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Remarks on $\mathcal{P}_\kappa\lambda$-combinatorics

by

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Abstract. We prove that $\{ x \in \mathcal{P}_\kappa\lambda \mid x \cap \kappa$ is almost x-ineffable$\}$ has $p_*(\mathrm{NIn}_{\kappa,\lambda^{<\kappa}})$-measure 1 and $\{ x \in \mathcal{P}_\kappa\lambda \mid x \cap \kappa$ is x-ineffable$\}$ has $\mathcal{I}$-measure 1, where $\mathcal{I}$ is the complete ineffable ideal on $\mathcal{P}_\kappa\lambda$. As corollaries, we show that λ-ineffability does not imply complete λ-ineffability and that almost λ-ineffability does not imply λ-ineffability.

In [6], Jech introduced the notion of λ-ineffability and almost λ-ineffability which are the $\mathcal{P}_\kappa\lambda$ generalizations of ineffability. Next, Johnson [8] introduced the notion of complete λ-ineffability. These properties can be characterized by certain ideals on $\mathcal{P}_\kappa\lambda$ (see [3]). By the definitions, it follows directly that λ-supercompact cardinals are completely λ-ineffable and that λ-ineffable cardinals are almost λ-ineffable. Johnson [8] showed that completely λ-ineffable cardinals are λ-ineffable.

Whether the converse implications also hold seems to be interesting. Concerning this, Abe [1] proved that almost λ-ineffability and λ-ineffability are equivalent if $\lambda > \kappa$ is an ineffable cardinal. It is not difficult to check that complete λ-ineffability does not imply λ-supercompactness. In this paper, we shall prove the following two theorems.

THEOREM 4.1. *If κ is $\lambda^{<\kappa}$-ineffable, then $\{x \in \mathcal{P}_\kappa\lambda \mid x \cap \kappa$ is almost x-ineffable$\} \in p_*(\mathrm{NIn}_{\kappa,\lambda^{<\kappa}})^*$, where p denotes the projection from $\mathcal{P}_\kappa\lambda^{<\kappa}$ to $\mathcal{P}_\kappa\lambda$.*

THEOREM 4.2. *Let $\mathcal{I}$ be a normal, $(\lambda^{<\kappa}, 2)$-distributive ideal on $\mathcal{P}_\kappa\lambda$. Then $\{x \in \mathcal{P}_\kappa\lambda \mid x \cap \kappa$ is x-ineffable$\} \in \mathcal{I}^*$.*

By using these theorems, we shall show that λ-ineffability does not imply complete λ-ineffability and that almost λ-ineffability does not imply λ-ineffability.

In the proofs of Theorems 4.1 and 4.2, we shall use the notion of strong normality (which was introduced by Carr [4]) and a certain correspondence

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between $\mathcal{P}_\kappa\lambda$ and $\mathcal{P}_\kappa\lambda^{<\kappa}$. The strong normality and this correspondence will be dealt with in Sections 2 and 3, respectively. The two theorems will be proved in Section 4.

The author got the idea of the correspondence between $\mathcal{P}_\kappa\lambda$ and $\mathcal{P}_\kappa\lambda^{<\kappa}$ from discussions with Prof. Y. Abe at Kanagawa University and would like to thank him.

1. Notation and terminology. Throughout this paper, κ denotes a regular uncountable cardinal, and λ a cardinal $\geq \kappa$. Let $\mathcal{J}$ be an ideal on a set S. Then $\mathcal{J}^*$ denotes the dual filter of $\mathcal{J}$ and $\mathcal{J}^+$ the set $\mathcal{P}(S) \setminus \mathcal{J}$. For any $X \subset S$, $\mathcal{J}^+ \restriction X$ denotes $\mathcal{J}^+ \cap \mathcal{P}(X)$. For any $f : S \to T$, $f_*(\mathcal{J})$ denotes the ideal $\{Y \subset T \mid f^{-1}Y \in \mathcal{J}\}$ on T.

Let A be a set such that $\kappa \leq |A|$. Then $\mathcal{P}_\kappa A$ is the set $\{x \subset A \mid |x| < \kappa\}$. For each $x \in \mathcal{P}_\kappa A$, $\widehat{x}$ denotes the set $\{y \in \mathcal{P}_\kappa A \mid x \subset y \,\&\, x \neq y\}$. $\mathrm{I}_{\kappa,A}$ denotes the ideal $\{X \subset \mathcal{P}_\kappa A \mid X \cap \widehat{y} = \emptyset$ for some $y \in \mathcal{P}_\kappa A\}$. An element of $\mathrm{I}^+_{\kappa,A}$ is called *unbounded*. A subset of $\mathcal{P}_\kappa A$ is called *club* if it is unbounded and closed under unions of increasing chains with length $< \kappa$. A subset X of $\mathcal{P}_\kappa A$ is called *stationary* if $X \cap C \neq \emptyset$ for any club subset C of $\mathcal{P}_\kappa A$. $\mathrm{NS}_{\kappa,A}$ denotes the ideal $\{X \subset \mathcal{P}_\kappa A \mid X$ is non-stationary$\}$. A function f from X $(\subset \mathcal{P}_\kappa A)$ to A is called *regressive* if $f(x) \in x$ for all $x \in X \setminus \{\emptyset\}$. For any indexed family $\{X_a \mid a \in A\}$ of subsets of $\mathcal{P}_\kappa A$, the *diagonal union* $\bigtriangledown_{a\in A} X_a$ and the *diagonal intersection* $\bigtriangleup_{a\in A} X_a$ are the sets $\{x \in \mathcal{P}_\kappa A \mid x \in X_a$ for some $a \in x\}$ and $\{x \in \mathcal{P}_\kappa A \mid x \in X_a$ for all $a \in x\}$, respectively. A κ-complete ideal on $\mathcal{P}_\kappa A$ is said to be *normal* if it contains $\mathrm{I}_{\kappa,A}$ and is closed under diagonal unions.

A subset $X \subset \mathcal{P}_\kappa A$ is said to be A-*ineffable*, *almost* A-*ineffable*, and A-*Shelah*, respectively, if

$$\forall f_x : x \to 2 \text{ (for } x \in X) \ \exists f : A \to 2 \ (\{x \in X \mid f_x \subset f\} \in \mathrm{NS}^+_{\kappa,A}),$$

$$\forall f_x : x \to 2 \text{ (for } x \in X) \ \exists f : A \to 2 \ (\{x \in X \mid f_x \subset f\} \in \mathrm{I}^+_{\kappa,A}),$$

$$\forall f_x : x \to x \text{ (for } x \in X) \ \exists f : A \to A \ \forall x \in \mathcal{P}_\kappa A \ \exists y \in X \cap \widehat{x} \ (f_y \restriction x = f \restriction x).$$

Following Carr [2], [3], define

$$\mathrm{NIn}_{\kappa,A} = \{X \subset \mathcal{P}_\kappa A \mid X \text{ is not } A\text{-ineffable}\},$$

$$\mathrm{NAIn}_{\kappa,A} = \{X \subset \mathcal{P}_\kappa A \mid X \text{ is not almost } A\text{-ineffable}\},$$

$$\mathrm{NSh}_{\kappa,A} = \{X \subset \mathcal{P}_\kappa A \mid X \text{ is not } A\text{-Shelah}\}.$$

Carr [2], [3] showed that these are normal ideals on $\mathcal{P}_\kappa A$ and that $\mathrm{NSh}_{\kappa,A} \subset \mathrm{NAIn}_{\kappa,A}$. A cardinal κ is said to be A-*ineffable* (*almost* A-*ineffable*, A-*Shelah*) if $\mathrm{NIn}_{\kappa,A}$ ($\mathrm{NAIn}_{\kappa,A}$, $\mathrm{NSh}_{\kappa,A}$) is proper.

Let $\mathcal{I}$ be an ideal on $\mathcal{P}_\kappa A$ and ϱ a cardinal. Then $\mathcal{I}$ is said to be $(\varrho, 2)$-*distributive* if for any $X \in \mathcal{I}^+$ and any family $\{\{X_{\alpha,0}, X_{\alpha,1}\} \mid \alpha < \varrho\}$ of disjoint partitions of X, there exist $X' \in \mathcal{I}^+ \restriction X$ and $f : \varrho \to 2$ such that $X' \setminus X_{\alpha,f(\alpha)} \in \mathcal{I}$ for all $\alpha < \varrho$. Note that this definition is equivalent to the usual definition of $(\varrho, 2)$- (or (ϱ, ϱ)-)distributivity given in [8]. Following Johnson [8], we say that κ is *completely A-ineffable* if there exists a proper, normal, $(|A|, 2)$-distributive ideal on $\mathcal{P}_\kappa A$. By using the following theorem [8, Theorem 5.1], she proved that completely A-ineffable cardinals are A-ineffable.

THEOREM 1.1. *For any ideal $\mathcal{I}$ on $\mathcal{P}_\kappa A$ containing $\mathrm{I}_{\kappa,A}$, the following statements are equivalent.*

(a) *$\mathcal{I}$ is normal and $(|A|, 2)$-distributive.*

(b) $\forall X \in \mathcal{I}^+\ \forall f_x : x \to 2$ *(for $x \in X$)* $\exists f : A \to A$ $(\{x \in X \mid f_x \subset f\} \in \mathcal{I}^+)$.

2. Strong normality. From now on, $\mathcal{I}$ denotes a proper, κ-complete ideal on $\mathcal{P}_\kappa\lambda$ containing $\mathrm{I}_{\kappa,\lambda}$. In this section, we shall consider the strong normality of ideals on $\mathcal{P}_\kappa\lambda$ which was introduced by Carr [4]. For $x, y \in \mathcal{P}_\kappa\lambda$, $x \prec y$ means that $x \subset y$ and $|x| < |\kappa \cap y|$. Following Carr [4], $\mathcal{I}$ is called *strongly normal* if

$$\forall X \in \mathcal{I}^+\ \forall a_x \prec x \text{ (for } x \in X) \ \exists a \in \mathcal{P}_\kappa\lambda\ (\{x \in X \mid a_x = a\} \in \mathcal{I}^+).$$

It is clear that strongly normal ideals are normal. Carr [4, Theorems 3.4, 3.5] showed that, under the assumption that $\lambda^{<\kappa} = \lambda$, the ideals $\mathrm{NIn}_{\kappa,\lambda}$, $\mathrm{NAIn}_{\kappa,\lambda}$ and $\mathrm{NSh}_{\kappa,\lambda}$ are strongly normal.

For $x \in \mathcal{P}_\kappa\lambda$, $\mathcal{Q}_x$ denotes the set $\mathcal{P}_{\kappa\cap x}x$ $(= \{t \subset x \mid t \prec x\})$. For any indexed family $\{X_t \mid t \in \mathcal{P}_\kappa\lambda\}$ of subsets of $\mathcal{P}_\kappa\lambda$, $\triangle_{t\in\mathcal{P}_\kappa\lambda} X_t$ denotes the set $\{x \in \mathcal{P}_\kappa\lambda \mid x \in X_t \text{ for all } t \prec x\}$, and $\triangledown_{t\in\mathcal{P}_\kappa\lambda} X_t$ the set $\{x \in \mathcal{P}_\kappa\lambda \mid x \in X_t$ for some $t \prec x\}$. We call $\triangle_{t\in\mathcal{P}_\kappa\lambda} X_t$ and $\triangledown_{t\in\mathcal{P}_\kappa\lambda} X_t$ the *strong diagonal intersection* and *union*, respectively, of $\{X_t \mid t \in \mathcal{P}_\kappa\lambda\}$. The following lemma is known [5] and can be easily verified.

LEMMA 2.1 *The following statements are equivalent.*

(a) *$\mathcal{I}$ is strongly normal.*

(b) *$\mathcal{I}$ is closed under strong diagonal unions.*

LEMMA 2.2. *If $\mathcal{I}$ is normal and $(\lambda, 2)$-distributive, then $\mathcal{I}$ is strongly normal.*

Proof. Let $X \in \mathcal{I}^+$ and $a_x \prec x$ for $x \in X$. For each $x \in X$, take $\beta_x \in x \cap \kappa$ such that $|a_x| \le |x \cap \beta_x|$. Since $\mathcal{I}$ is normal, we may assume that $\beta_x = \beta$ for all $x \in X$. For each $\alpha < \lambda$, set

$$Y_{\alpha,0} = \{x \in X \mid \alpha \in a_x\}, \qquad Y_{\alpha,1} = \{x \in X \mid \alpha \notin a_x\},$$
$$W_\alpha = \{Y_{\alpha,0}, Y_{\alpha,1}\} \cap \mathcal{I}^+.$$

Since W_α is an $\mathcal{I}$-partition of X for every $\alpha < \lambda$, there exist $g : \lambda \to 2$ and $Z \in \mathcal{I}^+$ such that

$$Z \subset X \quad \text{and} \quad Z \setminus Y_{\alpha,g(\alpha)} \in \mathcal{I} \quad \text{for all } \alpha < \lambda.$$

Set $Y = \triangle_{\alpha<\lambda} Y_{\alpha,g(\alpha)}$. Since $\mathcal{I}$ is normal, $Z \setminus Y \in \mathcal{I}$. So, $Y \in \mathcal{I}^+$. Set $A = g^{-1}\{0\}$. Then it is easy to see that $a_y = A \cap y$ for all $y \in Y$ and $|A| \le |\beta|$. So, $A \in \mathcal{P}_\kappa\lambda$. Set $Y_1 = Y \cap \widehat{A}$. Then $Y_1 \in \mathcal{I}^+$ and $a_y = A$ for all $y \in Y_1$. ∎

Define

$$\mathbf{S}(\mathcal{I}) = \{\mathop{\triangledown}_{t\in\mathcal{P}_\kappa\lambda} X_t \cup Y \mid \forall t \in \mathcal{P}_\kappa\lambda\ (X_t \in \mathcal{I})\,\&\, Y \in \mathcal{I}\}.$$

Lemma 2.3. *Suppose that κ is an inaccessible cardinal. Then $\mathbf{S}(\mathcal{I})$ is the smallest strongly normal ideal containing $\mathcal{I}$.*

Proof. Since it is clear that $\mathbf{S}(\mathcal{I})$ is an ideal, we only verify that $\mathbf{S}(\mathcal{I})$ is strongly normal. So, let $Y_t \in \mathbf{S}(\mathcal{I})$ (for $t \in \mathcal{P}_\kappa\lambda$). For each $t \in \mathcal{P}_\kappa\lambda$, take $X_{t,s} \in \mathcal{I}$ (for $s \in \mathcal{P}_\kappa\lambda$) and $A_t \in \mathcal{I}$ such that $Y_t \subset \triangledown_{s\in\mathcal{P}_\kappa\lambda} X_{t,s} \cup A_t$. For each $a \in \mathcal{P}_\kappa\lambda$, let $B_a = \bigcup_{s,t\subset a} X_{t,s} \cup A_a$. Since κ is inaccessible, $B_a \in \mathcal{I}$ for all $a \in \mathcal{P}_\kappa\lambda$. It is easy to check that

$$\mathop{\triangledown}_{t\in\mathcal{P}_\kappa\lambda} Y_t \subset \mathop{\triangledown}_{a\in\mathcal{P}_\kappa\lambda} B_a \cup (\mathcal{P}_\kappa\lambda \setminus \widehat{\omega}) \in \mathbf{S}(\mathcal{I}).$$

Corollary 2.4. *Suppose that κ is an inaccessible cardinal. Then* $\mathbf{S}(\mathrm{NS}_{\kappa,\lambda}) = \mathbf{S}(\mathrm{I}_{\kappa,\lambda})$. ∎

For each $\tau : \mathcal{P}_\kappa\lambda \to \mathcal{P}_\kappa\lambda$, $\mathrm{cl}(\tau)$ denotes the set $\{x \in \mathcal{P}_\kappa\lambda \mid x \ne \emptyset\,\&\, \forall t \prec x\ (\tau(t) \subset x)\}$.

Lemma 2.5. *Suppose that κ is an inaccessible cardinal. Let $X \subset \mathcal{P}_\kappa\lambda$. Then the following statements are equivalent.*

(a) $X \in \mathbf{S}(\mathrm{NS}_{\kappa,\lambda})$.

(b) *There exists $\tau : \mathcal{P}_\kappa\lambda \to \mathcal{P}_\kappa\lambda$ such that* $\mathrm{cl}(\tau) \cap X = \emptyset$.

Proof. (a)⇒(b). Let $X \in \mathbf{S}(\mathrm{NS}_{\kappa,\lambda})$. By the previous corollary, we can take $x_a \in \mathcal{P}_\kappa\lambda$ (for $a \in \mathcal{P}_\kappa\lambda$) and $b \in \mathcal{P}_\kappa\lambda$ such that

$$X \subset \mathop{\triangledown}_{a\in\mathcal{P}_\kappa\lambda} (\mathcal{P}_\kappa\lambda \setminus \widehat{x}_a) \cup (\mathcal{P}_\kappa\lambda \setminus \widehat{b}).$$

Let $\tau = \langle x_a \cup b \cup \omega \mid a \in \mathcal{P}_\kappa\lambda\rangle$. Then $\mathrm{cl}(\tau) \cap X = \emptyset$.

(b)⇒(a). Suppose $\tau : \mathcal{P}_\kappa\lambda \to \mathcal{P}_\kappa\lambda$ satisfies $\mathrm{cl}(\tau) \cap X = \emptyset$. For each $a \in \mathcal{P}_\kappa\lambda$, set $Y_a = \mathcal{P}_\kappa\lambda \setminus \tau(a)^\wedge$. Let $Y = \triangledown_{a\in\mathcal{P}_\kappa\lambda} Y_a$. Then $X \subset Y$ and $Y \in \mathbf{S}(\mathrm{I}_{\kappa,\lambda})$. ∎

The following lemma is not needed later. However, it seems to be interesting, because if κ is an inaccessible cardinal, then the set $X = \{x \in \mathcal{P}_\kappa\lambda \mid x\cap\kappa$ is an ordinal and $\mathrm{cof}(x\cap\kappa) = \omega\}$ satisfies $\{x \in X \mid X\cap\mathcal{Q}_x \in \mathrm{I}_{\kappa\cap x,x}\} \in \mathrm{NS}^+_{\kappa,\lambda}$.

LEMMA 2.6. *Suppose that κ is an inaccessible cardinal. Then*

$$\{x \in X \mid X \cap \mathcal{Q}_x \in \mathrm{I}_{\kappa\cap x, x}\} \in \mathbf{S}(\mathrm{NS}_{\kappa,\lambda}) \quad \textit{for any } X \subset \mathcal{P}_\kappa\lambda.$$

Proof. To get a contradiction, assume that there exists $X \subset \mathcal{P}_\kappa\lambda$ such that

$$Y = \{x \in X \mid X \cap \mathcal{Q}_x \in \mathrm{I}_{\kappa\cap x, x}\} \in \mathbf{S}(\mathrm{NS}_{\kappa,\lambda})^+.$$

For each $x \in Y$, take $a_x \in \mathcal{Q}_x$ such that $\widehat{a}_x \cap X \cap \mathcal{Q}_x = \emptyset$. Since $Y \in \mathbf{S}(\mathrm{NS}_{\kappa,\lambda})^+$, there exists $a \in \mathcal{P}_\kappa\lambda$ such that

$$Z = \{x \in Y \mid a_x = a\} \in \mathbf{S}(\mathrm{NS}_{\kappa,\lambda})^+.$$

Take $x, y \in Z$ such that $x \prec y$. Then $x \in X \cap \widehat{a}_y \cap \mathcal{Q}_y$. A contradiction. ∎

3. A correspondence between $\mathcal{P}_\kappa\lambda$ and $\mathcal{P}_\kappa\lambda^{<\kappa}$. From now on, we assume that κ is an inaccessible cardinal. Let $\theta = \lambda^{<\kappa}$ and $p : \mathcal{P}_\kappa\theta \to \mathcal{P}_\kappa\lambda$ denote the projection (i.e., $p(y) = y \cap \lambda$).

Take a bijection $h : \theta \to \mathcal{P}_\kappa\lambda$. Define $\pi = \pi(h) : \mathcal{P}_\kappa\lambda \to \mathcal{P}_\kappa\theta$ and $q = q(h) : \mathcal{P}_\kappa\theta \to \mathcal{P}_\kappa\lambda$ by

$$\pi(x) = h^{-1}\mathcal{Q}_x \quad \text{for each } x \in \mathcal{P}_\kappa\lambda,$$
$$q(y) = \bigcup h''y \quad \text{for each } y \in \mathcal{P}_\kappa\theta.$$

Set

$$C_h = \{y \in \mathcal{P}_\kappa\theta \mid \forall\alpha \in y\ (h(\alpha) \prec q(y))\ \&\ q(y) = p(y)\},$$

The following lemma can be easily verified.

LEMMA 3.1. (1) *$q\pi(x) = x$ for any $x \in \widehat{2}$ ($\subset \mathcal{P}_\kappa\lambda$).*

(2) *C_h is a club of $\mathcal{P}_\kappa\theta$ (so, $p''C_h = q''C_h$ is a club subset of $\mathcal{P}_\kappa\lambda$).*

(3) *$y \subset \pi q(y)$ for any $y \in C_h$.*

(4) *$Y \in \mathrm{I}_{\kappa,\theta}$ iff $\pi^{-1}Y \in \mathrm{I}_{\kappa,\lambda}$ for any $Y \subset \mathrm{rang}(\pi)$.* ∎

LEMMA 3.2. *There exist $x \in \mathcal{P}_\kappa\lambda$ and $y \in \mathcal{P}_\kappa\theta$ such that $\pi q\restriction(\mathrm{rang}(\pi)\cap\widehat{y})$ is the identity function and $\widehat{x} \subset q''(\mathrm{rang}(\pi) \cap \widehat{y})$.*

Proof. Take $\alpha < \theta$ such that $h(\alpha) = 2$. Then it is easy to see that $\pi q\restriction(\mathrm{rang}(\pi)\cap\{\alpha\}^\wedge)$ is the identity function and $\widehat{\omega} \subset q''(\mathrm{rang}(\pi)\cap\{\alpha\}^\wedge)$. ∎

COROLLARY 3.3. *Let $\mathcal{J}$ be an ideal on $\mathcal{P}_\kappa\theta$. If $\mathrm{rang}(\pi) \in \mathcal{J}^*$ and $\mathrm{I}_{\kappa,\theta} \subset \mathcal{J}$, then $\pi_*q_*(\mathcal{J}) = \mathcal{J}$.* ∎

LEMMA 3.4. *The following statements are equivalent.*

(a) *$\mathcal{I}$ is strongly normal.*

(b) *$\pi_*(\mathcal{I})$ is normal.*

Proof. (a)⇒(b). Assume that $\mathcal{I}$ is strongly normal. Let $Y_\alpha \in \pi_*(\mathcal{I})$ for $\alpha < \theta$. Set $Y = \triangledown_{\alpha<\theta} Y_\alpha$. For each $a \in \mathcal{P}_\kappa\lambda$, set $X_a = \pi^{-1}Y_{h^{-1}(a)} \in \mathcal{I}$. Set

$X = \triangledown_{a\in\mathcal{P}_\kappa\lambda} X_a$. Then $\pi^{-1}Y \subset X$. Since $X_a \in \mathcal{I}$ for all $a \in \mathcal{P}_\kappa\lambda$, it follows that $X \in \mathcal{I}$. So, $Y \in \pi_*(\mathcal{I})$.

(b)⇒(a). Assume that $\pi_*(\mathcal{I})$ is normal. Let $X_a \in \mathcal{I}$ for $a \in \mathcal{P}_\kappa\lambda$. Set $X = \triangledown_{a\in\mathcal{P}_\kappa\lambda} X_a$. For each $\alpha < \theta$, set $Y_\alpha = \pi'' X_{h(\alpha)}$. Set $Y = \triangledown_{\alpha<\theta} Y_\alpha$. Since $\pi^{-1}Y_\alpha = X_{h(\alpha)} \in \mathcal{I}$ for all $\alpha < \theta$, it follows that $Y \in \pi_*(\mathcal{I})$. Since $X \subset \pi^{-1}Y$, we conclude that $X \in \mathcal{I}$. ∎

Corollary 3.5. *If $\mathcal{I}$ is strongly normal, then* $\mathrm{NS}_{\kappa,\theta} \subset \pi_*(\mathcal{I})$. *In particular*, $\mathrm{NS}_{\kappa,\theta} \subset \pi_*(\mathbf{S}(\mathrm{NS}_{\kappa,\lambda}))$. ∎

Lemma 3.6. $Y \in \mathrm{NS}_{\kappa,\theta}$ *iff* $\pi^{-1}Y \in \mathbf{S}(\mathrm{NS}_{\kappa,\lambda})$ *for any* $Y \subset \mathrm{rang}(\pi)$.

Proof. The implication ⇒ follows immediately from the above corollary. To show the converse, let $Y \subset \mathrm{rang}(\pi)$ and $X = \pi^{-1}Y \in \mathbf{S}(\mathrm{NS}_{\kappa,\lambda})$. By Lemma 2.5, there exists $\tau : \mathcal{P}_\kappa\lambda \to \mathcal{P}_\kappa\lambda$ such that $\mathrm{cl}(\tau) \cap X = \emptyset$. Define $C \subset \mathcal{P}_\kappa\theta$ by

$$C = \{y \in C_h \mid \tau(h(\alpha) \cap \lambda) \subset p(y) \text{ for all } \alpha \in y\}.$$

Then C is a club subset of $\mathcal{P}_\kappa\theta$ and $C \cap Y = \emptyset$. So, $Y \in \mathrm{NS}_{\kappa,\theta}$. ∎

Lemma 3.7. $\mathrm{rang}(\pi) \in \mathrm{NSh}^*_{\kappa,\theta}$.

Proof. To get a contradiction, assume that $Y_0 = \mathcal{P}_\kappa\theta \setminus \mathrm{rang}(\pi) \in \mathrm{NSh}^+_{\kappa,\theta}$. Since C_h is a club, $Y = Y_0 \cap C_h \in \mathrm{NSh}^+_{\kappa,\theta}$. Since, for all $y \in Y$, we have $y \subset \pi(y \cap \lambda)$ and $y \neq \pi(y \cap \lambda)$, we can take a_y (for $y \in Y$) such that

$$a_y \prec y \cap \lambda \quad \text{and} \quad h^{-1}(a_y) \notin y \quad \text{for any } y \in Y.$$

Since κ is θ-Shelah and $\mathrm{cof}(\theta) \geq \kappa$, by the result of Johnson [8, Cor. 2.7], $\theta^{<\kappa} = \theta$. So, $\mathrm{NSh}_{\kappa,\theta}$ is strongly normal. Hence, there is $a \in \mathcal{P}_\kappa\lambda$ such that

$$Y' = \{y \in Y \mid a_y = a\} \in \mathrm{NSh}^+_{\kappa,\theta}.$$

Then $h^{-1}(a) \notin y$ for all $y \in Y'$. But this contradicts the fact that Y' is unbounded in $\mathcal{P}_\kappa\theta$. ∎

Theorem 3.8. *Let $Y \subset \mathcal{P}_\kappa\theta$ and $X = q^{-1}Y$. Then*:

(1) $Y \in \mathrm{NIn}^+_{\kappa,\theta}$ *iff*

$$\forall f_x : \mathcal{Q}_x \to 2 \ (\textit{for } x \in X)\ \exists f : \mathcal{P}_\kappa\lambda \to 2 \ (\{x \in X \mid f_x \subset f\} \in \mathbf{S}(\mathrm{NS}_{\kappa,\lambda})^+).$$

(2) $Y \in \mathrm{NAIn}^+_{\kappa,\theta}$ *iff*

$$\forall f_x : \mathcal{Q}_x \to 2 \ (\textit{for } x \in X)\ \exists f : \mathcal{P}_\kappa\lambda \to 2 \ (\{x \in X \mid f_x \subset f\} \in \mathrm{I}^+_{\kappa,\lambda}).$$

(3) $Y \in \mathrm{NSh}^+_{\kappa,\theta}$ *iff*

$$\forall f_x : \mathcal{Q}_x \to \mathcal{Q}_x \ (\textit{for } x \in X)\ \exists f : \mathcal{P}_\kappa\lambda \to \mathcal{P}_\kappa\lambda \textit{ such that } \forall x \in \mathcal{P}_\kappa\lambda\ \exists x' \in X \cap \widehat{x}\ (f_{x'} \restriction \mathcal{Q}_x = f \restriction \mathcal{Q}_x).$$

P r o o f. By Lemmas 3.2 and 3.7, we may assume that $Y \subset \mathrm{rang}(\pi)$ and $\pi q{\restriction}Y$ is the identity function.

(1⇒) Let $f_x : \mathcal{Q}_x \to 2$ for $x \in X$. Define $g_y : y \to 2$ (for $y \in Y$) by $g_y(\alpha) = f_{q(y)}(h(\alpha))$ for any $\alpha \in y$. Since $Y \in \mathrm{NIn}^+_{\kappa,\theta}$, there exists $g : \theta \to 2$ such that $Y_0 = \{y \in Y \mid g_y \subset g\} \in \mathrm{NS}^+_{\kappa,\theta}$. Set $X_0 = q''Y_0$. By Lemma 3.6, $X_0 \in \mathbf{S}(\mathrm{NS}_{\kappa,\lambda})^+$. Define $f : \mathcal{P}_\kappa\lambda \to 2$ by $f(t) = g(h^{-1}(t))$ for all $t \in \mathcal{P}_\kappa\lambda$. Then it is easy to see that $f_x \subset f$ for all $x \in X_0$. So, $\{x \in X \mid f_x \subset f\} \in \mathbf{S}(\mathrm{NS}_{\kappa,\lambda})^+$.

(1⇐) Let $g_y : y \to 2$ for $y \in Y$. Define $f_x : \mathcal{Q}_x \to 2$ (for $x \in X$) by $f_x(a) = g_{\pi(x)}(h^{-1}(a))$ for any $a \in \mathcal{Q}_x$. By the hypothesis, there exists $f : \mathcal{P}_\kappa\lambda \to 2$ such that $X_0 = \{x \in X \mid f_x \subset f\} \in \mathbf{S}(\mathrm{NS}_{\kappa,\lambda})^+$. Set $Y_0 = \pi''X_0$. By Lemma 3.6, $Y_0 \in \mathrm{NS}^+_{\kappa,\theta}$. Define $g : \theta \to 2$ by $g(\alpha) = f(h(\alpha))$ for all $\alpha < \theta$. Then it is easy to see that $g_y \subset g$ for all $y \in Y_0$. So, $\{y \in Y \mid g_y \subset g\} \in \mathrm{NS}^+_{\kappa,\theta}$.

(2), (3) Similar to (1). ∎

Theorem 3.9. *The following statements are equivalent.*

(a) *κ is completely θ-ineffable.*

(b) *There exists a proper, normal ideal $\mathcal{I}$ on $\mathcal{P}_\kappa\lambda$ which satisfies the $(\theta, 2)$-distributive law.*

P r o o f. (a)⇒(b). Assume that (a) holds. Take a proper normal ideal $\mathcal{J}$ on $\mathcal{P}_\kappa\theta$ such that $\mathcal{J}$ satisfies the $(\theta, 2)$-distributive law. Set $\mathcal{I} = q_*(\mathcal{J})$. Since $\mathrm{rang}(\pi) \in \mathcal{J}^*$, $\mathcal{I}$ is the desired ideal in (b).

(b)⇒(a). Let $\mathcal{I}$ be an ideal on $\mathcal{P}_\kappa\lambda$ which satisfies (b). Set $\mathcal{J} = \pi_*(\mathcal{I})$. Since $\mathcal{I}$ is strongly normal, $\mathcal{J}$ is the desired ideal in (a). ∎

4. Theorems. As in the previous section, θ denotes $\lambda^{<\kappa}$ and $p : \mathcal{P}_\kappa\theta \to \mathcal{P}_\kappa\lambda$ the projection. In this section, we prove the following theorems.

Theorem 4.1. $\{x \in \mathcal{P}_\kappa\lambda \mid x \cap \kappa$ *is almost x-ineffable*$\} \in p_*(\mathrm{NIn}_{\kappa,\theta})^*$.

Theorem 4.2. *Let $\mathcal{I}$ be a normal, $(\theta, 2)$-distributive ideal on $\mathcal{P}_\kappa\lambda$. Then* $\{x \in \mathcal{P}_\kappa\lambda \mid x \cap \kappa$ *is x-ineffable*$\} \in \mathcal{I}^*$.

For Theorem 4.2, in the case of original ineffability, Johnson [7, Cor. 4] proved a stronger result.

Theorem 4.1 has the following corollary.

Corollary 4.3. *Let κ be the least cardinal α such that α is almost α^+-ineffable. Then κ is not κ^+-ineffable.*

P r o o f. To get a contradiction, assume that κ is κ^+-ineffable. By a result of Johnson [8], $(\kappa^+)^{<\kappa} = \kappa^+$. So, $p_*(\mathrm{NIn}_{\kappa,\kappa^+})$ is proper. Since $\{x \in \mathcal{P}_\kappa\kappa^+ \mid |x| = (x \cap \kappa)^+\} \in p_*(\mathrm{NIn}_{\kappa,\kappa^+})^*$, by Theorem 4.1, there exists $x \in \mathcal{P}_\kappa\kappa^+$ such that $x \cap \kappa$ is almost x-ineffable and $|x| = (x \cap \kappa)^+$. Since $x \cap \kappa < \kappa$, this contradicts the choice of κ. ∎

By using a similar argument, the next corollary follows from Theorem 4.2.

Corollary 4.4. *Let κ be the least cardinal α such that α is α^+-ineffable. Then κ is not completely κ^+-ineffable.* ∎

First we prove Theorem 4.1. Before starting the proof, we show the following lemma.

Let $h:\theta\to\mathcal{P}_\kappa\lambda$ be a bijection, $\pi=\pi(h)$, and $q=q(h)$.

Lemma 4.5. *Let $X\in q_*(\mathrm{NIn}_{\kappa,\theta})^+$ and, for each $t\in\mathcal{P}_\kappa\lambda$, W_t be a family of disjoint subsets of X such that $|W_t|<\kappa$ and $X\setminus\bigcup W_t\in \mathrm{I}_{\kappa,\lambda}$. Then there exists $\sigma\in\prod_{t\in\mathcal{P}_\kappa\lambda}W_t$ such that*

$$\mathop{\triangle}_{t\in\mathcal{P}_\kappa\lambda}\sigma(t)\in\mathbf{S}(\mathrm{NS}_{\kappa,\lambda})^+.$$

Proof. Take an enumeration $\langle A_s\mid s\in\mathcal{P}_\kappa\lambda\rangle$ of $\bigcup_{t\in\mathcal{P}_\kappa\lambda}W_t$. For each $x\in X$, define $f_x:\mathcal{Q}_x\to 2$ by

$$f_x(s)=\begin{cases}0 & \text{if } x\in A_s,\\ 1 & \text{if } x\notin A_s.\end{cases}$$

By Theorem 3.8(3), there exists $f:\mathcal{P}_\kappa\lambda\to 2$ such that

$$Z=\{x\in X\mid f_x\subset f\}\in\mathbf{S}(\mathrm{NS}_{\kappa,\lambda})^+.$$

Claim 1. $\forall t\in\mathcal{P}_\kappa\lambda\ \forall A\in W_t\ (Z\setminus A\in\mathrm{NS}^+_{\kappa,\lambda}\Rightarrow Z\cap A\in\mathrm{NS}_{\kappa,\lambda})$.

Proof. Let $t\in\mathcal{P}_\kappa\lambda$ and $A\in W_t$ and $Z\setminus A\in\mathrm{NS}^+_{\kappa,\lambda}$. Take $s\in\mathcal{P}_\kappa\lambda$ such that $A=A_s$. Take $x\in\mathcal{P}_\kappa\lambda$ such that $s\in\mathcal{Q}_x$. Then, since $Z\setminus A\in\mathrm{NS}^+_{\kappa,\lambda}$, we have $(Z\setminus A)\cap\widehat{x}\neq\emptyset$. So, $f(s)=0$. Hence, $Z\cap A\cap\widehat{x}=\emptyset$.

Claim 2. $\forall t\in\mathcal{P}_\kappa\lambda\ \exists!\,A\in W_t\ (Z\setminus A\in\mathrm{NS}_{\kappa,\lambda})$.

Proof. Let $t\in\mathcal{P}_\kappa\lambda$. The uniqueness follow from the assumption that W_t is disjoint. The existence follows from Claim 1 and the fact that $Z\cap\bigcup W_t\in\mathbf{S}(\mathrm{NS}_{\kappa,\lambda})^+$.

By Claim 2, take $\sigma\in\prod_{t\in\mathcal{P}_\kappa\lambda}W_t$ such that $Z\setminus\sigma(t)\in\mathrm{NS}_{\kappa,\lambda}$ for any $t\in\mathcal{P}_\kappa\lambda$. Then σ is as required. ∎

Proof of Theorem 4.1. To get a contradiction, assume that

$$X=\{x\in\mathcal{P}_\kappa\lambda\mid x\cap\kappa\text{ is not almost }x\text{-ineffable}\}\in p_*(\mathrm{NIn}_{\kappa,\theta})^+.$$

Without loss of generality, we may assume that $q\pi\restriction X$ is the identity function on X and $p\restriction\pi''X=q\restriction\pi''X$. For each $x\in X$, take $f^x_t:t\to 2$ (for $t\in\mathcal{Q}_x$) such that

$$\forall f:x\to 2\ (\{t\in\mathcal{Q}_x\mid f^x_t\subset f\}\in\mathrm{I}_{\kappa\cap x,x}).$$

For each $t \in \mathcal{P}_\kappa\lambda$, define $A_t(e)$ (for $e \in {}^t2$) by $A_t(e) = \{x \in X \mid t \in Q_x \,\&\, f^x_t = e\}$, and set $W_t = \{A_t(e) \mid e \in {}^t2\}$. By Lemma 4.5, there exists $\sigma \in \prod_{t\in\mathcal{P}_\kappa\lambda} W_t$ such that

$$Z = \mathop{\triangle}_{t\in\mathcal{P}_\kappa\lambda} \sigma(t) \in \mathbf{S}(\mathrm{NS}_{\kappa,\lambda})^+.$$

For each $t \in \mathcal{P}_\kappa\lambda$, take $e_t \in {}^t2$ such that $\sigma(t) = A_t(e_t)$. Then

$$\forall x \in Z\ \forall t \in \mathcal{Q}_x\ (f^x_t = e_t).$$

Since $X \in p_*(\mathrm{NIn}_{\kappa,\theta})^+ \subset \mathrm{NIn}^+_{\kappa,\lambda}$, there exists $e : \lambda \to 2$ such that $X' = \{x \in X \mid e_x \subset e\} \in \mathrm{NS}^+_{\kappa,\lambda}$. Take $\tau : \mathcal{P}_\kappa\lambda \to \mathcal{P}_\kappa\lambda$ such that

$$\forall t \in \mathcal{P}_\kappa\lambda\ \exists s \in X'\ (t \subset s \prec \tau(t) \in X').$$

Since $Z \in \mathbf{S}(\mathrm{NS}_{\kappa,\lambda})^+$, there is $x \in Z$ such that $x \in \mathrm{cl}(\tau)$. Set $f = e{\restriction}x$. Then it is easy to see that $\{t \in \mathcal{Q}_x \mid f^x_t \subset f\} \in \mathrm{I}^+_{\kappa\cap x,x}$. But this contradicts the choice of $\{f^x_t \mid t \in \mathcal{Q}_x\}$. ∎

Next, we shall prove Theorem 4.2. The following lemma is an analogue of a result of Johnson [8, Theorem 5.1] and can be proved by a similar argument. But for the convenience of the reader, we give a proof.

LEMMA 4.6. *The following statements are equivalent.*

(a) *$\mathcal{I}$ is normal and satisfies the $(\theta, 2)$-distributive law.*

(b) *Whenever $X \in \mathcal{I}^+$ and $A_x \subset \mathcal{Q}_x$ (for $x \in X$), there exists $A \subset \mathcal{P}_\kappa\lambda$ such that $\{x \in X \mid A \cap \mathcal{Q}_x = A_x\} \in \mathcal{I}^+$.*

Proof. (a)⇒(b). For each $t \in \mathcal{P}_\kappa\lambda$, set

$$X_{t,0} = \{x \in X \mid t \in A_x\}, \quad X_{t,1} = \{x \in X \mid t \notin A_x\}, \quad W_t = \{X_{t,0}, X_{t,1}\}.$$

Take $g : \mathcal{P}_\kappa\lambda \to 2$ and $Z \in \mathcal{I}^+$ such that $Z \setminus X_{t,g(t)} \in \mathcal{I}$ for each $t \in \mathcal{P}_\kappa\lambda$. Set $A = g^{-1}\{0\}$ and $Z_1 = \triangle_{t\in\mathcal{P}_\kappa\lambda} X_{t,g(t)}$. It is easy to check that $A \cap \mathcal{Q}_x = A_x$ for all $x \in Z_1$. Since $\mathcal{I}$ is strongly normal, $Z \setminus Z_1 \in \mathcal{I}$. So, $Z_1 \in \mathcal{I}^+$.

(b)⇒(a). Normality can be easily proved. So, we must only show distributivity. Suppose that $X \in \mathcal{I}^+$ and W_t is an $\mathcal{I}$-partition of X with $|W_t| \le 2$, for each $t \in \mathcal{P}_\kappa\lambda$. Without loss of generality, we may assume that $W_t = \{X_{t,0}, X_{t,1}\}$ is a disjoint partition of X for all $t \in \mathcal{P}_\kappa\lambda$. For each $x \in X$, define $A_x = \{t \in \mathcal{Q}_x \mid x \in X_{t,0}\}$. By (b), there exists $A \subset \mathcal{P}_\kappa\lambda$ such that

$$X' = \{x \in X \mid A \cap \mathcal{Q}_x = A_x\} \in \mathcal{I}^+.$$

Define $g : \mathcal{P}_\kappa\lambda \to 2$ by

$$g(t) = \begin{cases} 0 & \text{if } x \in A,\\ 1 & \text{if } x \notin A.\end{cases}$$

We claim that $X' \setminus X_{t,g(t)} \in \mathcal{I}$ for all $t \in \mathcal{P}_\kappa\lambda$. So, let $t \in \mathcal{P}_\kappa\lambda$. Take $x \in \mathcal{P}_\kappa\lambda$ such that $t \in \mathcal{Q}_x$. Then it is easy to check that $(X' \setminus X_{t,g(t)}) \cap \widehat{x} = \emptyset$. Hence, $X' \setminus X_{t,g(t)} \in \mathrm{I}_{\kappa,\lambda} \subset \mathcal{I}$. ∎

LEMMA 4.7. *Suppose that $\mathcal{I}$ is $(\theta,2)$-distributive. Then*

$$\{x \in X \mid X \cap \mathcal{Q}_x \in \mathrm{NS}_{\kappa\cap x,x}\} \in \mathcal{I} \quad \textit{for any } X \subset \mathcal{P}_\kappa\lambda.$$

Proof. To get a contradiction, suppose that there exists $X \subset \mathcal{P}_\kappa\lambda$ such that

$$X_0 = \{x \in X \mid X \cap \mathcal{Q}_x \in \mathrm{NS}_{\kappa\cap x,x}\} \in \mathcal{I}^+.$$

For each $x \in X_0$, take $C_x \subset \mathcal{Q}_x$ such that C_x is club in $\mathcal{Q}_x$ and $C_x \cap X \cap \mathcal{Q}_x = \emptyset$. Since $\mathcal{I}$ satisfies the $(\theta,2)$-distributive law, by Lemma 4.6 there is $D \subset \mathcal{P}_\kappa\lambda$ such that

$$X_1 = \{x \in X_0 \mid C_x = D \cap \mathcal{Q}_x\} \in \mathcal{I}^+.$$

Then D is club in $\mathcal{P}_\kappa\lambda$. So, take $t, x \in D \cap X_1$ such that $t \prec x$. Then $t \in D \cap \mathcal{Q}_x = C_x$. But this contradicts the fact that $C_x \cap X \cap \mathcal{Q}_x = \emptyset$. ∎

Proof of Theorem 4.2. To get a contradiction, assume that

$$X = \{x \in \mathcal{P}_\kappa\lambda \mid x \cap \kappa \text{ is not } x\text{-ineffable}\} \in \mathcal{I}^+.$$

For each $x \in X$, take $f_t^x : t \to 2$ (for $t \in \mathcal{Q}_x$) such that

$$\forall f : x \to 2\ (\{t \in \mathcal{Q}_x \mid f_t^x \subset f\} \in \mathrm{NS}_{\kappa\cap x,x}).$$

For each $t \in \mathcal{P}_\kappa\lambda$, define $A_t(g) \subset \mathcal{P}_\kappa\lambda$ (for $g \in {}^t2$) by $A_t(g) = \{x \in X \mid t \in \mathcal{Q}_x \;\&\; f_t^x = g\}$ and set $W_t = \{A_t(g) \mid g \in {}^t2\} \cap \mathcal{I}^+$. Since W_t is an $\mathcal{I}$-partition of X for all $t \in \mathcal{P}_\kappa\lambda$, there exist $\sigma \in \prod_{t\in\mathcal{P}_\kappa\lambda} W_t$ and $X_0 \in \mathcal{I}^+$ such that

$$X_0 \subset X \quad \text{and} \quad X_0 \setminus \sigma(t) \in \mathcal{I} \quad \text{for all } t \in \mathcal{P}_\kappa\lambda.$$

Set $X_1 = \triangle_{t\in\mathcal{P}_\kappa\lambda}\sigma(t)$. Since $\mathcal{I}$ is strongly normal, $X_1 \in \mathcal{I}^+$. For each $t \in \mathcal{P}_\kappa\lambda$, take $g_t : t \to 2$ such that $\sigma(t) = A_t(g_t)$. Since $X_1 \in \mathcal{I}^+$, there exists $g : \lambda \to 2$ such that

$$X_2 = \{x \in X_1 \mid g_x \subset g\} \in \mathcal{I}^+.$$

By Lemma 4.7,

$$X_3 = \{x \in X_2 \mid X_2 \cap \mathcal{Q}_x \in \mathrm{NS}^+_{\kappa\cap x,x}\} \in \mathcal{I}^+.$$

Take $x \in X_3$. Then it is easy to check that $X_2 \cap \mathcal{Q}_x \subset \{t \in \mathcal{Q}_x \mid f_t^x \subset g{\restriction}x\}$. So, $\{t \in \mathcal{Q}_x \mid f_t^x \subset g{\restriction}x\} \in \mathrm{NS}^+_{\kappa\cap x,x}$. But this contradicts the choice of $\{f_t^x \mid t \in \mathcal{Q}_x\}$. ∎

References

[1] Y. Abe, *Notes on $\mathcal{P}_\kappa\lambda$ and $[\lambda]^\kappa$*, Tsukuba J. Math. 10 (1986), 155–163.

[2] D. Carr, *$\mathcal{P}_\kappa\lambda$-generalizations of weak compactness*, Z. Math. Logik Grundlag. Math. 31 (1985), 393–401.
[3] —, *The structure of ineffability properties of $\mathcal{P}_\kappa\lambda$*, Acta Math. Hungar. 47 (1986), 325–332.
[4] —, *$\mathcal{P}_\kappa\lambda$ partition relations*, Fund. Math. 128 (1987), 181–195.
[5] D. Carr and D. Pelletier, *Towards a structure theory for ideals on $\mathcal{P}_\kappa\lambda$*, in: Set Theory and its Applications, J. Steprāns and S. Watson (eds.), Lecture Notes in Math. 1401, Springer, 1987, 41–54.
[6] T. Jech, *Some combinatorial problems concerning uncountable cardinals*, Ann. Math. Logic 5 (1973), 165–198.
[7] C. A. Johnson, *Distributive ideals and partition relations*, J. Symbolic Logic 51 (1986), 617–625.
[8] —, *Some partition relations for ideals on $\mathcal{P}_\kappa\lambda$*, Acta Math. Hungar. 56 (1990), 269–282.

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