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The S^1-CW decomposition of the geometric realization of a cyclic set

by

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Abstract. We show that the geometric realization of a cyclic set has a natural, S^1-equivariant, cellular decomposition. As an application, we give another proof of a well-known isomorphism between cyclic homology of a cyclic space and S^1-equivariant Borel homology of its geometric realization.

0. Introduction. It is a rudimentary fact in simplicial topology that the geometric realization of a simplicial set is a CW complex. In this paper, we prove a similar result in case of a cyclic set. A cyclic set X_* is a simplicial set together with actions of cyclic groups Z_{n+1} on the sets of n-simplices. The actions are subject to compatibility relations with the simplicial structure of X_* (cf. [4] and [3]). The geometric realization $|X_*|$ of a cyclic set has a canonical circle action (cf. [3], Proposition 1.4). Let Fix denote the fixed point set of the action. Our first result is the following.

THEOREM 1. *The geometric realization of a cyclic set X_* is an S^1-CW complex.*

More precisely, in Section 2 below, we construct a filtration of $|X_*|$:

$$\text{Fix} \subseteq F_0 \subseteq F_1 \subseteq \ldots \subseteq |X_*| = \bigcup_{n=0}^{\infty} F_n, \tag{0.1}$$

which is the skeletal filtration of an S^1-CW complex in the sense of [13], p. 9. Our construction is a cyclic version of the cellular decomposition of the geometric realization of a simplicial set. The space F_n, for $n \geq 1$, is obtained from F_{n-1} by attaching S^1-equivariant cells $S^1/Z_m \times \Delta^n$. The

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0-cells of the CW complex $|X_*|$ consist of fixed points and free S^1-orbits. The last fact makes the filtration (0.1) inconvenient for a purpose we have in mind: calculation of equivariant homology of $|X_*|$ by spectral sequences. To remedy the disadvantage, we construct a space $|EX_*|$ which is S^1-homotopy equivalent to $|X_*|$ and has a natural filtration $\{G_n\}_{n\geq 0}$ similar to (0.1) but for which $G_0 = X_0 \times S^1$.

For any S^1-equivariant homology theory $h_*^{S^1}$, the filtration $\{G_n\}_{n\geq 0}$, in a standard manner, yields a spectral sequence which converges to $h_*^{S^1}(|EX_*|) = h_*^{S^1}(|X_*|)$. We first consider the spectral sequence in Borel homology. In this case, one easily obtains the following known fact.

Theorem 2 ([3], [9]). *If X_* is a cyclic space and R is a commutative ring with* 1, *then homology groups $H_*(ES^1 \times_{S^1} |X_*|, R)$ and $HC_*(R[X_*])$ are naturally isomorphic.*

Here $HC_*(R[X_*])$ stands for cyclic homology of the cyclic module $R[X_*]$, (cf. [8], Chap. 6).

We conclude the paper with remarks on Bredon homology of the geometric realization of a cyclic set. The filtration $\{G_n\}_{n\geq 0}$ leads to a spectral sequence in S^1-equivariant Bredon homology whose E^1 term seems to be accessible.

1. Notation and terminology. In the sequel, we use Section 1 of [3] as a basic reference for the theory of cyclic spaces. In particular, by a *cyclic space* we mean a functor $X_* : \Lambda^{\mathrm{op}} \to \mathrm{Top}$, where Λ denotes the cyclic category of A. Connes. As usual, d_i and s_i are the simplicial faces and degeneracies of X_*, i.e., $d_i = X_*(\partial_i)$ and $s_i = X_*(\sigma_i)$ where $\partial_i : [n-1] \to [n]$ and $\sigma_i : [n] \to [n-1]$ are the standard simplicial operators. We assume that the underlying simplicial space of X_* is proper, i.e., all degeneracy maps are cofibrations. We denote by $t_n : X_n \to X_n$ the value of X_* at the cyclic operator $\tau_n : [n] \to [n]$ in Λ which generates $Z_{n+1} = \mathrm{Aut}_\Lambda([n])^{\mathrm{op}}$. We denote by $s_n : X_{n-1} \to X_n$ the extra degeneracy map which is by definition $t_n^n s_0 = X_*(\sigma_0 \tau_n^n)$. It follows from relations in Λ that $s(X_n) = \bigcup_{i=0}^n s_i(X_{n-1})$ is a Z_{n+1}-subspace of X_n. Finally, we identify the fixed point set Fix of the canonical S^1-action on $|X_*|$ with the equalizer of the degeneracy maps $s_0, s_1 : X_0 \to X_1$ (cf. [6], p. 145), i.e., we have

$$\mathrm{Fix} = \{x \in X_0 : s_0 x = s_1 x = t_1 s_0 x\}. \tag{1.1}$$

2. The cellular decomposition

Theorem 1. *The geometric realization of a cyclic set X_* is an S^1-CW complex.*

Proof. We will use coends of functors $C^{\mathrm{op}} \times C \to \mathrm{Top}$, where C is a subcategory of Λ. Basic properties of these coends are summarized in Section 1 of [6]. For the geometric realization of X_* we have

$$(2.1)\quad \begin{aligned} |X_*| &= \int_{[n]\in\Delta} X_n \times \Delta^n = \int_{[n]\in\Delta}\int_{[m]\in\Lambda} X_m \times \mathrm{Hom}_\Lambda([n],[m]) \times \Delta^n \\ &= \int_{[m]\in\Lambda} X_m \times \int_{[n]\in\Delta} \mathrm{Hom}_\Lambda([n],[m]) \times \Delta^n \\ &= \int_{[m]\in\Lambda} X_m \times |\mathrm{Hom}_\Lambda(*,[m])| = \int_{[m]\in\Lambda} X_m \times (S^1 \times \Delta^m). \end{aligned}$$

The cyclic group Z_{m+1} acts on the m-simplices of the cocyclic space $S^1 \times \Delta^m$ of the last coend by the formula

$$(2.2)\qquad \tau_m(z, u_0, u_1, \ldots, u_m) = (ze^{-2\pi i u_0}, u_1, \ldots, u_m, u_0),$$

where $(z, u_0, u_1, \ldots, u_m) \in S^1 \times \Delta^m$ (cf. [3], Proposition 1.4(iii)). Note that one can use the last coend in (2.1) to define the S^1-action on $|X_*|$. In what follows, we will also use another action of Z_{m+1} on $S^1 \times \Delta^m$ which is given by

$$(2.3)\qquad \tau_m(z, u_0, u_1, \ldots, u_m) = (zw_{m+1}, u_1, \ldots, u_m, u_0),$$

where $\omega_{m+1} = \exp(-2\pi i/(m+1))$. Let $f_m : S^1 \times \Delta^m \to S^1 \times \Delta^m$ be defined by

$$(2.4)\qquad f_m(z, u_0, u_1, \ldots, u_m) = (z\omega_{m+1}^{l(u)}, u_0, u_1, \ldots, u_m),$$

where $l(u) = u_0 + 2u_1 + \ldots + (m+1)u_m$. It is straightforward to check that f_m is an equivariant map from the Z_{m+1}-space (2.2) to the Z_{m+1}-space (2.3) and that it induces a homeomorphism of Z_{m+1}-orbit spaces. In order to construct an equivariant, cellular decomposition of $|X_*|$, we use the canonical filtration of the coend (2.1) (cf. [6], pp. 146–147). We have

$$\int_{[m]\in\Lambda} X_m \times (S^1 \times \Delta^m) = \bigcup_{n\geq -1} F_n,$$

where $F_{-1} = \mathrm{Fix}$ and the spaces F_n, for $n \geq 0$, are defined by the pushout diagrams

$$(2.5)\qquad \begin{array}{ccc} \mathrm{Fix} \times S^1 & \xrightarrow{\text{incl.}} & X_0 \times S^1 \\ \downarrow{\scriptstyle \text{proj.}} & & \downarrow \\ \mathrm{Fix} & \longrightarrow & F_0 \end{array}$$

$$
\begin{array}{ccc}
s(X_n)\times_{Z_{n+1}}(S^1\times\Delta^n)\cup X_n\times_{Z_{n+1}}(S^1\times\partial\Delta^n) & \xrightarrow{\tau_n} & F_{n-1}\\
\downarrow & & \downarrow\\
X_n\times_{Z_{n+1}}(S^1\times\Delta^n) & \longrightarrow & F_n
\end{array}
\tag{2.6}
$$

The attaching map τ_n in (2.6) is induced by relations in Λ. For a standard, inductive argument which shows that τ_n is well defined, we refer the reader to [7], proof of Lemma 1. Using the maps (2.4), one obtains from (2.6) a similar pushout square with the Z_{n+1}-action (2.2) replaced by the action (2.3). We conclude that there exists a filtration

$$
\mathrm{Fix}\subseteq F_0\subseteq F_1\subseteq\ldots\subseteq|X_*|=\bigcup_{n=0}^{\infty}F_n. \tag{2.7}
$$

The space F_n is obtained from F_{n-1} by glueing to it a number of spaces $Z_{n+1}/Z_k\times_{Z_{n+1}}(S^1\times\Delta^n)=S^1\times_{Z_k}\Delta^n$ (orbit space in the action (2.3)), one space for any orbit $[x]\in(X_n\backslash s(X_n))/Z_{n+1}$, where $Z_k=\mathrm{stab}_{Z_{n+1}}(x)$.

Attaching maps of the glueing are defined on $S^1\times_{Z_k}\partial\Delta^n$. Note that passing to the first barycentric subdivision of Δ^n, one can obtain F_n from F_{n-1} by attaching to it standard equivariant cells $S^1/Z_m\times\Delta^n$. It follows that $|X_*|$ is an S^1-equivariant CW complex whose n-skeleton is F_n. ■

By (2.5) we have $F_0=\mathrm{Fix}\amalg(X_0\backslash\mathrm{Fix})\times S^1$, i.e., 0-dimensional cells of the equivariant cell decomposition of $|X_*|$ consist of fixed points and free S^1-orbits. This fact makes the filtration (2.7) inconvenient for computing equivariant (co)homology of $|X_*|$ by spectral sequences. Our next aim is to construct a space which is S^1-homotopy equivalent to $|X_*|$ and has a filtration similar to (2.7) but whose 0-th level consists of free orbits. The idea is to ignore degeneracies in the coend (2.1). Let Λ_{face} be the full subcategory of the cyclic category Λ generated by cofaces ∂_i and isomorphisms.

Definition 1. For a cyclic space X_*, define a new cyclic space EX_* to be the coend

$$
EX_*=\int_{[m]\in\Lambda_{\mathrm{face}}}X_m\times\mathrm{Hom}_\Lambda(*,[m]).
$$

Let $q_*:EX_*\to X_*$ be the projection map induced on coends by the inclusion $\Lambda_{\mathrm{face}}\subset\Lambda$. Then q_* is a cyclic map and therefore its geometric realization $q=|q_*|$ is an S^1-equivariant map (cf. [3]). Note that

$$
|EX_*|=\int_{[m]\in\Lambda_{\mathrm{face}}}X_m\times(S^1\times\Delta^m).
$$

Let $\{G_n\}_{n\geq 0}$ be the coend filtration of $|EX_*|$ (cf. [6], pp. 146–147). We have

$$
G_0\subseteq G_1\subseteq\ldots\subseteq|EX_*|=\bigcup_{n=0}^{\infty}G_n, \tag{2.9}
$$

$G_0 = X_0 \times S^1$ and the space G_n, for $n \geq 1$, is defined by the pushout square

$$\begin{array}{ccc} X_n \times_{Z_{n+1}} (S^1 \times \partial\Delta^n) & \longrightarrow & G_{n-1} \\ \downarrow & & \downarrow \\ X_n \times_{Z_{n+1}} (S^1 \times \Delta^n) & \longrightarrow & G_n \end{array}$$

PROPOSITION 1. *The S^1-map $q : |EX_*| \to |X_*|$ is a homotopy equivalence and it induces homotopy equivalences on all fixed point sets.*

Proposition 1, Theorem 1 and the equivariant Whitehead theorem (cf. [13]) give the following.

COROLLARY 1. *If X_* is a cyclic set, then the map q is an S^1-homotopy equivalence.*

Proof of Proposition 1. *Step* 1. First we show that q is an ordinary homotopy equivalence. To achieve this we resolve X_* by the two-sided bar construction $B_*(F, F, X_*)$ as in [8], proof of Theorem 5.12. Here F denotes the monad on the category of simplicial spaces defined by the free cyclic space functor (cf. [8], Definition 4.3). Recall that $B_*(F, F, X_*)$ is a simplicial cyclic space whose space of n-simplices is the cyclic space $F^{n+1}X_* = F(\ldots F(FX)\ldots)_*$ and whose simplicial faces and degeneracies are defined using structure maps of the cyclic, crossed simplicial group C_* and the cyclic space X_*. There exists a map of simplicial cyclic spaces $\varepsilon_* : B_*(F, F, X_*) \to X_*$ whose geometric realization $\varepsilon : |B_*(F, F, X_*)| \to |X_*|$ is a homotopy equivalence. One can adapt the proof of the last statement (cf. [14], Theorem 9.10) to show that ε_* induces an equivalence $|EB_*(F, F, X_*)| \to |EX_*|$. By commutativity of coends $|EB_*(F, F, X_*)| = |n \to EF^{n+1}X_*|$ and we see that it suffices to show that q is an equivalence for the free cyclic space FX_*. In this case, however, we have $|FX_*| = S^1 \times |X_*|$ (cf. [8], Theorem 5.3), and by definition $|EFX_*| = S^1 \times \|X_*\|$, where $\|X_*\|$ is the realization of the simplicial space X_* without using degeneracies (cf. [16], Appendix A). It is easy to check that under these identifications q becomes a product of id_{S^1} and the equivalence $\|X_*\| \to |X_*|$ from [16], Proposition A.1. This proves that q is an equivalence in the free case and therefore in general.

Step 2. To complete the proof we show that q induces equivalences on all fixed point sets. Since $EX_0 = X_0$, (1.1) implies that $|EX_*|^{S^1} = |X_*|^{S^1}$. To check that the map $q^{Z_k} : |EX_*|^{Z_k} \to |X_*|^{Z_k}$, for $k \geq 2$, is an equivalence, we use edgewise subdivisions of cyclic spaces (cf. [1], Section 1). Let $\mathrm{sd}_k X_*$ denote the kth subdivision of X_* and let $\mathrm{sd}_k^{Z_k} X_*$ be the Λ_k^{op}-space whose space of n-simplices is $\mathrm{sd}_k^{Z_k} X_n = X_{k(n+1)-1}^{Z_k}$. Here Λ_k denotes a category with the same morphisms and relations as Λ except for the relation $\tau_n^{n+1} = \mathrm{id}$ which is replaced in Λ_k by $\tau_n^{k(n+1)} = \mathrm{id}$ (cf. [1], Definition 1.5). There exists a natural homeomorphism $D_k : |\mathrm{sd}_k X_*| \to |X_*|$ (cf. [1], Lemma 1.1). Using relations in Λ and Λ_k, one can check that D_k induces a homeomorphism $|\mathrm{sd}_k^{Z_k} X_*| \to |X_*|^{Z_k}$ which we abusively also denote by D_k. As in the cyclic case, for a Λ_k^{op}-space Y_*, we define a new space $E_k Y_*$ to be the coend

$$E_k Y_* = \int_{[m]\in\Lambda_{k,\mathrm{face}}} Y_m \times \mathrm{Hom}_{\Lambda_k}(*, [m]).$$

Here $\Lambda_{k,\mathrm{face}}$ denotes the subcategory of Λ_k generated by the cofaces ∂_i and isomorphisms. Let $q_k : |E_k Y_*| \to |Y_*|$ be the geometric realization of the projection map induced on coends by the inclusion $\Lambda_{k,\mathrm{face}} \subset \Lambda_k$. An argument similar to the one given in the first part of the proof shows that q_k is an equivalence. In Step 1, one has to replace Λ by Λ_k and F by a monad given by the free Λ_k^{op}-space (cf. [8], Definition 4.3, where for G_* we take the crossed simplicial group of the category Λ_k). A link between the maps q^{Z_k} and q_k is provided by the commutative diagram

$$\begin{array}{ccc} |EX_*|^{Z_k} & \xrightarrow{q^{Z_k}} & |X_*|^{Z_k} \\ \uparrow{\scriptstyle D_k} & & \\ |\mathrm{sd}_k^{Z_k} EX_*| & & \uparrow{\scriptstyle D_k} \\ \| & & \\ |E_k \,\mathrm{sd}_k^{Z_k} X_*| & \xrightarrow{q_k} & |\mathrm{sd}_k^{Z_k} X_*|. \end{array} \tag{2.10}$$

The identification on the left side of (2.10) follows from an equality of spaces of m-simplices

$$\mathrm{sd}_k^{Z_k} EX_m = E_k \,\mathrm{sd}_k^{Z_k} X_m,$$

which one checks easily using relations in Λ and Λ_k. Consequently, q^{Z_k} is an equivalence, which finishes the proof. ∎

3. Applications to equivariant homology

THEOREM 2. *If X_* is a cyclic space and R is a commutative ring with* 1, *then there exists a natural isomorphism*

$$H_*(ES^1 \times_{S^1} |X_*|; R) \to HC_*(R[X_*]).$$

Proof. In what follows we assume that X_* is a cyclic set. Our argument extends to cyclic spaces by a standard use of singular chains (cf. [8], proof of Theorem 5.9). By Corollary 1, we have

$$ES^1 \times_{S^1} |X_*| \cong ES^1 \times_{S^1} |EX_*| = \int_{[m]\in\Lambda_{\mathrm{face}}} X_m \times (ES^1 \times \Delta^m),$$

where $ES^1 \times \Delta^*$ is a cocyclic space obtained from $S^1 \times \Delta^*$ by applying the Borel construction $ES^1 \times_{S^1} (-)$ degreewise. Let $\{G'_n\}_{n\ge 0}$ be the canonical filtration of the last coend, i.e.,

$$\int^{[m]\in\Lambda_{\mathrm{face}}} X_m \times (ES^1 \times \Delta^m) = \bigcup_{n=0}^{\infty} G'_n,$$

where $G'_0 = X_0 \times ES^1$ and, for $n \geq 1$, G'_n is defined by a pushout square

$$\begin{array}{ccc} X_n \times_{Z_{n+1}} (ES^1 \times \partial\Delta^n) & \longrightarrow & G'_{n-1} \\ \downarrow & & \downarrow \\ X_n \times_{Z_{n+1}} (ES^1 \times \Delta^n) & \longrightarrow & G'_n \end{array} \tag{3.1}$$

Using maps (2.4) from the proof of Theorem 1, we may choose for ES^1 in (3.1) the space EZ_{n+1} with its standard Z_{n+1}-action. As in the case of pushouts (2.6) this changes the attaching maps. The filtration $\{G'_n\}_{n\geq 0}$ yields a spectral sequence in homology which converges to

$$H_*(ES^1 \times_{S^1} |EX_*|; R) = H_*(ES^1 \times_{S^1} |X_*|; R).$$

The E^1 term of the sequence is

$$\begin{aligned} E^1_{n,m} &= H_{n+m}(EZ_{n+1} \times_{Z_{n+1}} (X_n \times \Delta^n), EZ_{n+1} \times_{Z_{n+1}} (X_n \times \partial\Delta^n); R) \\ &\cong H_{n+m}(Z_{n+1}, C^R_*(X_n \times \Delta^n, X_n \times \partial\Delta^n)), \end{aligned} \tag{3.2}$$

where the latter is the hyperhomology of Z_{n+1} with coefficients in cellular R-chains of the CW pair $(X_n \times \Delta^n,\ X_n \times \partial\Delta^n)$. The homotopy invariance of group hyperhomology (cf. [2], Proposition 5.2) implies that the last group in (3.2) is isomorphic to the homology group $H_m(Z_{n+1}, R[X_n])$. The Z_{n+1}-module structure on the free R-module $R[X_n]$ is induced by the Z_{n+1}-action on X_n accompanied by the sign $(-1)^n$.

The naturality of the two isomorphisms which we used above to identify $E^1_{n,m}$ with group homology implies that the spectral sequence of the filtration $\{G'_n\}_{n\geq 0}$ coincides with the spectral sequence from [8], Theorem 6.9, which converges to $HC_*(R[X_*])$. ■

We conclude the paper with some remarks on the spectral sequences induced by the coend filtration (2.9) in S^1-equivariant Bredon homology and $RO(S^1)$-graded cohomology.

For the definition and basic properties of the Bredon homology groups $H^G_*(X, A; \underline{M})$, where G is a topological group, (X, A) is a G-CW pair, and $\underline{M} : hO(G) \to \mathrm{Ab}$ is a functor with values in abelian groups, we refer the reader to [18] and [5]. Here $hO(G)$ denotes the homotopy category of the orbit category of G. Recall that on orbits one has

$$H^G_*(G/H; \underline{M}) = \begin{cases} \underline{M}(G/H) & \text{if } * = 0, \\ 0 & \text{otherwise,} \end{cases} \tag{3.3}$$

and that the Bredon homology $H_*^G(-;\underline{M})$ on the category of finite G-CW complexes is uniquely determined by its coefficient system $\underline{M}$.

PROPOSITION 2. *If X_* is a cyclic set and $\underline{M} : hO(S^1) \to \mathrm{Ab}$ is a coefficient system, then there exists a spectral sequence which converges to $H_*^{S^1}(|X_*|;\underline{M})$ and whose E^1 term is*

$$E^1_{n,m} = H^{Z_{n+1}}_{n+m}(X_n \times \Delta^n, X_n \times \partial\Delta^n; \underline{M}_n),$$

where $\underline{M}_n(-) = \underline{M}(S^1 \times_{Z_{n+1}} (-))$ is the restriction of $\underline{M}$ to orbits of Z_{n+1}.

Proof. The filtration (2.9) induces a spectral sequence in Bredon homology which converges to $H_*^{S^1}(|EX_*|;\underline{M}) = H_*^{S^1}(|X|;\underline{M})$ (cf. Proposition 1). The E^1 term of the sequence is

$$E^1_{n,m} = H^{S^1}_{n+m}(X \times_{Z_{n+1}} (S^1 \times \Delta^n), X \times_{Z_{n+1}} (S^1 \times \partial\Delta^n); \underline{M}), \tag{3.4}$$

where Z_{n+1} acts on $S^1 \times \Delta^n$ by the formula (2.3). Since by (3.3) the functors $H_*^{S^1}(S^1 \times_{Z_{n+1}} (-);\underline{M})$ and $H_*^{Z_{n+1}}(-;\underline{M}_n)$ have the same effect on orbits of Z_{n+1}, the uniqueness of Bredon homology implies that the group (3.4) is isomorphic to $H^{Z_{n+1}}_{n+m}(X_n \times \Delta^n, X_n \times \partial\Delta^n; \underline{M}_n)$. ■

Remark 1. The spectral sequence from Proposition 2 can be treated as a cyclic version of a sequence constructed by G. Segal for a simplicial space in any (co)homology theory (cf. [15], Proposition 5.1). In our case, however, the first differential

$$\begin{aligned} d^1_{n,m} : H^{Z_{n+1}}_{n+m}&(X_n \times \Delta^n, X_n \times \partial\Delta^n; \underline{M}_n) \\ &\to H^{Z_n}_{n+m-1}(X_{n-1} \times \Delta^{n-1}, X_{n-1} \times \partial\Delta^{n-1}; \underline{M}_{n-1}) \end{aligned}$$

is more difficult to handle than its simplicial counterpart. Its complexity is caused, in part, by the fact that the Bredon homology groups $H_k^{Z_{n+1}}(\Delta^n, \partial\Delta^n;\underline{M}_n)$ have torsion for many $k < n$. To see this one identifies the Z_{n+1}-space $\Delta^n/\partial\Delta^n$ with the one-point compactification S^{V_n} of the reduced, regular representation V_n of Z_{n+1}. If $\underline{M}$ is the constant coefficient system with value Z, then

$$H_k^{Z_{n+1}}(\Delta^n, \partial\Delta^n; \underline{M}_n) = \widetilde{H}_k(S^{V_n}/Z_{n+1}; Z).$$

Now, one can use Kawasaki's calculation of the integral (co)homology of generalized lens spaces (cf. [10]) to find that the homology of the orbit space S^{V_n}/Z_{n+1} has torsion in half of the dimensions. For more general coefficient systems one can use the results of Słomińska on Bredon cohomology of spheres in representations (cf. [17]).

Remark 2. Let $\underline{H}_G^*(X)$ (G a compact Lie group) be the $RO(G)$-graded cohomology of a based G-space X with coefficients in the Burnside ring functor (cf. [12]). For a cyclic set X_*, the coend filtration (2.9) induces a spectral sequence in the $RO(S^1)$-graded cohomology whose E^1 term is of the form

$$\underline{H}^{\alpha}_{Z_{n+1}}(X_n^+ \wedge S^{V_n}) = \underline{H}^{\alpha - V_n}_{Z_{n+1}}(X_n^+),$$

where + denotes a disjoint base point and α is a representation of Z_{n+1}. The spectral sequence reduces calculation of $\underline{H}^*_{S^1}(|X_*|)$ to $RO(Z_{n+1})$-graded cohomology of the Z_{n+1}-set X_n, which seems to be accessible (cf. [11], Section 2, where $\underline{H}^{\alpha}_{Z_p}(S^0)$ and $\underline{H}^{\alpha}_{Z_p}(Z_p^+)$ are given, for a prime number p).

We will elaborate on the last two remarks in a forthcoming paper.

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