
$$= (K[X_1, \ldots, X_n] \otimes \bigwedge(x_1, \ldots, x_n) \otimes \bigwedge(y_1, \ldots, y_n) \otimes K[Y_1, \ldots, Y_n], \delta),$$
$$\delta(X_i) = \delta(x_i) = 0, \quad i = 1, \ldots, n,$$
$$\delta(y_j) = f_j, \quad \delta(Y_j) = \sum_{i=1}^{n} \frac{\partial f_j}{\partial X_i} \otimes x_i, \quad j = 1, \ldots, n,$$
$$\deg(Y_i) = \deg(y_i) + 1, \quad \deg(x_i) = \deg(X_i) + 1, \quad i = 1, \ldots, n.$$

Now, it is enough to apply Lemma 3 to each y_i (the regularity of $f_1, \ldots, f_n$ guarantees the possibility of successive elimination). Finally,

$$H^*(\mathcal{H}, \delta) = H^*((K[X_1, \ldots, X_n] \otimes \bigwedge(x_1, \ldots, x_n)/(f_1, \ldots, f_n))$$
$$\otimes K[Y_1, \ldots, Y_n], \bar{\delta})$$

where $\bar{\delta}$ is induced by δ. Denote by A the algebra

$$A = (K[X_1, \ldots, X_n]/(f_1, \ldots, f_n)) \otimes \bigwedge(x_1, \ldots, x_n)$$
$$= H^*(\mathcal{A}, d) \otimes \bigwedge(x_1, \ldots, x_n).$$

(The latter equality is obtained by applying Lemma 3 again.) To finish the proof apply the Corollary of Lemma 2 to $A \otimes K[Y_1, \ldots, Y_n]$.

3. Hochschild and cyclic homology (proof of Theorem 2). To start with, we outline briefly the basic notions of Hochschild and cyclic homology. A more complete exposition can be found in [4].

As defined in [4], an *algebraic S^1-chain complex* $\widetilde{C}_* = (C_n, d_n, \beta_n)_{n \geq 0}$ of K-vector spaces is a chain complex $(C_*, d) = (C_n, d_n)_{n \geq 0}$ of K-vector spaces equipped with the linear maps $\beta = \{\beta_n : C_n \to C_{n+1}, n \geq 0\}$ (called an S^1-*action*) such that $\beta_{n+1}\beta_n = 0$ and $\beta_{n-1}d_n + d_{n+1}\beta_n = 0$, With $\widetilde{C}_*$, one associates the chain complex $({}_{\beta}C_*, {}_{\beta}d^*)$ defined by

$$({}_{\beta}C)_n = C_n + C_{n-2} + \ldots$$

and

$$(\,_\beta d)_n(x_n, x_{n-2}, \ldots) = (d(x_n) + \beta(x_{n-2}), d(x_{n-2}) + \beta(x_{n-4}), \ldots).$$

DEFINITION 1. The *cyclic homology* of $\widetilde{C}_* = (C_*, d, \beta)$ is the homology of the chain complex $(\,_\beta C_*, \,_\beta d^*)$:

$$HC_*(\widetilde{C}_*) = H_*(\,_\beta C_*, \,_\beta d^*).$$

One can extend the notion of S^1-chain complex to bigraded complexes. A *bigraded S^1-chain complex* $C = (C_{n,p}, d^I, d^E, \beta)$ is a collection of K-vector spaces $C_{n,p}$, $n \geq 0$, $p \geq 0$, and K-linear maps

$$d^I : C_{n,p} \to C_{n,p-1}, \quad d^E : C_{n,p} \to C_{n-1,p}, \quad \beta_{n,p} : C_{n,p} \to C_{n+1,p}$$

such that $(d^I)^2 = 0$, $(d^E)^2 = 0$, $\beta^2 = 0$, $\beta d^E + d^E \beta = 0$ and $\beta d^I + d^I \beta = 0$. For any such C one writes

$$(\text{Tot}\, C)_* = \Big(\bigoplus_{p+q=n} C_{p,q}, d^I + d^E, \beta \Big).$$

DEFINITION 2. The *cyclic homology* of the bigraded S^1-complex C is the cyclic homology of the associated total S^1-complex $(\text{Tot}\, C)_*$ (in the sense of Definition 1).

Now, let $(A, d) \in K\text{-ADG}_{(c)}$. We define

$$T(A)_{p,q} = \bigoplus_{i_0+i_1+\ldots+i_p=q} A_{i_0} \otimes A_{i_1} \otimes \ldots \otimes A_{i_p} \quad \text{for } p \geq 0,\ q \geq 0,$$

$$d^I(a_{i_0} \otimes \ldots \otimes a_{i_p}) = da_{i_0} \otimes a_{i_1} \otimes \ldots \otimes a_{i_p}$$

$$+ \sum_{l=1}^{p} (-1)^{i_0+\ldots+i_{l-1}} a_{i_0} \otimes \ldots \otimes da_{i_l} \otimes \ldots \otimes a_{i_p},$$

$$d^E(a_{i_0} \otimes \ldots \otimes a_{i_p}) = \sum_{l=0}^{p-1} (-1)^l a_{i_0} \otimes \ldots \otimes a_{i_l} a_{i_{l+1}} \otimes \ldots \otimes a_{i_p}$$

$$+ (-1)^{p+i_p(i_0+\ldots+i_{p-1})} a_{i_p} a_{i_0} \otimes \ldots \otimes a_{i_{p-1}},$$

$$\beta_{p,q} = (-1)^p (1 - T_{p+1}) \circ S_p \circ (1 + T_p + \ldots + T_p^p),$$

where

$$S_p(a_{i_0} \otimes \ldots \otimes a_{i_p}) = a_{i_0} \otimes \ldots \otimes a_{i_p} \otimes 1,$$

$$T_p(a_{i_0} \otimes \ldots \otimes a_{i_p}) = (-1)^{p+i_p(i_0+\ldots+i_{p-1})} a_{i_p} \otimes a_{i_0} \otimes \ldots \otimes a_{i_{p-1}}, \quad a_{i_j} \in A_{i_j}.$$

It can be verified that $(T(A)_{p,q}, d^I, d^E, \beta_{p,q})$ is a bigraded S^1-chain complex.

DEFINITION 3. By definition, the homology of $\text{Tot}(A)$,

$$H_*(\text{Tot}(A), d^I + d^E) = HH_*(A, d),$$

is called the *Hochschild homology* of $(A, d) \in K\text{-ADG}_{(c)}$. In the same way, the formula

$$HC_*(T(A), d^I, d^E, \beta) = HC_*(A, d)$$

defines the *cyclic homology* of (A, d).

These algebraic definitions are transformed into topological ones by the following procedure: denote by MX the Moore loop space on X, and by $C_*(MX)$ its algebra of singular K-chains.

DEFINITION 4 ([2], [7]). Put by definition

$$HH_*(X) = HH_*(C_*(MX)), \quad HC_*(X) = HC_*(C_*(MX)),$$

and call $HH_*(X)$ and $HC_*(X)$ respectively the *Hochschild* and *cyclic homology* of the topological space X.

We have already mentioned in the introduction Goodwillie's isomorphisms [7]:

$$H_*(X^{S^1}) \simeq HH_*(X), \quad H_*(ES^1 \times_{S^1} X^{S^1}) \simeq HC_*(X),$$

which permit us to use duality and consider below cohomology rather than homology. Our considerations use some notions of rational homotopy theory. We refer to [13] for details.

A graded differential algebra $(\mathcal{M}, d)$ is called *minimal* if $\mathcal{M}$ is a free graded commutative algebra

$$\mathcal{M} = K[W^{\text{even}}] \otimes \bigwedge(W^{\text{odd}}),$$

satisfying the following conditions:

(a) $W = \bigoplus_{\alpha \in I} W_\alpha$ (I is an ordered set);
(b) each W_α consists of homogeneous elements;
(c) for any $\alpha \in I$, $d(W_\alpha) \subset S(\bigoplus_{\beta < \alpha} W_\beta)$ ($S(K)$ denotes the subalgebra generated by K).

DEFINITION 5 ([13]). (i) Let $(A, d) \in K\text{-ADG}_{(c)}$. A minimal algebra $(\mathcal{M}_A, D)$ is said to be a *minimal model* of (A, d) if there exists a homomorphism of graded differential algebras $\varrho : (\mathcal{M}_A, D) \to (A, d)$ inducing isomorphism in cohomology,

$$\varrho^* : H^*(\mathcal{M}_A, D) \to H^*(A, d).$$

(ii) Let X be a topological space, and $A_Q : \Im \to \mathbb{Q}\text{-ADG}_{(c)}$ be a functor from the category $\Im$ of simplicial sets to the category $\mathbb{Q}\text{-ADG}_{(c)}$ constructed in [13] (that is, satisfying the simplicial de Rham theorem). A minimal model of the algebra $A_{\mathbb{Q}}(S_*(X)) \in \mathbb{Q}\text{-ADG}_{(c)}$ is called a *minimal model of the topological space X* and is denoted by

$$\mathcal{M}_X = \mathcal{M}_{A_{\mathbb{Q}}(S_*(X))}.$$

Here $S_*(X)$ is the simplicial set of singular simplexes of X.

The proof of Theorem 2 is based on the following result of Burghelea and Vigué-Poirrier [4]:

THEOREM (B–V). *Let X be any simply connected topological space with minimal model $(\mathcal{M}_X, d) = (\bigwedge(V), d)$. Then*

$$HH^*(X) \simeq H^*(\mathcal{H}, \delta)$$

where $(\mathcal{H}, \delta)$ is given by (2) (or by the remark below (2)).

Proof of Theorem 2

LEMMA 4. *Let X be any topological space satisfying the assumptions of Theorem 2. Then $\mathcal{M}_X$ is as in (1).*

P r o o f. Clearly, the K-ADG$_{(c)}$-morphism

$$\varrho : K[X_1, \ldots, X_n] \otimes \bigwedge(y_1, \ldots, y_n) \to K[X_1, \ldots, X_n]/(f_1, \ldots, f_n),$$
$$\varrho(X_i) = X_i, \quad \varrho(y_j) = 0, \quad i, j = 1, \ldots, n,$$

induces isomorphism in cohomology by Lemma 3. This proves Lemma 4.

Now, by the (B–V) theorem, $HH^*(X) \simeq H^*(\mathcal{H}, \delta)$, where $(\mathcal{H}, \delta)$ is obtained from $(\mathcal{M}_X, d)$ of the form (1). Applying Theorem 1 to $(\mathcal{H}, \delta)$ yields (3).

4. Applications of Theorem 2: Hochschild homology of some homogeneous spaces. Recall some facts relating to cohomology of homogeneous spaces. Let G be a compact connected Lie group, and H be its closed subgroup. In the sequel, the Lie algebras of Lie groups G, H, ... are denoted by the corresponding small letters g, h, ... Let $W \leq \mathrm{GL}(V)$ be a discrete subgroup of $\mathrm{GL}(V)$ generated by reflections. Let $K[V]$ denote the symmetric algebra over the vector space V. Consider the extension of the W-action to $K[V]$ and denote the ring of W-invariants by $K[V]^W$. In particular, consider a maximal torus T of a Lie group G, its Weyl group $W(G, T)$ and the algebra

$$\mathbb{Q}[t_\mathbb{Q}]^{W(G,T)}$$

(here $t_\mathbb{Q}$ denotes the $\mathbb{Q}$-structure on t, that is, $t_\mathbb{Q} = \{v \in t^\mathbb{C} : \alpha(v) \in \mathbb{Q}$ for any α in the root system $R(g^\mathbb{C}, t^\mathbb{C})\}$). The well-known Chevalley theorem implies

$$(13) \qquad \mathbb{Q}[t_\mathbb{Q}]^{W(G,T)} \simeq \mathbb{Q}[f_1, \ldots, f_n],$$

where the f_i are algebraically independent generators. Consider the homogeneous space G/H and choose maximal tori T and T' in G and H in such a way that $T' \subset T$. Consider also the algebra of invariants

$$\mathbb{Q}[t'_\mathbb{Q}]^{W(H,T')} \simeq \mathbb{Q}[u_1, \ldots, u_s].$$

Denote by $\bigwedge(V)$ the exterior algebra over V. If a base $x_1, \ldots, x_k$ of V is chosen, we also use the notation $\bigwedge(x_1, \ldots, x_k)$. If V is a graded vector space, the vectors x_i have odd degrees, $\deg(x_i) = 2l_i - 1$. As usual, $\bigwedge_k(V)$ denotes the subspace of all elements of degree k.

It is well known that

$$H^*(G) \simeq \bigwedge(x_1, \ldots, x_n), \quad n = \mathrm{rank}(G),$$

where the x_i are primitive elements in $H^*(G)$.

DEFINITION 6. The algebra $(C', d') \in \mathbb{Q}\text{-ADG}_{(c)}$ of the form

(14) $$(C', d') = (\mathbb{Q}[t'_{\mathbb{Q}}]^{W(H,T')} \otimes \bigwedge(x_1, \ldots, x_n), d),$$

(15) $$d(u) = 0 \quad \text{for any } u \in \mathbb{Q}[t'_{\mathbb{Q}}]^{W(H,T')},$$

$$d(x_i) = f_i|'_t = \widetilde{f}_i(u_1, \ldots, u_s),$$

where f_i $(i = 1, \ldots, n = \mathrm{rank}(G))$ are defined by (13), is called a *Cartan algebra* of the homogeneous space G/H.

R e m a r k 1. To obtain the above definition in the form (14)–(15) it is enough to combine the isomorphism in [8, p. 565] and the definition of Koszul's complex in [8, p. 420].

R e m a r k 2. It was proven in [1], [8] that

$$H^*(M, \mathbb{Q}) \simeq H^*(C', d')$$

if $M = G/H$ with G a reductive Lie group.

EXAMPLE 1 (Poincaré polynomial $P_{HH^*(X)}(t)$ for $X = \mathrm{SU}(3)/T$). Let $X = \mathrm{SU}(3)/T$ be the flag manifold of the group $\mathrm{SU}(3)$ (T is its maximal torus). Use the general theory described above. Introduce the coordinates X_1, X_2, X_3 in t satisfying the condition $X_1 + X_2 + X_3 = 0$. Then the polynomials

$$f_1 = X_1^2 + X_2^2 + X_3^2, \quad f_2 = X_1^3 + X_2^3 + X_3^3,$$

are $W(\mathrm{SU}(3))$-invariant and by direct calculation one obtains (after calculating the minimal model)

$$H^*(X) = \mathrm{span}(u_1, u_2 : u_1^3 = u_2^3 = 0, \ u_1 u_2 = -(u_1 + u_2)^2, \ u_1^2 u_2 = -u_1 u_2^2).$$

Then the equivalence classes of $\sum_{i=1}^n (\partial f_j / \partial X_i) x_i$ in $H^*(X) \otimes \bigwedge(x_1, x_2)$ have representatives

$$a_1 = 2u_1 x_1 + u_1 x_2 + u_2 x_1 + 2u_2 x_1, \quad a_2 = 2u_1^2 x_1 + u_1^2 x_2 + u_2^2 x_1 + 2u_2^2 x_2.$$

By the Hirsch formula,

$$P_{\mathrm{SU}(3)/T}(t) = \frac{(1 - t^4)(1 - t^6)}{(1 - t^2)^2} = 1 + 2t^2 + 2t^4 + t^6$$

and therefore

$$P_{H^*(X)\otimes\bigwedge(x_1,x_2)}(t) = P_{H^*(X)}(t) \cdot (1+t^3)^2$$
$$= 1 + 2t^2 + 2t^3 + 2t^4 + 4t^5 + 2t^6$$
$$+ 4t^7 + 2t^8 + 2t^9 + 2t^{10} + t^{12}.$$

Applying (5) to $H^*(X) \otimes \bigwedge(x_1, x_2)$ and a_1, a_2, one can calculate directly all dimensions of the factor algebra in (5) in this particular case.

Degree	Additive generators	Dimensions			
		case 1	case 2	case 3	case 4
2	u_1, u_2	2	0	2	2
3	x_1, x_2	2	0	0	0
4	u_1^2, u_2^2	2	0	2	2
5	$u_1 x_1, u_2 x_2, u_1 x_2, u_2 x_1$	3	1	1	1
6	$u_1^2 u_2, x_1 x_2$	2	2	2	2
7	$u_1^2 x_1, u_1^2 x_2, u_2^2 x_1, u_2^2 x_2$	2	1	1	1
8	$u_1 x_1 x_2, u_2 x_1 x_2$	1	1	1	1
9	$u_1^2 x_1 u_2, u_1^2 u_2 x_2$	1	0	0	0
10	$u_1^2 x_1 x_2, u_2^2 x_1 x_2$	1	0	0	0
12	$u_1^2 u_2 x_1 x_2$	0	0	0	0

The table gives the explicit expression for the Poincaré series

$$P_{HH^*(X)}(t) = 1 + 2t^2 + 2t^3 + 3t^5 + 2t^6 + 2t^7 + t^8 + t^9 + t^{10}$$
$$+ \left(\frac{1}{1-t^4} - 1\right)(4t^2 + 4t^4 + 2t^5 + 4t^6 + 2t^7 + 2t^8)$$
$$+ \left(\frac{1}{(1-t^4)^2} - 1\right)(t^5 + 2t^6 + t^7 + t^8).$$

R e m a r k. In the case $H^*(X) = K[X]/(X^{n+1})$ our procedure gives the same result as in [19] because annihilators are calculated automatically and one obtains the algebra

$$(K[X] \otimes \bigwedge(x)/(X^{n+1}, X^n x)) \otimes K[Y]$$

(as in Addendum to [20]).

EXAMPLE 2. Let $X = G_2/T$. Introduce the coordinates X_1, X_2, X_3 in t satisfying $X_1 + X_2 + X_3 = 0$. Then the polynomials

$$f_1 = X_1^2 + X_2^2 + X_3^2, \quad f_2 = X_1^6 + X_2^6 + X_3^6$$

are G_2-invariant and by direct calculation one obtains

$$H^*(X) = \text{span}(u_1, u_2 : u_1^2 + u_2^2 = -u_1 u_2, u_1^3 = u_2^3 = -u_1^2 u_2 - u_1 u_2^2,$$
$$u_1^5 u_2 = -u_1 u_2^5, \ u_1^3 u_2^3 = 0, u_1^4 + u_2^4 = -u_1^2 u_2^2, \ u_1^5 + u_2^5 = -u_1 u_2^4).$$

The equivalence classes of $\sum_{i=1}^{n}(\partial f_j/\partial X_i)x_i$ in $H^*(X) \otimes \bigwedge(x_1, x_2)$ have representatives

$$a_1 = 2u_1x_1 + u_2x_1 + u_1x_2 + 2u_2x_1, \quad a_2 = 6u_1^5x_1 + 6u_2^5x_2 + 6(u_1+u_2)^5(x_1+x_2).$$

By the Hirsch formula,

$$P_X(t) = \frac{(1-t^4)(1-t^{12})}{(1-t^2)^2} = (1+t^2)(1+t^2+t^4+t^6+t^8+t^{10})$$

and

$$\begin{aligned}
P_{H^*(X)\otimes\wedge(x_1,x_2)}(t) &= P_X(t) \cdot (1+t^3)^2 \\
&= 1 + 2t^2 + 2t^3 + 2t^4 + 4t^5 + 3t^6 \\
&\quad + 4t^7 + 4t^8 + 4t^9 + 4t^{10} + 4t^{11} + 3t^{12} \\
&\quad + 4t^{13} + 2t^{14} + 2t^{15} + 2t^{16} + t^{18}.
\end{aligned}$$

Applying (5) to $H^*(X) \otimes \bigwedge(x_1, x_2)$ and a_1, a_2, one can calculate directly all dimensions of the factor algebra (see Example 1).

Degree	Additive generators	Dimensions			
		case 1	case 2	case 3	case 4
2	u_1, u_2	2	0	2	2
3	x_1, x_2	2	0	0	0
4	u_1^2, u_2^2	2	0	2	2
5	$u_1x_1, u_2x_2, u_2x_1, u_2x_2$	4	1	1	1
6	$x_1x_2, u_1^2u_2, u_2^2u_1$	3	3	2	2
7	$u_1^2x_1, u_1^2x_2, u_2^2x_1, u_2^2x_2$	2	1	1	1
8	$u_1x_1x_2, u_2x_1x_2, u_1^4, u_2^4$	3	2	2	2
9	$u_1^2u_2x_1, u_1^2u_2x_2, u_2^2u_1x_1, u_2^2u_1x_2$	3	2	1	1
10	$u_1^2x_1x_2, u_2^2x_1x_2, u_1^5, u_2^5$	3	2	2	2
11	$u_1^4x_1, u_2^4x_2, u_1^4x_2, u_2^4x_1$	2	1	1	1
12	$u_1^2u_2x_1x_2, u_2^2u_1x_1x_2, u_1^3x_1x_2$	1	1	0	0
13	$u_1^5x_1, u_2^5x_2, u_1^5x_2, u_2^5x_1$	2	1	0	0
14	$u_1^4x_1x_2, u_2^4x_1x_2$	1	0	0	0
15	$u_1^5u_2x_1, u_1^5u_2x_2$	1	0	0	0
16	$u_1^4u_2x_1x_2, u_2^4u_1x_1x_2$	1	0	0	0
18	$u_1^4u_2^2x_1x_2$	0	0	0	0

The table gives the explicit expression for the Poincaré series

$$P_{HH^*(X)}(t) = 1 + 2t^2 + 2t^3 + 4t^4 + 3t^5 + 3t^6 + 2t^7 + 3t^8 + 3t^9 + 3t^{10} + 2t^{11}$$

$$+ t^{12} + 2t^{13} + t^{14} + t^{15} + t^{16}$$

$$+ \left(\frac{1}{1 - t^4} - 1 \right) (4t^2 + 4t^4 + 2t^5 + 4t^6 + 2t^7 + 4t^8 + 2t^9 + 4t^{10} + 2t^{11})$$

$$+ \left(\frac{1}{(1 - t^4)^2} - 1 \right) (t^5 + 3t^6 + t^7 + 2t^8 + 2t^9 + 2t^{10} + t^{11} + t^{12} + t^{13}).$$

5. Quasifree and non-quasifree cyclic homology: proof of Theorems 3–5. Let $K[\alpha]$ be the graded free commutative algebra generated by α with $\deg \alpha = 2$ over a field K. Suppose that a K-graded vector space P^* is endowed with the structure of a $K[\alpha]$-module by a map $\nabla : K[\alpha] \otimes P^* \to P^*$. Clearly, the existence of ∇ is equivalent to the existence of a K-linear map $S : P^* \to P^{*+2}$ defined by the condition $\nabla(\alpha^P \otimes x) = S^P(x)$.

DEFINITION 7 ([19]). The $K[\alpha]$-module (P^*, S) is called: (a) *free* if S is injective; (b) *trivial* if S is zero; (c) *quasifree* if it is the direct sum of a free and a trivial module.

Consider the Connes long exact sequence for the cyclic and Hochschild cohomology of a topological space X:

$$\ldots \to HH^n(X) \to HC^n(X) \xrightarrow{S} HC^{n+2}(X) \to HH^{n+1}(X) \to \ldots$$

Since $HC^*\{\mathrm{pt}\} = K[\alpha]$, the operator S defines a structure of a $K[\alpha]$-graded module on $HC^*(X)$.

DEFINITION 8 ([17]). The cyclic cohomology $HC^*(X, K)$ is said to be *quasifree* if $HC^*(X, K)$ is quasifree as a $K[\alpha]$-module in the sense of Definition 7.

It was conjectured by D. Burghelea and M. Vigué-Poirrier that any topological space whose cohomology is a polynomial algebra truncated by a regular sequence, is a space with quasifree cyclic homology. This conjecture is valid even under more general assumption of formality, as was proved recently by M. Vigué-Poirrier [18] (in this sense our proof is weaker, but explains the phenomenon in the case of truncated polynomial algebras). Our version (Theorem 3) is presented as an example of the use of (5).

Proof of Theorem 3. As usual, for the cohomology algebra $H^*(A, d)$ of any $(A, d) \in K\text{-ADG}_{(c)}$, the symbol $H_+(A, d)$ denotes the subspace $\bigoplus_{p>0} H^p(A, d)$. As in Theorem 2, consider the minimal model of X, $\mathcal{M}_X = (\bigwedge(V), d)$ and introduce the graded differential algebra $(\mathcal{R}, \mathcal{D})$ by the formulae

$$\mathcal{R} = K[\alpha] \otimes \bigwedge(V) \otimes \bigwedge(\overline{V}),$$
$$\mathcal{D}\alpha = 0, \quad \mathcal{D}(u) = \delta(u) + \alpha\beta(u), \quad u \in \bigwedge(V) \otimes \bigwedge(\overline{V}).$$

Consider the short exact sequence

$$0 \to \mathcal{R} \overset{l_\alpha}{\to} \mathcal{R} \overset{p}{\to} \mathcal{H} \to 0,$$

where $l_\alpha(r) = \alpha \otimes r$ for $r \in \mathcal{R}$, and $p : \mathcal{R} \to \mathcal{H}$ is the projection $p(\alpha^k \otimes c) = 0$ $(k \geq 1)$, $p(1 \otimes c) = c$. As usual, one obtains the long exact sequence

(16) $\ldots \to H^*(\mathcal{H}, \delta) \overset{B}{\to} H^*(\mathcal{R}, \mathcal{D}) \overset{J}{\to} H^{*+2}(\mathcal{R}, \mathcal{D}) \overset{I}{\to} H^{*+1}(\mathcal{H}, \delta) \to \ldots$

where B is the connecting homomorphism, J is induced by l_α, and I is induced by p. By direct calculation one obtains

$$B([y]) = [1 \otimes \beta(y)] \in H^*(\mathcal{R}, \mathcal{D}), \quad [y] \in H^*(\mathcal{H}, \delta).$$

Let $B' = I \circ B$. Clearly, $(B')^2 = 0$ and $\deg B' = -1$, therefore it is possible to introduce the complex $(H^*(\mathcal{H}, \delta), B')$.

PROPOSITION 1 ([19]). (i) *The following isomorphisms are valid*:

$$HH^*(X) \simeq H^*(\mathcal{H}, \delta), \quad HC^*(X) \simeq H^*(\mathcal{R}, \mathcal{D}).$$

(ii) *The operator S in the Connes exact sequence can be identified with the operator J in* (16).

Proposition 1 allows us to use $H^*(\mathcal{R}, \mathcal{D})$ instead of $HC^*(X)$. In what follows we shall use the Hodge decompositions for Hochschild and cyclic homology [4].

PROPOSITION 2 ([4]). (i) *Both Hochschild and cyclic homology of a commutative graded differential K-algebra (A, d) have natural decompositions*

$$HH^n(A, d) = \bigoplus_{p \geq 0} HH^n(A, d)^{(p)},$$

$$HC^n(A, d) = HC_n(K) \oplus HC_n^{(p)}(A, d), \quad HC_n^{(p)}(A, d) = 0, \quad p > n + 1.$$

(ii) *For the Connes exact sequence the following equality holds*:

$$S^p(x) = 0 \quad \text{for any } x \in HC^*(A)^{(p)}.$$

Denote by $\widetilde{HC}^*(X, K)$ the reduced cyclic cohomology [4]. Since for any augmented graded commutative differential algebra (A, d),

$$HC^*(A, d) = HC^*(K) \oplus \widetilde{HC}^*,$$

applying the above proposition to the chain algebra $C_*(MX)$ of the Moore loop space, one obtains the equality

$$\widetilde{HC}^n(X) = \bigoplus_{p=0}^{n+2} (HC^n)^{(p)}.$$

Proposition 2 and the last formula imply

PROPOSITION 3. (i) $J|_{\widetilde{HC}^*(X)}$ *is nilpotent*; (ii) $(\widetilde{HC}^*)^0 = 0$.

The following lemma can be proved by direct calculation.

LEMMA 5. *There is a short exact sequence*

$$0 \to \operatorname{im} J \cap \widetilde{HC}^* / \operatorname{im} J^2 \cap \widetilde{HC}^* \xrightarrow{a} H_*(\widetilde{HH}^*, B') \xrightarrow{b} (\ker J \cap \operatorname{im} J) \cap \widetilde{HC}^* \to 0$$

where the maps a, b are defined by

$$a([J(x)]) = [I(x)], \quad b([y]) = B(y).$$

P r o o f. See [14], [15].

R e m a r k. The above lemma is due to R. Krasauskas.

LEMMA 6. $HC^*(X, K)$ *is quasifree if and only if* $J|_{\widetilde{HC}^*} = 0$.

P r o o f. Follows directly from Proposition 3.

LEMMA 7. $HC^*(X, K)$ *is quasifree if and only if*

$$(17) \qquad\qquad H_+(H_+(\mathcal{H}, \delta), \beta_*) = 0.$$

P r o o f. By Lemma 6, $HC^*(X, K)$ is quasifree if and only if $J|_{\widetilde{HC}^*} = 0$. By Lemma 5, the last equality is equivalent to $H_*(\widetilde{HH}^*, B') = 0$.

Now, to prove the lemma it is sufficient to notice that

$$B'|_{\widetilde{HH}^*} = \beta_*|_{H_+(\mathcal{H}, \delta)}, \quad \widetilde{HH}^* = H_+(\mathcal{H}, \delta).$$

To finish the proof of Theorem 3 it is necessary to calculate (17) directly for $(\mathcal{H}, \delta)$ defined by (12). Obviously, (17) can be rewritten in the form

$$(18) \qquad\qquad H_+(\widetilde{\mathcal{H}}^+, \beta_*) = 0,$$

where $\widetilde{\mathcal{H}}$ is defined by (3), $\widetilde{\mathcal{H}}^+$ denotes the subalgebra in $\widetilde{\mathcal{H}}$ generated by the positive degrees and β_* is induced by β (apply Lemma 3). Consider the quotient algebra

$$\widetilde{\mathcal{H}}_1 = K[X_1, \ldots, X_n] \otimes \bigwedge(x_1, \ldots, x_n)/(f_1, \ldots, f_n, \beta(f_1), \ldots, \beta(f_n)).$$

Suppose first that $\beta(f) \in (\beta(f_1), \ldots, \beta(f_n))$ for some $f \in K[X_1, \ldots, X_n] \otimes \bigwedge(x_1, \ldots, x_n)$, that is,

$$\beta(f) = a_1\beta(f_1) + \ldots + a_n\beta(f_n),$$

with a_i being "polynomials" in the variables X_j, x_k. Set

$$\widetilde{f} = f - (-1)^{\deg(a_1)}a_1 f_1 - \ldots - (-1)^{\deg(a_n)}a_n f_n.$$

Then, by direct computation, one obtains $\beta(\widetilde{f}) \in (f_1, \ldots, f_n)$. Thus, calculating cohomology of (12) one can always choose f in such a way that

$$(19) \qquad\qquad \beta(f) \in (f_1, \ldots, f_n).$$

Now we prove (18). Observe that $\widetilde{\mathcal{H}}_1$ can be represented in the form

$$(\widetilde{\mathcal{H}}_1, \beta_*) = (\operatorname{span}(u_1, \ldots, u_n) \otimes \bigwedge(x_1, \ldots, x_n)/(v_1, \ldots, v_n), \beta_*),$$

where $\beta_*(u_i) = x_i$, the u_i are generators of $\mathrm{span}(u_1, \ldots, u_n)$ satisfying the relations determined by the ideal $(f_1, \ldots, f_n)$, and $v_i = \sum_{j=1}^{n} (\partial f_i / \partial X_j) \cdot x_j|_{(u_1, \ldots u_n)}$ (we substitute $(u_1, \ldots, u_n)$ for X_j and take into consideration their relations). Then (19) can be expressed as

$$\beta_*(f) = 0, \quad f \in \mathrm{span}(u_1, \ldots, u_n) \otimes \bigwedge(x_1, \ldots, x_n).$$

Let us prove by induction the implication

(20) $$\beta_*(f) = 0 \Rightarrow f = \beta_*(g) + q_1 v_1 + \ldots + q_n v_n$$

for some $g, q_i \in \mathrm{span}(u_1, \ldots, u_n) \otimes \bigwedge(x_1, \ldots, x_n)$. Use induction on the number of variables i generating $\mathrm{span}(u_1, \ldots, u_n) \otimes \bigwedge(x_1, \ldots, x_n)$, containing f. Let $i = 1$. Then any cocycle with respect to β_* is of the form $u_1^k x_1$. If $u_1^{k+1} x_1 \neq 0$, then

$$f = \beta_*\left(\frac{1}{k+1} u_1^{k+1} x_1\right).$$

If $u_1^{k+1} x_1 = 0$, then

$$\begin{aligned}
X_1^{k+1} \in (f_1, \ldots, f_n) &\Rightarrow X_1^{k+1} = a_1 f_1 + \ldots + a_n f_n \\
&\Rightarrow \beta(X_1^{k+1}) = (k+1) X_1^k x_1 \\
&\qquad = \beta(a_1) f_1 + \ldots + \beta(a_n) f_n \\
&\qquad\quad + (-1)^{\deg(a_1)} a_1 \beta(f_1) \\
&\qquad\quad + \ldots + (-1)^{\deg(a_n)} a_n \beta(f_n) \\
&\Rightarrow (k+1) u_1^k x_1 = \bar{a}_1 v_1 + \ldots + \bar{a}_n v_n
\end{aligned}$$

(the $\bar{a}_i$ are the images of a_i with appropriate coefficients). Thus in both cases (20) is valid for $i = 1$. Suppose that (20) is satisfied for $i \leq n - 1$. Then for arbitrary $u \in \mathrm{span}(u_1, \ldots, u_n) \otimes \bigwedge(x_1, \ldots, x_n)$ we consider

$$u = g_1 + g_2 u_n^l + g_3^t x_n + g_4 x_n, \quad g_i = g_i(u_1, \ldots, u_{n-1}, \ldots, x_{n-1}),$$

and rewrite the equality $\beta_*(u) = 0$ directly. We have to consider the following possibilities for l and t:

$$1)\ l > 1,\ l - 1 \neq t, \quad 2)\ l > 1,\ l - 1 = t, \quad 3)\ l = 1,\ t \geq 0.$$

Each case should be considered separately, but calculations do not differ essentially, therefore we reproduce them in detail only in the first case. Then

$$\beta_*(g_1) = 0, \ \beta_*(g_2) u_n^l = 0, \ g_2 u_n^{l-1} = 0, \ \beta_*(g_3) u_n^t = 0, \ \beta_*(g_4) = 0.$$

By the induction hypothesis one can eliminate g_1 and g_4. Moreover, $g_2 u_n^{l-1} = 0$ eliminates the term $g_2 u_n^l$. The equality $\beta_*(g_3) u_n^t = 0$ gives two possibilities: either $g_3 u_n^{t+1}$ is zero, or not. In the first case

$$\beta_*(g_3 u_n^{t+1}) = \beta_*(g_3) u_n^{t+1} + (-1)^{\deg(g_3)} (t+1) g_3 u_n^t x_n,$$

which implies

$$g_3 u_n^t x_n = \beta_* \left(\frac{(-1)^{\deg(g_3)}}{t+1} g_3 u_n^{t+1} \right).$$

In the second case $g_3 X_n^{t+1} \in (f_1, \ldots, f_n)$ and applying exactly the same argument as in the case $i = 1$, one obtains

$$g_3 u_n^t x_n = \bar{a}_1 v_1 + \ldots + \bar{a}_n v_n,$$

which implies (20).

In case 2), by the same technique one obtains the equality

$$u = g_1 + \beta_* \left(\frac{(-1)^{\deg(g_2)}}{l} g_3 u_n^l \right) + g_4 x_n,$$

from which the assertion follows. Case 3) does not differ from the previous one.

Now, to finish the proof it is enough to notice that $\beta_*(Y_{i_k}) = 0$ and that β_* preserves all annihilators in (5) because of the evident equality $\beta(a\beta(g)) = \beta(a)\beta(g)$. Our argument was inductive and one can observe that all equalities in the reasoning above remain unchanged under the assumption that they belong to any annihilator (note that $g_3 u_n^t \notin \mathrm{Ann}(v_i)$ implies $x_n \beta(f_i) \in (f_1, \ldots, f_n)$, from which the contradiction with the regularity condition can be derived very easily). The proof of the theorem is complete.

P r o o f   o f   T h e o r e m 4. Consider $(\mathcal{M}_X, d)$ as defined in Theorem 4. Consider the ideal $I = (f_1, \ldots, f_m)$. Choose a maximal regular subsequence, say $f_1, \ldots, f_s$.

First we show that because $H^*(X)$ is finite-dimensional, $s \geq n$. Denote by $V(I)$ the affine algebraic variety of I and by $\mathrm{Rad}(I)$ its radical. If $\dim V(I) = 0$, then $\mathrm{Rad}(I)$ is a complete intersection and using [11, pp. 134–135] it is easy to derive that $I = (f_1, \ldots, f_n, f_{n+1}, \ldots, f_m)$, where $f_1, \ldots, f_n$ is a regular sequence, thus $s \geq n$. If $\dim V(I) > 0$, then there exists an infinite sequence of polynomials $q_1, q_2, \ldots$, which are linearly independent on $V(I)$ and therefore, mod I. They are cocycles. Suppose that $\sum \alpha_i q_i = d(b)$. Then, clearly, $b = \sum p_i y_i$ and thus $d(b) \in I$, contrary to the above remark. Thus $[q_i]$ are independent cohomology classes and $H^*(X)$ is not finite-dimensional. On the other hand, it is impossible to obtain $s > n$, because by the Macaulay theorem [11] the maximal length $d(I)$ of a regular sequence of any ideal I is equal to its height $h(I)$. But for the polynomial ring $K[X_1, \ldots, X_n]$, $h(I) \leq n$, and thus $s \leq d(I) = h(I) \leq n$. Thus $s = n$ and so $I = (f_1, \ldots, f_n, f_{n+1}, \ldots, f_m)$, where $f_1, \ldots, f_n$ is a regular sequence. If $m = n$, then I is a complete intersection [11] and $\mu(I) = \mu(I/I^2)$ ([11]).

Regularity implies that $HC^*(X)$ is quasifree by Theorem 3. Then the following possibilities remain:

(i) $f_{n+1}, \ldots, f_m$ are all in $(f_1, \ldots, f_n)$,
(ii) at least one of them is not, say $f_{n+1} \notin (f_1, \ldots, f_n)$.

In case (i) the well known derivation change

$$d'(y_j) = \begin{cases} f_j, & j = 1, \ldots, n, \\ f_j + \sum_{k=1}^n p_k f_k, & j = n+1, \ldots, m, \end{cases}$$

allows us to replace $(\mathcal{M}_X, d)$ by

$$(\mathcal{M}'_X, d') = (K[X_1, \ldots, X_n] \otimes \bigwedge(y_1, \ldots, y_n) \otimes \bigwedge(y_{n+1}, \ldots, y_m), d'),$$
$$d'(X_i) = d(X_i), \quad d'(y_i) = d(y_i) \quad (i = 1, \ldots, n),$$
$$d'(y_{n+1}) = \ldots = d'(y_m) = 0,$$

and one easily notices that the "trivial part" $(\bigwedge(y_{n+1}, \ldots, y_m), d = 0)$ does not affect the considerations of Theorem 3, and again $\mu(I) = \mu(I/I^2)$ and $HC^*(X)$ is quasifree.

In case (ii), $f_{n+1} \notin (f_1, \ldots, f_n)$, but the sequence is not regular. Observe that the sequence $f_1, \ldots, f_i, \ldots$ contains a minimal system of generators of I (see [11, p. 109]). Therefore finally $I = (f_1, \ldots, f_n, f_{n+1}, \ldots, f_{n+k})$, $\mu(I) = n + k$. Now, suppose that $\mu(I) > \mu(I/I^2)$. Then a generator, say f_{n+2}^2, can be expressed as a combination of the others. Now, use the following inequality, which is valid for any Noetherian ring R (see [11]):

PROPOSITION 4. (i) $h(I) \le \mu(I/I^2) \le \mu(I) \le \mu(I/I^2) + 1$.
(ii) If $\mu(I/I^2) > \dim R$, then $\mu(I) = \mu(I/I^2)$.

This proposition gives $\mu(I/I^2) \le n$ and $\mu(I) \le n + 1$. Thus, the assumptions of Theorem 4 imply $k = 1$, $I = (f_1, \ldots, f_n, f_{n+1})$ and $f_{n+1}^2 \in (f_1, \ldots, f_n)$. By Theorem 3, $HC^*(X)$ is quasifree if and only if $H_+(H_+(\mathcal{H}, \delta), \beta_*) = 0$.

Consider now $(\mathcal{H}, \delta)$ obtained as in (12), and take, as in the previous cases,

$$(\widetilde{\mathcal{H}}, \widetilde{\delta}) = ((K[X_1, \ldots, X_n]/(f_1, \ldots, f_n) \otimes K[Y_1, \ldots, Y_n])$$
$$\otimes \bigwedge(y_{n+1}) \otimes K[Y_{n+1}], \widetilde{\delta}).$$

Consider the element $u = f_{n+1} y_{n+1}$ and its equivalence class in $\widetilde{\mathcal{H}}$,

$$\widetilde{u} = \widetilde{f}_{n+1} y_{n+1}.$$

Since $f_{n+1}^2 \in (f_1, \ldots, f_n)$, $[\widetilde{u}]$ is a cohomology class in $H^*(\widetilde{\mathcal{H}}, \widetilde{\delta})$ ($[\widetilde{u}] \neq 0$, which can be verified directly).

It can be proved by direct calculation that $\beta_*[u] = 0$, but $[u] \neq \beta_*[v]$. Indeed,

$$\beta(\widetilde{u}) = \beta(f_{n+1})y_{n+1} + f_{n+1}Y = \delta(y_{n+1}Y)$$

and thus $\beta_*[u] = 0$. On the other hand,

$$u \neq \delta(v) + \beta(w), \quad \text{where } \delta w = 0,$$

because the left hand side does not contain free variables x_i and the right hand side does contain them if $\beta(w) \neq 0$. If $\beta(w) = 0$, then u is a coboundary, contrary to the remark above. Thus $H_+(H_+(\mathcal{H}, \delta), \beta_*) \neq 0$ and $HC^*(X)$ is not quasifree by Theorem 3. The proof of Theorem 4 is complete.

Proof of Theorem 5. Consider the case

$$M = \mathrm{SU}(6)/\mathrm{SU}(3) \times \mathrm{SU}(3)$$

and calculate its Cartan algebra (C, d) by the methods described in Section 3. In our case

$$(C, d) = (\mathbb{R}[t']^{W(A_1 \times A_2)} \otimes \bigwedge(y_1, \ldots, y_5), d),$$
$$d(y_j) = \widetilde{f}_j = f_j|_{t'}, \quad j = 1, \ldots, 5,$$

where $d = 0$ on the first factor of the tensor product, and f_j and $\widetilde{f}_j$ are determined by the Chevalley isomorphism:

$$\mathbb{R}[t]^{W(A_5)} \simeq \mathbb{R}[f_1, \ldots, f_5], \quad \mathbb{R}[\tau']^{W(A_1 \times A_2)} \simeq \mathbb{R}[X_1, X_2, Y_1, Y_2].$$

Using the explicit expressions for f_j, X_i, Y_i (see [6]), one obtains

$$f_j(Z_1, \ldots, Z_6) = Z_1^{j+1} + \ldots + Z_6^{j+1}, \quad Z_1 + \ldots + Z_6 = 0, \quad j = 1, \ldots, 6,$$
$$Z_i = X_1^{i+1} + X_2^{i+1} + X_3^{i+1}, \quad X_1 + X_2 + X_3 = 0, \quad i = 1, 2,$$
$$Y_i = X_4^{i+1} + X_5^{i+1} + X_6^{i+1}, \quad X_4 + X_5 + X_6 = 0, \quad i = 1, 2.$$

Then, up to scalar multiples which are not important in our considerations,

$$\widetilde{f}_1 = X_1 + Y_1, \quad \widetilde{f}_2 = X_2 + Y_2, \quad \widetilde{f}_3 = X_1^2 + Y_2^2,$$
$$\widetilde{f}_4 = X_1 X_2 + Y_1 Y_2, \quad \widetilde{f}_5 = X_1^3 + Y_1^3 + X_2^2 + X_3^2,$$

and finally

$$(C, d) = (\mathbb{R}[X_1, X_2, Y_1, Y_2] \otimes \bigwedge(y_1, \ldots, y_5), d),$$
$$d(X_i) = d(Y_i) = 0, \quad i = 1, 2,$$
$$d(y_1) = X_1 + Y_1, \quad d(y_2) = X_2 + Y_2, \quad d(y_3) = X_1^2 + Y_1^2,$$
$$d(y_4) = X_1 X_2 + Y_1 Y_2, \quad d(y_5) = X_1^3 + Y_1^3 + X_2^2 + X_3^2,$$

where the degrees of the variables are

$$\deg(X_1) = 4, \quad \deg(Y_1) = 4, \quad \deg(X_2) = \deg(Y_2) = 6,$$
$$\deg(y_1) = 3, \ldots, \deg(y_5) = 11.$$

Observe that (C, d) is not a minimal algebra. Nevertheless, (C, d) is a free graded commutative algebra and we can apply Sullivan's method for constructing the minimal model of (C, d). In what follows we apply the calculations of [13]. Denote by V the linear span of $\{X_1, X_2, Y_1, Y_2, y_1, \ldots, y_5\}$ and introduce a new derivation d' on V by the formula

$$d' : V^n \to V^{n+1}, \quad d' = \pi \circ d,$$
$$\pi : V^{n+1} \oplus L^{++}(V)^{n+1} \to V^{n+1}$$

($L^{++}(V)$ is the ideal of decomposable elements in $\bigwedge(V)$, and π is the projection onto the first summand). Then

$$d'(X_i) = d'(Y_i) = 0 \quad (i = 1, 2),$$
$$d'(y_1) = X_1 + Y_1, \quad d'(y_2) = X_2 + Y_2, \quad d'(y_j) = 0 \quad (j > 2).$$

Consider the direct sums

$$V = \operatorname{im} d' \oplus V' \oplus W, \quad \operatorname{im} d' \oplus V' = \ker d'.$$

Then, obviously,

$$\ker d' = L(X_1, X_2, Y_1, Y_2, y_3, y_4, y_5),$$
$$\operatorname{im} d' = L(X_1 + Y_1, X_2 + Y_2)$$

(L denotes linear span). Thus

$$L(X_1, X_2, Y_1, Y_2, y_3, y_4, y_5) = L(X_1 + Y_1, X_2 + Y_2) + L(Y_1, Y_2) + L(y_3, y_4, y_5),$$

which implies

$$V' = L(Y_1, Y_2) \oplus L(y_3, y_4, y_5), \quad W = L(y_1, y_2),$$
$$W' = d(W) = L(d(y_1), d(y_2)) = L(X_1 + Y_1, X_2 + Y_2).$$

Consider the algebra

$$\overline{C} = \mathbb{R}[X_1 + Y_1, X_2 + Y_2] \otimes \bigwedge(y_1, y_2) = \bigwedge(W' \oplus W)$$

and the ideal $\langle \overline{C}^+ \rangle$ in $\bigwedge(V)$ generated by $\overline{C}^+$ (the elements of positive degrees). It is easy to calculate that

$$(\bigwedge(V'), d) \simeq (\bigwedge(V)/\langle \overline{C}^+ \rangle, d) \simeq (\mathbb{R}[X_1, X_2] \otimes \bigwedge(y_3, y_4, y_5), D) = (\mathcal{M}, D),$$
$$D(X_i) = 0 \quad (i = 1, 2), \quad D(y_3) = X_1^2, \quad D(y_4) = X_1 X_2, \quad D(y_5) = X_2^2$$

(up to scalars). By Sullivan's theorem [13], $(\mathcal{M}, D)$ is a minimal model for M. Now, clearly, for the ideal $I = (X_1^2, X_1 X_2, X_2^2)$ we have

$$\mu(I) = 3, \quad \mu(I/I^2) = 2,$$

and therefore Theorem 4 completes the proof.

In the case $M = \operatorname{Sp}(20)/\operatorname{SU}(6)$ one applies the same calculation to obtain the minimal model

$$(\mathcal{M}_M, d) = (\mathbb{R}[X_1, X_2] \otimes \bigwedge(y_1, y_2, y_3, y_4, y_5, y_6, y_7), d),$$

$$d(X_i) = 0, \quad d(y_1) = X_1 X_2, \quad d(y_2) = X_1^2, \quad d(y_3) = X_2^4.$$

Obviously $(X_1 X_2)^2 \in (X_1^2, X_2^4)$, and therefore again

$$\mu(I) > \mu(I/I^2).$$

This completes the proof of Theorem 5.

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