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Decomposing Baire class 1 functions into continuous functions

by

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Abstract. It is shown to be consistent that every function of first Baire class can be decomposed into $\aleph_1$ continuous functions yet the least cardinal of a dominating family in ${}^{\omega}\omega$ is $\aleph_2$. The model used in the one obtained by adding ω_2 Miller reals to a model of the Continuum Hypothesis.

1. Introduction. In [1] the authors consider the following question: What is the least cardinal κ such that every function of first Baire class can be decomposed into κ continuous functions? This cardinal κ will be denoted by $\mathfrak{dec}$. The authors of [1] were able to show that $\mathrm{cov}(\mathbb{K}) \leq \mathfrak{dec} \leq \mathfrak{d}$ and asked whether these inequalities could, consistently, be strict. By $\mathrm{cov}(\mathbb{K})$ is meant the least number of closed nowhere dense sets required to cover the real line and $\mathfrak{d}$ denotes the least cardinal of a dominating family in ${}^{\omega}\omega$. In [5] it was shown that it is consistent that $\mathrm{cov}(\mathbb{K}) \neq \mathfrak{dec}$. In this paper it will be shown that the second inequality can also be made strict. The model where $\mathfrak{dec}$ is different from $\mathfrak{d}$ is the one obtained by adding ω_2 Miller—sometimes known as super-perfect or rational-perfect—reals to a model of the Continuum Hypothesis. It is somewhat surprising that the model used to establish the consistency of the other inequality, $\mathrm{cov}(\mathbb{K}) \neq \mathfrak{dec}$, is a slight modification of the iteration of super-perfect forcing.

By ${}^{\underset{\smile}{\omega}}\omega$ we denote $\bigcup_{n\in\omega}\{{}^{n}\omega : n \in \omega\}$. As usual, a *tree* will be defined to mean an initial subset of ${}^{\underset{\smile}{\omega}}\omega$ under $\subseteq$. So if T is a tree and $t \in T$ then $t\restriction k \in T$ for each $k \in \omega$. Also, $T\langle t\rangle$ will be defined to be $\{s \in T : s \subseteq t$ or $t \subseteq s\}$. If t and s are both finite sequences then $s \wedge t$ is defined by

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declaring that $\mathrm{dom}(s \wedge t) = |\mathrm{dom}(t)| + |\mathrm{dom}(s)|$ and

$$s \wedge t(i) = \begin{cases} s(i) & \text{if } i \in \mathrm{dom}(s), \\ t(i - |\mathrm{dom}(s)|) & \text{if } i \notin \mathrm{dom}(s). \end{cases}$$

If $t \in T \subseteq {}^{\omega}\omega$ and $i \in \omega$ then $t \wedge i$ is defined to be $t \wedge \{(0,i)\}$ and $i \wedge t$ is defined to be $\{(0,i)\} \wedge t$. Finally, $\overline{T} = \{f \in {}^{\omega}\omega : (\forall n \in \omega)(f \restriction n \in T)\}$ and closure in other spaces is denoted similarly.

Definition 1.1. If $T \subseteq {}^{\omega}\omega$ is a tree then $\beta(T)$ is defined to be the set of all $t \in T$ such that $|\{n \in \omega : t \wedge n \in T\}| = \aleph_0$. A tree $T \subseteq {}^{\omega}\omega$ is said to be *super-perfect* if for each $t \in T$ there is some $s \in \beta(T)$ such that $t \subseteq s$ and if $|\{n \in \omega : t \wedge n \in T\}| \in \{1, \aleph_0\}$ for each $t \in T$. The set of all super-perfect trees will be denoted by $\mathbb{S}$.

For each $T \in \mathbb{S}$ there is a natural way to assign a mapping $\theta_T : {}^{\omega}\omega \to \beta(T)$ such that:

- θ_T is one-to-one and onto $\beta(T)$,
- $s \subseteq t$ if and only if $\theta(s) \subseteq \theta(t)$,
- $s \le_{\mathrm{Lex}} t$ if and only if $\theta(s) \le_{\mathrm{Lex}} \theta(t)$.

Notice that $\theta_T(\emptyset)$ is the root of T. Using the mapping θ_T, it is possible to define a refinement of the ordering on $\mathbb{S}$.

Definition 1.2. Define $T \prec_n S$ if both S and T are in $\mathbb{S}$, $T \subseteq S$ and $\theta_T \restriction {}^{n}\omega = \theta_S \restriction {}^{n}\omega$.

It should be clear that the ordering $\prec_n$ satisfies Axiom A. The proof of the main result of this paper will use a fusion based on a sequence of the orderings $\prec_n$. Notice that while $\prec_n$ can be used in the same way as the analogous ordering for Sacks reals in the case of adding a single real, it is not as easy to deal with in the context of iterations. The chief difficulty is that $\prec_n$ requires deciding an infinite amount of information because branching is infinite. This conflicts with the usual goal of fusion arguments which decide only a finite amount of information at a time.

2. Iterated super-perfect reals. It will be shown that iterating ω_2 times the partial orders $\mathbb{S}$ with countable support over a ground model where $2^{\aleph_0} = \aleph_1$ yields a model where $\mathfrak{d} = \aleph_2$ and $\mathfrak{dec} = \aleph_1$. The fact that $\mathfrak{d} = \aleph_2$ is well known [3]. The fact that $\mathfrak{dec} = \aleph_1$ is an immediate consequence of the following result.

Lemma 2.1. *Suppose that $\xi \in \omega_2 + 1$, $\mathbb{S}_\xi$ is the iteration with countable support of the partial orders $\mathbb{S}$ and G is $\mathbb{S}_\xi$-generic over V. Then for any $x \in [0,1]$ in $V[G]$ and any Borel function $H : [0,1] \to [0,1]$ in $V[G]$ there is a Borel set $X \in V$ such that $x \in X$ and $H \restriction X$ is continuous.*

Saying that $X \in V$ means, of course, that the real coding the Borel set X belongs to the model V. In order to prove Lemma 2.1 it will be useful to employ a different interpretation of iterated super-perfect forcing. The next sequence of definitions will be used in doing this. If G is $\mathbb{S}_\xi$-generic over some model $\mathfrak{M}$ then there is a natural way to assign a mapping $\Gamma : \xi \cap \mathfrak{M} \to {}^\omega\omega$ such that $\mathfrak{M}[G] = \mathfrak{M}[\Gamma]$. On the other hand, given $\Gamma : \mathfrak{M} \cap \xi \to {}^\omega\omega$ we define $G_\Gamma(\mathfrak{M})$ to be the set

$$\{q \in \mathfrak{M} \cap \mathbb{S}_\xi : \\ (\forall k \in \omega)(\forall A \in [\mathfrak{M} \cap \xi]^{<\aleph_0})(\exists p \le q)(\forall \alpha \in A)(p \restriction \alpha \Vdash_{\mathbb{S}_\alpha} \text{“}\Gamma(\alpha)\restriction k \in p(\alpha)\text{”})\}$$

and we say that Γ is $\mathbb{S}_\xi$-*generic* over $\mathfrak{M}$ if and only if G_Γ is $\mathbb{S}_\xi$-generic over $\mathfrak{M}$. Note that if G is $\mathbb{S}_\xi$-generic over $\mathfrak{M}$ and $\Gamma : \mathfrak{M} \cap \xi \to {}^\omega\omega$ is its associated function then $G_\Gamma(\mathfrak{M}) = G$. This will be used without further comment to identify $\mathbb{S}_\xi$-generic sets over $\mathfrak{M}$ with elements of $({}^\omega\omega)^{\mathfrak{M}\cap\xi}$. Whenever a topology on $({}^\omega\omega)^X$ is mentioned, the product topology is intended.

Definition 2.1. If $p \in \mathbb{S}_\xi$ and $\Lambda \in [\xi]^{\le\aleph_0}$ then define $S(\Lambda, p)$ to be the set of all functions $\Gamma : \Lambda \to {}^\omega\omega$ such that for all $k \in \omega$ and for all finite subsets $A \subseteq \Lambda$ there is $q \le p$ such that $q \Vdash_{\mathbb{S}_\xi}$ “$\Gamma(\alpha)\restriction k \in q(\alpha)$” for all $\alpha \in A$.

Definition 2.2. Given a countable elementary submodel $\mathfrak{M} \prec H((2^{\aleph_0})^+)$ and $p \in \mathbb{S}_\xi$ define p to be *strongly* $\mathbb{S}_\xi$-*generic* over $\mathfrak{M}$ if and only if

- each $\Gamma \in S(\mathfrak{M} \cap \xi, p)$ is $\mathbb{S}_\xi$-generic over $\mathfrak{M}$,
- if ψ is a statement of the $\mathbb{S}_\xi$-forcing language using only parameters from $\mathfrak{M}$, then $\{\Gamma \in S(\mathfrak{M} \cap \xi, p) : \mathfrak{M}[\Gamma] \models \psi\}$ is a clopen set in $S(\mathfrak{M} \cap \xi, p)$.

A set $X \subseteq ({}^\omega\omega)^\alpha$ will be defined to be *large* by induction on α.

Definition 2.3. If $\alpha = 1$ then X is large if X is a super-perfect tree. If α is a limit then X is large if the projection of X to $({}^\omega\omega)^\beta$ is large for every $\beta \in \alpha$. If $\alpha = \beta + 1$ then X is large if there is a large set $Y \subseteq ({}^\omega\omega)^\beta$ such that $X = \bigcup_{y \in Y} \{y\} \times X_y$ and each X_y is a large subset of ${}^\omega\omega$.

From large closed sets it is possible to obtain, in a natural way, conditions in $\mathbb{S}_\xi$.

Definition 2.4. If $X \subseteq ({}^\omega\omega)^\alpha$ is a large closed set then define $p_X \in \mathbb{S}_\alpha$ by letting $p_X(\eta)$ be the $\mathbb{S}_\eta$ name for the subset $T \subseteq {}^\omega\omega$ such that if $\Gamma : \alpha \to {}^\omega\omega$ is $\mathbb{S}_\alpha$-generic then

$$T = \{f \in {}^\omega\omega : (\exists h)(\Gamma\restriction\eta \cup \{(\eta, f)\} \cup h \in X)\}$$

Observe that, if $X \subseteq ({}^\omega\omega)^\alpha$ is large and closed, it follows that $p_X \in \mathbb{S}_\alpha$. The following result provides a partial converse to this observation.

Lemma 2.2. *If* $p \in \mathbb{S}_\xi$ *and* $\mathfrak{M} \prec H((2^{\aleph_0})^+)$ *is a countable elementary submodel containing* p *then there is* $q \le p$ *such that* q *is strongly* $\mathbb{S}_\xi$*-generic over* $\mathfrak{M}$.

Proof. The proof consists of merely repeating the proof that the countable support iteration of proper partial orders is proper and checking the assertions in this special case. Only a sketch will be given and the reader should consult [4] for details.

The proof is by induction on ξ. If $\xi = 1$ then a standard fusion argument applied to an enumeration $\{D_n : n \in \omega\}$ of all dense subsets of $\mathbb{S}$ provides the result. In particular, there is a sequence $\{T_i : i \in \omega\}$ such that $T_{i+1} \prec_i T_i$, $T_0 = T$ and $T_i\langle\theta_{T_i}(\sigma)\rangle \in D_{i-1}$ for each $\sigma : i \to \omega$. The condition $T_\omega = \bigcap_{i\in\omega} T_i$ has the desired property. The fact that if ψ is a statement of the $\mathbb{S}_\xi$-forcing language using only parameters from $\mathfrak{M}$, then $\{\Gamma \in S(\mathfrak{M}, T_\omega) : \mathfrak{M}[\Gamma] \models \psi\}$ is a clopen set is obvious because $S(1, T_\omega) = \overline{T}_\omega$.

If $\xi = \mu + 1$ then use the induction hypothesis to find $q' \le p\restriction\xi$ such that q' is strongly $\mathbb{S}_\mu$-generic over $\mathfrak{M}$. Then, in particular, q' is $\mathbb{S}_\mu$-generic over $\mathfrak{M}$ and so, if G contains q' and is $\mathbb{S}_\mu$-generic over V, it is also generic over $\mathfrak{M}$. Therefore $\mathfrak{M}[G]$ is an elementary submodel in $V[G]$ and it is possible to choose an enumeration $\{D_n : n \in \omega\}$ of all dense subsets of $\mathbb{S}$ which are members of $\mathfrak{M}[G]$. It is therefore possible to choose, in $\mathfrak{M}[G]$, as in the case $\xi = 1$, a sequence $\{T_i : i \in \omega\}$ such that $T_{i+1} \prec_i T_i$ and that $T_i\langle\theta_{T_i}(\sigma)\rangle \in D_{i-1}$ for each $\sigma : i \to \omega$. The condition $T_\omega = \bigcap_{i\in\omega} T_i$ is then strongly $\mathbb{S}$-generic over $\mathfrak{M}[G]$. Notice that, while T_ω does not itself have a name in $\mathfrak{M}$, each T_n does have a name and so there are enough objects in $\mathfrak{M}[G]$ to construct T_ω.

In order to see that $q = q' * T_\omega$ is strongly $\mathbb{S}_\xi$-generic over $\mathfrak{M}$ suppose that $\Gamma \in S(\mathfrak{M} \cap \xi, q)$. Obviously $\Gamma\restriction\mu \in S(\mathfrak{M} \cap \mu, q')$ and therefore $\mathfrak{M}[\Gamma]$ is an elementary submodel. Hence, by genericity, $T_{i+1} \prec_i T_i$, $T_0 = T$ and $T_i\langle\theta_{T_i}(\sigma)\rangle \in D_{i-1}$ and so it follows that $\bigcap\{T_i : i \in \omega\}$ is a strongly $\mathbb{S}$-generic condition over $\mathfrak{M}[G]$. Hence $\Gamma(\xi)$ is $\mathbb{S}$-generic over $\mathfrak{M}[G]$ and so Γ is $\mathbb{S}_\xi$-generic over $\mathfrak{M}$.

Just as in the case $\xi = 1$, it is easy to use the induction hypothesis to see that if ψ is a statement of the $\mathbb{S}_\xi$-forcing language using only parameters from $\mathfrak{M}$, then $\{\Gamma \in S(\mathfrak{M} \cap \xi, q) : \mathfrak{M}[\Gamma] \models \psi\}$ is a clopen set.

Finally, suppose that ξ is a limit ordinal. If it has uncountable cofinality then there is nothing to do because of the countable support of the iteration. So assume that $\{\mu_n : n \in \omega\}$ is an increasing sequence of ordinals cofinal in ξ. Let $\{D_n : n \in \omega\}$ enumerate all dense subsets of $\mathfrak{M}$ and choose a sequence of conditions $\{p_i : i \in \omega\}$ such that

- $p_i\restriction\mu_i$ is strongly $\mathbb{S}_{\mu_i}$-generic over $\mathfrak{M}$,
- $p\restriction\mu_i \Vdash_{\mathbb{S}_{\mu_i}}$ "$p_i\restriction(\xi \setminus \mu_i) \in D_i/G$" (this is an abbreviation for the more

precise statement

$$p\restriction\mu_i \Vdash_{\mathbb{S}_{\mu_i}} \text{“}(\exists q \in G \cap \mathbb{S}_{\mu_i})(q * p_i\restriction(\xi \setminus \mu_i) \in D_i)\text{”}$$

and will be used later as well),

- $p_i\restriction(\xi \setminus \mu_i)$ belongs to $\mathfrak{M}$,
- $p\restriction\mu_i \Vdash_{\mathbb{S}_{\mu_i}}$ “$p_{i+1}\restriction(\mu_{i+1} \setminus \mu_i)$ is $\mathbb{S}_{\mu_{i+1}\setminus\mu_i}$-generic over $\mathfrak{M}[G]$”,
- $p_{i+1} \le p_i$.

Notice that the statement that $p_i\restriction(\xi \setminus \mu_i) \in D_i/G$ can be expressed in $\mathfrak{M}$ and so if $\Gamma \in S(\mathfrak{M} \cap \mathbb{S}_{\mu_i}, p_i\restriction\mu_i)$ then $p_i\restriction(\xi \setminus \mu_i) \in D_i/\Gamma$. From this it easily follows that letting $p_\omega = \lim_{n\in\omega} p_n$ yields a strongly $\mathbb{S}_\xi$-generic condition over $\mathfrak{M}$.

To see that if ψ is a statement of the $\mathbb{S}_\xi$-forcing language using only parameters from $\mathfrak{M}$, then $\{\Gamma \in S(\mathfrak{M} \cap \xi, p_\omega) : \mathfrak{M}[\Gamma] \models \psi\}$ is a clopen set, observe that to any such ψ there corresponds the dense subset of $\mathbb{S}_\xi$ consisting of all conditions which decide ψ. Any such dense set is therefore D_n for some $n \in \omega$. It follows that if $\Gamma \in S(\mathfrak{M}\cap\xi, p_\omega)$ then the interpretation of $p_n\restriction(\xi \setminus \mu_n)$ in $\mathfrak{M}[\Gamma\restriction\mu_n]$ decides the truth value of ψ because $p_n\restriction\mu_n$ is strongly $\mathbb{S}_{\mu_n}$-generic over $\mathfrak{M}$. From the induction hypothesis it follows that there is a clopen set $U \subseteq S(\mathfrak{M} \cap \mu_n, p_n\restriction\mu_n)$ such that for each $\Gamma' \in U$ the model $\mathfrak{M}[\Gamma']$ is such that the interpretation of $p_n\restriction(\xi \setminus \mu_n)$ in $\mathfrak{M}[\Gamma\restriction\mu_n]$ decides the truth value of ψ. Let U^* be the lifting of U to $S(\mathfrak{M} \cap \xi, p_\omega)$—in other words, $\Gamma \in U^*$ if and only if $\Gamma\restriction\mu_n \in U$. Since the interpretation of $p_\omega\restriction(\xi \setminus \mu_n)$ in $\mathfrak{M}[\Gamma\restriction\mu_n]$ is a stronger condition than the interpretation of $p_n\restriction(\xi \setminus \mu_n)$ in $\mathfrak{M}[\Gamma\restriction\mu_n]$, it follows that $U^* \subseteq S(\mathfrak{M} \cap \xi, p_\omega)$ is the desired clopen set. ∎

DEFINITION 2.5. A subset $X \subseteq {}^n\omega$ is said to be a *full subset* if $X \neq \emptyset$ and for each $x \in X$ and $i \in n$ there is $A \in [\omega]^{\aleph_0}$ such that for all $m \in A$ there is $x_m \in X$ such that $x_m\restriction i = x\restriction i$ and $x_m(i) = m$.

LEMMA 2.3. *If $F : {}^n\omega \to [0,1]$ is a one-to-one function then there is a full subset $T \subseteq {}^n\omega$ such that the image of T under F is discrete.*

Proof. Proceed by induction on n to prove the following stronger assertion: If $F : {}^n\omega \to [0,1]$ is one-to-one then there is a full subset $T \subseteq {}^n\omega$, there is $f \in {}^\omega\omega$ and there is $x \in [0,1]$ such that

A. for any descending sequence $\{U_i : i \in \omega\}$ of neighbourhoods of x such that $\operatorname{diam}(U_{n+1}) \cdot f(\lceil 1/\operatorname{diam}(U_n)\rceil) < 1$ and for each $X \in [\omega]^{\aleph_0}$ the set $\{t \in T : F(t) \in \bigcup_{i\in X}(U_i \setminus \overline{U}_{i+1})\}$ is a full subset.

The case $n = 1$ is easy. Choose $A \in [\omega]^{\aleph_0}$ such that $\{F(\emptyset \wedge i) : i \in A\}$ converges to $x \in [0,1]$. Let $f \in {}^\omega\omega$ be any increasing function such that for each $m \in \omega$ there is some $j \in A$ such that $1/m > |F(\emptyset \wedge j)| > 1/f(m)$. Let $T = \{\emptyset \wedge i : i \in A\}$.

Now let $F : {}^{n+1}\omega \to [0,1]$ be one-to-one. Use the induction hypothesis to find, for each $m \in \omega$, full subsets $T_m \subseteq {}^n\omega$ such that the image of F restricted to

$$\{x \in {}^{n+1}\omega : (\exists t \in T_m)(x = \emptyset \wedge m \wedge t)\}$$

is a discrete family and Condition **A** is witnessed by $f_m \in {}^\omega\omega$ and $x_m \in [0,1]$. There are two cases to consider depending on whether or not there is $Z \in [\omega]^{\aleph_0}$ such that $\{x_m : m \in Z\}$ are all distinct.

Case 1. Assume that there is $Z \in [\omega]^{\aleph_0}$ such that $\{x_m : m \in Z\}$ are all distinct. It is then possible to assume that there is some $x \in [0,1]$ such that $\lim_{m\in Z} x_m = x$ and that, without loss of generality, $x_m > x_{m+1} > x$. As in the case $n = 1$, it is possible to find $f \in {}^\omega\omega$ such that for any descending sequence $\{U_i : i \in \omega\}$ of neighbourhoods of x such that $\operatorname{diam}(U_{n+1}) \cdot f(\lceil 1/\operatorname{diam}(U_n)\rceil) < 1$ and for each $X \in [\omega]^{\aleph_0}$ the set $\{m \in \omega : x_m \in \bigcup_{i\in X}(U_i \setminus \overline{U}_{i+1})\}$ is infinite. Notice that each $U_i \setminus \overline{U}_{i+1}$ is open, so it follows from Condition **A** that $\{t \in T_m : F(m \wedge t) \in U_i \setminus \overline{U}_{i+1}\}$ is a full subset provided that $x_m \in U_i \setminus \overline{U}_{i+1}$. Hence,

$$\bigcup\{\{t \in T_m : F(\langle m\rangle \wedge t) \in U_i \setminus \overline{U}_{i+1}\} : x_m \in U_i \setminus \overline{U}_{i+1}\}$$

is a full subset provided that $\operatorname{diam}(U_{n+1}) \cdot f(\lceil 1/\operatorname{diam}(U_n)\rceil) < 1$ and $X \in [\omega]^{\aleph_0}$. Let $T = \{t \in {}^{n+1}\omega : (\exists t' \in T_{t(0)})(t = t(0) \wedge t')\}$. Then T, f and x satisfy Condition **A**.

Case 2. In this case there exists $x \in [0,1]$ such that $x_m = x$ for all but finitely many $m \in \omega$. Let $f \in {}^\omega\omega$ be such that $f \geq^* f_m$ for all $m \in \omega$. Let

$$T = \{t \in {}^{n+1}\omega : (\exists t' \in T_{t(0)})(t = t(0) \wedge t' \text{ and } x_{t(0)} = x)\}.$$

To see that this works, suppose that $\{U_i : i \in \omega\}$ is a descending sequence of neighbourhoods of x such that $\operatorname{diam}(U_{i+1}) \cdot f(\lceil 1/\operatorname{diam}(U_i)\rceil) < 1$ and suppose that $X \in [\omega]^{\aleph_0}$.

Let $X = \bigcup_{j\in\omega} X_j$ be a partition of X into infinite subsets. It may be assumed that $f(i) \geq f_m(i)$ for all $i \in X_m$. By the induction hypothesis it follows that $\{t \in T_m : F(t) \in \bigcup_{i\in X_m}(U_i \setminus \overline{U}_{i+1})\}$ is a full subset of ${}^n\omega$ for each $m \in \omega$ because $f \geq^* f_m$. Hence $\{t \in T : F(t) \in \bigcup_{i\in X}(U_i \setminus \overline{U}_{i+1})\}$ is a full subset of ${}^{n+1}\omega$. ∎

Although this fact will not be used, it should be noted that Lemma 2.3 can be generalized to arbitrary well founded trees.

If $X \subseteq ({}^\omega\omega)^\alpha$ is large then for each $e : \beta \to {}^\omega\omega$ let X_e represent the set of all $f : \alpha \setminus \beta \to {}^\omega\omega$ such that $e \cup f \in X$. Note that if $h \in X$ then for every $\beta \in \alpha$, $X_{h\restriction\beta}$ is a large subset of $({}^\omega\omega)^{\alpha\setminus\beta}$. Moreover, the projection $X_{h\restriction\beta}$ to $({}^\omega\omega)^{\delta\setminus\beta}$ is large provided that $\beta \in \delta$. This set will be denoted by $\pi_\delta(X_{f\restriction\beta})$. Note that $\pi_{\beta+1}(X_{f\restriction\beta})$ is the closure of a super-perfect tree

$T_{X,f,\beta}$, and so $\theta_{T_{X,f,\beta}} : {}^{\omega}\omega \to T_{X,f,\beta}$ is an isomorphism. This induces a natural isomorphism from ${}^{\alpha}({}^{\omega}\omega)$ to the open sets of X, which will be denoted by Φ_X.

LEMMA 2.4. *Suppose* $\alpha \in \omega_1$, $\mathfrak{M}$ *is a countable elementary submodel*, $q \in \mathbb{S}_\alpha$ *and* $F : S(\mathfrak{M} \cap \alpha, q) \to \mathbb{R}$ *is continuous and satisfies*

B. *for each* $\beta \in \alpha$ *and each* $e \in ({}^{\omega}\omega)^{\beta}$, *if* $S(\mathfrak{M} \cap \alpha, q)_e \neq \emptyset$, *then the range of* F *restricted to* $S(\mathfrak{M} \cap \alpha, q)_e$ *is uncountable.*

Then there is a large closed set $X \subseteq S(\mathfrak{M} \cap \alpha, q)$ *such that* $F \restriction X$ *is one-to-one and*, *moreover*, $F \restriction X$ *is a homeomorphism onto its range.*

Proof. For $\tau, \tau' \in {}^{\alpha}({}^{\omega}\omega)$ define $\tau \le \tau'$ if and only if $\tau(\sigma) \subseteq \tau'(\sigma)$ for each σ in the domain of τ, and define τ_1 and τ_2 to be incompatible if there is no τ' such that $\tau_1 \le \tau'$ and $\tau_2 \le \tau'$. To begin, let $\{\tau_i : i \in \omega\}$ enumerate a subset of ${}^{\alpha}({}^{\omega}\omega)$ which forms a tree base for $S(\mathfrak{M} \cap \alpha, q)$—in other words, if i and j are in ω then either $\tau_i < \tau_j$, $\tau_j < \tau_i$ or τ_i and τ_j are incompatible; moreover, $\{\Phi_{S(\mathfrak{M}\cap\alpha,q)}(\tau_i) : i \in \omega\}$ is a base for $S(\mathfrak{M} \cap \alpha, q)$. It may also be assumed that if $\tau_i < \tau_j$ then $i \le j$ and that for each $k \in \omega$ there is a unique ϱ and some $i \in k$ such that $\tau_k(\mu) = \tau_i(\mu)$ if $\mu \neq \varrho$ and $\tau_k(\varrho) = \tau_i(\varrho) \wedge W$ for some integer W. Let $X_0 = S(\mathfrak{M} \cap \alpha, q)$. Construct by induction a sequence $\{(X_k, \{U_i : i \in k\} : k \in \omega\}$ such that:

(a) X_k is a large and closed subset of $({}^{\omega}\omega)^{\alpha}$,
(b) each U_i is an open subset of $\mathbb{R}$,
(c) $F(\Phi_{X_k}(\tau_i)) \subseteq U_i$,
(d) $\Phi_{X_{k+1}}(\tau_i) = \Phi_{X_k}(\tau_i) \cap X_{k+1}$ if $i < k$,
(e) $\overline{U}_i \cap \overline{U}_j = \emptyset$ if τ_i and τ_j are incompatible,
(f) $U_i \subseteq U_j$ if $\tau_j < \tau_i$,
(g) if $\tau_i < \tau_j$ then $\overline{U}_j \cap \overline{F(\Phi_{X_k}(\tau_i) \setminus \Phi_{X_k}(\tau_j))} = \emptyset$,
(h) X_k satisfies Condition **B** for each $k \in \omega$.

If this can be accomplished then let $X = \bigcap_{k\in\omega} X_k$. It follows that X is large and closed because, by (d), branching is eventually preserved at each node. Moreover, $F \restriction X$ is also one-to-one because of the choice of the U_i satisfying (e) for each $i \in \omega$. To see that F is a homeomorphism onto its range suppose that $V \subseteq X$ is an open set and that z belongs to the image of V under F. This means that there is some $i \in \omega$ and z' such that $z' \in \Phi_X(\tau_i) \subseteq V$ and $F(z') = z$. It follows that $z \in U_i \cap F(X)$ and so it suffices to show that $U_i \cap F(X) = F(\Phi_X(\tau_i))$. Clearly, (c) implies that $U_i \cap F(X) \supseteq F(\Phi_X(\tau_i))$. On the other hand, if $w \in U_i \cap F(X)$ then there is some $w' \in X$ such that $F(w') = w$. Since $w \in U_i$ it follows that $w' \in \Phi_{X_k}(\tau_i)$ for each $k \ge i$ because $\{\Phi_{X_k}(\tau_j) : j \in \omega\}$ is a tree base. Hence $w \in F(\Phi_X(\tau_i))$.

To perform the induction, use the hypothesis on $\{\tau_i : i \in k\}$ to choose a maximal τ_i below τ_k. Hence there is a unique ϱ such that $\tau_k(\mu) = \tau_i(\mu)$ if $\mu \neq \varrho$ and $\tau_k(\varrho) = \tau_i(\varrho) \wedge W$ for some integer W. The open set U_k will be chosen so that $\overline{U}_k \subseteq U_i$ and this will guarantee that if τ_j is incompatible with τ_i then $\overline{U}_k \cap \overline{U}_j = \emptyset$. The hypothesis on $\{\tau_i : i \in k\}$ also implies that there is no $j \in k$ such that $\tau_k < \tau_j$. Moreover, if $\tau_i < \tau_j$ then $\overline{F(\Phi_{X_k}(\tau_i) \setminus \Phi_{X_k}(\tau_j))} \cap \overline{U}_j = \emptyset$.

To satisfy Condition (g), let $\{\delta_m : m \in a\}$ enumerate, in increasing order, the domain of τ_i together with the unique ordinal ϱ and define $H : {}^a\omega \to \mathbb{R}$ as follows. Choose $y_s \in {}^{\alpha}({}^{\omega}\omega)$ so that for each $s \in {}^a\omega$:

- $y_s \in \Phi_{X_k}(\tau_i \wedge s)$ where, in this context, $\tau_i \wedge s$ is defined by $(\tau_i \wedge s)(\delta_m) = \tau_i(\delta_m) \wedge s(m)$,
- if $s \restriction j = s' \restriction j$ then $y_s \restriction \delta_j = y_{s'} \restriction \delta_j$,
- if $s \neq s'$ then $F(y_s) \neq F(y_{s'})$.

This is easily done using Condition **B** to satisfy the last two conditions. Finally, define $H(s) = F(y_s)$ and observe that this is one-to-one.

Now use Lemma 2.3 to find a full subset $T \subseteq {}^a\omega$ such that $H \restriction T$ has discrete image, and furthermore, this is witnessed by $\{\mathcal{V}_t : t \in T\}$. Shrinking T by a finite amount, if necessary, it may be assumed that

$$\Phi_{X_k}(\tau_j) \cap \Phi_{X_k}(\tau_i \wedge s) = \emptyset \quad \text{ for all } s \in T \text{ and } j \in k$$

because $a \geq 1$. Let

$$X_{k+1} = (X_k \setminus \Phi_{X_k}(\tau_i)) \cup \Big(\bigcup \{\Phi_{X_k}(\tau_i \wedge s) : s \in T\}\Big) \cup \Big(\bigcup \{\Phi_{X_k}(\tau_j) : \tau_i \leq \tau_j\}\Big)$$

and define $U_k = \mathcal{V}_{\overline{t}} \cap U_i$ where $\overline{t} \in T$ is lexicographically the first element of T. It is an easy matter to verify that all of the induction hypotheses are satisfied. ∎

To finish the proof of Lemma 2.1 suppose that $\xi \in \omega_2 + 1$ and $\mathbb{S}_\xi$ is the iteration with countable support of the partial orders $\mathbb{S}$. Suppose also that $p \Vdash_{\mathbb{S}_\xi}$ "$x \in [0,1]$" and

$$p \Vdash_{\mathbb{S}_\xi} \text{"}H : [0,1] \to [0,1] \text{ is a Borel function"}.$$

Let $\eta \in \omega_2$ be such that x occurs for the first time in the model $V[G \cap \mathbb{S}_\eta]$. Let $\mathfrak{M}$ be a countable elementary submodel of $H((2^{\aleph_0})^+)$ containing p and the names x and H. It follows from Lemma 2.2 that it is possible to find $q \leq p$ which is strongly $\mathbb{P}_\eta$-generic over $\mathfrak{M}$. Let $F : S(\mathfrak{M} \cap \xi, q) \to [0,1]$ be defined by $F(\Gamma) = x_\Gamma$ or, in other words, $F(\Gamma)$ is the interpretation of x in $\mathfrak{M}[\Gamma]$. It follows from the second clause of Definition 2.2 that F is a continuous function. Moreover, because it is assumed that x does not belong to any model $\mathfrak{M}[G \cap \mathbb{S}_\mu]$ where $\mu \in \eta$, it follows that Condition **B** of Lemma 2.4 is satisfied by F. Using this lemma, and the fact that $\eta \cap \mathfrak{M}$ has countable order type, it is possible to find $q' \le q$ such that $\mathrm{dom}(q) = \mathrm{dom}(q')$ and $F{\restriction}S(\mathfrak{M} \cap \eta, q')$ is a homeomorphism onto its range.

Now let X be the image of $S(\mathfrak{M} \cap \eta, q')$ under the mapping F. An inspection of the definition of $S(\mathfrak{M} \cap \eta, q')$ reveals it to be a Borel set. Since $F{\restriction}S(\mathfrak{M} \cap \eta, q')$ is one-to-one, it follows that X is also Borel. Obviously $q' \Vdash_{\mathbb{S}_{\omega_2}}$ "$x \in X$". Because the name H belongs to $\mathfrak{M}$ and F is one-to-one on X, it is possible to define a mapping $H' : X \to [0,1]$ by letting $H'(z)$ be the interpretation of $H(x)$ in $\mathfrak{M}[F^{-1}(z)]$. Obviously $q' \Vdash_{\mathbb{S}_{\omega_2}}$ "$H(x) = H'(x)$".

All that remains to be shown is that H' is continuous. To see this, let $z \in X$. Then there is some $\Gamma \in S(\mathfrak{M} \cap \eta, q'')$ such that $z = F(\Gamma) = x_\Gamma$. For any interval with rational end-points, (p,q), the statement $\psi_{p,q}$ which asserts that $H(x) \in (p,q)$ has all of its parameters in $\mathfrak{M}$. Moreover, $\mathfrak{M}[\Gamma] \models H(x) = H(x_\Gamma) = H'(z)$. For each interval with rational end-points containing $H'(z)$, (p,q), there is therefore an open neighbourood $U_{p,q}$ of Γ such that $\mathfrak{M}[\Gamma'] \models \psi_{p,q}$ for each $\Gamma' \in U_{p,q}$. Since $F{\restriction}S(\mathfrak{M} \cap \eta, q'')$ is a homeomorphism, it follows that the image of any $U_{p,q}$ under F is an open neighbourhood $U^*_{p,q}$ of z. Now, if $\overline{z} \in U^*_{p,q}$ then $\overline{z} = x_{\Gamma'}$ for some $\Gamma' \in U_{p,q}$, and therefore $\mathfrak{M}[\Gamma'] \models \psi_{p,q}$. This means that the interpretation of $H(x)$ in $\mathfrak{M}[\Gamma']$ belongs to (p,q). Hence the image of $U^*_{p,q}$ under H' is contained in (p,q) and so H' is continuous.

3. Remarks. The proof presented here can also be generalized, without difficulty, to apply to the iteration of ω_2 Laver reals as well super-perfect reals. The notion of a large set has its obvious analogue which can be used to deal with the iteration. In the single step case use the proof that a Laver real is minimal [2]. The only difference is that, for a Laver condition T, the "frontiers" of [2] should be used in place of the images of $\theta_T{\restriction}^n\omega$. In fact, the proof of the preceding section can be viewed as a generalization of the fact that adding super-perfect real adds a minimal real in the sense that the structure of the iterated model is shown to depend very predictably on the generic reals added.

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