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Universal spaces in the theory of transfinite dimension, II

by

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Abstract. We construct a family of spaces with "nice" structure which is universal in the class of all compact metrizable spaces of large transfinite dimension ω_0, or, equivalently, of small transfinite dimension ω_0; that is, the family consists of compact metrizable spaces whose transfinite dimension is ω_0, and every compact metrizable space with transfinite dimension ω_0 is embeddable in a space of the family. We show that the least possible cardinality of such a universal family is equal to the least possible cardinality of a dominating sequence of irrational numbers.

1. Introduction. In Part I of the paper we have proved that there is no universal space in the class of all compact metrizable spaces X with $\operatorname{Ind} X = \omega_0$, or equivalently, with $\operatorname{ind} X = \omega_0$ (see [3], Proposition 4.11, or Lemma 2.3 of this paper). That class will be denoted by $\mathcal{D}$. We have also shown that there is no universal space in the class of all separable metrizable spaces X with $\operatorname{Ind} X = \omega_0$, to be denoted by $\mathcal{C}$. In this part we introduce the notion of a universal family which is a generalization of the notion of a universal space, and we study universal families for $\mathcal{C}$ and $\mathcal{D}$.

1.1. Definition. Let $\mathcal{C}$ be a class of topological spaces. A family $\mathcal{A}$ of spaces belonging to $\mathcal{C}$ is said to be a *universal family* in $\mathcal{C}$ if every space in $\mathcal{C}$ is embeddable in a space belonging to $\mathcal{A}$.

Universal families can play a role similar to that played by universal spaces. Universal families of small cardinality consisting of spaces with "nice" structure are of particular interest.

In Sections 4 and 5 we construct a universal family $\mathcal{A}$ in $\mathcal{D}$ consisting of spaces with "nice" structure. Since every separable metrizable space X has a compactification Z such that $\operatorname{Ind} Z = \operatorname{Ind} X$ (see [5], and [6] for the proof), the family $\mathcal{A}$ is also universal in $\mathcal{C}$. In Section 6 we estimate the least possible cardinality of a universal family in $\mathcal{D}$; by the above compactification theorem, it is equal to the least possible cardinality of a universal family in $\mathcal{C}$.

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In order to formulate our result, we have to recall a few notions.

Irrational numbers can be viewed as sequences of natural numbers. We denote by $\mathbb{N}^{\omega_0}$ the set of irrational numbers, i.e., the set of all sequences $\sigma = (\sigma(k))_{k=0}^{\infty}$ of natural numbers. On $\mathbb{N}^{\omega_0}$ we consider the relation $\le^*$ defined by letting

$$\sigma \le^* \tau \quad \text{if} \quad \sigma(k) \le \tau(k) \text{ for all but a finite number of } k \in \mathbb{N}\,.$$

A subset $D \subseteq \mathbb{N}^{\omega_0}$ cofinal in $\mathbb{N}^{\omega_0}$ is said to be *dominating*, i.e., D is dominating if for every $\sigma \in \mathbb{N}^{\omega_0}$, there exists a $\tau \in D$ such that $\sigma \le^* \tau$. We set $\mathfrak{d} = \min\{|D| : D \subseteq \mathbb{N}^{\omega_0}$ is a dominating sequence$\}$.

One can prove that

$$\aleph_1 \le \mathfrak{d} \le \mathfrak{c}\,;$$

one can also prove that each of the following formulae is consistent with the axioms of set theory:

$$\aleph_1 = \mathfrak{d} = \mathfrak{c}\,,$$
$$\aleph_1 = \mathfrak{d} < \mathfrak{c}\,,$$
$$\aleph_1 < \mathfrak{d} < \mathfrak{c}\,,$$
$$\aleph_1 < \mathfrak{d} = \mathfrak{c}\,.$$

For a deeper discussion and the proofs of the above statements we refer the reader to E. K. van Douwen's survey [1].

Section 6 contains the proof of the equality

$$\min\{|\mathcal{A}| : \mathcal{A} \text{ is a universal family in } \mathcal{D}\} = \mathfrak{d}\,,$$

which, in particular, gives

$$\min\{|\mathcal{A}| : \mathcal{A} \text{ is a universal family in } \mathcal{C}\} = \mathfrak{d}\,.$$

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2. Three lemmas. For a topological space and its closed subset A, we denote by X/A the quotient space obtained by identifying A to a point (see [4], Example 2.4.12); we denote this point by a, and the natural quotient mapping by q.

2.1. Lemma. *Let X be a compact metrizable space, and A its closed subset. Let ε be a positive real number. If $f : X \to I^n$ has the property that $f|A$ is an ε-mapping, then there exist $m > n$ and $g : X \to I^{m-n}$ such that the diagonal $f \bigtriangleup g$ is an ε-mapping and*

$$(f \bigtriangleup g)^{-1}(I^n \times \{(0, \ldots, 0)\}) = A\,.$$

Proof. Since X/A is a compact metrizable space, there exists an embedding

$$h = (h_1, h_2, \ldots) : X/A \to I^{\aleph_0} = I \times I \times \ldots$$

Let $h_0 : X/A \to I$ be a function such that $(h_0)^{-1}(0) = \{a\}$; set $\phi = (h_0, h_0 \cdot h_1, h_0 \cdot h_2, \ldots)$. It is easy to check that $\phi : X/A \to I \times I^{\aleph_0}$ is also an embedding, and therefore $f \triangle (\phi \circ q) : X \to I^n \times I \times I^{\aleph_0}$ is an ε-mapping. By compactness of X, so is $f \triangle (p_{m-n} \circ \phi \circ q)$ for sufficiently large m, where $p_{m-n} : I \times I^{\aleph_0} \to I \times I^{m-n}$ denotes the projection, i.e., $p_{m-n}((x_1, x_2, x_3, \ldots)) = (x_1, x_2, x_3, \ldots, x_{m-n+1})$ for $x_1 \in I$ and $(x_2, x_3, \ldots) \in I^{\aleph_0}$. Consider an m with this property.

Set $g = p_{m-n} \circ \phi \circ q$. Then $(f \triangle g)^{-1}(I^n \times \{(0, \ldots, 0)\}) = g^{-1}((0, \ldots, 0)) = (h_0 \circ q)^{-1}(0) = q^{-1}(a) = A$.

2.2. Lemma. *Let m and n be natural numbers such that $n \geq 2m + 2$. Let X be a compact metrizable space, and A and B its closed subspaces with* $\operatorname{Ind} B \leq m$. *Then there exists $f : X \to I^n$ such that $f|B - A$ is an embedding, and $f^{-1}((0, \ldots, 0)) = A$.*

Proof. Consider the quotient mapping $q : X \to X/A$. Since we have $\operatorname{Ind}(\{a\} \cup q(B)) \leq m$ and $n - 1 \geq 2m + 1$, there exists an embedding $h : \{a\} \cup q(B) \to I^{n-1}$. Let h^* be an extension of h onto X/A, and $g : X/A \to I$ a function such that $g^{-1}(0) = \{a\}$. Since I^{n-1} is embeddable in the geometrical boundary $\operatorname{bd} I^n$ of I^n, and $\operatorname{bd} I^n$ is homogeneous, we can assume that $\operatorname{bd} I^n$ is the range of h^*, and $h^*(a) = (0, \ldots, 0)$. Let $\phi : (\operatorname{bd} I^n) \times I \to I^n$ be an embedding with $\phi(x, 0) = x$ for every $x \in \operatorname{bd} I^n$.

It is easy to check that $f = \phi \circ (h^* \triangle g) \circ q$ has the required properties.

Note that the assumption $n \geq 2m + 2$ in Lemma 2.2 can be replaced by $n \geq 2m + 1$. The proof under this weaker assumption is similar to that of Corollaries 2.5 and 2.7 in [7]. However, we will only need the lemma in the form given above.

2.3. Lemma. *A compact metrizable space satisfies* $\operatorname{Ind} X \leq \omega_0$ *if and only if for every pair of distinct points $x, y \in X$ there exists a finite-dimensional partition L between x and y.*

Proof. The necessity is obvious. To show the sufficiency, we first prove that for any $x \in X$ and any neighbourhood $U \subseteq X$ of x, there exists an open set $V \subseteq X$ such that $\operatorname{Ind} \operatorname{bd} V < \omega_0$, i.e., $\operatorname{ind} X \leq \omega_0$.

For every $y \in X - U$, consider a finite-dimensional partition L_y between x and y; let U_y and V_y be disjoint open subsets of X such that $x \in U_y$, $y \in V_y$, and $L_y = X - (U_y \cup V_y)$. Then $X - U \subseteq \bigcup\{V_y : y \in X - U\}$; by compactness of $X - U$, there exists a finite family $\mathcal{V} \subseteq \{V_y : y \in X - U\}$

such that $X - U \subseteq \bigcup \mathcal{V}$. It follows immediately that

$$V = X - \bigcup \{\operatorname{cl} V_y : V_y \in \mathcal{V}\}$$

is an open subset of X with $x \in V \subseteq U$, and

$$\operatorname{bd} V \subseteq \bigcup \{\operatorname{bd} V_y : V_y \in \mathcal{V}\} \subseteq \bigcup \{L_y : V_y \in \mathcal{V}\};$$

since $\mathcal{V}$ is finite,

$$\operatorname{Ind} \operatorname{bd} V \le \max\{\operatorname{Ind} L_y : V_y \in \mathcal{V}\} < \omega_0 .$$

Now, let $A \subseteq X$ be an arbitrary closed set, and $U \subseteq X$ an open set containing A. For every $x \in A$, consider an open set $V_x \subseteq X$ such that $x \in V_x \subseteq U$ and $\operatorname{Ind} \operatorname{bd} V_x < \omega_0$. By compactness of A, there exists a finite family $\mathcal{V} \subseteq \{V_x : x \in A\}$ such that $A \subseteq \bigcup \mathcal{V}$. Then $V = \bigcup \mathcal{V}$ is an open subset of X such that $A \subseteq V \subseteq U$ and $\operatorname{bd} V \subseteq \bigcup \{\operatorname{bd} V_x : V_x \in \mathcal{V}\}$; since $\mathcal{V}$ is finite,

$$\operatorname{Ind} \operatorname{bd} V \le \max\{\operatorname{Ind} \operatorname{bd} V_x : V_x \in \mathcal{V}\} < \omega_0 .$$

3. The structure of spaces X with $\operatorname{Ind} X \le \omega_0$. In this section we shall prove that compact metrizable spaces of a certain structure have large transfinite dimension not greater than ω_0 (see Theorem 3.2); actually, it turns out (see Theorem 5.1) that each compact metrizable space X with $\operatorname{Ind} X \le \omega_0$ has that structure.

Let $\{M_k, r_j^k\}$ be an inverse sequence; then M denotes the inverse limit of $\{M_k, r_j^k\}$, and $r_k : M \to M_k$, for $k \in \mathbb{N}$, denotes the projection. Each family $\{M_k, r_k^{k+1}\}$, where $r_k^{k+1} : M_{k+1} \to M_k$, determines an inverse sequence $\{M_k, r_j^k\}$; to wit, it suffices to set $r_j^k = r_j^{j+1} \circ \ldots \circ r_{k-1}^k$ for $k > j$ and r_k^k equal to the identity mapping of M_k. For simplicity, we shall also call each such family $\{M_k, r_k^{k+1}\}$ an inverse sequence.

The next lemma is a technical one and will only be used in the proofs of Theorem 3.2 and Lemma 6.1.

3.1. Lemma. *Let n be a fixed natural number, and let $x, y \in M$ be distinct. Suppose there is a $j \in \mathbb{N}$ such that for every $k \ge j$, there exist pairwise disjoint subsets U_k, V_k and L_k of M_k, where U_k, V_k are open and L_k is closed, which satisfy the following conditions*:

(3.1) $\quad M_k = U_k \cup V_k \cup L_k$,

(3.2) $\quad \operatorname{Ind} L_k \le n$,

(3.3) $\quad r_j(x) \in U_j$ *and* $r_j(y) \in V_j$, (3.4) $(r_k^{k+1})^{-1}(U_k) \subseteq U_{k+1}$ *and* $(r_k^{k+1})^{-1}(V_k) \subseteq V_{k+1}$.

Then there exists an at most n*-dimensional partition in* M *between* x *and* y.

Proof. Define

$$L = \bigcap_{k=j}^{\infty} r_k^{-1}(L_k), \quad U = \bigcup_{k=j}^{\infty} r_k^{-1}(U_k), \quad \text{and} \quad V = \bigcup_{k=j}^{\infty} r_k^{-1}(V_k).$$

From (3.1) it follows directly that $M = U \cup V \cup L$. Clearly, U, V are open and L is closed in M. Since $U_k \cap V_k = \emptyset$, we have $U \cap V = \emptyset$ by (3.4), and since $U_k \cap L_k = \emptyset = L_k \cap V_k$, we also have $U \cap L = \emptyset = L \cap V$. By (3.3), $x \in U$ and $y \in V$, and thus L is a partition in M between x and y.

Since $L_{k+1} \cap U_{k+1} = \emptyset = V_{k+1} \cap L_{k+1}$, by (3.1) and (3.4), $r_k^{k+1}(L_{k+1}) \subseteq L_k$, so we can regard $r_k^{k+1}|L_{k+1}$ as a mapping to L_k; thus L coincides with $\varprojlim\{L_k, r_k^{k+1}|L_{k+1}\}$, and so $\operatorname{Ind} L \le n$ by the theorem on the dimension of the limit of an inverse sequence (see [2], Theorem 1.13.4) and (3.2).

3.2. Theorem. *Let* $\{M_k, r_k^{k+1}\}$ *be an inverse sequence of finite-dimensional compact metrizable spaces* M_k *in which all bonding mappings* r_k^{k+1} *are retractions. Suppose that for every* $k \in \mathbb{N}$, *there exist a covering* $\mathcal{A}_k$ *of* M_k *and a metric* ϱ_k *on* M_k *with the following properties*:

(3.5) *the sets* $A_{k+1} - M_k$, *where* $A_{k+1} \in \mathcal{A}_{k+1}$, *are open in* M_{k+1} *and pairwise disjoint*,

(3.6) *for every* $A_{k+1} \in \mathcal{A}_{k+1}$, *there exists an* $A_k \in \mathcal{A}_k$ *such that* $r_k^{k+1}(A_{k+1}) \subseteq A_k$,

(3.7) *for every* $i \in \mathbb{N}$, *the sequence of real numbers* $(\sup\{\operatorname{diam}_{\varrho_i} r_i^k(A_k) : A_k \in \mathcal{A}_k\})_{k=i+1}^{\infty}$ *converges to* 0.

Then $\operatorname{Ind} M \le \omega_0$.

Proof. First for every $A_k \in \mathcal{A}_k$, we define $A_k^{(j)} \in \mathcal{A}_j$ for $j \le k$ in such a way that

(3.8) $r_j^k(A_k) \subseteq A_k^{(j)}$,

(3.9) $(A_k^{(i)})^{(j)} = A_k^{(j)}$ whenever $j \le i \le k$.

For instance, one can define $A_k^{(k-1)}$ to be an arbitrary member A_{k-1} of $\mathcal{A}_{k-1}$ such that $r_{k-1}^k(A_k) \subseteq A_{k-1}$ for $A_k \in \mathcal{A}_k$ (its existence is guaranteed by (3.6)), and then set by induction $A_k^{(j)} = (A_k^{(j+1)})^{(j)}$. Of course, we put $A_k^{(k)} = A_k$.

We can now begin the proof of $\operatorname{Ind} M \le \omega_0$. Since M, as the inverse limit of a sequence of compact spaces, is compact, it suffices to find a finite-dimensional partition between any two distinct points $x, y \in M$ (see Lemma 2.3). To this end, we shall apply Lemma 3.1.

Take the smallest $i \in \mathbb{N}$ such that $r_i(x) \neq r_i(y)$ and define $\varepsilon = \varrho_i(r_i(x), r_i(y))$. By (3.7), there exists a $j > i$ such that

$$\operatorname{diam}_{\varrho_i} r_i^j(A_j) < \varepsilon/3 \quad \text{for each } A_j \in \mathcal{A}_j\,. \tag{3.10}$$

Let

$$\begin{aligned} U_i &= \{z \in M_i : \varrho_i(z, r_i(x)) < \varepsilon/3\}\,,\\ V_i &= \{z \in M_i : \varrho_i(z, r_i(x)) > 2\varepsilon/3\}\,,\\ L_i &= \{z \in M_i : \varepsilon/3 \le \varrho_i(z, r_i(x)) \le 2\varepsilon/3\}\,, \end{aligned}$$

and $U_j = (r_i^j)^{-1}(U_i)$, $V_j = (r_i^j)^{-1}(V_i)$, $L_j = (r_i^j)^{-1}(L_i)$.

Put

$$\mathcal{U}_k = \{A_k \in \mathcal{A}_k : A_k^{(j)} \cap V_j = \emptyset\}\,, \qquad \mathcal{V}_k = \{A_k \in \mathcal{A}_k : A_k^{(j)} \cap V_j \neq \emptyset\}$$

for $k \ge j$, and

$$U_k = \Big(\bigcup \mathcal{U}_k\Big) - L_j\,, \quad V_k = \Big(\bigcup \mathcal{V}_k\Big) - L_j\,, \quad L_k = L_j$$

for $k > j$; since the bonding mappings are retractions, L_j is a subset of M_k for $k > j$.

We first check that

$$U_k, V_k \text{ and } L_k \text{ are pairwise disjoint.} \tag{3.11}$$

Obviously, $U_j \cap V_j = \emptyset$. Suppose, on the contrary, that $U_k \cap V_k \neq \emptyset$ for a $k > j$. Then there exists a $z \in (A_k \cap B_k) - L_j$ for some $A_k \in \mathcal{U}_k$ and $B_k \in \mathcal{V}_k$. By (3.5), we have $z \in M_{k-1}$ and since r_{k-1}^k is a retraction, we conclude, using (3.8), that

$$z \in A_k \cap B_k \cap M_{k-1} \subseteq A_k^{(k-1)} \cap B_k^{(k-1)}\,.$$

Observe that $A_k^{(k-1)} \in \mathcal{U}_{k-1}$ and $B_k^{(k-1)} \in \mathcal{V}_{k-1}$; indeed, as $A_k \in \mathcal{U}_k$ ($B_k \in \mathcal{V}_k$) we have $A_k^{(j)} \cap V_j = \emptyset$ ($B_k^{(j)} \cap V_j \neq \emptyset$); hence by (3.9), $(A_k^{(k-1)})^{(j)} \cap V_j = A_k^{(j)} \cap V_j = \emptyset$ ($(B_k^{(k-1)})^{(j)} \cap V_j = B_k^{(j)} \cap V_j \neq \emptyset$), and so $A_k^{(k-1)} \in \mathcal{U}_{k-1}$ ($B_k^{(k-1)} \in \mathcal{V}_{k-1}$). In the same manner we can show by induction that $z \in A_k^{(j)} \cap B_k^{(j)}$ and $A_k^{(j)} \in \mathcal{U}_j$, $B_k^{(j)} \in \mathcal{V}_j$.

Consequently, $A_k^{(j)} \cap V_j = \emptyset$ and $z \notin V_j$; however, $z \notin L_j$, and so, by the definition of U_j, V_j, L_j, we conclude that $z \in U_j$; thus $B_k^{(j)} \cap U_j \neq \emptyset$. On the other hand, $B_k^{(j)} \cap V_j \neq \emptyset$ (since $B_k^{(j)} \in \mathcal{V}_j$).

This shows that

$$\varrho_i(r_i^j(z_1), r_i(x)) < \varepsilon/3 \quad \text{and} \quad \varrho_i(r_i^j(z_2), r_i(x)) > 2\varepsilon/3$$

for some $z_1, z_2 \in B_k^{(j)}$, contrary to (3.10). Thus $U_k \cap V_k = \emptyset$. That $U_k \cap L_k = \emptyset = L_k \cap V_k$ follows directly from the definition of U_k, V_k, L_k.

We now show that for $n = \operatorname{Ind} M_j$,

(3.12) U_k, V_k and L_k satisfy conditions (3.1)–(3.4) of Lemma 3.1.

Condition (3.1) is obvious for $k = j$. Take $k > j$. Since

$$\begin{aligned} M_k \supseteq U_k \cup V_k \cup L_k &= \Big[\Big(\bigcup \mathcal{U}_k\Big) - L_j\Big] \cup \Big[\Big(\bigcup \mathcal{V}_k\Big) - L_j\Big] \cup L_j \\ &\supseteq \Big(\bigcup \mathcal{U}_k\Big) \cup \Big(\bigcup \mathcal{V}_k\Big) = \bigcup \mathcal{A}_k = M_k\,, \end{aligned}$$

(3.1) also holds for $k > j$.

Conditions (3.2) and (3.3) follow immediately from the definitions of U_k, V_k and L_k.

Let $z \in M_{k+1}$ and $r_k^{k+1}(z) \in U_k$. Take an $A_{k+1} \in \mathcal{A}_{k+1}$ such that $z \in A_{k+1}$ and suppose that $A_{k+1} \in \mathcal{V}_{k+1}$.

Then $A_{k+1}^{(j)} \cap V_j \neq \emptyset$, so $(A_{k+1}^{(k)})^{(j)} \cap V_j \neq \emptyset$ (see (3.9)) and $A_{k+1}^{(k)} \in \mathcal{V}_k$. If $k = j$, then, by (3.8), $r_k^{k+1}(z) \in A_{k+1}^{(j)}$, therefore also $A_{k+1}^{(j)} \cap U_j \neq \emptyset$, contrary to (3.10). If $k > j$, then $A_{k+1}^{(k)} \subseteq V_k \cup L_k$ and, by (3.8), $r_k^{k+1}(z) \in V_k \cup L_k$; but we know (see (3.11)) that $(V_k \cup L_k) \cap U_k = \emptyset$, and therefore $r_k^{k+1}(z) \notin U_k$, a contradiction. Thus $A_{k+1} \in \mathcal{U}_{k+1}$.

This clearly forces $A_{k+1} \subseteq U_{k+1} \cup L_{k+1}$. We have $z \notin L_{k+1}$, because otherwise $z \in L_{k+1} = L_k$, and hence $r_k^{k+1}(z) = z \in L_k \cap U_k = \emptyset$ (see (3.11)). Thus $z \in A_{k+1} - L_{k+1} \subseteq U_{k+1}$, and the first part of (3.4) is proved.

The second part of (3.4) can be shown similarly, and thus the proof of (3.12) is complete.

We see at once that

(3.13) L_k is a closed subset of M_k.

In order to check that the assumptions of Lemma 3.1 are satisfied, it remains to show that

(3.14) U_k and V_k are open subsets of M_k.

We prove this for U_k by induction on k; the same argument works for V_k.

For $k = j$, (3.14) is evident. Assume that (3.14) holds for numbers less than some $k > j$.

First, we verify that

(3.15) $U_k \cap M_{k-1} = U_{k-1}$.

Indeed,

$$U_{k-1} \subseteq M_{k-1} \cap (r_{k-1}^k)^{-1}(U_{k-1}) \subseteq M_{k-1} \cap U_k$$

(see (3.12) and (3.4); recall that r_{k-1}^k is a retraction). On the other hand, if there were $z \in (U_k \cap M_{k-1}) - U_{k-1}$, we would have

$$z \in (M_{k-1} - U_{k-1}) \cap U_k \subseteq (L_{k-1} \cup V_{k-1}) - (L_k \cup V_k) \subseteq (L_k \cup V_{k-1}) - (L_k \cup V_{k-1}) = \emptyset$$

(recall that $L_k = L_{k-1}$ and $V_{k-1} \subseteq V_k$; see (3.12) and (3.4)).

We now return to the inductive proof. Take a $z \in U_k$. If $z \in M_{k-1}$, then $z \in U_{k-1}$ by (3.15); then $(r^k_{k-1})^{-1}(U_{k-1})$ is a neighbourhood of z by the inductive assumption, and it follows from (3.12) and (3.4) that $(r^k_{k-1})^{-1}(U_{k-1}) \subseteq U_k$. If $z \notin M_{k-1}$, then $z \in A_k - M_{k-1}$ for some $A_k \in \mathcal{U}_k$; by (3.5), $A_k - M_{k-1}$ is a neighbourhood of z, and since $L_k \subseteq M_{k-1}$, this neighbourhood is contained in U_k.

We have thus verified that the assumptions of Lemma 3.1 are satisfied. Consequently, there exists a finite-dimensional partition (more precisely, a partition of dimension not greater than $n = \operatorname{Ind} M_j$) between x and y in the space M.

4. The spaces I^σ_a. In this section, for any increasing sequence $\sigma = (\sigma(k))^\infty_{k=0}$ of positive integers, and any sequence $a = (a(k))^\infty_{k=1}$ of real numbers such that $1/2 < a(k) < 1$ for every k and $\prod^\infty_{k=1} a(k) = 0$ we construct a compact metrizable space I^σ_a with $\operatorname{Ind} I^\sigma_a = \omega_0$. For our purposes it suffices to restrict attention to any fixed sequence $a = (a(k))^\infty_{k=1}$ which has the above properties, except for Section 6, where we will additionally need the condition

$$(*) \qquad \prod_{k=1}^{\infty} 2a(k) < \infty\,.$$

Thus the reader can assume that $a = (1/2 + 1/2^{k+1})^\infty_{k=1}$; it is easy to see that this sequence has all the required properties.

From now on, σ stands for an increasing sequence of positive integers, and a for a sequence of real numbers such that $1/2 < a(k) < 1$ and $\prod^\infty_{k=1} a(k) = 0$; for a given σ, we denote by $\sigma|k$ the sequence $\sigma(0), \sigma(1), \ldots, \sigma(k-1)$; in particular, $\sigma|0$ is the empty sequence. We denote by $S^a_{\sigma(k)}$ the set of all sequences

$$\gamma : \{1, 2, \ldots, \sigma(k)\} \to \{0, 1 - a(k+1)\}\,,$$

i.e., the set of all sequences of $\sigma(k)$ elements equal to either 0 or $1 - a(k+1)$; S_0 consists of the empty sequence. Let

$$S^a_{\sigma|0} = S_0, \qquad S^a_{\sigma|k} = S^a_{\sigma(0)} \times S^a_{\sigma(1)} \times \ldots \times S^a_{\sigma(k-1)}\,, \qquad S^a_\sigma = \bigcup_{k=0}^{\infty} S^a_{\sigma|k}\,.$$

For every sequence $s = (\gamma_0, \gamma_1, \ldots, \gamma_{k-1}) \in S^a_{\sigma|k}$ and $m < k$, let $s|m$ stand for the sequence $(\gamma_0, \gamma_1, \ldots, \gamma_{m-1}) \in S^a_{\sigma|m}$.

Fix σ and a. First, we construct by induction an inverse sequence $\{I_a^{\sigma|k}, r_a^{\sigma|k+1}\}$, where $r_a^{\sigma|k+1} : I_a^{\sigma|k+1} \to I_a^{\sigma|k}$; simultaneously, we define a covering $\{I_s : s \in S^a_{\sigma|k}\}$ of $I_a^{\sigma|k}$ by $\sigma(k)$-dimensional cubes for $k = 0, 1, \ldots$

Let $I_a^{\sigma|0}$ be the $\sigma(0)$-dimensional cube $I^{\sigma(0)}$ and $I_s = I^{\sigma(0)}$ for the unique $s \in S^a_{\sigma|0}$. Assume that we have already defined $I_a^{\sigma|k}$ and its covering $\{I_s : s \in S^a_{\sigma|k}\}$.

For every $s \in S^a_{\sigma|k}$, we identify I_s and the standard $\sigma(k)$-dimensional cube $I^{\sigma(k)}$. For $t = (s, \gamma) \in S^a_{\sigma|k+1}$, set $I'_t = \{(x_1, \ldots, x_{\sigma(k)}) \in I_s : \gamma(i) \le x_i \le \gamma(i) + a(k+1) \text{ for } i \le \sigma(k)\}$; that is, I'_t is a smaller cube placed in a corner of I_s such that the length ratio of their edges is $a(k+1)$.

Roughly speaking, we glue a $\sigma(k+1)$-dimensional cube, denoted here by I_t, along its $\sigma(k)$-dimensional face, which is identified with I'_t, to every cube $I'_t \subseteq I_a^{\sigma|k}$, where $t \in S^a_{\sigma|k+1}$, in such a way that the sets $I_t - I'_t$ are pairwise disjoint. Our $I_a^{\sigma|k+1}$ is the space so obtained.

Precisely, the space $I_a^{\sigma|k+1}$ can be defined as follows. Let Q_t, where $t \in S^a_{\sigma|k+1}$, be a copy of the $(\sigma(k+1) - \sigma(k))$-dimensional cube $I^{\sigma(k+1)-\sigma(k)}$. Set

$$I_t = \{(y, \{y_s : s \in S^a_{\sigma|k+1}\}) \in I_a^{\sigma|k} \times \mathbb{P}\{Q_s : s \in S^a_{\sigma|k+1}\} : y \in I'_t \text{ and } y_s = (0, \ldots, 0) \text{ for } s \ne t\}$$

for $t \in S^a_{\sigma|k+1}$, and

$$I_a^{\sigma|k+1} = \bigcup\{I_t : t \in S^a_{\sigma|k+1}\}\,.$$

It is easily seen that the covering $\{I_s : s \in S^a_{\sigma|k+1}\}$ consists of $\sigma(k+1)$-dimensional cubes.

The orthogonal projections of the $\sigma(k+1)$-dimensional cubes I_s onto their $\sigma(k)$-dimensional faces I'_s, where $s \in S^a_{\sigma|k+1}$, determine a retraction of $I_a^{\sigma|k+1}$ onto $I_a^{\sigma|k}$; denote it by $r_a^{\sigma|k+1}$. More precisely,

$$r_a^{\sigma|k+1}((y, \{y_s : s \in S^a_{\sigma|k+1}\})) = (y, \{z_s : s \in S^a_{\sigma|k+1}\})\,,$$

where $z_s = (0, \ldots, 0)$ for $s \in S^a_{\sigma|k+1}$, for every $(y, \{y_s : s \in S^a_{\sigma|k+1}\})$.

Thus, the inductive construction of $\{I_a^{\sigma|k}, r_a^{\sigma|k+1}\}$ is complete.

In Fig. 4.1 the first steps in constructing $\{I_a^{\sigma|k}, r_a^{\sigma|k+1}\}$, where $\sigma(k) = k+1$ for $k = 0, 1, \ldots$, and $a(k) = 1/2 + 1/2^{k+1}$ for $k = 1, 2, \ldots$, are exhibited.

Let

$$I_a^\sigma = \varprojlim\{I_a^{\sigma|k}, r_a^{\sigma|k+1}\}\,.$$

We list several properties of the space I_a^σ.

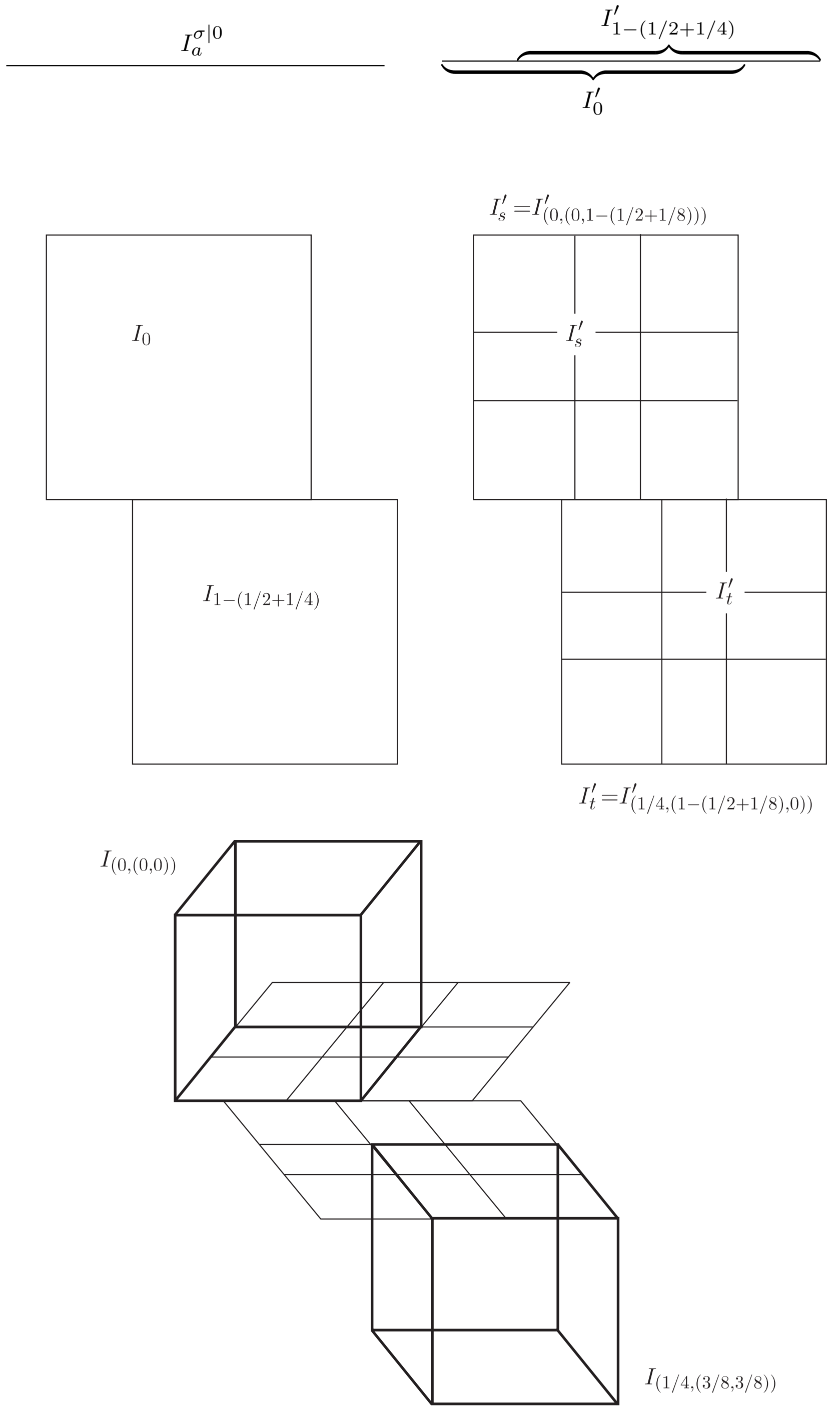
$I_a^{\sigma|0}$
$I'_{1-(1/2+1/4)}$
I'_0
$I'_s = I'_{(0,(0,1-(1/2+1/8)))}$
I_0
I'_s
$I_{1-(1/2+1/4)}$
I'_t
$I'_t = I'_{(1/4,(1-(1/2+1/8),0))}$
$I_{(0,(0,0))}$
$I_{(1/4,(3/8,3/8))}$

Fig. 4.1

It is easy to check that for $k = 0, 1, \ldots$, $I_a^{\sigma|k+1}$ is a closed subspace of $I_a^{\sigma|k} \times \mathbb{P}\{Q_s : s \in S^a_{\sigma|k+1}\}$; thus the spaces $I_a^{\sigma|k}$ are compact and metrizable, and so is I_a^σ. Since $a(k+1) > 1/2$, we have $I_s = \bigcup\{I'_t : t \in S^a_{\sigma|k+1}$ and $t|k = s\}$ for every $s \in S^a_{\sigma|k}$; in particular, $I_a^{\sigma|k} = \bigcup\{I'_t : t \in S^a_{\sigma|k+1}\}$. Hence

$$I_s \subseteq \bigcup\{I_t : t \in S^a_{\sigma|k+1} \text{ and } t|k = s\} \quad \text{for every } s \in S^a_{\sigma|k}\,, \tag{4.1}$$

$$I_a^{\sigma|k} \subseteq I_a^{\sigma|k+1} \quad \text{for } k = 0, 1, \ldots \tag{4.2}$$

Since $a(k+1) > 1/2$, we also have

$$\bigcap\{I_t : t \in S^a_{\sigma|k+1} \text{ and } t|k = s\} \neq \emptyset \quad \text{for every } s \in S^a_{\sigma|k}\,. \tag{4.3}$$

It follows immediately from the definition that the bonding mappings $r_a^{\sigma|k+1}$ are retractions; thus we can assume that

$$I_a^{\sigma|k} \subseteq I_a^\sigma \quad \text{for } k = 0, 1, \ldots \tag{4.4}$$

From the definition it also follows that

$$r_a^{\sigma|k+1}(I_s) = I'_s \subseteq I_{s|k} \quad \text{for every } s \in S^a_{\sigma|k+1}\,. \tag{4.5}$$

For $k = 0, 1, \ldots$, there exists a metric $\varrho^a_{\sigma,k}$ on $I_a^{\sigma|k}$ with the property that

$$\begin{aligned} &\operatorname{diam}_{\varrho^a_{\sigma,k}} r_a^{\sigma|m} \circ r_a^{\sigma|m-1} \circ \ldots \circ r_a^{\sigma|k+1}(I_s) \\ &\quad \le \Big(\prod_{l=k+1}^{m} a(l)\Big) \operatorname{diam}_{\varrho^a_{\sigma,k}} I_a^{\sigma|k} \quad \text{for any } m > k \text{ and } s \in S^a_{\sigma|m}\,. \end{aligned} \tag{4.6}$$

The metrics $\varrho^a_{\sigma,k}$ can be defined by induction in the following way: $\varrho^a_{\sigma,0}$ is the standard Euclidean metric on $I_a^{\sigma|0} = I^{\sigma(0)}$, and $\varrho^a_{\sigma,k+1}$ is the restriction to $I_a^{\sigma|k+1}$ of the Cartesian product metric of $I_a^{\sigma|k} \times \mathbb{P}\{Q_s : s \in S^a_{\sigma|k+1}\}$, where $I_a^{\sigma|k}$ is equipped with the metric $\varrho^a_{\sigma|k}$ and every Q_s is equipped with the standard Euclidean metric.

We now prove that for each increasing sequence σ of natural numbers, $\operatorname{Ind} I_a^\sigma = \omega_0$.

Since $I_a^{\sigma|k}$, and hence I_a^σ, contains a $\sigma(k)$-dimensional cube and $\sigma(k) \to \infty$ as $k \to \infty$, we have $\operatorname{Ind} I_a^\sigma \ge \omega_0$.

The opposite inequality follows from Theorem 3.2 applied to $M_k = I_a^{\sigma|k}$ for $k \in \mathbb{N}$; we take $\{I_s : s \in S^a_{\sigma|k}\}$ for $\mathcal{A}_k$. It is clear that (3.5) is satisfied; (3.6) follows from (4.5), and (3.7) follows from (4.6) and the equality $\prod_{k=1}^\infty a(k) = 0$.

5. Every space X with $\operatorname{Ind} X \le \omega_0$ is embeddable in some I_a^σ. Let $a = (a(k))_{k=1}^\infty$ be an arbitrary sequence of real numbers such that $1/2 < a(k) < 1$ for each $k \in \mathbb{N}$ and $\prod_{k=1}^\infty a(k) = 0$.

5.1. THEOREM. *Every compact metrizable space X with* $\operatorname{Ind} X \le \omega_0$ *is embeddable in I_a^σ for some increasing sequence σ of natural numbers.*

Proof. Fix a metric on X. We shall define inductively an increasing sequence $\sigma = (\sigma(k))_{k=0}^\infty$ of natural numbers and a sequence $(h_k)_{k=0}^\infty$ of mappings, where $h_k : X \to I_a^{\sigma|k}$, such that

$$(5.1)_k \qquad r_a^{\sigma|k} \circ h_k = h_{k-1}$$

for every $k > 0$ and

$$(5.2)_k \qquad h_k \text{ is a } 1/(k+1)\text{-mapping.}$$

Simultaneously, we shall define families $\{G_s : s \in S_{\sigma|k+1}^a\}$ of pairwise disjoint open subsets of X (for the definition of $S_{\sigma|k+1}^a$, see Section 4) satisfying the following conditions:

$$(5.3)_k \qquad h_k(G_s) \subseteq I'_s \quad \text{for every } s \in S_{\sigma|k+1}^a,$$

$$(5.4)_k \qquad h_k \Big| X - \bigcup \{G_s : s \in S_{\sigma|k+1}^a\} \text{ is an embedding.}$$

By Lemma 2.1, there exist an $m \in \mathbb{N}$ and a 1-mapping $g_0 : X \to I^m$. Since $\operatorname{Ind} X \le \omega_0$, there exists a finite-dimensional partition L_i in X between

$$E_0^i = g_0^{-1}(\{(z_1, \ldots, z_m) \in I^m : z_i \le 1 - a(1)\})$$

and

$$E_{1-a(1)}^i = g_0^{-1}(\{(z_1, \ldots, z_m) \in I^m : z_i \ge a(1)\}),$$

for $i = 1, \ldots, m$; let G_0^i and $G_{1-a(1)}^i$ be disjoint open subsets of X such that

$$E_0^i \subseteq G_0^i, \quad E_{1-a(1)}^i \subseteq G_{1-a(1)}^i, \quad \text{and} \quad X - L_i = G_0^i \cup G_{1-a(1)}^i\,.$$

Let

$$\sigma(0) = \max\{2(\operatorname{Ind} L_i) + 1 : i = 1, \ldots, m\} + m\,.$$

Consider a mapping $f_0 : X \to I^{\sigma(0)-m}$ such that $f_0 | \bigcup_{i=1}^m L_i$ is an embedding and

$$f_0(X) \subseteq \{(z_{m+1}, z_{m+2}, \ldots, z_{\sigma(0)}) \in I^{\sigma(0)-m} : \\ z_i \le a(1) \text{ for } i = m+1, m+2, \ldots, \sigma(0)\}\,,$$

and set $h_0 = g_0 \vartriangle f_0$; obviously, $h_0 : X \to I^{\sigma(0)} = I_a^{\sigma(0)}$. The family $\{G_s : s \in S_{\sigma|1}^a\}$ is defined by letting

$$G_s = G_{\gamma(1)}^1 \cap G_{\gamma(2)}^2 \cap \ldots \cap G_{\gamma(m)}^m$$

for $s = \gamma = (\gamma(i))_{i=0}^{\sigma(0)} \in S_{\sigma|1}^a$ such that $\gamma(i) = 0$ for $i > m$, and $G_s = \emptyset$ for other $s \in S_{\sigma|1}^a$; observe that this family consists of pairwise disjoint open subsets of X.

It is a simple matter to verify that conditions $(5.2)_k$–$(5.4)_k$ are satisfied.

Assume that the numbers $\sigma(0) < \sigma(1) < \ldots < \sigma(k)$, the mappings h_k, and the families $\{G_s : s \in S^a_{\sigma|k+1}\}$ of pairwise disjoint subsets of X are defined and satisfy $(5.1)_k$–$(5.4)_k$.

Fix an $s \in S^a_{\sigma|k+1}$. The set $I'_s \subseteq I^{\sigma|k}_a$ is closed, and so $h_k(\mathrm{cl}\, G_s) \subseteq I'_s$ by $(5.3)_k$. Since the sets G_s are open and pairwise disjoint, $\mathrm{bd}\, G_s \subseteq X - \bigcup\{G_t : t \in S^a_{\sigma|k+1}\}$. Hence, by $(5.4)_k$, $h_k|\mathrm{bd}\, G_s$ is an embedding of $\mathrm{bd}\, G_s$ in I'_s.

By Lemma 2.1 applied to the space $\mathrm{cl}\, G_s$ and to its closed subset $\mathrm{bd}\, G_s$, there exist a natural number $m > \mathrm{Ind}\, I'_s = \sigma(k)$ and a mapping $g_s : \mathrm{cl}\, G_s \to I^{m-\sigma(k)}$ such that

(5.5) $$(h_k|\mathrm{cl}\, G_s) \vartriangle g_s \text{ is a } 1/(k+2)\text{-mapping},$$

(5.6) $$((h_k|\mathrm{cl}\, G_s) \vartriangle g_s)^{-1}(I'_s \times \{(0,\ldots,0)\}) = \mathrm{bd}\, G_s\,.$$

Since $S^a_{\sigma|k+1}$ is finite, one can assume that m does not depend on s.

Let

$$E^i_{s,0} = ((h_k|\mathrm{cl}\, G_s) \vartriangle g_s)^{-1}(\{(z_1,\ldots,z_m) \in I'_s \times I^{m-\sigma(k)} : z_i \le 1-a(k+2)\})\,,$$

$$E^i_{s,1-a(k+2)} = ((h_k|\mathrm{cl}\, G_s) \vartriangle g_s)^{-1}(\{(z_1,\ldots,z_m) \in I'_s \times I^{m-\sigma(k)} : z_i \ge a(k+2)\})\,.$$

Since $\mathrm{Ind}\,\mathrm{cl}\, G_s \le \mathrm{Ind}\, X \le \omega_0$, there exists a finite-dimensional partition L^i_s in the space $\mathrm{cl}\, G_s$ between $E^i_{s,0}$ and $E^i_{s,1-a(k+2)}$; let $G^i_{s,0}$ and $G^i_{s,1-a(k+2)}$ be disjoint open subsets of $\mathrm{cl}\, G_s$ such that $E^i_{s,0} \subseteq G^i_{s,0}$, $E^i_{s,1-a(k+2)} \subseteq G^i_{s,1-a(k+2)}$, and $\mathrm{cl}\, G_s - L^i_s = G^i_{s,0} \cup G^i_{s,1-a(k+2)}$.

Set $\sigma(k+1) = \max\{2(\mathrm{Ind}\, L^i_s) + 2 : i = 1,\ldots,m \text{ and } s \in S^a_{\sigma|k+1}\} + m$.

By Lemma 2.2 applied to the space $\mathrm{cl}\, G_s$ and its closed subsets $A = \mathrm{bd}\, G_s$ and $B = \bigcup_{i=1}^m L^i_s$, there exists a mapping $f_s : \mathrm{cl}\, G_s \to I^{\sigma(k+1)-m}$ such that

(5.7) $$f_s\Big|\bigcup_{i=1}^m L^i_s - \mathrm{bd}\, G_s \text{ is an embedding},$$

(5.8) $$f_s^{-1}((0,\ldots,0)) = \mathrm{bd}\, G_s\,;$$

obviously, one can additionally assume that

(5.9) $$f_s(\mathrm{cl}\, G_s) \subseteq \{(z_{m+1}, z_{m+2},\ldots, z_{\sigma(k+1)}) \in I^{\sigma(k+1)-m} : z_i \le a(k+2) \text{ for } i = m+1, m+2, \ldots, \sigma(k+1)\}\,.$$

Let $h_s = (h_k|\mathrm{cl}\, G_s) \vartriangle g_s \vartriangle f_s$; then

$$h_s : \mathrm{cl}\, G_s \to I'_s \times I^{m-\sigma(k)} \times I^{\sigma(k+1)-m} = I'_s \times I^{\sigma(k+1)-\sigma(k)}\,.$$

Identifying in the natural way $I'_s \times I^{\sigma(k+1)-\sigma(k)}$ and I_s, we can assume that I_s is the range of h_s.

By (5.6) and (5.8), we have $h_s(x) = h_k(x)$ for every $x \in \mathrm{bd}\, G_s$ and $s \in S^a_{\sigma|k+1}$; thus the mappings h_s, $s \in S^a_{\sigma|k+1}$, and $h_k|X - \bigcup\{G_s : s \in S^a_{\sigma|k+1}\}$ are compatible. Denote by h_{k+1} the combination $(\nabla\{h_s : s \in S^a_{\sigma|k+1}\})\nabla(h_k|X - \bigcup\{G_s : s \in S^a_{\sigma|k+1}\})$; obviously, $h_{k+1} : X \to I^{\sigma|k+1}_a$.

Since the sets $I_s - I'_s$ are pairwise disjoint, we have

$$h^{-1}_{k+1}(I_s - I'_s) = G_s \quad \text{for every } s \in S^a_{\sigma|k+1} \tag{5.10}$$

(see (5.6) and (5.8)).

We now show that h_{k+1} satisfies $(5.1)_{k+1}$ and $(5.2)_{k+1}$.

From the definition of h_{k+1} it follows immediately that $h_{k+1}(x) = h_k(x)$ for $x \in X - \bigcup\{G_s : s \in S^a_{\sigma|k+1}\}$; since $r^{\sigma|k+1}_a$ is a retraction, $r^{\sigma|k+1}_a(h_{k+1}(x)) = h_k(x)$ for these points. On the other hand, if $x \in G_s$ for some $s \in S^a_{\sigma|k+1}$, then $r^{\sigma|k+1}_a(h_{k+1}(x)) = r^{\sigma|k+1}_a(h_s(x)) = r^{\sigma|k+1}_a(((h_k|\operatorname{cl} G_s) \vartriangle g_s \vartriangle f_s)(x)) = (h_k|\operatorname{cl} G_s)(x) = h_k(x)$. Thus we have shown that $(5.1)_{k+1}$ is satisfied.

In order to show $(5.2)_{k+1}$ take an arbitrary point $z \in I^{\sigma|k+1}_a$. If $z \in I^{\sigma|k}_a \subseteq I^{\sigma|k+1}_a$, then $h^{-1}_{k+1}(z) \subseteq X - \bigcup\{G_s : s \in S^a_{\sigma|k+1}\}$ by (5.10); since $h_k(x) = h_{k+1}(x)$ for every $x \in X - \bigcup\{G_s : s \in S^a_{\sigma|k+1}\}$, by $(5.2)_k$, $h^{-1}_{k+1}(z)$ is either empty or a one-point set. If $z \notin I^{\sigma|k}_a$, then $z \in I_s - I'_s$ for some $s \in S^a_{\sigma|k+1}$, and so $h^{-1}_{k+1}(z) \subseteq G_s$ by (5.10); thus

$$h^{-1}_{k+1}(z) = h^{-1}_s(z) = ((h_k|\operatorname{cl} G_s) \vartriangle g_s \vartriangle f_s)^{-1}(z)\,,$$

and so $\operatorname{diam} h^{-1}_{k+1}(z) \le 1/(k+2)$ by (5.5). We have shown that $(5.2)_{k+1}$ is satisfied.

Define the family $\{G_t : t \in S^a_{\sigma|k+2}\}$ by letting

$$G_t = G_s \cap G^1_{s,\gamma(1)} \cap G^2_{s,\gamma(2)} \cap \ldots \cap G^m_{s,\gamma(m)}$$

for $t = (s,\gamma) \in S^a_{\sigma|k+2}$ such that $\gamma(i) = 0$ for $i = m+1, m+2, \ldots, \sigma(k+1)$, and $G_t = \emptyset$ for other $t \in S^a_{\sigma|k+2}$.

It is easily seen that the sets G_t are open, pairwise disjoint, and

$$X - \bigcup\{G_t : t \in S^a_{\sigma|k+2}\} = \Big(X - \bigcup\{G_s : s \in S^a_{\sigma|k+1}\}\Big) \cup \bigcup\{L^i_s : i = 1, \ldots, m \text{ and } s \in S^a_{\sigma|k+1}\}\,. \tag{5.11}$$

We now show $(5.3)_{k+1}$–$(5.4)_{k+1}$. Take a $t = (s,\gamma) \in S^a_{\sigma|k+2}$. If $\gamma(i) = 1 - a(k+2)$ for some $i > m$, then $G_t = \emptyset$, and so $(5.3)_{k+1}$ is satisfied. Assume therefore that $\gamma(i) = 0$ for $i = m+1, m+2, \ldots, \sigma(k+1)$. Since $G^i_{s,\gamma(i)} \cap E^i_{s,1-a(k+2)-\gamma(i)} = \emptyset$ for $i = 1, \ldots, m$, we have

$$((h_k|\operatorname{cl} G_s) \vartriangle g_s)(G_t) \subseteq \{(z_1, \ldots, z_n) \in I'_s \times I^{m-\sigma(k)} : \gamma(i) \le z_i \le \gamma(i) + a(k+2) \text{ for } i = 1, \ldots, m\}\,;$$

hence, by (5.9),

$$h_{k+1}(G_t) = ((h_k|\mathrm{cl}\, G_s) \vartriangle g_s \vartriangle f_s)(G_t) \subseteq I'_t\,,$$

and therefore $(5.3)_{k+1}$ is satisfied for every $s \in S^a_{\sigma|k+2}$.

In order to prove $(5.4)_{k+1}$ it suffices to show that $h_{k+1}|X - \bigcup\{G_s : s \in S^a_{\sigma|k+2}\}$ is 1-1. Put

$$C = X - \bigcup\{G_s : s \in S^a_{\sigma|k+1}\}\,,$$
$$D = \Big(\bigcup\{L^i_s : s \in S^a_{\sigma|k+1},\ i = 1, \ldots, m\}\Big) \cap \Big(\bigcup\{G_s : s \in S^a_{\sigma|k+1}\}\Big)\,;$$

observe that $X - \bigcup\{G_t : t \in S^a_{\sigma|k+2}\} = C \cup D$ (see (5.11)).

Consider a pair of distinct points $x, y \in X - \bigcup\{G_t : t \in S^a_{\sigma|k+2}\}$. If one of them, say x, belongs to C, and the other to D, then $h_{k+1}(x) = h_k(x) \in I^{\sigma|k}_a$, whereas $h_{k+1}(y) \notin I^{\sigma|k}_a$ (see (5.10)); if $x, y \in C$, then $h_{k+1}(x) = h_k(x) \neq h_k(y) = h_{k+1}(y)$ by $(5.4)_k$. If $x, y \in (\bigcup_{i=1}^m L^i_s) \cap G_s$ for some $s \in S^a_{\sigma|k+1}$, then $h_{k+1}(x) = h_s(x) \neq h_s(y) = h_{k+1}(y)$ by (5.7). If $x \in G_s$ and $y \in G_t$ for some distinct $s, t \in S^a_{\sigma|k+1}$, then $h_{k+1}(x) \in I_s - I'_s$ and $h_{k+1}(y) \in I_t - I'_t$ (see (5.10)); since $(I_s - I'_s) \cap (I_t - I'_t) = \emptyset$, we have $h_{k+1}(x) \neq h_{k+1}(y)$. Thus $(5.4)_{k+1}$ is satisfied.

The inductive construction of the mappings $h_0, h_1, \ldots$ is complete.

Since the sequence $(h_k)_{k=0}^\infty$ satisfies $(5.1)_k$ for $k = 1, 2, \ldots$, it determines a mapping $h : X \to I^\sigma_a$; from $(5.2)_k$, $k = 0, 1, \ldots$, it follows that h is 1-1, and so it is a homeomorphic embedding by compactness of X.

5.2. Corollary. *For every sequence* $a = (a(k))_{k=1}^\infty$ *such that* $1/2 < a(k) < 1$ *for each* $k \in \mathbb{N}$ *and* $\prod_{k=1}^\infty a(k) = 0$ *the family* $\{I^\sigma_a : \sigma$ *is an increasing sequence of natural numbers*$\}$ *is universal in the class of all compact metrizable spaces* X *with* $\mathrm{Ind}\, X = \omega_0$.

6. Cardinalities of universal families. Denote by $\mathcal{D}$ the class of all compact metrizable spaces X with $\mathrm{ind}\, X = \omega_0$; let

$$\mathfrak{m} = \min\{|\mathcal{A}| : \mathcal{A} \text{ is a universal family in } \mathcal{D}\}\,.$$

In this section we show that

$$\mathfrak{m} = \mathfrak{d}\,. \tag{6.1}$$

The proof will be preceded by two lemmas. We denote by $\mathrm{st}(Y)$ the star of a set $Y \subseteq I^{\sigma|j}_a$ with respect to the covering $\{I_t : t \in S^a_{\sigma|j}\}$, that is,

$$\mathrm{st}(Y) = \bigcup\{I_t : t \in S^a_{\sigma|j} \text{ and } Y \cap I_t \neq \emptyset\}\,.$$

6.1. Lemma. *Let j be a natural number*, *and let A be a subspace of I^σ_a. If the image of A under the projection of I^σ_a onto $I^{\sigma|j}_a$ is not contained in* *any* $\mathrm{st}(I_s)$, *where* $s \in S^a_{\sigma|j}$, *then for some* $x, y \in A$, *there exists an at most* $\sigma(j)$*-dimensional partition in* A *between* x *and* y.

Proof. Denote by A_j the image of A under the projection of I^σ_a onto $I^{\sigma|j}_a$. Since $A \neq \emptyset$ and $I^{\sigma|j}_a = \bigcup\{I_s : s \in S^a_{\sigma|j}\}$, it follows that $A_j \cap I_s \neq \emptyset$ for some $s \in S^a_{\sigma|j}$; take an $x \in A$ whose image under the above projection—to be denoted by x_j—belongs to I_s. By assumption, there exists a $y \in A$ whose image, say y_j, belongs to $I^{\sigma|j}_a - \mathrm{st}(I_s)$. Let

$$U_j = I^{\sigma|j}_a - \bigcup\{I_t : t \in S^a_{\sigma|j} \text{ and } I_t \cap I_s = \emptyset\},$$
$$V_j = I^{\sigma|j}_a - \mathrm{st}(I_s) \quad \text{and} \quad L_j = I^{\sigma|j}_a - (U_j \cup V_j)\,.$$

The sets U_j and V_j are open, and $x_j \in U_j, y_j \in V_j$; hence L_j is a partition in $I^{\sigma|j}_a$ between x_j and y_j. Of course,

$$\mathrm{Ind}\, L_j \le \mathrm{Ind}\, I^{\sigma|j}_a \le \sigma(j)\,.$$

From the definitions it follows immediately that none of the sets I_t, where $t \in S^a_{\sigma|j}$, intersects U_j and V_j simultaneously.

Set

$$\mathcal{U}_k = \{I_t : t \in S^a_{\sigma|k} \text{ and } I_{t|j} \cap I_s \neq \emptyset\}\,,$$
$$\mathcal{V}_k = \{I_t : t \in S^a_{\sigma|k} \text{ and } I_{t|j} \cap I_s = \emptyset\} \quad \text{for } k \ge j\,,$$

and

$$U_k = \Big(\bigcup \mathcal{U}_k\Big) - L_j, \quad V_k = \Big(\bigcup \mathcal{V}_k\Big) - L_j, \quad \text{and} \quad L_k = L_j \quad \text{for } k > j\,.$$

A reasoning similar to that in the proof of Theorem 3.2 shows that the sets U_k, V_k, L_k satisfy the assumptions of Lemma 3.1 for $n = \mathrm{Ind}\, L_j \le \sigma(j)$; therefore there exists an at most $\sigma(j)$-dimensional partition L in I^σ_a between x and y.

The set $A \cap L$ is a partition in A between x and y, and $\mathrm{Ind}(A \cap L) \le \mathrm{Ind}\, L \le \sigma(j)$.

From now on, we will only be concerned with sequences $a = (a(k))^\infty_{k=1}$ which additionally satisfy the following condition:

$$(*) \qquad \prod_{k=1}^{\infty} 2a(k) < \infty$$

(see the beginning of Section 4).

6.2. Lemma. *Let* σ *and* τ *be increasing sequences of natural numbers*; *let* $n_k = \min\{i = 0, 1, \ldots : \tau(k) \le \sigma(k+i)\}$ *for* $k \in \mathbb{N}$. *Then* I^τ_a *is embeddable in* I^σ_a *if and only if the sequence* $(n_k)^\infty_{k=1}$ *is bounded.*

P r o o f. Assume that there exists an $n \in \mathbb{N}$ such that $n_k \le n$ for every k. Define σ_n by letting

$$\sigma_n(k) = \sigma(k+n) \quad \text{for } k = 0, 1, \ldots;$$

then $\tau(k) \le \sigma_n(k)$ for every k.

From the definitions of $I_a^\tau, I_a^{\sigma_n}$, and I_a^σ (see Section 4) it follows that I_a^τ is embeddable in $I_a^{\sigma_n}$, and for every $s \in S^a_{\tau|n}$, $I_a^{\sigma_n}$ is homeomorphic to the subspace of I_a^σ consisting of all points which are mapped into $\bigcup\{I_t : t \in S^a_{\sigma|i}$ and $t|n = s\}$ under the projection of I_a^σ onto $I_a^{\sigma|i}$ for $i \ge n$. Thus I_a^τ is embeddable in I_a^σ.

Suppose now, on the contrary, that there exists an embedding $h : I_a^\tau \to I_a^\sigma$ and the sequence $(n_k)_{k=1}^\infty$ is not bounded.

Fix an $m \in \{0, 1, \ldots\}$; let $\varrho = \varrho^a_{\sigma,m}$ be a metric on $I_a^{\sigma|m}$ satisfying (4.6). Take $k > m+2$ and $s \in S^a_{\tau|k}$. Since $h(I_s)$ is a $\tau(k)$-dimensional cube, the dimension of every partition in $h(I_s)$ is not less than $\tau(k) - 1$.

By Lemma 6.1 and the inequality $\sigma(k + n_k - 2) < \tau(k) - 1$, the image of $h(I_s)$ under the projection of I_a^σ onto $I_a^{\sigma|k+n_k-2}$ is contained in $\mathrm{st}(I_t)$ for some $t \in S^a_{\sigma|k+n_k-2}$; thus the diameter of the image of $h(I_s)$ under the projection of I_a^σ onto $I_a^{\sigma|m}$ is not greater than

$$3 \cdot \Big(\prod_{i=m+1}^{k+n_k-2} a(i)\Big) \cdot \mathrm{diam}_\varrho\, I_a^{\sigma|m}$$

(see (4.6)). Since the above estimate holds for every $s \in S^a_{\tau|k}$, we conclude, by (4.1) and (4.3), that the diameter of every $h(I_t)$, where $t \in S^a_{\tau|k-1}$, under the projection of I_a^σ onto $I_a^{\sigma|m}$ is not greater than

$$3 \cdot 2 \cdot \Big(\prod_{i=m+1}^{k+n_k-2} a(i)\Big) \cdot \mathrm{diam}_\varrho\, I_a^{\sigma|m}\,.$$

We continue in this fashion to deduce that the diameter of the image of $h(I_a^{\tau|0}) = h(I_s)$, where s is the unique element of $S^a_{\tau|0}$, under the projection of I_a^σ onto $I_a^{\sigma|m}$ is not greater than

$$\begin{aligned} 3 \cdot 2^k \cdot \Big(\prod_{i=m+1}^{k+n_k-2} a(i)\Big) \cdot \mathrm{diam}_\varrho\, I_a^{\sigma|m} &= 3 \cdot 2^{m-n_k+2} \cdot \prod_{i=m+1}^{k+n_k-2} 2a(i) \\ &\le 3 \cdot 2^{m-n_k+2} \cdot \prod_{i=1}^{\infty} 2a(i)\,. \end{aligned}$$

Since $\prod_{i=1}^\infty 2a(i) < \infty$ (see $(*)$) and $(n_k)_{k=1}^\infty$ is not bounded, the image of $h(I_a^{\tau|0})$ under the projection of I_a^σ to $I_a^{\sigma|m}$ has to be a one-point set.

As m is an arbitrary natural number we conclude that $h(I_a^{\tau|0})$ is also a one-point set, which contradicts the assumption that h is an embedding.

We can now prove (6.1). Let $D \subseteq \mathbb{N}^{\omega_0}$ be a dominating set of cardinality $\mathfrak{d}$; obviously, we can assume that each element of D is an increasing sequence. Set $\mathcal{A} = \{I_a^\sigma : \sigma \in D\}$. In order to prove $\mathfrak{m} \le \mathfrak{d}$, it suffices to show that

(6.2) $\mathcal{A}$ is a universal family in $\mathcal{D}$.

In Section 4, we have shown that for every increasing sequence σ of natural numbers, I_a^σ is a compact metrizable space with $\operatorname{ind} I_a^\sigma = \omega_0$, and so I_a^σ belongs to $\mathcal{D}$. If X is a compact metrizable space with $\operatorname{ind} X \le \omega_0$, then, by Theorem 5.1, X is embeddable in I_a^σ for some increasing sequence $\sigma \in \mathbb{N}^{\omega_0}$. Take $\tau \in D$ such that $\sigma \le^* \tau$; then there exists an $n \in \{0, 1, \ldots\}$ such that $\sigma(k) \le \tau(k)$ for $k \ge n$ (see Section 1, the definition of $\le^*$). Since $\sigma(k) \le \sigma(k+n) \le \tau(k+n)$ for $k = 0, 1, \ldots,$ I_a^σ is embeddable in I_a^τ by Lemma 6.2, and so X is embeddable in I_a^τ. The proof of (6.2) is complete.

Consider now a family $\mathcal{A}$ of cardinality $\mathfrak{m}$ universal in the class $\mathcal{D}$; by Theorem 5.1, we can assume that $\mathcal{A}$ consists of the spaces I_a^σ, where a stands for a sequence with property $(*)$. For every σ such that $I_a^\sigma \in \mathcal{A}$ and $n = 0, 1, \ldots,$ define the sequence σ_n by letting

$$\sigma_n(k) = \sigma(n+k) \quad \text{for } k = 0, 1, \ldots,$$

and put

$$D = \{\sigma_n : I_a^\sigma \in \mathcal{A} \text{ and } n = 0, 1, \ldots\}.$$

In order to prove $\mathfrak{d} \le \mathfrak{m}$, it suffices to show that D is a dominating sequence.

Let $\tau \in \mathbb{N}^{\omega_0}$ be an arbitrary sequence. Take an increasing sequence $\theta \in \mathbb{N}^{\omega_0}$ such that $\tau(k) \le \theta(k)$ for every $k = 0, 1, \ldots$ The space I_a^θ is embeddable in some $I_a^\sigma \in \mathcal{A}$; thus by Lemma 6.2, there exists a natural number n such that $\theta(k) \le \sigma(n+k)$ for every k. Hence

$$\tau(k) \le \theta(k) \le \sigma(n+k) = \sigma_n(k) \quad \text{for } k = 0, 1, \ldots,$$

which completes the proof.

From (6.1) it follows that $\aleph_0 < \mathfrak{m} \le \mathfrak{c}$ (see Section 1); however, the inequalities $\aleph_0 < \mathfrak{m}$ and $\mathfrak{m} \le \mathfrak{c}$ can be proved in a more direct way.

Indeed, since the cardinality of the family of all closed subspaces of the Hilbert cube is $\mathfrak{c}$, we have $\mathfrak{m} \le \mathfrak{c}$.

On the other hand, the existence of a countable universal family $\mathcal{A}$ in the class $\mathcal{D}$ would contradict Theorem 5.1 of [8]. Namely, the one-point compactification of the sum of topological spaces $\bigoplus \mathcal{A}$ would be a universal space for compact metrizable spaces X with $\operatorname{ind} X \le \omega_0$.

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