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Recursive expansions

by

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Abstract. Let $\mathcal{A}$ be a recursive structure, and let ψ be a recursive infinitary Π_2 sentence involving a new relation symbol. The main result of the paper gives syntactical conditions which are necessary and sufficient for every recursive copy of $\mathcal{A}$ to have a recursive expansion to a model of ψ, provided $\mathcal{A}$ satisfies certain decidability conditions. The decidability conditions involve a notion of rank. The main result is applied to prove some earlier results of Metakides–Nerode and Goncharov. In these applications, the ranks turn out to be low, but there are examples in which the rank takes arbitrary recursive ordinal values.

0. Introduction. Let $\mathcal{A}$ be a recursive L-structure. Let ψ be a recursive infinitary Π_2 sentence in the language $L \cup \{P\}$, where P is a new relation symbol. (Roughly speaking, a recursive infinitary formula is an infinitary formula with recursive disjunctions and conjunctions.) We consider in this paper the question of whether every recursive structure $\mathcal{B} \cong \mathcal{A}$ has an expansion to a recursive model of ψ. We look for an answer involving syntactical conditions on the structure $\mathcal{A}$.

In [AK], we considered the related question of whether every (not necessarily recursive) $\mathcal{B} \cong \mathcal{A}$ has an expansion to a model of ψ which is recursive *relative* to $\mathcal{B}$. We established syntactical conditions necessary and sufficient for this. The conditions in [AK] assert the existence of a *formally Σ_1^0 expansion family for ψ on $\mathcal{A}$*. Here we shall give a slightly different definition, of a *recursive expansion family for ψ on $\mathcal{A}$*. The two notions are equivalent, in the sense that the existence of one easily implies the existence of the other. If there is a recursive expansion family for ψ on $\mathcal{A}$, then by the result of [AK], every recursive $\mathcal{B} \cong \mathcal{A}$ has a recursive expansion satisfying ψ. We show here that if there is no such family and if, in addition, $\mathcal{A}$ satisfies certain decidability conditions, then there exists a recursive $\mathcal{B} \cong \mathcal{A}$ with no

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recursive expansion satisfying ψ. Thus, for $\mathcal{A}$ sufficiently decidable, we have the solution to our problem. For every recursive structure $\mathcal{B} \cong \mathcal{A}$ to have an expansion to a recursive model of ψ, a necessary and sufficient condition is the existence of a recursive expansion family for ψ on $\mathcal{A}$.

The decidability conditions on $\mathcal{A}$ cannot be omitted. In [AKS], there is an example of a recursive structure $\mathcal{A}$ and a recursive infinitary Π_2 sentence ψ such that for all recursive $\mathcal{B} \cong \mathcal{A}$, $\mathcal{B}$ has recursive expansion satisfying ψ, but for some non-recursive $\mathcal{B} \cong \mathcal{A}$, $\mathcal{B}$ has no expansion to a model of ψ which is recursive relative to $\mathcal{B}$. The decidability conditions involve a kind of rank, which depends on $\mathcal{A}$ and ψ. If some object in a certain class fails to have ordinal rank, then under appropriate decidability conditions, we obtain a recursive expansion family. If all objects in the class have ordinal rank, then there is a recursive ordinal bound, and under some decidability conditions, there is a recursive $\mathcal{B} \cong \mathcal{A}$ with no recursive expansion satisfying ψ. The structure $\mathcal{B}$ is obtained by a finite injury priority construction. There is a connection between rank and number of injuries to a given requirement. In particular, if the ranks are bounded by some finite n, then a given requirement receives attention at most n times after the last injury involving higher priority requirements, as the rank is reduced step by step to 0.

In §1, we define rank and prove the result described above. We also give special result for the case where the ranks are bounded by 1. In §2, we give two applications of the special result. The first is a result of Metakides and Nerode [MN], saying that there is a recursive algebraically closed field having infinite transcendence degree but with no infinite recursive algebraically independent set. The second application is a weak version of a result of Goncharov [G], on "recursively categorical" structures. In §3, we construct some rather complicated examples in which the ranks are arbitrary recursive ordinals. The remainder of the present section contains some definitions.

Let L be a recursive language, and let $\mathcal{A}$ be an L-structure with universe ω. Let P be an r-placed relation symbol. A *P-formula* is a consistent formula $\varrho(\mathbf{x})$ which is either $\top$ (logically valid) or a finite conjunction of formulas of the forms $P(x_{i_1}, \ldots, x_{i_r})$ or $\neg P(x_{i_1}, \ldots, x_{i_r})$. A *$P$-sentence* $\varrho(\mathbf{a})$ on $\mathcal{A}$ is the result of replacing each variable in a P-formula $\varrho(\mathbf{x})$ by a constant naming an element of $\mathcal{A}$. Let ψ be a recursive infinitary Π_2 sentence of the language $L \cup \{P\}$. Then ψ has the form $\bigwedge\!\!\!\bigwedge_i \forall \mathbf{x}_i \bigvee\!\!\!\bigvee_j \exists \mathbf{y}_{ij}\, \delta_{ij}(\mathbf{x}_i, \mathbf{y}_{ij})$, where$\delta_{ij}(\mathbf{x}_i, \mathbf{y}_{ij})$ is open. We can re-arrange ψ so that each $\delta_{ij}(\mathbf{x}_i, \mathbf{y}_{ij})$ is a conjunction of atomic formulas and negations of atomic formulas, and only variables, not more complicated terms, occur in the conjuncts which involve P. Then $\delta_{ij}(\mathbf{x}_i, \mathbf{y}_{ij}) = \alpha_{ij}(\mathbf{x}_i, \mathbf{y}_{ij}) \,\&\, \varrho_{ij}(\mathbf{x}_i, \mathbf{y}_{ij})$, where $\alpha_{ij}(\mathbf{x}_i, \mathbf{y}_{ij})$ is an open L-formula and $\varrho_{ij}(\mathbf{x}_i, \mathbf{y}_{ij})$ is a P-formula. Let $\sigma(\mathbf{a})$ be a P-sentence on $\mathcal{A}$, and let $\mathbf{b} \subseteq \mathbf{a}$. We say that $\sigma(\mathbf{a})$ *decides* P on $\mathbf{b}$ if for each r-tuple $\mathbf{d}$ in $\mathbf{b}$, either $\sigma(\mathbf{a}) \vdash P(\mathbf{d})$ or $\sigma(\mathbf{a}) \vdash \neg P(\mathbf{d})$. (This means that for each r-tuple

$\mathbf{d}$ from $\mathbf{b}$, either $P(\mathbf{d})$ or $\neg P(\mathbf{d})$ is a conjunct in $\sigma(\mathbf{a})$.) Suppose I is a finite subset of ω. We say that $\sigma(\mathbf{a})$ *gives evidence for* ψ *on* I *and* $\mathbf{b}$ if for all $i \in I$ and $\mathbf{c}$ in $\mathbf{b}$, there exist $j \in \omega$ and $\mathbf{d}$ in $\mathbf{a}$ such that $\mathcal{A} \vDash \alpha_{ij}(\mathbf{c}, \mathbf{d})$ and $\sigma(\mathbf{a}) \vdash \varrho_{ij}(\mathbf{c}, \mathbf{d})$. An *expansion family* for ψ on $\mathcal{A}$, with parameters $\mathbf{c} \in \mathcal{A}$, is a set $\mathfrak{F}$of pairs $(\varphi(\mathbf{c}, \mathbf{x}), \varrho(\mathbf{x}))$ such that $\varphi(\mathbf{c}, \mathbf{x})$ is an existential L-formula, $\varrho(\mathbf{x})$ is a P-formula, and the following conditions hold:

(1) if $(\varphi(\mathbf{c}, \mathbf{x}), \varrho(\mathbf{x})) \in \mathfrak{F}$, then there exists $\mathbf{a} \in \mathcal{A}$ such that $\mathcal{A} \vDash \varphi(\mathbf{c}, \mathbf{a})$,
(2) if $(\varphi(\mathbf{c}, \mathbf{x}), \varrho(\mathbf{x})) \in \mathfrak{F}$ and $\mathcal{A} \vDash \varphi(\mathbf{c}, \mathbf{a})$, then for any finite $I \subseteq \omega$ and $\mathbf{a}_1 \supseteq \mathbf{a}$, there exist $(\varphi'(\mathbf{c}, \mathbf{x}), \varrho'(\mathbf{x})) \in \mathfrak{F}$ and $\mathbf{a}_2 \supseteq \mathbf{a}_1$ such that $\mathcal{A} \vDash \varphi'(\mathbf{c}, \mathbf{a}_2)$, $\varrho'(\mathbf{a}_2) \vdash \varrho(\mathbf{a})$, and $\varrho'(\mathbf{a}_2)$ decides P on $\mathbf{a}_1$ and gives evidence for ψ on I and $\mathbf{a}_1$.

A *recursive expansion family* for ψ on $\mathcal{A}$ is an expansion family $\mathfrak{F}$ which is r.e.

As we mentioned above, the formally Σ_1^0 expansion families of [AK] are equivalent to these recursive expansion families, in the sense that a family of one kind can easily be converted into one of the other kind.

1. Ranks and expansions. Throughout this section, $\mathcal{A}$ is an L-structure with universe ω and ψ is a recursive infinitary Π_2 sentence of the language $L \cup \{P\}$, as described in §0. Let R_M be the set of all P-sentences on $\mathcal{A}$ for which there is no relation P on $\mathcal{A}$ such that $(\mathcal{A}, P) \vDash \varrho(\mathbf{a}) \,\&\, \psi$. We say that R is a set *suitable for rank* 0 if $R \subseteq R_M$ and for any P-sentences ϱ, ϱ' on $\mathcal{A}$, if $\varrho' \vdash \varrho$ and $\varrho \in R$, then $\varrho' \in R$. The set R_M is suitable for rank 0. So is the set R_0 which consists of all P-sentences on $\mathcal{A}$ such that for some finite $I \subseteq \omega$ and $\mathbf{a}_1 \supseteq \mathbf{a}$, there is no P-sentence $\varrho'(\mathbf{a}_2)$ such that $\varrho'(\mathbf{a}_2) \vdash \varrho(\mathbf{a})$ and $\varrho'(\mathbf{a}_2)$ decides P on $\mathbf{a}_1$ and gives evidence for ψ on I and $\mathbf{a}_1$.

Let R be a set suitable for rank 0. Let $\mathbf{c}$ be a finite sequence of elements of $\mathcal{A}$ and let $\varrho(\mathbf{a})$ be a P-sentence on $\mathcal{A}$. The *R-rank* of $\varrho(\mathbf{a})$ over $\mathbf{c}$, denoted by $R(\varrho(\mathbf{a})|\mathbf{c})$, is defined as follows:

(1) $R(\varrho(\mathbf{a})|\mathbf{c}) = 0$ if $\varrho(\mathbf{a}) \in R$,
(2) $R(\varrho(\mathbf{a})|\mathbf{c}) = \alpha$, for $\alpha > 0$, if for all $\beta < \alpha$, $R(\varrho(\mathbf{a})|\mathbf{c}) \neq \beta$, and there exist finite $\mathbf{a}_1 \supseteq \mathbf{a}$ and $I \subseteq \omega$ such that for any P-sentence $\varrho'(\mathbf{a}_2)$ on $\mathcal{A}$ such that $\varrho'(\mathbf{a}_2) \vdash \varrho(\mathbf{a})$ and $\varrho'(\mathbf{a}_2)$ decides P on $\mathbf{a}_1$ and gives evidence for ψ on I and $\mathbf{a}_1$, and for all existential L-formulas $\varphi(\mathbf{c}, \mathbf{x})$ such that $\mathcal{A} \vDash \varphi(\mathbf{c}, \mathbf{a}_2)$, there exists $\mathbf{a}_2'$ such that $\mathcal{A} \vDash \varphi(\mathbf{c}, \mathbf{a}_2')$ and $R(\varrho'(\mathbf{a}_2')|\mathbf{c}) < \alpha$. We say that$\mathbf{a}_1$ and I *witness* the rank (more precisely, they witness the fact that $R(\varrho(\mathbf{a})|\mathbf{c}) \leq \alpha$).

We write $R(\varrho(\mathbf{a})|\mathbf{c}) = \infty$ if there is no ordinal α such that $R(\varrho(\mathbf{a})|\mathbf{c}) = \alpha$.

The lemma below gives relations between ranks obtained from different sets suitable for rank 0.

Lemma 1.1. *Suppose* $R_1 \subseteq R_2$, *where both sets are suitable for rank* 0.

(a) *If* $R_1(\varrho(\mathbf{a})|\mathbf{c}) \le \alpha$, *then* $R_2(\varrho(\mathbf{a})|\mathbf{c}) \le \alpha$.
(b) *If* $R_2(\varrho(\mathbf{a})|\mathbf{c}) = \infty$, *then* $R_1(\varrho(\mathbf{a})|\mathbf{c}) = \infty$.

The proof of (a) is an easy induction on α, and (b) follows from (a).
The next lemma gives some basic properties of ranks.

Lemma 1.2. *Let* R *be a set suitable for rank* 0.

(a) *If* ϱ *and* ϱ' *are* P*-sentences on* $\mathcal{A}$ *and* $\varrho' \vdash \varrho$, *then* $R(\varrho'|\mathbf{c}) \le R(\varrho|\mathbf{c})$, *if the ranks are ordinals*; *and if* $R(\varrho'|\mathbf{c}) = \infty$, *then* $R(\varrho|\mathbf{c}) = \infty$.
(b) *If* $\mathbf{c}' \supseteq \mathbf{c}$ *and* ϱ *is a* P*-sentence on* $\mathcal{A}$, *then* $R(\varrho|\mathbf{c}) \le R(\varrho|\mathbf{c}')$, *if the ranks are ordinals*; *and if* $R(\varrho|\mathbf{c}) = \infty$, *then* $R(\varrho|\mathbf{c}') = \infty$.
(c) *If* $R(\varrho|\mathbf{c}) = \alpha$, *then for each* $\beta < \alpha$, *there exists* ϱ' *such that* $\varrho' \vdash \varrho$ *and* $R(\varrho'|\mathbf{c}) = \beta$.
(d) *If* ϱ *is a* P*-sentence on* $\mathcal{A}$, *then the following are equivalent*:

(i) *there is some finite* $\mathbf{c} \in \mathcal{A}$ *such that* $R(\varrho|\mathbf{c}) = 0$,
(ii) $R(\varrho|\emptyset) = 0$,
(iii) *for all* $\mathbf{c} \in \mathcal{A}$, $R(\varrho|\mathbf{c}) = 0$.

The proofs of all of these statements are easy. It is for (a) that we need the fact that if $\varrho' \vdash \varrho$ and $\varrho \in R$, then $\varrho' \in R$.

It is obvious that for the given structure $\mathcal{A}$ and sentence ψ, and any set R suitable for rank 0, exactly one of the following holds.

Case A. There exist $\mathbf{c}$ and ϱ such that $R(\varrho|\mathbf{c}) = \infty$.
Case B. For all $\mathbf{c}$ and ϱ, $R(\varrho|\mathbf{c})$ has ordinal value.

The next lemma says that the existence of an expansion family puts us in Case A.

Lemma 1.3. *Let* R *be a set suitable for rank* 0. *If* $\mathfrak{F}$ *is an expansion family for* ψ *on* $\mathcal{A}$, *with parameters* $\mathbf{c}$, *then for* $(\varphi(\mathbf{c},\mathbf{x}), \varrho(\mathbf{x})) \in \mathfrak{F}$ *and all* $\mathbf{a}$ *such that* $\mathcal{A} \vDash \varphi(\mathbf{c},\mathbf{a})$, $R(\varrho(\mathbf{a})|\mathbf{c}) = \infty$.

Proof. Suppose not. Let α be the least ordinal such that for some $(\varphi(\mathbf{c},\mathbf{x}), \varrho(\mathbf{x})) \in \mathfrak{F}$, there exists $\mathbf{a}$ such that $\mathcal{A} \vDash \varphi(\mathbf{c},\mathbf{a})$ and $R(\varrho(\mathbf{a})|\mathbf{c}) = \alpha$. The expansion family allows us to construct P such that $(\mathcal{A}, P) \vDash \psi \,\&\, \varrho(\mathbf{a})$, so $\alpha > 0$. Then there exist $I \subseteq \omega$ and $\mathbf{a}_1$ witnessing the fact that $R(\varrho(\mathbf{a})|\mathbf{c}) = \alpha$. By the definition of expansion family, there exist $(\varphi'(\mathbf{c},\mathbf{x}'), \varrho'(\mathbf{x}')) \in \mathfrak{F}$ and $\mathbf{a}_2$ such that $\mathcal{A} \vDash \varphi'(\mathbf{c},\mathbf{a}_2)$, $\varrho'(\mathbf{a}_2) \vdash \varrho(\mathbf{a})$, and $\varrho'(\mathbf{a}_2)$ decides P on $\mathbf{a}_1$ and gives evidence for ψ on I and $\mathbf{a}_1$. By the definition of rank and the choice of I_1 and $\mathbf{a}_1$, there exists $\mathbf{a}_2'$ such that $\mathcal{A} \vDash \varphi'(\mathbf{c},\mathbf{a}_2')$ and $R(\varrho'(\mathbf{a}_2)|\mathbf{c}) < \alpha$, a contradiction.

The next lemma says that if our set suitable for rank 0 includes R_0 and we are in Case A, then there exists an expansion family.

LEMMA 1.4. *Let R be a set suitable for rank* 0, *where $R_0 \subseteq R$. Suppose that for some $\mathbf{c}$ and $\varrho_0(\mathbf{a}_0)$, $R(\varrho_0(\mathbf{a}_0)|\mathbf{c}) = \infty$. Let $\mathfrak{F}$ be the set of pairs $(\varphi(\mathbf{c},\mathbf{x}),\varrho(\mathbf{x}))$ such that $\varphi(\mathbf{c},\mathbf{x})$ is an existential L-formula, $\varrho(\mathbf{x})$ is a P-formula, there exists $\mathbf{a}$ such that $\mathcal{A} \vDash \varphi(\mathbf{c},\mathbf{a})$ and $\varrho(\mathbf{a}) \vdash \varrho_0(\mathbf{a}_0)$, and for all $\mathbf{a}'$ such that $\mathcal{A} \vDash \varphi(\mathbf{c},\mathbf{a}')$, $R(\varrho(\mathbf{a}')|\mathbf{c}) = \infty$. Then $\mathfrak{F}$ is an expansion family for ψ on $\mathcal{A}$.*

Proof. The fact that $\mathfrak{F}$ is an expansion family follows easily from the statement below.

CLAIM. *If $R(\varrho(\mathbf{a})|\mathbf{c}) = \infty$, then for all finite $I \subseteq \omega$ and $\mathbf{a}_1$, there exist an existential L-formula $\varphi(\mathbf{c},\mathbf{x})$ and a P-sentence $\varrho'(\mathbf{a}_2)$ such that $\mathcal{A} \vDash \varphi(\mathbf{c},\mathbf{a}_2)$, $\varrho'(\mathbf{a}_2) \vdash \varrho(\mathbf{a})$, $\varrho'(\mathbf{a}_2)$ decides P on $\mathbf{a}_1$ and gives evidence for ψ on I and $\mathbf{a}_1$, and for all $\mathbf{a}_2'$ such that $\mathcal{A} \vDash \varphi(\mathbf{c},\mathbf{a}_2')$, $R(\varrho'(\mathbf{a}_2')|\mathbf{c}) = \infty$.*

Proof of claim. Suppose the statement fails, and take I and $\mathbf{a}_1$ witnessing the failure. Let C be the set of triples $(\mathbf{a}_2,\varrho'(\mathbf{x}),\varphi(\mathbf{c},\mathbf{x}))$ such that $\varphi(\mathbf{c},\mathbf{x})$ is an existential L-formula and $\varrho'(\mathbf{x})$ is a P-formula such that $\mathcal{A} \vDash \varphi(\mathbf{c},\mathbf{a}_2)$,$\varrho'(\mathbf{a}_2) \vdash \varrho(\mathbf{a})$, and $\varrho'(\mathbf{a}_2)$ decides P on $\mathbf{a}_1$ and gives evidence for ψ on $I$and $\mathbf{a}_1$. Since $R_0 \subseteq R$ and $R(\varrho(\mathbf{a})|\mathbf{c}) = \infty$, we have $\varrho(\mathbf{a}) \notin R$, so $\varrho(\mathbf{a}) \notin R_0$. Therefore, $C \neq \emptyset$. By our assumptions, for each $(\mathbf{a}_2,\varrho'(\mathbf{x}),\varphi(\mathbf{c},\mathbf{x})) \in C$, there exists $\mathbf{a}_2'$ such that $\mathcal{A} \vDash \varphi(\mathbf{c},\mathbf{a}_2')$ and $R(\varrho'(\mathbf{a}_2')|\mathbf{c})$ is an ordinal. Let $\beta(\mathbf{a}_2,\varrho'(\mathbf{x}),\varphi(\mathbf{c},\mathbf{x}))$ be the least such ordinal. Let α be least such that for all $(\mathbf{a}_2,\varrho'(\mathbf{x}),\varphi(\mathbf{c},\mathbf{x})) \in C$, $\beta(\mathbf{a}_2,\varrho'(\mathbf{a}_2),\varphi) < \alpha$. Since $C \neq \emptyset$, $\alpha > 0$. Then I and $\mathbf{a}_1$ witness that $R(\varrho(\mathbf{a})|\mathbf{c}) \leq \alpha$, a contradiction.

LEMMA 1.5. *Let R be a set suitable for rank* 0, *where $R_0 \subseteq R$. Then the following are equivalent*:

(1) *there exists an expansion family $\mathfrak{F}$ for ψ on A, with parameters* $\mathbf{c}$,
(2) $R(\top|\mathbf{c}) = \infty$,
(3) *there exists $\varrho(\mathbf{a})$ such that $R(\varrho(\mathbf{a})|\mathbf{c}) = \infty$.*

Proof. Lemma 1.2 gives (2)⇔(3). Lemma 1.3 gives (1)⇒(2), and Lemma 1.4 gives (3)⇒(1).

Remark. It follows from Lemma 1.5 that if there exists a set $R \supseteq R_0$ suitable for rank 0 such that R-rank takes only ordinal values, then the same is true for all sets $R \supseteq R_0$ suitable for rank 0.

LEMMA 1.6. *Let $R \supseteq R_0$ be a set suitable for rank* 0, *and suppose R-rank takes only ordinal values. If $\mathcal{A}$ is recursive*, *then there is a recursive ordinal bounding the values. In general*, *there is an ordinal bound in the least admissible set over $\mathcal{A}$.*

Proof. We suppose that $\mathcal{A}$ is recursive. By the remark above, R_0-rank takes only ordinal values. Then by Lemma 1.1(a), it is enough to show that there is a recursive ordinal bounding the values for R_0-rank. If A is the least admissible set, then $\mathcal{A}$ is an element of A. By Lemma 1.2(c), the set of ordinals $R_0(\varrho|\mathbf{c})$ is closed downwards, so it is an ordinal, say β. We suppose $\beta \geq \omega_1^{\mathrm{CK}}$, hoping for a contradiction.

Case 1: Suppose $\beta = \omega_1^{\mathrm{CK}}$. Let C be the set of pairs $(\varrho, \mathbf{c})$ such that ϱ is a P-sentence on $\mathcal{A}$ and $\mathbf{c}$ is a finite sequence from $\mathcal{A}$. We consider some formulas in the language of set theory. There exist Σ_1 and Π_1 formulas, both elements of A, with parameter $\mathcal{A}$, saying (in the structure $(A, \in)$) that $(\varrho, \mathbf{c}) \in C$ and $R_0(\varrho|\mathbf{c}) = \gamma$. (It is here that we need R_0; the existence of these formulas would not be clear for R_M.) By Σ_1 replacement, there is a set $B \in A$ such that for all $(\varrho, \mathbf{c}) \in C$, $R_0(\varrho|\mathbf{c}) \in B$. Then B contains all recursive ordinals, a contradiction.

Case 2: Suppose $\beta > \omega_1^{\mathrm{CK}}$. There exist ϱ_0 and $\mathbf{c}$ such that $R_0(\varrho_0|\mathbf{c}) = \omega_1^{\mathrm{CK}}$. Let I and $\mathbf{a}_1$ witness the rank. Let C be the set of triples $(\varrho(\mathbf{u}), \mathbf{a}_2, \varphi(\mathbf{c}, \mathbf{u}))$ such that $\varrho(\mathbf{u})$ is a P-formula, $\varrho(\mathbf{a}_2) \vdash \varrho_0$, $\varrho(\mathbf{a}_2)$ decides P on $\mathbf{a}_1$ and gives evidence for ψ on I and $\mathbf{a}_1$, and $\varphi(\mathbf{c}, \mathbf{u})$ is an existential L-formula such that $\mathcal{A} \models \varphi(\mathbf{c}, \mathbf{a}_2)$. There is a Σ_1 formula defining (in $(A, \in)$) the function F on C where $F(\varrho(\mathbf{u}), \mathbf{a}_2, \varphi(\mathbf{c}, \mathbf{u}))$ is the first β such that for some $\mathbf{a}_2' \in \mathcal{A}$, $\mathcal{A} \models \varphi(\mathbf{c}, \mathbf{a}_2')$ and $R_0(\varrho(\mathbf{a}_2')) = \beta$. (Here again we need R_0.) By Σ_1 replacement, there exists $B \in A$ such that B contains all elements of $\mathrm{ran}(F)$. However, $\mathrm{ran}(F)$ is cofinal in ω_1^{CK}, a contradiction.

LEMMA 1.7. *Let R be a set suitable for rank* 0. *Suppose that R-rank takes only ordinal values*, *and α* (*a recursive ordinal*) *is the supremum of the values. Suppose further that*

(1) *the relation $R(\varrho|\mathbf{c}) = \beta$ is r.e. uniformly in β for $\beta < \alpha$*,

(2) *given ϱ, $\mathbf{c}$ such that $R(\varrho|\mathbf{c}) \leq \beta$, where $0 < \beta \leq \alpha$, we can find I and$\mathbf{a}_1$ witnessing the rank.*

Then there is a recursive $\mathcal{B} \cong \mathcal{A}$ such that for all recursive P, $(\mathcal{B}, P) \not\models \psi$.

Proof. Let B be an infinite recursive set of constants, to be used for the universe of $\mathcal{B}$ (we could take $B = \omega$). We determine $\mathcal{B}$ from a function F such that $\mathcal{B} \cong_F \mathcal{A}$. The construction proceeds in stages. At stage s, we will have determined a finite function f_s (tentatively part of F), and we will have enumerated a finite part δ_s of $D(\mathcal{B})$ such that f_s makes δ_s true in $\mathcal{A}$. For each $e \in \omega$, we have a requirement

$$R_e: \quad \text{if } \varphi_e = \chi_P, \text{ then } (\mathcal{B}, P) \models \neg\psi.$$

We describe the strategy for meeting requirement R_e. Let g_0 be the part of f_s being protected for higher priority requirements at the stage s when we start on R_e. Say g_0 maps $\mathbf{b}_0$ to $\mathbf{a}_0$. On the assumption that φ_e may be the characteristic function of a relation P, let ϱ_0 be the P-sentence carrying the information which φ_e has given about P up to stage s. Suppose

$R(\varrho_0|\mathbf{a}_0) = \alpha_0$. If $\alpha_0 = 0$, there is nothing to do. Suppose $\alpha_0 > 0$, and let I_1 and $\mathbf{a}_1$ witness the rank. Extend f_s to f'_s, adding $\mathbf{a}_1$ to the range, and let g_1 be the restriction of f'_s such that g_1 maps some $\mathbf{b}_1$ onto $\mathbf{a}_1$. Now, protect g_1, working on requirements of lower priority, until a stage t (if any) at which we have produced f_t and δ_t such that $f_t \supseteq g_1$, f_t takes some $\mathbf{b}_2$ to $\mathbf{a}_2 \supseteq \mathbf{a}_1$, the constants mentioned in δ_t are all in $\mathbf{b}_2$, and f_t makes δ_t true in $\mathcal{A}$, and φ_e has produced $\varrho_1(\mathbf{b}_2)$ such that $\varrho_1(\mathbf{a}_2)$ decides P on $\mathbf{a}_1$ and gives evidence for ψ on I_1 and$\mathbf{a}_1$. By the definition of rank, there exists $\mathbf{a}'_2 \supseteq \mathbf{a}_0$ such that $\mathcal{A} \models \delta_t(\mathbf{a}'_2)$ and $R(\varrho_1(\mathbf{a}'_2)|\mathbf{c}) = \alpha_1$ for some $\alpha_1 < \alpha_0$. Let g_2 map $\mathbf{b}_2$ to $\mathbf{a}'_2$.

We have succeeded in reducing the rank. If $\alpha_1 > 0$, then let g_2, $\mathbf{b}_2$, $\mathbf{a}'_2$, and $\varrho_1(\mathbf{a}'_2)$ play the role of g_0, $\mathbf{b}_0$, $\mathbf{c}$, and ϱ_0 above. The process continues until either φ_e fails to force our next move or we have g_{2n} mapping $\mathbf{b}_{2n}$ to$\mathbf{a}'_{2n}$ such that $R(\varrho_n(\mathbf{a}'_{2n})|\mathbf{c}) = 0$. This takes care of requirement R_e. It is clear that we can meet all of the requirements to produce the desired F and $\mathcal{B}$.

We shall gather our various lemmas into a single theorem. In order to keep the statement relatively short, we state the decidability conditions first.

Decidability conditions

Case A. Suppose that $R(\varrho|\mathbf{c})$ takes only ordinal values, and α is the supremum of the values. Suppose further that

(a) the relation $R(\varrho|\mathbf{c}) = \beta$ is r.e. uniformly in β, for $\beta < \alpha$, and

(b) given ϱ, $\mathbf{c}$, and $0 < \beta \le \alpha$ such that $R(\varrho|\mathbf{c}) \le \beta$, we can find I and $\mathbf{a}_1$ witnessing the rank.

Case B. Suppose that for some $\varrho_0(\mathbf{a}_0)$ and $\mathbf{c}$, $R(\varrho(\mathbf{a}_0)|\mathbf{c}) = \infty$. Let $\mathfrak{F}$ be the set of pairs $(\varphi(\mathbf{c},\mathbf{x}), \varrho(\mathbf{x}))$ such that $\varphi(\mathbf{c},\mathbf{x})$ is an existential L-formula, $\varrho(\mathbf{x})$ is a P-formula, there exists $\mathbf{a}$ such that $\mathcal{A} \models \varphi(\mathbf{c},\mathbf{a})$ and $\varrho(\mathbf{a}) \vdash \varrho_0(\mathbf{a}_0)$, and for all $\mathbf{a}'$ such that $\mathcal{A} \models \varphi(\mathbf{c},\mathbf{a}')$, $R(\varrho(\mathbf{a}')|\mathbf{c}) = \infty$, and suppose that $\mathfrak{F}$ is r.e.

THEOREM 1.8. *Suppose $\mathcal{A}$ is a recursive L-structure and ψ is a recursive infinitary Π_2 sentence in the language $L \cup \{P\}$. Let R be suitable for rank* 0, *where $R_0 \subseteq R$. Then under the decidability conditions above*, *the following are equivalent*:

(1) *for all recursive $\mathcal{B} \cong \mathcal{A}$, there is a recursive P such that $(\mathcal{B}, P) \models \psi$*,
(2) *there is a recursive expansion family for ψ on $\mathcal{A}$.*

Proof. Showing that (2)⇒(1) does not require the decidability conditions. The proof is in [AK]. We prove (1)⇒(2) in the two cases. In Case A, there is no expansion family, by Lemma 1.5, so we have ¬(2). By Lemma 1.6, there is a recursive bound on the ranks. By Lemma 1.7, under the decidability conditions, there is a recursive $\mathcal{B} \cong \mathcal{A}$ with no recursive P such that $(\mathcal{B}, P) \models \psi$, so we have $\neg(1)$. In Case B, Lemma 1.4 yields a specific expansion family $\mathfrak{F}$, and the decidability conditions say that this is a recursive expansion family.

Remark. In the proof of Theorem 1.8, the assumption that $R_0 \subseteq R$ was only needed in Case B.

The result below is obtained by interpreting Theorem 1.8 in Case A, where 1 is a bound on the rank. We drop the assumption $R_0 \subseteq R$.

THEOREM 1.9. *Suppose $\mathcal{A}$ is a recursive L-structure and ψ is a recursive infinitary Π_2 sentence in the language $L \cup \{P\}$. Let R be an r.e. set suitable for rank 0. Suppose that for all $\mathbf{c}$ in $\mathcal{A}$ and all P-sentences ϱ on $\mathcal{A}$, we can find finite $I \subseteq \omega$ and $\mathbf{a}_1$ in $\mathcal{A}$ such that if $\varrho'(\mathbf{a}_2) \vdash \varrho$ and $\varrho'(\mathbf{a}_2)$ decides P on $\mathbf{a}_1$ and gives evidence for ψ on I and $\mathbf{a}_1$, then for any existential L-formula φ such that $\mathcal{A} \models \varphi(\mathbf{c}, \mathbf{a}_2)$, there exists $\mathbf{a}_2'$ such that $\mathcal{A} \models \varphi(\mathbf{c}, \mathbf{a}_2')$ and $\varrho'(\mathbf{a}_2) \in R$. Then there exists a recursive $\mathcal{B} \cong \mathcal{A}$ with no recursive P such that $(\mathcal{B}, P) \models \psi$.*

In [V], there are some results related to Theorem 1.9, giving conditions under which a given recursive structure $\mathcal{A}$ has a recursive copy $\mathcal{B}$ with no Δ^0_α relation P such that $(\mathcal{B}, P) \models \psi$.

2. Familiar examples.

As a first example, we obtain the following result of Metakides and Nerode [MN].

THEOREM 2.1 (Metakides–Nerode). *There is a recursive algebraically closed field of infinite transcendence degree in which no infinite algebraically independent set is r.e.* (*The field can be taken to have any desired characteristic.*)

Proof. Let L be the language of fields, and let $\mathcal{A}$ be a recursive field of infinite transcendence degree (with the desired characteristic). We produce a recursive $\mathcal{B} \cong \mathcal{A}$ such that no infinite independent subset of $\mathcal{B}$ is recursive. Since every infinite r.e. set has an infinite recursive subset, $\mathcal{B}$ is the field we want. Let ψ say of a unary relation symbol P that it is an infinite algebraically independent set. We can take ψ to be the conjunction of a recursive infinitary Π_1 sentence saying that P is independent, and a recursive infinitary Π_2 sentence saying that P is infinite. The idea is to use the fact that if $\mathcal{A} \models \varphi(\mathbf{c}, \mathbf{b})$, where $\varphi(\mathbf{c}, \mathbf{x})$ is existential (or not), then there is some $\mathbf{b}'$ in the algebraic closure of $\mathbf{c}$ such that $\mathcal{A} \models \varphi(\mathbf{c}, \mathbf{b}')$.

We shall apply Theorem 1.9, taking R to be R_M. Note that $\varrho(\mathbf{a}) \in R_M$ iff $\varrho(\mathbf{a})$ puts into P elements which are dependent. Since the set of polynomial equations showing algebraic dependence is r.e., so is R_M. Let $\mathbf{c} \in \mathcal{A}$. Let N be the number of elements in $\mathbf{c}$. Choose a finite $I \subseteq \omega$ such that for any P-sentence $\varrho(\mathbf{a})$ deciding P on $\mathbf{c}$ and giving evidence for ψ on I and $\mathbf{c}$, $\mathbf{a}$

has at least $N+1$ elements d for which $\varrho(\mathbf{a}) \vdash P(d)$. For any L-formula $\varphi(\mathbf{c}, \mathbf{x})$ such that $\mathcal{A} \models \varphi(\mathbf{c}, \mathbf{a})$, there exists $\mathbf{b}$ such that $\mathcal{A} \models \varphi(\mathbf{c}, \mathbf{b})$ and $\mathbf{b}$ has at most N independent elements (we can choose $\mathbf{b}$ in the algebraic closure of $\mathbf{c}$). Then $\varrho(\mathbf{b}) \in R_M$. We are now in a position to apply Theorem 1.9, and we obtain the desired $\mathcal{B}$.

As a second example, we obtain a weak version of a result of Goncharov [G] on "recursive categoricity". A recursive structure $\mathcal{A}$ is said to be *recursively categorical* if for all recursive $\mathcal{B} \cong \mathcal{A}$, there is a recursive isomorphism f from $\mathcal{A}$ onto $\mathcal{B}$. For a given $\mathcal{A}$, let $\mathcal{A}^* = (A_1 \cup A_2, \mathcal{A}_1, \mathcal{A}_2)$, where $\mathcal{A}_1$ and $\mathcal{A}_2$ are copies of $\mathcal{A}$, with disjoint universes A_1, A_2, respectively, and there are recursive isomorphisms from $\mathcal{A}$ onto $\mathcal{A}_1$, $\mathcal{A}_2$. There is a recursive infinitary Π_2 sentence ψ saying of a new binary relation symbol P that it is an isomorphism from $\mathcal{A}_1$ onto $\mathcal{A}_2$. Note that $\mathcal{A}$ is recursively categorical iff for every recursive $\mathcal{B}^* \cong \mathcal{A}^*$, there is a recursive relation P such that $(\mathcal{B}^*, P) \models \psi$.

THEOREM 2.2. *Let $\mathcal{A}$ be a structure for which the existential diagram is recursive. Suppose that for each $\mathbf{c}$ in $\mathcal{A}$, we can find $\mathbf{a}$ such that for any existential formula $\varphi(\mathbf{c}, \mathbf{x})$ such that $\mathcal{A} \models \varphi(\mathbf{c}, \mathbf{a})$, there exists $\mathbf{a}'$ such that $\mathcal{A} \models \varphi(\mathbf{c}, \mathbf{a}')$ and there is an existential formula θ such that $\mathcal{A} \models \theta(\mathbf{c}, \mathbf{a}) \leftrightarrow \neg\theta(\mathbf{c}, \mathbf{a}')$. Then $\mathcal{A}$ is not recursively categorical.*

Proof. Let $\mathcal{A}^*$ and ψ be as above, and suppose that there are recursive isomorphisms from $\mathcal{A}$ onto $\mathcal{A}_1$ and $\mathcal{A}_2$. We apply Theorem 1.9, taking R to be R_0. For any $\mathbf{c}$ in $\mathcal{A}^*$, any P-sentence ϱ on $\mathcal{A}^*$, and any finite $I \subseteq \omega$ and $\mathbf{a}$ in $\mathcal{A}^*$, we can decide, using the existential diagram of $\mathcal{A}$, whether there exists ϱ' such that $\varrho' \vdash \varrho$, and ϱ' decides P on $\mathbf{a}$ and gives evidence for ψ on I and $\mathbf{a}$. From this, it follows that R_0 is r.e.

Given $\mathbf{c} \in \mathcal{A}^*$, we show how to find some finite $I \subseteq \omega$ and $\mathbf{a}$ in $\mathcal{A}^*$ such that if $\varrho(\mathbf{a}')$ decides P on $\mathbf{a}$ and gives evidence for ψ on I and $\mathbf{a}$, then for any existential formula $\varphi(\mathbf{c}, \mathbf{x})$ such that $\mathcal{A}^* \models \varphi(\mathbf{c}, \mathbf{a}')$, there exists $\mathbf{a}''$ such that $\mathcal{A}^* \models \varphi(\mathbf{c}, \mathbf{a}'')$ and $R_0(\varrho(\mathbf{a}'')|\mathbf{c}) = 0$. Let $\mathbf{c}_1$, $\mathbf{c}_2$ be the parts of $\mathbf{c}$ in A_1, A_2, respectively. Take $\mathbf{a}$ in A_1 such that for any existential formula φ_1 such that $\mathcal{A}_1 \models \varphi_1(\mathbf{c}_1, \mathbf{a})$, there exists $\mathbf{b}_1$ such that $\mathcal{A}_1 \models \varphi_1(\mathbf{c}_1, \mathbf{b}_1)$ but for some existential formula $\theta(\mathbf{c}_1, \mathbf{x})$, $\mathcal{A}_1 \models \theta(\mathbf{c}_1, \mathbf{a}) \leftrightarrow \neg\theta(\mathbf{c}_1, \mathbf{b}_1)$. Take I such that if $\varrho(\mathbf{a}')$ gives evidence for ψ on $\mathbf{a}$, then ϱ puts $\mathbf{c}_1\mathbf{a}$ into the domain of P. Let $\mathbf{a}' = \mathbf{a}_1'\mathbf{a}_2'$, where $\varrho(\mathbf{a}')$ gives evidence for ψ on $\mathbf{a}$, and $\mathbf{a}_1'$, $\mathbf{a}_2'$ are the parts of $\mathbf{a}'$ in A_1, A_2 ($\mathbf{a}_1' \supseteq \mathbf{a}$). Let $\mathbf{d}\mathbf{a}_2$ be the part of $\mathbf{c}_2\mathbf{a}_2'$ which, according to ϱ, is the P-image of $\mathbf{c}_1\mathbf{a}$.

It may be that for some existential formula θ, $\mathcal{A}_1 \models \theta(\mathbf{c}_1, \mathbf{a})$ iff $\mathcal{A}_2 \not\models \theta(\mathbf{d}, \mathbf{a}_2)$. In this case, $\varrho(\mathbf{a}') \in R_0$. Suppose that the existential formulas satisfied by $\mathbf{c}_1\mathbf{a}$ in $\mathcal{A}_1$ match those satisfied by $\mathbf{d}\mathbf{a}_2$ in $\mathcal{A}_2$. Let $\varphi(\mathbf{c}, \mathbf{x})$ be an existential formula such that $\mathcal{A}^* \models \varphi(\mathbf{c}, \mathbf{a}')$. There are existential formulas

$\varphi_1(\mathbf{c}_1, \mathbf{x}_1)$, $\varphi_2(\mathbf{c}_2, \mathbf{x}_2)$ such that $\mathcal{A}_i \models \varphi_i(\mathbf{c}_i, \mathbf{a}'_i)$ and for $\mathbf{b}' = \mathbf{b}'_1\mathbf{b}'_2$ such that $\mathcal{A}_i \models \varphi_i(\mathbf{c}_i, \mathbf{b}'_i)$, we have $\mathcal{A}^* \models \varphi(\mathbf{c}, \mathbf{b}')$. Take $\mathbf{b}'_1$ such that $\mathcal{A}_1 \models \varphi_1(\mathbf{c}_1, \mathbf{b}'_1)$ and if $\mathbf{b}$ is the part of $\mathbf{b}'_1$ corresponding to $\mathbf{a}$, the existential formulas satisfied by $\mathbf{c}_1\mathbf{a}$ and $\mathbf{c}_1\mathbf{b}_1$ in $\mathcal{A}_1$ do not match. If $\mathbf{a}'' = \mathbf{b}'_1\mathbf{a}'_2$, then $\varrho(\mathbf{a}'') \in R_0$.

We are in a position to apply Theorem 1.9. We obtain a recursive $\mathcal{B}^* \cong \mathcal{A}^*$ with no recursive P such that $(\mathcal{B}^*, P) \models \psi$. Therefore, $\mathcal{A}$ is not recursively categorical.

3. Examples of arbitrarily large recursive rank. In this section, we shall show that for arbitrarily large recursive ordinals α, there exist examples $\mathcal{A}$ and ψ such that $\mathcal{A}$ is a recursive structure and ψ is a recursive infinitary Π_2 sentence for which $R_M(\top|\emptyset) = \alpha$. In each case, the structure $\mathcal{A}$ will be a tree, isomorphic to a subtree of $\omega^{<\omega}$, with added unary predicates, and ψ will say of a new unary predicate P that it is a path. The tree structure will be given by a predecessor function p, where for the node a at level 0, $p(a) = a$. Subtrees are closed under the predecessor function. For each node a, let $l(a)$ denote the level of a. Our trees grow downwards, so when we say that c *lies below* b, we mean that for some n, $p^n(c) = b$. (We allow $n = 0$ and $c = b$.)

There are different possibilities for the sentence ψ saying that P is a path, and the choice of sentence may influence the rank. We shall take ψ to be the sentence $\bigwedge\!\!\!\bigwedge_n \psi_n$, for

$$\psi_0 = \forall x_1\, \forall x_2\, [\delta_0(x_1) \,\&\, \delta_1(x_1, x_2)],$$

where

$$\delta_0(x_1) = P(x_1) \to P(p(x_1)) \quad \text{and}$$
$$\delta_1(x_1, x_2) = P(x_1) \,\&\, P(x_2) \,\&\, p(x_1) = p(x_2) \to x_1 = x_2,$$

and

$$\psi_n = \exists x\, \delta_n(x), \quad \text{where}$$
$$\delta_n(x) = P(x) \,\&\, p^{(n-2)}(x) \neq p^{(n-1)}(x) \,\&\, p^{(n-1)}(x) = p^{(n)}(x).$$

In dealing with labeled trees T, it will be convenient to have ranks assigned to the nodes. We shall prove that the new ranks agree with R_M-ranks, and then use the new ranks.

Here is the definition of rank for nodes a in a tree T. We write $R(a)$ for the rank of a.

(1) $R(a) = 0$ if there is no path through a,
(2) for $\alpha > 0$, $R(a) = \alpha$ if
  (a) $R(a) \neq \beta$ for any $\beta < \alpha$, and
  (b) there is a level N such that $N \geq l(a)$ and for all b below a at level N, for each finite subtree of T containing b, there is an

isomorphic subtree such that if b' is the element corresponding to b, then $R(b') < \alpha$. We say that N *witnesses* the rank (more precisely, N witnesses that $R(a) \leq \alpha$).

We say that the tree T has *rank* α if the top element has rank α.

LEMMA 3.1. *Let $\mathcal{A}$ be a labeled tree, let* $\mathbf{a}$ *be a finite sequence from $\mathcal{A}$, and let a be an element of* $\mathbf{a}$ *such that $l(a)$ is maximal for elements of the subtree generated by* $\mathbf{a}$. *Suppose $\varrho(\mathbf{a})$ is a P-sentence on $\mathcal{A}$ saying that a and the elements generated by it are in P (and not putting any other elements into P). Then $R(a) \leq \alpha$ iff $R_M(\varrho(\mathbf{a})|\emptyset) \leq \alpha$.*

Proof. For $\alpha = 0$, the statement is clear. Let $\alpha > 0$, and suppose the statement holds for all $\beta < \alpha$. We show that if $R(a) \leq \alpha$, witnessed by N, then $R_M(\varrho(\mathbf{a})|\emptyset) \leq \alpha$, witnessed by $I = \{0, \ldots, N\}$ and the subtree $\mathbf{a}_1$ generated by a. Suppose $\varrho'(\mathbf{a}_2) \vdash \varrho(\mathbf{a})$, where $\varrho'(\mathbf{a}_2)$ decides P on $\mathbf{a}_1$ and gives evidence for ψ on I and $\mathbf{a}_1$, and $\mathcal{A} \models \varphi(\mathbf{a}_2)$, where φ is existential. Let b be an element of $\mathbf{a}_2$ at level N below a, say b is first in the sequence $\mathbf{a}_2$. Then $\mathcal{A} \models \exists \mathbf{u}\, \varphi(b, \mathbf{u})$. There is a finite subtree $\mathbf{b}$ containing $\mathbf{a}_2$ and any additional witnesses needed to make $\exists \mathbf{u}\, \varphi(b, \mathbf{u})$ true in $\mathcal{A}$. By hypothesis, there is a subtree $\mathbf{b}'$ isomorphic to $\mathbf{b}$ such that if b' is the element of $\mathbf{b}'$ corresponding to b, then $R(b') < \alpha$. Let $\mathbf{a}_2'$ be the sequence in $\mathbf{b}'$ corresponding to $\mathbf{a}_2$. We have $\mathcal{A} \models \varphi(\mathbf{a}_2')$. Let $\mathbf{b}^*$ be the subtree generated by b' and let $\varrho^*(\mathbf{b}^*)$ say that the elements of $\mathbf{b}^*$ are all in P. By the induction hypothesis, we have $R_M(\varrho^*(\mathbf{b}^*)|\emptyset) \leq R(b')$, and by Lemma 1.2, $R_M(\varrho'(\mathbf{a}_2')|\emptyset) \leq R(\varrho^*(\mathbf{b}^*)|\emptyset)$, so $R(\varrho'(\mathbf{a}_2')|\emptyset) < \alpha$. Therefore, I and $\mathbf{a}_1$ witness that $R_M(\varrho(\mathbf{a})|\emptyset) \leq \alpha$.

Now, suppose $R_M(\varrho(\mathbf{a})|\emptyset) \leq \alpha$, witnessed by $I = \{0, \ldots, N\}$ and $\mathbf{a}_1$. We may assume that $N \geq l(a)$. We show that $R(a) \leq \alpha$, witnessed by N. Let b lie below a at level N, and let $\mathbf{b}$ be a subtree containing $\mathbf{a}_1$ and b such that $l(b)$ is maximal in $\mathbf{b}$. Let $\varrho'(\mathbf{b}) \vdash \varrho(\mathbf{a})$, where $\varrho'(\mathbf{b})$ decides P on $\mathbf{a}_1$, putting into P just the elements $p^{(n)}(b)$. Let $\varphi(\mathbf{b})$ be an open formula describing the subtree $\mathbf{b}$. By hypothesis, there exists $\mathbf{b}'$ such that $\mathcal{A} \models \varphi(\mathbf{b}')$ and $R_M(\varrho'(\mathbf{b}')|\emptyset) < \alpha$. Then $\mathbf{b}'$ is a subtree isomorphic to $\mathbf{b}$. By the induction hypothesis, if b' is the element corresponding to b, then $R(b') \leq R_M(\varrho'(\mathbf{b}')|\emptyset)$. Therefore, N witnesses that $R(a) \leq \alpha$.

The next result says that for each recursive ordinal α, there is a recursive labeled tree T such that if a is the top node and $\varrho(a)$ is the P-sentence on T putting just a into P, then $R_M(\varrho(a)|\emptyset) = \alpha$.

THEOREM 3.2. *For each recursive ordinal α, there is a recursive labeled tree T with rank α.*

Proof. The proof will have the following organization. Lemma 3.3 below says that from a tree with certain abstract features, we can derive a tree having rank α. We then concentrate on producing a tree having the abstract features called for in the lemma, and such that the derived tree is recursive.

LEMMA 3.3. *Let P be a tree, isomorphic to a subtree of $\omega^{<\omega}$, with each node carrying a label of the form β for $\beta \le \alpha$ or (β, γ) for $\gamma < \beta \le \alpha$. Suppose that the tree satisfies the following conditions*:

(1) *for any node a with label 0 or $(\_, 0)$, the subtree consisting of all nodes under a has only finitely many levels*,

(2) *if $0 < \beta < \alpha$, then for any node a with label β or $(\_, \beta)$, a has successors with labels β or $(\_, \beta)$*,

(3) *if $0 < \beta < \alpha$, then for any node a with label β or $(\_, \beta)$, there is a level $N \ge l(a)$ such that for all b at level N below a, if b has label β or $(\_, \beta)$, then for all n, there is some b' such that $p(b') = p(b)$, b' has label γ or $(\_, \gamma)$ for some $\gamma < \beta$, and the first n levels under b are copied under b' in such a way that labels δ stay δ if $\delta \le \gamma$ and change to (δ, γ) if $\gamma < \delta$, and labels $(\_, \delta)$ stay $(\_, \delta)$ if $\delta \le \gamma$ and become $(\_, \gamma)$ if $\delta > \gamma$*,

(4) *if $0 < \gamma < \alpha$, then there exists n such that if $\delta < \gamma$, then under a node with label δ or $(\_, \delta)$ there can be no chain of $\ge n$ nodes with labels β or $(\beta, \_)$ for $\beta > \gamma$*,

(5) *the top node has label α.*

Let T be obtained from P by dropping the labels in favor of unary relations U_β for $\beta \le \alpha$, where $a \in U_\beta$ in T iff a had label β or $(\beta, \_)$ in P. Then T has rank α.

Proof. First, we show that for each a with label β or $(\_, \beta)$, $R(a) \le \beta$. The proof is by induction on β. For $\beta = 0$, the statement follows from (1). Let $\beta > 0$, and suppose the statement holds for $\gamma \le \beta$. If a has label β or $(\_, \beta)$, then by (3) and the induction hypothesis, $R(a) \le \beta$. Next, we show that for each a with label β or $(\_, \beta)$, $R(a) \ne \gamma$ for any $\gamma < \beta$. For $\gamma = 0$, if a has label β or $(\_, \beta)$ for $\beta > 0$, then by (2), $R(a) \ne 0$. Suppose $\gamma > 0$ and the statement holds for $\delta < \gamma$. Let a have label β or $(\_, \beta)$, and suppose $R(a) = \gamma$, where $\gamma < \beta$, hoping for a contradiction. Say N witnesses the rank. By (2), there is some b below a at level N such that b has label β or $(\_, \beta)$. By (4), we have n such that a node b' with label δ or $(\_, \delta)$ for $\delta < \gamma$ cannot have under it a chain of $\ge n$ elements with labels of the form ν or $(\nu, \_)$ for $\nu > \gamma$. Applying (2) again, we get a chain of n nodes from b down to some c, all with labels of the form β or (β', β) for $\beta' > \beta$. Since N witnesses that $R(a) = \gamma$, there is a copy of this chain, say from b' to c', where $R(b') = \delta < \gamma$. The labels on the copied chain are of the form $(\beta, \_)$ or $(\beta', \_)$ for $\beta' > \beta$. However, by the induction hypothesis, the label on b' cannot be $(\beta, \_)$ or $(\beta', \_)$. We have shown that the rank of a node in T is

β if the label is β or $(\_,\beta)$. By (5), the label on the top node of P is α, so T has rank α. This completes the proof of the lemma.

Now, to prove Theorem 3.2, it is enough to produce P satisfying the five conditions in Lemma 3.3, such that the derived tree T is recursive. For each β, let T_β be a tree such that the top node has label β, and for any $\gamma < \beta$ and any node a with label γ, a has infinitely many successors with label δ for each $\delta \leq \gamma$, and every successor of a has one of the labels $\delta \leq \gamma$. We shall arrive at the desired tree P through an infinite sequence of approximations P_s, for $s \in \omega$.

Each node $a \in P_s$ is identified with a task. At stage s, if we are working on task $a \in P_s$, we break the task down into infinitely many sub-tasks, identified with triples (a,b,n), where $a,b \in P_s$ and $n \in \omega$. If, at stage t, we are working on a sub-task (a,b,n), then we will normally extend the tree, giving P_{t+1} infinitely many nodes which were not in P_t. These new nodes give rise to new tasks.

We pause here to describe a scheme for making sure that all of the tasks receive attention. Fix a recursive list of the elements of $\omega^{<\omega}$, such that $\emptyset$ appears first, and σ appears before any successor σn. We shall attach the tasks to elements of $\omega^{<\omega}$, and visit the nodes of $\omega^{<\omega}$ in order, attending to the associated tasks as needed. In carrying out certain tasks, we make it unnecessary to do certain others. To start off, we list the elements of P_0 (representing tasks) so that the top node appears first and no node appears before its predecessor, and we assign these in order to the elements n at level 1 of $\omega^{<\omega}$. When we attend to the task associated with $\sigma \in \omega^{<\omega}$, thereby creating new tasks, we shall form a list of the new tasks and attach these to the elements $\sigma n \in \omega^{<\omega}$, in order. Thus, although we never have the full list of tasks, our partial list always gives at least the next task to work on.

At stage s, having attended to finitely many tasks, we determine a tree P_s, a list L_s of tasks still needing attention, and a set N_s of notes restraining certain choices of levels and elements. We also determine a finite part of the diagram of the final structure, but we shall put off the description of this. We now describe P_s, T_s, and N_s. Let $P_0 = T_\alpha$, let L_0 be the list (of elements of P_0) described above, and let $N_0 = \emptyset$. Suppose we are at stage $s+1$, having determined P_s, L_s, and N_s at stage s. We work on the first task from L_s.

Case 1: Suppose the task is represented by a, where $a \in P_s$. The node a will have a label of the form β or $(\_,\beta)$ in P_s. If $\beta = 0$, there is nothing to do. We let $P_{s+1} = P_s$, $L_{s+1} = L_s - \{a\}$, and $N_{s+1} = N_s$. Now, suppose $\beta > 0$. We choose a level $L(a)$, where $L(a) \geq l(a)$ and $L(a)$ is also below any levels mentioned in N_s. The levels mentioned in N_s are those chosen before and those with elements labeled by pairs. We record in N_{s+1} the fact that $L(a)$ has now been chosen. In N_s, we have notes about elements as well as about levels, and we put into N_{s+1} a note that we have worked on a. For each node b under a at level $L(a)$ such that b has label β, we shall have sub-tasks (a,b,n) for all $n \in \omega$. We put these into a list and attach them to the elements of $\omega^{<\omega}$ which are successors of the one to which a was attached. For any b' between a and b, the task associated with b' is made unnecessary. Then L_{s+1} consists of the remaining elements of L_s, plus the new tasks (a,b,n). We let $P_{s+1} = P_s$.

Case 2: Suppose the task is represented by a triple (a,b,n), where $a,b \in P_s$, $n \in \omega$. If the label on a is β or $(\_,\beta)$, then b will have label β. Let τ_n consist of the first n levels under b in P_s. We shall choose a node b' such that $p(b) = p(b')$ and b' has label $\delta = g(\beta,n)$, where $g(\beta,n)$ is a recursive function to be described below. We first fix a recursive list of pairs (β,n) for $0 < \beta \le \alpha$, $n \in \omega$. Then $g(\beta,n)$ is defined as follows:

(1) if $\beta = \delta + 1$, then for all n, $g(\beta,n) = \delta$,

(2) if β is a limit ordinal, the notation we have in mind for β picks out an increasing sequence $(\beta_k)_{k\in\omega}$ with limit β. Then $g(\beta,n) = \beta_k$ for the first k such that $k \ge n$ and we do not have $\beta_k \le g(\varrho,m) \le \beta$ for any pair (ϱ,m) that comes before (β,n) on the list.

The following lemma is not difficult to prove. It has been used (without being explicitly stated) in [AJK].

Lemma 3.4. *If* $0 < \gamma < \alpha$, *there are only finitely many pairs* (β,n) *such that* $g(\beta,n) < \gamma < \beta$.

Let us return to the task (a,b,n). We choose b' such that b' has label $\delta = g(\beta,n), p(b') = p(b)$, and b' is not one of the elements that N_s tells us to avoid. (These will include the nodes previously chosen, those corresponding to tasks previously worked on, and certain other nodes related to an enumeration of the diagram of our final structure.) Having chosen b', we record the choice in N_{s+1}.

We copy τ_n under b', changing some of the labels. We change the label δ on b' to (β,δ). If c in τ_n corresponds to c' in the copy, then

(1) a label ϱ on c becomes (ϱ,δ) on c' if $\varrho > \delta$, and stays ϱ if $\varrho \le \delta$ (necessarily, $\varrho,\delta \le \beta$),

(2) (ϱ,σ) on c becomes (ϱ,δ) if $\sigma \ge \delta$, and stays (ϱ,σ) if $\sigma \le \delta$ (necessarily, $\varrho,\sigma \le \beta$).

Now, we may add further nodes under the copy τ'_n of τ_n. Let c' be terminal in τ'_n. If c' has label $(\_,\delta)$ or δ, then we hang under c' a copy of T_δ, with c' at the top. Note that if $\delta = 0$, this means adding nothing.

We have said how to form the tree P_{s+1}, and what to record in N_{s+1}. There are new tasks corresponding to the new nodes. We make a list of these, making sure as always that no node appears before its predecessor if that is new, and we attach the new tasks to elements of $\omega^{<\omega}$ which are successors of the element associated with (a, b, n). Let L_{s+1} be the list of tasks, with (a, b, n) removed and the new nodes added.

Let P_ω be the "limit" of the labeled trees P_s. The tree P_ω is the union of the trees P_s, and the label on any node in P_ω is the last one it took in a P_s. (Note that the label on a node is changed at most once, so there is a last label.) We put unary relations U_β on P_ω to form T as described above.

We must make T recursive. At any stage, we will have enumerated only a finite part of the diagram. We must not change the label on a node a, say from γ to (β, γ), after we have put a sentence $U_\gamma(a)$ into the diagram of T. The only time we change the label on a is when we choose a or some node above it as a place to put a copy. If at stage s, we add a sentence to the diagram mentioning some element for the first time, then we put into N_s a note not to choose this element or any above it as a place to put a copy. It is clear that in this way, we make the structure T recursive.

LEMMA 3.5. *Suppose $a, b \in P_\omega$, where b lies below a. If a has label β, then b has label β' or $(\beta', \_)$ where $\beta' \le \beta$. If a has label (β, γ), then b has label γ', where $\gamma' \le \gamma$, or (β', γ'), where $\beta' \le \beta$ and $\gamma' \le \gamma$.*

Proof. It is enough to show by induction on s that the statements hold in P_s. The statement holds for P_0. Suppose it holds for P_s. The only change from P_s to P_{s+1} comes when we copy some τ_n from under a node b with label β to a location under some b' with label $\gamma < \beta$, and then develop the copy τ'_n. The changes in labels and the labels on the new part of the tree below τ'_n keep the statement true.

We must show that the ranks behave as they should. To do this, it is enough to verify the properties from Lemma 3.3.

PROPERTY (1). *If a has label 0 or $(\_, 0)$, then there are only finitely many levels with nodes below a.*

This is true in P_0 and in each P_s. If it fails in P_ω, then there must be some s such that a maximal chain in P_s of length ≥ 2 is extended in P_{s+1}. Say the chain from a to b has labels 0 or $(\_, 0)$, where b is terminal in P_s. We could only extend the chain by putting a copy under b. However, we cannot choose b, since $p(b)$ has a label 0 or $(\_, 0)$, so the node c with $p(c) = p(b)$ also has label 0 or $(\_, 0)$, and there was no task involving such a c.

PROPERTY (2). *If $0 < \beta < \alpha$, then for any node a with label β or $(\_, \beta)$, a has successors with labels β or $(\_, \beta)$.*

This is true in P_0 and all P_s. Any node with a label of the form β or $(\_, \beta)$ for $\beta > 0$ has successors with labels of the form β or $(\_, \beta)$.

PROPERTY (3). *If $0 < \beta < \alpha$, then for any node a with label β or $(\_, \beta)$, there is a level $N \geq l(a)$ such that for all b at level N below a, if b has label β or $(\_, \beta)$, then for all n, there is some b' such that $p(b') = p(b)$, b' has label γ or $(\_, \gamma)$ for some $\gamma < \beta$, and the first n levels under b are copied and put under b' in such a way that labels δ stay δ if $\delta \leq \gamma$ and change to (δ, γ) if $\gamma < \delta$, and labels $(\_, \delta)$ stay $(\_, \delta)$ if $\delta \leq \gamma$ and become $(\_, \gamma)$ if $\delta > \gamma$.*

We acted on certain nodes to make this true and crossed off those for which it was unnecessary.

PROPERTY (4). *For each γ such that $0 < \gamma < \alpha$, there exists N such that if $\delta < \gamma$, then under a node with label δ or $(\_, \delta)$ there can be no chain of $\geq N$ nodes with labels β or $(\beta, \_)$ for $\beta > \gamma$.*

The function g was defined so that there are only finitely many pairs (β, n) such that $\gamma < \beta$ and $g(\beta, n) < \gamma$. Take N greater than any n in these pairs. Suppose that for some node a with label δ or $(\_, \delta)$, there is a chain of length $\geq N$ below a in which all nodes have labels β or $(\beta, \_)$, for $\beta > \gamma$. Suppose s is first such that P_s includes such a chain. If a has label δ or $(\_, \delta)$, then no node below a can have label β, but copying could give rise to nodes with labels $(\beta, \_)$. Pairs only occur in copied sections of the tree, or at the tops of such.

Say at stage s, we copied some σ from under a node e, extended the copy to form σ', and put σ' under e', and suppose b is in the copy σ' (possibly $b = e'$). Then $p(e) = p(e')$, and e' lies at or above a. The full chain between b and c must have been in σ', since if some d between b and c were terminal in σ', then in P_{s+1}, the tree under d would be a copy of $T_{\delta'}$ for some $\delta' \leq \delta$, and later developments would leave d with a label $\varrho \leq \delta$ or (ϱ', ϱ) for $\varrho', \varrho \leq \delta$. Now, σ consists of $\geq N$ levels. Since $g(\beta, n) \geq \gamma$ for $n \geq N$, we could not have chosen the node e' unless it had label $\gamma' \geq \gamma$. Let a^*, b^*, c^* be the nodes in σ corresponding to a, b, c in σ'. Since $\gamma' > \delta$, a^* must have label $(\_, \delta)$. The labels on the chain between b^* and c^* must be the same as for the chain from b to c. In modifying the labels when we pass from σ to σ', we could not give b a label (β, δ') for $\delta' < \gamma$ unless b^* had a label β or (β, δ') in σ. This contradicts the assumption that s is the first stage at which we introduced a chain of $\geq N$ elements with labels β or $(\beta, \_)$ under a node with a label (β, δ) for $\delta < \gamma$.

PROPERTY (5). *The top node has label α.*

We made this true in P_0 and never changed the label.

We have now shown the existence of a tree with the features called for in Lemma 3.3, such that the derived tree T is recursive. This completes the proof of Theorem 3.2.

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