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On partitions of lines and space

by

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Dedicated to the memory of K. Kuratowski and W. Sierpiński

Abstract. We consider a set, L, of lines in $\mathbb{R}^n$ and a partition of L into some number of sets: $L = L_1 \cup \ldots \cup L_p$. We seek a corresponding partition $\mathbb{R}^n = S_1 \cup \ldots \cup S_p$ such that each line l in L_i meets the set S_i in a set whose cardinality has some fixed bound, ω_τ. We determine equivalences between the bounds on the size of the continuum, $2^\omega \le \omega_\theta$, and some relationships between p, ω_τ and ω_θ.

In 1951, Sierpiński [S2] showed that the continuum hypothesis is equivalent to the following: for the partition of the lines in $\mathbb{R}^3$ parallel to one of the coordinate axes into the disjoint sets L_1, L_2, and L_3, where L_i consists of all lines parallel to the ith axis, there is a partition of $\mathbb{R}^3$ into disjoint sets, S_1, S_2, and S_3, such that any line in L_i meets at most finitely many points in S_i. He also showed that the corresponding statement for $\mathbb{R}^4$, using L_1, L_2, L_3 and L_4 and four sets S_1, S_2, S_3 and S_4, is equivalent to $2^\omega \le \omega_2$. Also, the corresponding statement for $\mathbb{R}^2$, using sets of lines L_1 and L_2 and sets S_1 and S_2, is false. He obtained analogous results by replacing "finite" by "countable". Thus, CH is equivalent to the assertion that $\mathbb{R}^2$ can be divided into two disjoint sets S_1 and S_2 with each line in L_i meeting S_i in a countable set [S1]. He showed that the countable version for $\mathbb{R}^3$ with three sets is equivalent to $2^\omega \le \omega_2$. These theorems were generalized by Kuratowski [Ku] and Sikorski [Si]. Erdős [Er] raised the issue of whether these results could be further strengthened by considering partitions of all lines rather than just those lines parallel to some coordinate axis. Davies

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[D1] showed that an analogous result is obtained if one partitions the lines in $\mathbb{R}^k$, $k \geq 2$, which are parallel to one of $L_1, \ldots, L_p$, where $L_1, \ldots, L_p$ are fixed pairwise non-parallel lines (and one partitions the lines according to which L_i it is parallel to). This result was extended by Simms [Sm2], who considered translates of linear subspaces instead of just lines. Simms' result also generalizes Sikorski's result, and gives best possible bounds for the type of partitions it considers. Davies [D2] later removed the restriction that the lines in $\mathbb{R}^k$ be partitioned in the special manner referred to above. Bagemihl [B] has also extended some of these results. See Simms [Sm1] for an extensive historical survey.

We develop a general framework within which these theorems can be obtained as corollaries. Our framework deals with arbitrary partitions of all lines (or planes, or more general objects) and not necessarily special partitions or families of lines. As we shall see, the central issues are the number of sets of lines in the partition, the allowed size of the intersection of a line in a given set with the corresponding set in the decomposition of the space, and the value of the continuum. Galvin and Gruenhage [GG], and independently Bergman and Hrushovski (cf. Proposition 19 of [BH]), have previously obtained results which imply special cases of some of our results. In particular, those results yield (1.1)⇒(1.2) for the case $\bar\theta = 0$ and $p = s + 2$. Corollary 8 of this paper also follows from [D2] and unpublished results of [GG]. In the last part of this paper, we deal with some perhaps surprising phenomena arising from infinite partitions. In particular, we show that some interesting set-theoretic properties come into play.

We should mention that some of the key ideas of our arguments go back to combinatorial arguments of Erdős and Hajnal [EH], and thank Fred Galvin for bringing to our attention some of his earlier work.

1. The main result. Let us establish some conventions for this paper. If t is a positive integer, then $\operatorname{card}(A) = |A| \leq \omega_{-t}$ means A is finite. If $\theta = \bar\theta + s$, where $\bar\theta > 0$ is a limit ordinal and s is an integer, and t is an integer with $t > s$, then $|A| \leq \omega_{\theta - t}$ means $|A| < \omega_\theta$.

THEOREM 1. *Let θ be an ordinal, $\theta = \bar\theta + s \geq 1$, where $\bar\theta$ is* 0 *or a limit ordinal, and let $s \in \omega$. The following statements are equivalent*:

(1.1) $2^\omega \leq \omega_\theta$.

(1.2) *For each $n \geq 2$ and for each partition of L, the set of all lines in $\mathbb{R}^n$, into $p \geq 2$ disjoint sets, $L = L_1 \cup L_2 \cup \ldots \cup L_p$, there is a partition of $\mathbb{R}^n$ into p disjoint sets, $\mathbb{R}^n = S_1 \cup S_2 \cup \ldots \cup S_p$, such that each line in L_i meets S_i in a set of size $\leq \omega_{\theta - p + 1}$.*

(1.3) *For some $n \geq 2$, some p, with $s + 2 \geq p \geq 2$, and some non-parallel lines $l_1, \ldots, l_p$ in $\mathbb{R}^n$, if we let L_i be the set of all lines in $\mathbb{R}^n$*

parallel to l_i, then there is a partition $\mathbb{R}^n = S_1 \cup \ldots \cup S_p$ such that every line in L_i meets S_i in a set of size $\leq \omega_{\theta-p+1}$.

Before we prove Theorem 1, let us make some comments and derive some corollaries.

Remark. Note that statement (1.2) implies statement (1.3) for all $p \geq 2$, not just for those p satisfying $s+2 \geq p \geq 2$. However, if $p > s+2$, then from the facts that $2^\omega \leq \omega_{\theta+(p-s-2)}$ and (1.1) implies (1.2), we get the conclusion of (1.2) which means $|l \cap S_i| < \omega_{\bar{\theta}}$ if $l \in L_i$, $i = 1, \ldots, p$. Thus, we cannot possibly derive (1.1) from even the statement of (1.2) for $p > s+2$ since $2^\omega \leq \omega_{\theta+(p-s-2)}$ does not imply $2^\omega \leq \omega_\theta$.

Remark. The fact that (1.3) implies (1.1) is Davies' theorem. We include a proof for the sake of completeness.

The first corollary yields Sierpiński's theorem as a special case and answers question a) in [Er].

COROLLARY 1. *The following are equivalent*:

(i) *CH, the continuum hypothesis, holds*: $2^\omega = \omega_1$.

(ii) *If the lines in $\mathbb{R}^3$ are decomposed into three sets L_i $(i = 1, 2, 3)$, then there exists a decomposition of $\mathbb{R}^3$ into three sets S_i such that the intersection of each line of L_i with the corresponding set S_i is finite.*

Proof. Take $\theta = 1$, $n = 3$ and $p = 3$ in Theorem 1. Then each line in L_i meets S_i in a set of size at most $\omega_{\theta-p+1} = \omega_{-1}$, which by our convention means finite. ∎

The second corollary yields the Bagemihl–Davies theorem [Sm1, p. 127] as a special case and notes that the condition that we be in $\mathbb{R}^3$ in Corollary 1 is not necessary. This also answers question b) in [Er].

COROLLARY 2. *The following are equivalent*:

(i) $2^\omega = \omega_1$.

(ii) *If the lines in $\mathbb{R}^2$ are decomposed into three sets L_i $(i = 1, 2, 3)$, then there exists a decomposition of $\mathbb{R}^2$ into three sets S_i such that the intersection of each line of L_i with the corresponding set S_i is finite.*

Proof. Take $\theta = 1$, $n = 2$ and $p = 3$ in Theorem 1. ∎

The next corollary is a theorem of Kuratowski [Ku].

COROLLARY 3. *Let $n \in \omega$, and $\bar{\theta}$ be a limit ordinal or zero. The following two statements are equivalent*:

(i) $2^\omega < \omega_{\bar{\theta}}$.

(ii) *There is a partition of $\mathbb{R}^{n+1}$, $\mathbb{R}^{n+1} = S_1 \cup \ldots \cup S_{n+1}$, such that $|l \cap S_i| < \omega_{\bar{\theta}}$ whenever l is parallel to the i-th axis.*

P r o o f. As Kuratowski mentions, the case $n = 0$ is easy. If $n > 0$, take $\theta = \bar{\theta} + (n-1)$ and $p = n+1$ in Theorem 1. Thus, $2^\omega \le \omega_\theta < \omega_{\bar{\theta}}$ if and only if $\mathbb{R}^{n+1} = \bigcup_{i=1}^{n+1} S_i$, where $|l \cap S_i| \le \omega_{\theta-p+1} = \omega_{\bar{\theta}}$ provided l is parallel to the ith axis. This means, by our convention, that $|l \cap S_i| < \omega_{\bar{\theta}}$, as required. ■

R e m a r k. As noted by Kuratowski, a special case of Corollary 3 is the following: $2^\omega \le \omega_n$ if and only if $\mathbb{R}^{n+1} = \bigcup_{i=1}^{n+1} S_i$, where $|l \cap S_i| < \omega$ provided l is parallel to the ith axis. It also follows from Theorem 1 that the condition of working in $\mathbb{R}^{n+1}$ can be dispensed with, as well as the requirement that the lines be parallel to a coordinate axis (though the lines must satisfy the conditions stated in (1.3) to get the implication (ii)⇒(i)). This yields Davies' theorem:

COROLLARY 4 (Davies). *Let $n \ge 2$, and let $l_1, \ldots, l_p$, $p \ge 2$, be non-parallel lines in $\mathbb{R}^n$. Then the following are equivalent*:

(i) $2^\omega \le \omega_\theta$.

(ii) *There is a partition $\mathbb{R}^n = \bigcup_{i=1}^{p} S_i$ of the points in $\mathbb{R}^n$ such that for every line l parallel to l_i, $|l \cap S_i| \le \omega_{\theta-p+1}$.*

P r o o f o f T h e o r e m 1. We introduce an auxiliary proposition $Q(p)$ for integer $p \ge 2$.

PROPOSITION $Q(p)$. *For each ordinal θ, if A is a set of lines and points in $\mathbb{R}^n$ of size at most ω_θ, and the set of lines in A, which we call L, is divided into k disjoint sets, $L = L_1 \cup \ldots \cup L_k$, where $k \ge p$, and if f is a function with domain S, the set of points in A, such that for all $x \in S$, $f(x) \subseteq \{1, \ldots, k\}$ and $|f(x)| \le k - p$, then there is a partition of S into k sets, $S = S_1 \cup \ldots \cup S_k$, such that for each $x \in S$*:

a) $x \notin S_a$, *if* $a \in f(x)$.

b) *Each line l in L_i meets at most $\omega_{\theta-p+1}$ points in S_i.*

We think of $f(x)$ as being forbidden "colors" for x. Thus, the hypothesis of $Q(p)$ requires there to be at least p non-forbidden colors for each point $x \in A$. Note that $Q(p)$ for all $p \ge 2$ yields (1.1)⇒(1.2) of Theorem 1 by taking $k = p$ and f the function with constant value $\emptyset$.

We establish $Q(p)$, working in ZFC, by induction on p. So, assume first that $p = 2$. Let A be a set of points and lines in $\mathbb{R}^n$ of size $\le \omega_\theta$, for some ordinal θ (we allow $\theta = 0$). Let $L = L_1 \cup \ldots \cup L_k$ be a partition of the lines in A with $k \ge p$, and let f be as in the statement of $Q(p)$. We define the partition $S = S_1 \cup \ldots \cup S_k$ of the points in A as required. Let $\{l_1^\alpha\}, \ldots, \{l_k^\alpha\}$ and $\{x^\alpha\}$, $\alpha < \omega_\theta$, enumerate the lines in $L_1, \ldots, L_k$, and the points of S, respectively. We inductively decide to which S_i we add x^α. Suppose we are at step $\alpha < \omega_\theta$ and we have decided for all $\beta < \alpha$ to which S_i we add x^β. Consider the following cases.

Case I. For some $1 \le i \le k$ such that $i \notin f(x^\alpha)$, and all $\beta < \alpha$, $x^\alpha \notin l_i^\beta$. In this case add x^α to S_i (choose i arbitrarily if the above is satisfied for more than one i).

Case II. For all $1 \le i \le k$ with $i \notin f(x^\alpha)$, x^α lies on some $l_i^{\beta(i)}$, with $\beta(i) < \alpha$. Let $\beta(i)$ in fact be the least such ordinal $< \alpha$. Let $i_0 \notin f(x^\alpha)$ be such that $\beta(i_0) \ge \beta(i)$ for all $i \notin f(x^\alpha)$. We then add x^α to S_{i_0}.

Thus, we have defined a partition $S = S_1 \cup \ldots \cup S_k$. Fix now a line $l_i^\delta \in L$. We show that $|S_i \cap l_i^\delta| \le \omega_{\theta-p+1} = \omega_{\theta-1}$ (this means, by our convention, that $|S_i \cap l_i^\delta| < \omega_\theta$). First, we need only consider those points x^α with $\alpha > \delta$, since there are $< \omega_\theta$ points x^α with $\alpha \le \delta$. If x^α were put in S_i by virtue of Case I, then x^α would not lie on l_i^δ. Suppose then that x^α, $\alpha > \delta$, is put in S_i by virtue of Case II. Thus, $\beta_j(\alpha)$ is defined for each $j \notin f(x^\alpha)$. Since x^α is put into S_i, we have $\beta_i(\alpha) \ge \sup\{\beta_j(\alpha) : j \notin f(x^\alpha)\}$. If $\beta_i(\alpha) > \delta$, then by definition, $x^\alpha \notin l_i^\delta$. Thus, we need only consider x^α for which $\delta \ge \beta_i(\alpha) \ge \sup\{\beta_j(\alpha) : j \notin f(x^\alpha)\}$. There are $< \omega_\theta$ possibilities for the set $\{\beta_j(\alpha)\}$. Since $k - f(x^\alpha)$ has at least two elements, and two lines determine a point, it follows that the set of such x^α has size $< \omega_\theta$. This completes the proof of $Q(2)$.

Note, in particular, that $Q(2)$ holds when $\theta = 0$, that is, when A is countable. However, for countable A, $Q(2)$ easily implies $Q(p)$ for all $p \ge 2$ as well (since in this case $|l \cap S_i| \le \omega_{\theta-p+1}$ means the same thing, i.e., $l \cap S_i$ is finite, for all $p \ge 2$).

Before giving the inductive step in the proof of $Q(p)$, we introduce a basic definition.

Definition. If A is a collection of lines and points in $\mathbb{R}^n$, we call A *good* if it satisfies the following:

a) For any two distinct points $x, y \in A$, the line determined by x and y is also in A.

b) For any two distinct intersecting lines in A, the point of intersection is also in A.

Clearly, for any infinite set A of lines and points in $\mathbb{R}^n$, the good set generated by A has the same cardinality as A.

Assume now that $Q(p)$ holds, and we show $Q(p+1)$. Let A be a collection of lines L and points S in $\mathbb{R}^n$ with size ω_θ, and let $L = L_1 \cup \ldots \cup L_k$ be a partition of L with $k \ge p+1$. We may assume $\theta \ge 1$ by our note above. Without loss of generality, we may also assume that A is good. Let $f : S \to \{1, \ldots, k\}$ be given with $|f(x)| \le k - (p+1) = k - p - 1$. Express

A as an increasing union, $A = \bigcup_{\alpha<\omega_\theta} A_\alpha$, where each A_α is good, and $|A_\alpha| \le \omega_{\theta-1}$. We call a line $l \in A_\alpha$ "new" if $l \in A_\alpha \setminus \bigcup_{\beta<\alpha} A_\beta$, and otherwise call l "old" (relative to α). We label the points of A_α as new and old in the same fashion. We define at step α the partition of S_α, the set of new points in A_α, $S_\alpha = S_1^\alpha \cup \ldots \cup S_k^\alpha$. Suppose we are at step $\alpha < \omega_\theta$. Enumerate $L_{\alpha,i}$, the new lines of L_i in A_α, and points of A_α into type $\omega_{\sigma(\alpha)} < \omega_\theta$, say $l_{\alpha,i}^\beta$, x_α^β, $\beta < \omega_\sigma$, $1 \le i \le k$. Note that for each $\beta < \omega_{\sigma(\alpha)}$, each x_α^β lies on at most one old line, since each A_δ is good. Thus, for $\beta < \omega_{\sigma(\alpha)}$, set $f_\alpha(x_\alpha^\beta) = f(x_\alpha^\beta) \cup \{j\}$, where j is such that x_α^β lies on an old line in L_j if one exists, and otherwise set $f_\alpha(x_\alpha^\beta) = f(x_\alpha^\beta)$. Thus, f_α maps the new points of A_α into $\{1,\ldots,k\}$ and $|f_\alpha(x)| \le k-p$. Now, by the induction hypothesis $Q(p)$ applied to $\omega_{\sigma(\alpha)}$, we may partition the points in S_α, $S_\alpha = S_1^\alpha \cup \ldots \cup S_k^\alpha$, so that any new line $l_{\alpha,i}^\beta$ intersects at most $\omega_{\sigma(\alpha)-p+1}$ points from S_i^α, and $x_\alpha^\beta \notin S_a^\alpha$ for any $a \in f_\alpha(x_\alpha^\beta)$. Note that $\omega_{\sigma(\alpha)-p+1} \le \omega_{\theta-p}$.

This defines our partition of S. To show this partition works, fix a line l in A, say $l \in L_i$. Let α be the least such that $l \in L_{\alpha,i}$, so that l is a new line at step α. We must show that $\le \omega_{\theta-p}$ points in $S_i = \bigcup_{\gamma<\omega_\theta} S_i^\gamma$ lie on l. First, any point x_γ^δ, for $\gamma > \alpha$, in S_i cannot lie on l, since then $i \in f_\gamma(x_\gamma^\delta)$, but, by construction, $x_\gamma^\delta \notin S_i$. So, we may assume $\gamma \le \alpha$. Now, there is at most one point x_γ^δ for $\gamma < \alpha$ on the line l, since otherwise l would not be new at α. Thus, we need only consider points of the form x_α^δ, $\delta < \omega_{\sigma(\alpha)}$. However, from the definition of the set S_i^α, $\le \omega_{\sigma(\alpha)-p+1}$ of these points lie on $l \in L_{\alpha,i}$.

Thus, each line in L_i intersects $\le \omega_{\theta-p}$ points of S_i. Since $f_\alpha(x_\alpha^\beta) \supset f(x_\alpha^\beta)$ for all $x_\alpha^\beta \in S$, we also have $x_\alpha^\beta \notin S_a$ if $a \in f(x_\alpha^\beta)$. This completes the proof of the proposition $Q(p+1)$ and, as mentioned, the proof that (1.1) implies (1.2).

Since (1.2) clearly implies (1.3), it only remains to prove (1.3) implies (1.1). Assume now (1.3) holds, with $\theta = \bar{\theta} + s$ and $2 \le p \le s+2$. Towards a contradiction, assume $2^\omega \ge \omega_{\theta+1}$. Let $l_1,\ldots,l_p$ and $L_1,\ldots,L_p$ and $S_1,\ldots,S_p$ be as in (1.3). For each i, $2 \le i \le p$, let v_i be a vector parallel to l_i with $\|v_i\| = 1$. We construct sets $B_1,\ldots,B_p$ as follows. Let $B_1 \subseteq l_1 = \{x_0 + tv_1 : t \in \mathbb{R}\}$ be any set of size $\omega_{(\theta-p+1)+1} \ge \omega_{\bar{\theta}}$. Assume $1 \le i \le p-1$ and B_i has been defined with $|B_i| = \omega_{(\theta-p+1)+i} < 2^\omega$. Let D_i be the set of all distances between two distinct points of B_i. So, $|D_i| = |B_i|$. Let C_{i+1} be a subset of $\mathbb{R}$ such that $|C_{i+1}| = \omega_{(\theta-p+1)+(i+1)}$ and $(C_{i+1} - C_{i+1}) \cap D_i = \emptyset$ (where $A - B := \{a-b : a \in A,\ b \in B\}$). Let $B_{i+1} = \bigcup_{c\in C_{i+1}} [cv_{i+1} + B_i] = \bigcup_{x\in B_i} [x + \bigcup_{c\in C_{i+1}} cv_{i+1}]$. Thus, B_{i+1} consists of $\omega_{(\theta-p+1)+(i+1)}$ translates of B_i in the direction of l_{i+1}. Also, notice that these translates of B_i form a pairwise disjoint family. Finally, since $2^\omega \ge \omega_{\theta+1} = \omega_{(\theta-p+1)+p}$, B_p is defined.

Consider first B_{p-1}. Since $|B_{p-1}| = \omega_\theta$, and since each line parallel to l_p through a point of B_{p-1} intersects S_p in at most $\omega_{\theta-p+1}$ points, $|S_p \cap B_p| \le \omega_\theta$. But, since B_p consists of $\omega_{\theta+1}$ disjoint translates of B_{p-1}, there is some $c_p \in C_p$ such that $c_p v_p + B_{p-1} \subseteq S_1 \cup \ldots \cup S_{p-1}$. If $p = 2$, stop; otherwise, continue. So, in general, suppose $3 \le j \le p$ and we have produced numbers $c_i \in C_i$, for $j \le i \le p$, such that $e_j + B_{j-1} \subseteq S_1 \cup \ldots \cup S_{j-1}$, where $e_j = c_p v_p + c_{p-1} v_{p-1} + \ldots + c_j v_j$. Now, $e_j + B_{j-1} = \bigcup_{c \in C_{j-1}} [e_j + c v_{j-1} + B_{j-2}]$, the translates in this union being pairwise disjoint, and $|C_{j-1}| = \omega_{\theta-p+j}$. Since S_{j-1} contains at most $\omega_{\theta-p+j-1}$ points of this union, there is some $c_{j-1} \in C_{j-1}$ such that $e_j + c_{j-1} v_{j-1} + B_{j-2} \subseteq S_1 \cup \ldots \cup S_{j-2}$. Finally, we have $\widetilde{B}_1 = e_2 + B_1 = c_p v_p + c_{p-1} v_{p-1} + \ldots + c_2 v_2 + B_1 \subseteq S_1$. As $\widetilde{B}_1$ is a translate of B_1, $|\widetilde{B}_1| = |B_1| = \omega_{\theta-p+2}$. But, also, $\widetilde{B}_1$ is a subset of the line through $x_0 + e_2$ parallel to l_1. Thus, $|\widetilde{B}_1| \le \omega_{\theta-p+1}$. This is a contradiction. ■

Further generalizations are possible. The only properties of lines that were used in the preceding argument were that two distinct lines determine at most one point and two distinct points determine a line. We generalize this as follows.

Definition. Let $H \subseteq \mathcal{P}(\mathbb{R}^n)$ be a family of subsets of $\mathbb{R}^n$. Let r and s be positive integers. We say that H is (r,s) *finitely determined* if the following are satisfied:

(1) The intersection of any r distinct elements of H is finite.

(2) For any s distinct points in $\mathbb{R}^n$, there are at most finitely many $h \in H$ which contain all those points.

Example. The set H of all circles in $\mathbb{R}^n$ $(n \ge 2)$ is $(2,3)$ finitely determined.

Example. The set H of all hyperplanes in $\mathbb{R}^n$ perpendicular to a coordinate axis is $(n,1)$ finitely determined.

Somewhat more generally still, we introduce the notion of a partition being (r,s) finitely determined.

Definition. Given $H \subseteq \mathcal{P}(\mathbb{R}^n)$, we say a partition $H = H_1 \cup \ldots \cup H_k$ (k can be infinite here) is (r,s) *finitely determined* if:

(1) The intersection of r distinct elements of H lying in different H_i is finite.

(2) For any s distinct points in $\mathbb{R}^n$, there are at most finitely many $h \in H$ containing these s points.

Note that if $H \subseteq \mathcal{P}(\mathbb{R}^n)$ is an (r,s) finitely determined family of sets, then any partition of H is (r,s) finitely determined.

Theorem 1 may be generalized as follows, where our convention is still in force.

THEOREM 2. *Let $\theta \geq 1$ be an ordinal. The following are equivalent*:

(2.1) $2^\omega \leq \omega_\theta$.

(2.2) *For each positive integer t, for each $n \geq 1$, and for any $r \geq 2$, $s \geq 1$, if $H = H_1 \cup \ldots \cup H_p$ is an (r,s) finitely determined partition of some $H \subseteq \mathcal{P}(\mathbb{R}^n)$ into $p = t(r-1)+1$ disjoint sets, then there is a partition of $\mathbb{R}^n$, $\mathbb{R}^n = S_1 \cup \ldots \cup S_p$, such that $|h \cap S_i| \leq \omega_{\theta-t}$ for all $h \in H_i$, $1 \leq i \leq p$.*

Proof. The proof that (2.1) implies (2.2) is similar to that of Theorem 1. As there, we formulate an auxiliary proposition, $R(t)$, for $t \geq 1$, which we prove in ZFC by induction on t.

PROPOSITION $R(t)$. *For each ordinal θ, k, and integers $n \geq 1$, $r \geq 2$, $s \geq 1$, if $H = H_1 \cup \ldots \cup H_k$ is an (r,s) finitely determined partition of $H \subseteq \mathcal{P}(\mathbb{R}^n)$ into $k \geq t(r-1)+1$ pieces, then if $A \subseteq H \cup \mathbb{R}^n$ is a set consisting of some elements of H and points, S, of $\mathbb{R}^n$ with $|A| \leq \omega_\theta$, and f is a function from S into $\mathcal{P}(\{1,\ldots,k\})$ such that for all $x \in S$ we have $|f(x)| \leq k - [t(r-1)+1]$, then there is a partition of S into k sets, $S = S_1 \cup \ldots \cup S_k$, such that for each $x \in S$, $x \notin S_a$ if $a \in f(x)$, and if $h \in H_i \cap A$, then $|h \cap S_i| \leq \omega_{\theta-t}$, for $1 \leq i \leq k$.*

The proof for $t = 1$ proceeds exactly as the proof of Theorem 1 for $p = 2$. Again, the determination of which set x^α should be placed into breaks into two cases. In the first case, for some $i \notin f(x^\alpha)$ we have $x^\alpha \notin h_i^\gamma$ for all $\gamma < \alpha$, and x^α is placed in some S_i with i in this set. For the x^α in the second case, one obtains a function $x^\alpha \to (\beta(i_1(\alpha)), \ldots, \beta(i_g(\alpha)))$, where $g \geq r$ and the $i_j(\alpha)$ list the i's such that x^α lies on some h_i^γ, with $\gamma < \alpha$, and $\beta(i_j(\alpha))$ is the least such γ. This function is not necessarily one-to-one as in Theorem 1, but, from the first condition of being (r,s) finitely determined, the function is finite-to-one. This is sufficient for the argument.

Note, as in Theorem 1, that if $\theta = 0$, then $R(1)$ easily implies $R(t)$ for all t. Thus, we may assume in the inductive step that $\theta \geq 1$.

The inductive step for obtaining $R(t+1)$ from $R(t)$ is similar to that for $Q(p)$. Perhaps it should be noted that in obtaining $R(t+1)$ from $R(t)$, one builds, as before, an increasing transfinite sequence of "good" sets, A_α, with $A = \bigcup_{\alpha<\omega_\theta} A_\alpha$ and $|A_\alpha| < \omega_\theta$. A_α being *good* now means that if $h_1, \ldots, h_r$ are elements of distinct sets $A_\alpha \cap H_j$, then $\bigcap h_i \subseteq A_\alpha$, and for any s distinct points of A_α, the finitely many elements of H containing these points are in A_α. Since the partition of H is (r,s) finitely determined, the cardinality of the good set generated by an infinite set does not increase. The argument then proceeds as before.

To prove (2.2) implies (2.1), take $t = 1$ and $n = 2$. Let H_i be the set of lines parallel to the ith axis. So, the partition is $(2, 1)$ finitely determined. Applying (2.2) to this family, we have $p = r = 2$ and $t = p - 1$. So, there is a partition $\mathbb{R}^2 = S_1 \cup S_2$ such that $|h \cap S_i| \le \omega_{\theta-t} = \omega_{\theta-p+1}$. Since (1.3) implies (1.1), $2^\omega \le \omega_\theta$. ∎

Remark. Since a partition of lines is $(2, 2)$ finitely determined, Theorem 1 follows from Theorem 2, by taking $r = 2$, in which case $\theta - t = \theta - p + 1$.

Corollary 5 (Sikorski). *The continuum hypothesis is equivalent to the following statement. The points in $\mathbb{R}^3$ can be partitioned into three sets S_1, S_2 and S_3 such that each plane perpendicular to the x_i axis meets S_i in at most countably many points.*

Proof. If $H =$ planes in $\mathbb{R}^3$ perpendicular to a coordinate axis, then H is $(3, 1)$ finitely determined. Now, take $\theta = 1 = t$ in Theorem 2. The proof of the converse may be found in [Si] or done directly. Of course, our proof also works for any partition of the planes in $\mathbb{R}^3$ which is $(3, s)$ finitely determined for some s. ∎

As another example, consider the analog of Corollary 4 where "countable" is replaced by "finite". We first show that four "colors" are not sufficient (note: Lemma 1 and one direction of Corollary 6 follow from Theorem 5.9 of [Sm], but are included here for the sake of completeness).

Lemma 1. *There are four unit vectors, v_1, v_2, v_3 and v_4, in $\mathbb{R}^3$ such that if $H_i = \{h : h$ is a plane with normal $v_i\}$, then the partition $H_1 \cup \ldots \cup H_4$ is $(3, 1)$ finitely determined, and yet there is no partition $\mathbb{R}^3 = S_1 \cup \ldots \cup S_4$ such that $|h \cap S_i| < \omega_0$ for all $h \in H_i$.*

Proof. Let v_i, $i = 1, 2, 3$, be the canonical unit basis vectors for $\mathbb{R}^3$. Let $v_4 = (0, -\sqrt{2}/2, \sqrt{2}/2)$. Let $A_1, A_2 \subseteq \mathbb{R}$ with $|A_1| = \omega_0$ and $|A_2| = \omega_1$, and let $A_3 = \mathbb{Q}$, the rationals. Let $G \subseteq \mathbb{R}$ be such that $|G| = \omega_1$ and $(G - G) \cap \mathbb{Q} = \{0\}$. Let $W = \{(0, t, t) : t \in G\}$. Let $B = A_1 \times A_2 \times A_3$ and $E = B + W$. The following claim suffices to finish the proof of the lemma.

Claim. *For each $u = (u_1, u_2, u_3) \in \mathbb{R}^3$, $E + u \not\subseteq S_1 \cup \ldots \cup S_4$, where each S_i meets each plane with normal v_i in a finite set.*

Proof of Claim. Fix u, and assume such sets S_i exist. For each $y \in A_2$, let $E_y = [A_1 \times \{y\} \times A_3] + W + u$. Then $E + u = \bigcup_{y \in A_2} E_y$. To see that these sets are disjoint, notice that otherwise we would have $(a_1, y_1, q_1) + w_1 = (a_2, y_2, q_2) + w_2$, with $w_1 \ne w_2$. But this would imply $t_1 - t_2 \in \mathbb{Q}$ for some two distinct elements of G. Now, for each $x_1 \in A_1$, the plane $x = x_1 + u_1$ meets only finitely many points of S_1. Thus, $S_1 \cap (E + u)$ is countable and so there is some $y_0 \in A_2$ such that $E_{y_0} \subseteq S_2 \cup S_3 \cup S_4$. For each $(x, z) \in A_1 \times \mathbb{Q}$, the plane $h(x, y_0, z)$ passing through $(x, y_0, z) + u$ with normal v_4 meets only finitely many points of S_4. But $E_{y_0} = \bigcup_{w\in W}[(A_1 \times \{y_0\} \times A_3) + w + u]$ and the sets in this union are disjoint. So, there is some $w_0 \in W$ such that $(A_1\times\{y_0\}\times A_3)+w_0+u \subseteq S_2\cup S_3$. But this set lies in a plane with normal v_2. So, only finitely many points of this set are in S_2. Thus, there is some $z_0 \in \mathbb{Q}$ such that $(A_1\times\{y_0\}\times\{z_0\})+w_0+u \subseteq S_3$. But this set is an infinite subset of a plane with normal v_3 and S_3 meets this plane in a finite set. ∎

COROLLARY 6. *The continuum hypothesis is equivalent to the following statement. If H is a set of planes in $\mathbb{R}^3$ and $H = H_1\cup\ldots\cup H_5$ is a partition of H which for some s is $(3, s)$ finitely determined, then there is a partition $\mathbb{R}^3 = S_1\cup\ldots\cup S_5$ such that each plane in H_i meets S_i in a finite set. More generally, the hypothesis $2^\omega \le \omega_n$ is equivalent to the above statement, where $H = H_1\cup\ldots\cup H_5$ is replaced by $H = H_1\cup\ldots\cup H_{2n+3}$.*

Proof. If $2^\omega = \omega_1$, take $\theta = 1$ and $t = 2$ and apply Theorem 2. To prove the converse in this case, assume $2^\omega \ge \omega_2$. Let us follow the same notation used in the proof of Lemma 1. Let v_5 be a unit vector, $v_5 \ne v_i$, $1 \le i \le 4$, and $H_5 = \{h : h \text{ is a plane normal to } v_5\}$. Let $F \subseteq \{x : \langle x, v_5\rangle = 0\}$ such that $(F - F)\cap(E - E) = \{0\}$ and $|F| = \omega_2$. Let $S_1,\ldots,S_5$ be the required partition of $\mathbb{R}^3$. Set $M = \bigcup_{f\in F} E + f = \bigcup_{e\in E} e + F$, the sets in each union being disjoint. For each $e \in E$, $|S_5\cap(e+F)| < \omega_0$. So, $|S_5\cap F| \le \omega_1$. Thus, there is a vector $f \in F$ such that $E + f \subseteq S_1\cup\ldots\cup S_4$. This contradicts the Claim in the proof of Lemma 1.

The argument for this direction can be strengthened slightly. We may take $F \subseteq \{\alpha x : \alpha \in \mathbb{R}\}$, where $\langle x, v_5\rangle = 0$. Let $v_6 \ne v_1,\ldots,v_4$ be perpendicular to v_5, and define H_6 accordingly. Let $G \subseteq \{\alpha y : \alpha \in \mathbb{R}\}$, where $\langle y, v_6\rangle = 0$, be such that $|G| = \omega_2$ and $(G - G)\cap(M - M) = \{0\}$. Set $N = \bigcup_{g\in G} M + g$. It is easy to check then that if $\mathbb{R}^3 = S_1\cup\ldots\cup S_6$ is a partition of $\mathbb{R}^3$, for some $f \in F$, $g \in G$ we have $E + f + g \subseteq S_1\cup\ldots\cup S_4$, a contradiction. Thus, the following statement implies the continuum hypothesis: for every partition $H = H_1\cup\ldots\cup H_6$ of planes which is $(3, s)$ finitely determined for some s, there is a partition $\mathbb{R}^3 = S_1\cup\ldots\cup S_6$ with each plane in H_i meeting S_i in a finite set.

If $2^\omega = \omega_n$, apply Theorem 2 with $\theta = n$ and $t = n + 1$ to obtain one direction. The converse direction (which follows from Simms) can be obtained by extending the above argument, assuming $2^\omega \ge \omega_{n+1}$, and using vectors $v_5, v_6,\ldots,v_{2n+4}, v_{2n+5}$. This, in fact, gives the stronger result that the stated partition property using $2n + 4$ sets H_i, S_i implies $2^\omega \le \omega_n$. The details are left to the reader. ∎

Theorem 2 may be refined in several different ways. For some families $H \subseteq \mathbb{R}^n$, the value of p in (2.2) of Theorem 2 is not the best possible. For example, in $\mathbb{R}^4$, for each $\Lambda = \{i_1, i_2\} \subseteq \{1, 2, 3, 4\}$ with $i_1 \ne i_2$, let H_Λ consist of all planes of the form $x_{i_1} = a_1$ and $x_{i_2} = a_2$, where $a_1, a_2 \in \mathbb{R}$.

Notice that $H = \bigcup_\Lambda H_\Lambda$ is a $(4,3)$ finitely determined partition of some planes into 6 sets. Sikorski [Si] showed, as a particular case of a general theorem, that there is a corresponding partition $\mathbb{R}^4 = \bigcup S_\Lambda$ such that if $h \in H_\Lambda$, then $h \cap S_\Lambda$ is finite. A direct application of Theorem 2 requires partitioning $\mathbb{R}^4$ into 7 sets. One can refine Theorem 2, however, to obtain Sikorski's theorem.

Theorem 5.9 of Simms [Sm2] extends Sikorski's result by obtaining the best possible value of p (in the notation of our Theorem 2) in the case where H is the family of translates of a fixed finite number of subspaces of $\mathbb{R}^n$, and the elements h of H are partitioned according to which subspace they are a translate of. His results are stated in terms of the least integer n such that the collection of subspaces is "n-good". In fact, we may refine the argument of Theorem 2 to obtain Simms' result, and also allow general partitions of the family H. We briefly sketch the argument.

Let Π be a finite set of linear subspaces of $\mathbb{R}^n$, for some $n \geq 2$. Let H be the family of translates of these subspaces. That is, every $h \in H$ is of the form $h = V + u$, where $V \in \Pi$ and $u \in \mathbb{R}^n$. Following Simms, we say that Π is *t-good* if for every linear ordering $\prec$ of Π, there is a subset $\mathcal{S}$ of Π of size t such that for all $V \in \Pi$, $\bigcap\{V' \preceq V : \neg\exists V'' \in \mathcal{S}$ such that $V' \prec V'' \prec V\}$ is finite. We thus have:

Corollary 7. *Let $n \geq 2$, and $\theta \geq 1$ be an ordinal. The following are equivalent*:

(1) $2^\omega \leq \omega_\theta$.

(2) *For every non-empty set Π of size k of non-trivial linear subspaces V of $\mathbb{R}^n$ which is t-good, if $H = \{V + u : V \in \Pi,\ u \in \mathbb{R}^n\}$ is partitioned into k sets $H = H_1 \cup \ldots \cup H_k$, then there is a partition $\mathbb{R}^n = S_1 \cup \ldots \cup S_k$ such that for every $h \in H_i$, $|h \cap S_i| \leq \omega_{\theta-t}$.*

(3) *There is a non-empty set $\Pi = \{V_1, \ldots, V_k\}$ of non-trivial linear subspaces of $\mathbb{R}^n$ which is not $(t+1)$-good and for which there is a partition $\mathbb{R}^n = S_1 \cup \ldots \cup S_k$ such that $\forall 1 \leq i \leq k$ $\forall u \in \mathbb{R}^n$ $|(V_i + u) \cap S_i| \leq \omega_{\theta-t}$.*

Remark. The fact that (3) implies (1) is half of Theorem 5.9 of [Sm2], and will not be proven here. The special case of (1)⇒(2) for the partition of (3) is the other half of that theorem.

Sketch of proof of Corollary 7. Assume $2^\omega \leq \omega_\theta$, and let Π and $H = H_1 \cup \ldots \cup H_k$ be as in (2) above. As in the proof of Theorem 2, we prove in ZFC an auxiliary proposition $R(t)$ (which suffices to prove the corollary).

Proposition $R(t)$. *Let θ be an ordinal, $n \geq 2$, $t \geq 1$, $k \geq 1$ be integers, $\Pi = \{V_1, \ldots, V_k\}$ be a set of non-trivial subspaces of $\mathbb{R}^n$ which is t-good, $H = H_1 \cup \ldots \cup H_k$ be a partition of $H = \{V + u : V \in \Pi, u \in \mathbb{R}^n\}$, and let*

$A \subseteq \mathbb{R}^n \cup H$ be a set of size $\le \omega_\theta$. Then there is a partition $S = A \cap \mathbb{R}^n = S_1 \cup \ldots \cup S_k$ such that for all $h \in A \cap H_i$, $|h \cap S_i| \le \omega_{\theta-t}$.

If $t = 1$, then the hypothesis that Π is 1-good simply says that $\bigcap_{i=1}^k V_i$ is finite. It follows that the intersection of any k distinct elements of H is also finite. Thus, the given partition of H is $(k, 1)$ finitely determined. Theorem 2 then finishes this case. Since Π being t-good implies Π is t'-good for all $t' \le t$, we see that $R(t)$ also holds for all t when $\theta = 0$. So, we may assume $\theta \ge 1$. Likewise, in proving $R(t)$ we may assume that $\theta = \bar{\theta} + (t-1)$ for some ordinal $\bar{\theta}$. We call a set $A \subseteq \mathbb{R}^n \cup H$ *good* provided: (1) for any $h_1, \ldots, h_q \in A \cap H$, if $\bigcap_{i=1}^q h_i$ is finite, then $\bigcap_{i=1}^q h_i \subseteq A$, and (2) for any $x \in \mathbb{R}^n \cap A$, the finitely many $h \in H$ which contain x also lie in A.

Without loss of generality, we may assume A is good, and $|A| = \omega_\theta$. Write $A = \bigcup_{\alpha_1 < \omega_\theta} A_{\alpha_1}$ as an increasing union, where each A_{α_1} is good and has size $\le \omega_{\theta-1}$. Similarly, we write each A_{α_1} as an increasing union $A_{\alpha_1} = \bigcup_{\alpha_2 < \omega_{\theta-1}} A_{\alpha_1,\alpha_2}$, where each A_{α_1,α_2} is good of size $\le \omega_{\theta-2}$. Continuing, we define good sets $A_{\alpha_1,\alpha_2,\ldots,\alpha_{t-1}}$ for all $\alpha_1 < \omega_\theta, \ldots, \alpha_{t-1} < \omega_{\theta-(t-2)}$, such that each $A_{\alpha_1,\alpha_2,\ldots,\alpha_{t-1}}$ has size $\le \omega_{\theta-t+1} = \omega_{\bar{\theta}}$. Write also $A_{\alpha_1,\ldots,\alpha_{t-1}} = \bigcup_{\alpha_t < \omega_{\bar{\theta}}} A_{\alpha_1,\ldots,\alpha_t}$, where each $A_{\alpha_1,\ldots,\alpha_t}$ has size $< \omega_{\bar{\theta}}$ but is not necessarily good (if $\bar{\theta} \ge 1$, then we may make these sets good as well). For each point x (or $h \in H$) in A and ordinals $\alpha_1, \ldots, \alpha_i$, $i \le t$, we say that x (or h) is *new relative to* $\alpha_1, \ldots, \alpha_i$ provided for all $j \le i$, $x \in A_{\alpha_1,\ldots,\alpha_j} - \bigcup_{\beta < \alpha_j} A_{\alpha_1,\ldots,\alpha_{j-1},\beta}$. There is clearly a unique sequence $\alpha_1 = \alpha_1(x), \ldots, \alpha_t = \alpha_t(x)$ such that x is new relative to $\alpha_1, \ldots, \alpha_t$.

For $x \in A$, we now describe the S_i into which we place x. Let $\alpha_1 = \alpha_1(x), \ldots, \alpha_t = \alpha_t(x)$. Let $h_1^1, \ldots, h_1^{a(1)}$ enumerate the $h \in H \cap A$ on which x lies which are old relative to α_1. Let $h_2^1, \ldots, h_2^{a(2)}$ be those $h \in H \cap A$ on which x lies which are new relative to α_1 but old relative to α_1, α_2, and continuing, $h_t^1, \ldots, h_t^{a(t)}$ those $h \in H \cap A$ on which x lies which are new relative to $\alpha_1, \ldots, \alpha_{t-1}$ but old relative to $\alpha_1, \ldots, \alpha_t$. Clearly, $a(1) + \ldots + a(t) \le k$. If there is some "color" $1 \le i \le k$ not taken on by any of the h_j^l, put x into one such S_i. Note that this includes the case where $a(1) + \ldots + a(t) < k$. Otherwise, let $h_1^1, \ldots, h_1^{a(1)}, h_2^1, \ldots, h_2^{a(2)}, \ldots, h_t^1, \ldots, h_t^{a(t)}$ correspond to the subspaces $W_1, \ldots, W_k$ of Π (so, $W_1, \ldots, W_k$ is a permutation of $V_1, \ldots, V_k$). This determines a linear ordering $\prec = \prec(x)$ of Π. By t-goodness, there are $b(1) < \ldots < b(t)$ such that for all $0 \le j < t$, $\bigcap_{m=b(j)}^{b(j+1)} W_m$ is finite (where we interpret $b(0)$ as 1). Note first that $b(1) > a(1)$, as otherwise $h_1^1 \cap \ldots \cap h_1^{a(1)}$ would be finite. This would contradict the fact that x is new relative to α_1, and all of the A_β are good. Without loss of generality, we may assume that $b(1) = a(1) + 1$. It then follows by similar reasoning that $b(2) > a(2)$, and again we may assume that $b(2) = a(2) + 1$. Continuing, we may assume that $b(t-1) = a(t-1) + 1$. Thus, $h_t^1 \cap \ldots \cap h_t^{a(t)}$ is finite. Also, by our above remarks, we may assume that $a(1) + \ldots + a(t) = k$, and each color $1 \le i \le k$ is taken on exactly once in the sequence $h_1^1, \ldots, h_t^{a(t)}$ (that is, for each i, there is exactly one h in this sequence which lies in H_i). For each $1 \le j \le a(t)$, let $\beta(h_t^j) < \omega_{\bar\theta}$ be the ordinal such that h_t^j is new relative to $\alpha_1, \ldots, \alpha_{t-1}, \beta(h_t^j)$. Finally, put x into S_i, where $h_t^l \in H_i$ and l is such that $\beta(h_t^l) \ge \sup\{\beta(h_t^j) : 1 \le j \le a(t)\}$.

To show this works, fix an $h \in H_i \cap A$. We show that $|h \cap S_i| < \omega_{\bar\theta}$. Suppose $|h \cap S_i| \ge \omega_{\bar\theta}$. Let $\alpha_1 = \alpha_1(h), \ldots, \alpha_t = \alpha_t(h)$, i.e., h is new relative to $\alpha_1, \ldots, \alpha_t$. If $x \notin A_{\alpha_1}$, and x lies on h, then by definition of our coloring, $x \notin S_i$. There are no old (relative to α_1) points x which lie on h, since the A_β are good. Thus there must be $\ge \omega_{\bar\theta}$ points $x \in S_i$ which are new at α_1 which lie on h. Continuing, we see that $\ge \omega_{\bar\theta}$ points $x \in S_i$ which are new at $\alpha_1, \ldots, \alpha_{t-1}$ lie on h. There are $< \omega_{\bar\theta}$ points in $A_{\alpha_1, \ldots, \alpha_t}$, hence we need only consider x new at $\alpha_1, \ldots, \alpha_{t-1}, \beta$, where $\beta > \alpha_t$. If such an x lies on h, then the values of the $\beta(h_t^j)$, $1 \le j \le a(t)$, as computed for x, are all $\le \alpha_t$ from the definitions of the $\beta(h_t^j)$ and our coloring. Since $h_t^1 \cap \ldots \cap h_t^{a(t)}$ is finite, it follows that there are $< \omega_{\bar\theta}$ such x, a contradiction. ∎

2. Infinite partitions. In this section we consider results related to partitions of lines and points into infinitely many pieces. The analog of Theorem 2 becomes the following.

Theorem 3. (ZFC) *Let $n \ge 1$. For any $r \ge 2$, $s \ge 1$ and any (r, s) finitely determined partition $H = \bigcup_{k<\omega} H_k$ of $H \subseteq \mathcal{P}(\mathbb{R}^n)$, there is a partition $\mathbb{R}^n = \bigcup_{k<\omega} S_k$ such that $|h \cap S_i| < \omega$ for all $i < \omega$ and $h \in H_i$.*

Proof. First, one proves in ZFC, by induction on $\eta \in ON$, the following proposition:

Proposition $P(\eta)$. *If A is a collection of elements of H and points in $\mathbb{R}^n$, $|A| \le \omega_\eta$, and $A \cap H = \bigcup_{n<\omega} A_n$ is a partition which is (r, s) finitely determined, and if f is a function with domain S = points in A such that $\forall x \in S\ f(x) \subseteq \omega$, $|f(x)| < \omega$, then there is a partition $S = \bigcup_{n<\omega} S_n$ such that each $h \in A_n$ intersects S_n in a finite set, and, for all $x \in S$, $x \notin S_a$ for any $a \in f(x)$.*

Notice that $P(2^\omega)$ implies Theorem 3.

In proving $P(\eta)$, we may assume that A is good, that is, if $h_1, \ldots, h_r$ lie in different A_n, then $\bigcap_{i=1}^r h_i \subseteq A$ and if points $x_1, \ldots, x_s$ are in A, then so are the finitely many h in H which contain them. Note that $P(0)$ is essentially trivial (see the proof of Corollary 9 below). For $\eta \ge 1$, the proof that $P(\eta)$ holds is broken into cases depending on whether η is successor or limit. In each case, we write A as an increasing union of good sets, the argument then being essentially identical to those given earlier. ∎

As a special case of Theorem 3, we have:

COROLLARY 8. (ZFC) *If the lines L in $\mathbb{R}^n$ $(n \geq 2)$ are partitioned into ω disjoint pieces $L = \bigcup_{k<\omega} L_k$, then there is a partition $\mathbb{R}^n = \bigcup_{k<\omega} S_k$ such that each line $l \in L_i$ meets S_i in a finite set, for all $i \in \omega$.*

Still further generalizations are possible. For example, we may define $H \subseteq \mathcal{P}(\mathbb{R}^n)$ as being (r, s, κ) *determined* (or a partition being (r, s, κ) *determined*) where κ is an infinite cardinal as before, except that we now require that the intersection of r distinct elements of H (or the intersection of r elements of H lying in different H_n) has size $\leq \kappa$, and for any s distinct points at most κ many $h \in H$ contain these points. Then we have:

THEOREM 4. (ZFC) *Let $n \geq 1$, $r \geq 2$, $s \geq 1$ be integers, κ an infinite cardinal. Let $H \subseteq \mathcal{P}(\mathbb{R}^n)$ be (r, s, κ) determined. Then for any partition $H = \bigcup_{\alpha<\kappa} H_\alpha$ into κ disjoint sets, there is a partition $\mathbb{R}^n = \bigcup_{\alpha<\kappa} S_\alpha$ of $\mathbb{R}^n$ into κ disjoint sets such that $|h \cap S_\alpha| < \omega$ for all $h \in H_\alpha$.*

The proof is a trivial generalization of that of Theorem 3; just start with good sets of size κ.

Of course, since the last theorem and previous corollary are proved in ZFC only, their conclusions imply no bound on the continuum.

Corollary 8 may be modified in a curious manner, which reintroduces set-theoretic connections. The case $p = 0$ follows from Davies [D2].

COROLLARY 9. *Suppose m is a positive integer and $2^\omega \leq \omega_m$. If the set L of lines in $\mathbb{R}^n$, $n \geq 2$, is partitioned into ω sets, $L = \bigcup_{k<\omega} L_k$, then there is a partition $\mathbb{R}^n = \bigcup_{k<\omega} S_k$ such that any line in L_k meets S_k in a set of size at most $m+1$. More generally, if $2^\omega \leq \omega_m$, and the lines are partitioned into ω_p sets, $L = \bigcup_{\alpha<\omega_p} L_\alpha$, for $p \in \omega$, then we may partition the points, $\mathbb{R}^n = \bigcup_{\alpha<\omega_p} S_\alpha$, so that $|L_\alpha \cap S_\alpha| \leq m - p + 1$ for all $\alpha < \omega_p$.*

S k e t c h o f p r o o f. We show by induction on $m \geq 0$ (working in ZFC) that if A is a good set of lines and points in $\mathbb{R}^n$ $(n \geq 2)$ of size $\leq \omega_m$, $L = \bigcup_{\alpha<\omega_p} L_\alpha$ is a partition of the lines in A, and f is a function which assigns to each $x \in S = A \cap \mathbb{R}^n$ a finite subset of ω_p, then we may partition the set S of points in A, $S = \bigcup_{\alpha<\omega_p} S_\alpha$, so that each line $l \in L_\alpha$ intersects S_α in a set of size at most $m - p + 1$ and that for all $x \in S$, $x \notin S_\alpha$ for all $\alpha \in f(x)$. For $m \leq p$, the result is trivial (assign colors to the points of S in a one-to-one manner avoiding the forbidden colors). If A is a good set of size ω_m, $m > p$, write $A = \bigcup_{\beta<\omega_m} A_\beta$, with each A_β good of size $\leq \omega_{m-1}$. For $\beta < \omega_m$, consider the new points of A_β. Each such point lies on at most one old line. For each such point, let $\bar{f}(x) = f(x) \cup \{j\}$, where x lies on an old line in L_j if one exists (otherwise set $\bar{f}(x) = f(x)$). By induction, we may partition the new points at β so that for any $x \in S_\alpha$ new at β, x lies on at most $m-p$ lines new at β in L_α, and also $\alpha \notin \bar{f}(x)$. Since any line new at β has at most one old point which lies on it, it is easy to see that this partition of S works. ∎

Considering the converse direction to Corollary 9 leads to some interesting questions. For example, assuming CH, given a partition $L = \bigcup_{k<\omega} L_k$ of the lines in $\mathbb{R}^n$, we may partition the points, $\mathbb{R}^n = \bigcup_{k<\omega} S_k$, so that for each $l \in L_i$, $|l \cap S_i| \le 2$. Davies showed in [D2], answering a question of [Er], that we may not always get $|l \cap S_i| \le 1$, even assuming CH. We will strengthen this result in the next section. It is natural to ask, then, whether this partition property implies CH, or has any strength beyond ZF at all.

Question. Is it true (or consistent) in ZFC that if the lines in the plane are partitioned into countably many sets, $L = \bigcup_{k<\omega} L_k$, then we may partition the points, $\mathbb{R}^2 = \bigcup_{k<\omega} S_k$, so that for each $l \in L_i$, $|l \cap S_i| \le 2$? Is the analogous statement for $\mathbb{R}^n$ true (or consistent)? More generally, do the converse implications to Corollary 9 hold?

3. An ordinal partition property. We begin by considering the question of whether the hypothesis $2^\omega \le \omega_m$ is necessary in Corollary 9. Suppose that $2^\omega = \omega_2$. The argument of Corollary 9 shows that we still have the "two-point" partition property (i.e., for each line $l \in L_i$, $|l \cap S_i| \le 2$) provided we have the following:

$(*)$ *For every set $A \subseteq L \cup \mathbb{R}^n$ of lines and points in $\mathbb{R}^n$ of size ω_1, and any partition $A \cap L = \bigcup_{k<\omega} L_k$ of the lines in A, there is a partition $A \cap \mathbb{R}^n = \bigcup_{k<\omega} S_k$ such that for each line $l \in L_k$, $|l \cap S_k| \le 1$.*

Our previous argument showed that $(*)$ fails assuming CH, but it is not immediately clear $(*)$ fails assuming just ZFC. We show below, however, that this is the case. We first reformulate $(*)$ into purely set-theoretic partition properties. Consider the following partition statements about ω_1 (F. Galvin pointed out to us that the properties $P(\omega_1), Q(\omega_1)$ were introduced earlier in [EGH], where they were shown to be false assuming CH):

$P(\omega_1)$ *For every partition $P : (\omega_1)^2 \to \omega$, there is an $h : \omega_1 \to \omega$ such that for all $\alpha < \beta < \omega_1$, if $P(\alpha, \beta) = i$, then at least one of $h(\alpha)$, $h(\beta) \ne i$.*

Lemma 2. (ZFC) $(*) \Leftrightarrow P(\omega_1)$.

Proof. Assuming $(*)$, let $P : (\omega_1)^2 \to \omega$ be a partition. Let $B \subseteq \mathbb{R}^n$ be an independent set of size ω_1, i.e., no three points of A are colinear. Let

$A = B \cup L$, where L is the set of lines through two points of B. Applying now $(*)$ to A produces an $h : \omega_1 \to \omega$ as required by $P(\omega_1)$, identifying ω_1 with B. Assuming $P(\omega_1)$, let A, L be as in the statement of $(*)$. Let $\{x_\alpha\}$ enumerate the points of A. Define $P : \omega_1 \to \omega$ by $P(\alpha, \beta) = i$ iff the line between x_α and x_β lies in L_i. Applying $P(\omega_1)$ then produces an $h : \omega_1 \to \omega$. This defines a corresponding partition of the x_α which easily works. ∎

Note that it makes sense to consider $P(\omega_1)$ in just ZF. We reformulate $P(\omega_1)$ in a more suggestive manner of usual partition type properties:

$Q(\omega_1)$ *For any partition* $Q : (\omega_1)^2 \to \omega$, *we may write* $\omega_1 = \bigcup_{k<\omega} A_k$ *so that for all* k, $Q([A_k]^2)$ *is co-infinite.*

Note that in $Q(\omega_1)$ there is no loss of generality in assuming the A_k are disjoint.

Lemma 3. (ZF) $P(\omega_1) \Leftrightarrow Q(\omega_1)$.

Proof. Assume first $P(\omega_1)$, and let $Q : (\omega_1)^2 \to \omega$ be given. Let $r : \omega \to \omega$ be onto with $r^{-1}(i)$ infinite for all $i \in \omega$. Let $P(\alpha, \beta) = r(Q(\alpha, \beta))$. Let $h : \omega_1 \to \omega$ be as given by $P(\omega_1)$ for P. Let $A_k = \{\alpha < \omega_1 : h(\alpha) = k\}$. Then, for $\alpha, \beta \in A_k$, $r(Q(\alpha, \beta)) \neq k$, hence $Q(\alpha, \beta) \notin r^{-1}(k)$. Assume now $Q(\omega_1)$, and let $P : (\omega_1)^2 \to \omega$ be given. Let $\{A_k : k \in \omega\}$ be as given by $Q(\omega_1)$ for the partition P. Let $n_0, n_1, \ldots$ be distinct integers such that $n_k \notin P([A_k]^2)$ for all k. Let $h(\alpha) = n_k$ for all $\alpha \in A_k$. This easily works. ∎

The following theorem of Todorčević (see Section 4 of [To]) immediately implies that $Q(\omega_1)$ is false in ZFC.

Theorem (Todorčević). *Assume ZFC. There is a partition* $c : [\omega_1]^2 \to \omega$ *such that* $c([C]^2) = \omega$ *for all uncountable* $C \subseteq \omega_1$.

Corollary 10. (ZFC) $(*)$, $P(\omega_1)$, $Q(\omega_1)$ *are all false.*

From the failure of $P(\omega_1)$, it follows (in ZFC) that there is a partition $L = \bigcup_{k<\omega} L_k$ of a set L of lines in $\mathbb{R}^2$, with $|L| = \omega_1$, such that for every partition $\mathbb{R}^2 = \bigcup_{k<\omega} S_k$ we have $|l \cap S_n| \geq 2$ for some n and $l \in L_n$ (cf. the proof of Lemma 2). This strengthens a result of Davies mentioned earlier.

Todorčević's theorem is proved in ZFC, and thus it remains possible that $Q(\omega_1)$ (or, equivalently, $P(\omega_1)$) is consistent with ZF. We in fact show that $Q(\omega_1)$ is a theorem of AD, and thus holds in $L(\mathbb{R})$ assuming ZFC + large cardinal axioms. In fact, we show a much stronger version of $Q(\omega_1)$ under these hypotheses. Consider the following strengthening of $Q(\omega_1)$:

$Q^s(\omega_1)$ *For every partition* $Q : [\omega_1]^2 \to \omega$, *we may write* $\omega_1 = \bigcup_{k<\omega} A_k$ *where* $Q([A_k]^2)$ *is finite for all* $k \in \omega$.

R e m a r k. It is easy to see directly that $Q^s(\omega_1)$ fails in ZFC (let $\{x_\alpha : \alpha < \omega_1\}$ be an ω_1 sequence of distinct elements of 2^ω, and let $Q(\alpha,\beta) =$ least i such that $\alpha(i) \neq \beta(i)$).

THEOREM 5. (ZF + AD + DC) *$Q^s(\omega_1)$ holds.*

P r o o f. We sketch two proofs. The first uses only the theory of indiscernibles for $L(x)$, $x \in \mathbb{R}$. The second uses the analysis of measures on ω_1 of [J1]. The second proof, however, extends to cardinals other than ω_1.

Let $Q : [\omega_1]^2 \to \omega$ be given. From AD, there is an $x \in \mathbb{R}$ such that $Q \in L(x)$. Let $C = \{\xi_\alpha : \alpha \in ON\}$ be the canonical closed unbounded set of (Silver) indiscernibles for $L(x)$. Below, τ, σ denote terms in the language of set theory with x as a parameter. Let τ be a term such that $Q = \tau^{L(x)}(\xi_0,\ldots,\xi_n,\beta_0,\ldots,\beta_m)$, where $\xi_0 < \ldots < \xi_n < \omega_1 \le \beta_0 < \ldots < \beta_m \in C$. For each $\alpha < \omega_1$ we canonically choose a representation $\alpha = \sigma(\alpha)^{L(x)}(\theta_0(\alpha),\ldots,\theta_{m(\alpha)}(\alpha))$, for some term $\sigma(\alpha)$ and $\theta_0(\alpha) < \ldots < \theta_{m(\alpha)}(\alpha) < \omega_1$ in C. For each $\alpha < \omega_1$, let $p(\alpha) \in \omega$ be an integer which codes the term $\sigma(\alpha)$, $m(\alpha)$, and the manner in which the two sequences of ordinals $(\xi_0,\ldots,\xi_n)$, $(\theta_0(\alpha),\ldots,\theta_{m(\alpha)}(\alpha))$ are interlaced (including which of them are equal). For $\alpha,\beta < \omega_1$, let $q(\alpha,\beta) \in \omega$ be an integer which codes how the two sequences of ordinals $\vec\theta(\alpha)$, $\vec\theta(\beta)$ are interlaced. Let $A_k = \{\alpha < \omega_1 : p(\alpha) = k\}$. To see this works, fix $k \in \omega$, and consider $Q\restriction[A_k]^2$. Note that $q([A_k]^2)$ is finite. It thus suffices to show that if $\alpha < \beta$, $\gamma < \delta$ are in A_k, and $q(\alpha,\beta) = q(\gamma,\delta)$, then $Q(\alpha,\beta) = Q(\gamma,\delta)$. However, from the fact that $\alpha,\beta,\gamma,\delta \in A_k$ and $q(\alpha,\beta) = q(\gamma,d)$, it follows that the manner in which $(\xi_0,\ldots,\xi_n)$, $\vec\theta(\alpha)$, and $\vec\theta(\beta)$ are interlaced is the same as that for the sequences $(\xi_0,\ldots,\xi_n)$, $\vec\theta(\gamma)$, $\vec\theta(\delta)$. It thus follows by indiscernibility that $Q(\alpha,\beta) = (\tau^{L(x)}(\xi_0,\ldots,\xi_n,\beta_0,\ldots,\beta_m))(\sigma^{L(x)}(\vec\theta(\alpha)),\sigma^{L(x)}(\vec\theta(\beta))) = (\tau^{L(x)}(\xi_0,\ldots,\xi_n,\beta_0,\ldots,\beta_m))(\sigma^{L(x)}(\vec\theta(\gamma)),\sigma^{L(x)}(\vec\theta(\delta))) = Q(\gamma,\delta)$.

For the second proof, fix again $Q : [\omega_1]^2 \to \omega$. Let $\mathcal{I} \subseteq \mathcal{P}(\omega_1)$ be the countably additive ideal consisting of all $A \subseteq \omega_1$ such that $A \subseteq \bigcup_{k<\omega} S_k$ where each $S_k \subseteq \omega_1$ is such that $Q([S_k]^2)$ is finite. Assume by way of contradiction that $\mathcal{I}$ is a proper ideal (i.e., $\omega_1 \notin \mathcal{I}$). By Kunen, from AD, any countably additive ideal on an ordinal $\kappa < \Theta$ can be extended to a measure (i.e., countably additive ultrafilter) on κ. (Proof: Let μ be the Martin measure on the Turing degrees $\mathcal{D}$. By the coding lemma, let $\pi : \mathbb{R} \to \mathcal{I}$ be onto. For $d \in \mathcal{D}$, set $\varrho(d) =$ least $\alpha < \kappa$ not in $\bigcup_{x\in d} \pi(x)$. This is well-defined since $\mathcal{I}$ is proper. Then $\varrho(\mu)$ is a measure on κ giving measure 0 to all $I \in \mathcal{I}$, where $\varrho(\mu)(A) = 1$ iff $\mu(\{d \in \mathcal{D} : \varrho(d) \in A\}) = 1$.)

The claim of Section 2 of [J1] analyzes, assuming AD, all measures ν on ω_1. The result (somewhat restated) is that there is a function $f : [\omega_1]^m \to \omega_1$ for some $m \in \omega$ such that for all $B \subseteq \omega_1$, $\nu(B) = 1$ iff there is a c.u.b. $C \subseteq \omega_1$

such that $f(\delta_1, \ldots, \delta_m) \in B$ for all $\delta_1 < \ldots < \delta_m \in C$. By applying the finite exponent partition property on ω_1 (with exponent $2m$) finitely many times, we get a c.u.b. $C \subseteq \omega_1$ such that for all pairs of increasing sequences of length m from C, $(\alpha_1, \ldots, \alpha_m)$, $(\beta_1, \ldots, \beta_m)$, the value $P(f(\alpha_1, \ldots, \alpha_m), f(\beta_1, \ldots, \beta_m))$ depends only on the manner of interlacing of the two sequences. This C, however, then defines a measure one set with respect to ν on which Q takes only finitely many values, a contradiction. ∎

Corollary 11. *$P(\omega_1)$, $Q(\omega_1)$ are consistent with ZF.*

Finally, we state without proof some extensions of the above ordinal partition properties. For cardinals κ, δ, let $P(\kappa, \delta)$ be the statement that for any partition $P : [\kappa]^2 \to \delta$, there is an $h : \kappa \to \delta$ such that for any $\alpha < \beta < \kappa$, at least one of $h(\alpha), h(\beta)$ is different from $P(\alpha, \beta)$. Let $Q(\kappa, \delta)$ be the statement that for any $Q : [\kappa]^2 \to \delta$, we may write $\kappa = \bigcup_{\lambda<\delta} A_\lambda$ where for each λ, $\delta - Q([A_\lambda]^2)$ is infinite. Let also $Q^s(\kappa, \delta)$ be as $Q(\kappa, \delta)$ except that we write $\kappa = \bigcup_{k<\omega} A_k$, and we require each $Q([A_k]^2)$ to be finite.

The same argument given before shows that $\forall \kappa, \delta\ (P(\kappa, \delta) \Leftrightarrow Q(\kappa, \delta))$. Also, in $Q(\kappa, \delta)$, we may replace "$\delta - Q([A_k]^2)$ is infinite" by "$\delta - Q([A_k]^2)$ has size δ". The second proof given above for $Q^s(\omega_1)$ when combined with the analysis of measures on $\boldsymbol{\delta}^1_{2n+1}$ (see [J2] for the case $n = 1$) yields:

Theorem 6. (ZF + AD + DC) *For all $\delta < \boldsymbol{\delta}^1_{2n+1}$, $Q^s(\boldsymbol{\delta}^1_{2n+1}, \delta)$ holds.*

It is again easy to see that $Q^s(\kappa, \delta)$ fails in ZFC for all uncountable κ and infinite δ. We believe, however, that the Steel–Van Wesep–Woodin forcing [W] for recovering ω_1-DC can be used to show the following: $(\mathrm{ZFC} + \mathrm{AD}^{L(\mathbb{R})})$ There is a model of $\mathrm{ZF} + \omega_1\text{-DC} + \forall \delta < \boldsymbol{\delta}^1_{2n+1}\ (Q^s(\boldsymbol{\delta}^1_{2n+1}, \delta)$ holds). Thus, $Q^s(\kappa, \delta)$ is consistent with small amounts of choice.

As S. Todorčević pointed out to us, one can show that $Q(\omega_1)$, and hence $Q^s(\omega_1)$, have consistency strength beyond ZFC. In fact, $Q(\omega_1)$ implies ω_1 is inaccessible to L. For if not, then for some $x \in \mathbb{R}$, $\omega_1 = \omega_1^{L(x)}$. Let $c : [\omega_1]^2 \to \omega$ be the Todorčević partition defined in $L(x)$. Applying Q, let $A \subseteq \omega_1, |A| = \omega_1$ be such that $c([A]^2)$ is co-infinite. The proof of Todorčević's theorem shows that in $L(x, A)$, c retains the property that $c([B]^2) = \omega$ for all $B \subseteq \omega_1$ of size ω_1. This contradicts $c([A]^2)$ being co-infinite.

The failure of $P(\omega_1)$ in ZFC rules out one approach for showing the "two-point" partition property (as in Corollary 9) in ZFC, or even from $2^\omega = \omega_2$. The original question, stated at the end of Section 2, however, remains. Note, however, that the consistency of $\mathrm{ZF} + \neg\mathrm{CH} + Q(\omega_1)$ shows that the "ordinal version" of the two-point partition problem is consistent with $\mathrm{ZF} + \neg\mathrm{CH}$. Here "lines" refers to subsets of ω_2 satisfying the usual properties, i.e., two ordinals less than ω_2 determine a line, and two distinct lines intersect in at most one ordinal.

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