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# On absolute retracts of $\omega^*$

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**Abstract.** An extremally disconnected space is called an absolute retract in the class of all extremally disconnected spaces if it is a retract of any extremally disconnected compact space in which it can be embedded. The Gleason spaces over dyadic spaces have this property. The main result of this paper says that if a space $X$ of $\pi$-weight $\omega_1$ is an absolute retract in the class of all extremally disconnected compact spaces and $X$ is homogeneous with respect to $\pi$-weight (i.e. all non-empty open sets have the same $\pi$-weight), then $X$ is homeomorphic to the Gleason space over the Cantor cube $\{0,1\}^{\omega_1}$.

**Introduction.** As usual, a subset $Y$ of a space $X$ is a *retract* of $Y$ whenever there exists a continuous mapping $r : X \to Y$ such that $r|Y$ is the identity. $X$ is an *absolute retract* of $Y$, briefly $X \in \mathrm{AR}(Y)$, if $X$ can be embedded in $Y$ and every subset of $Y$ which is homeomorphic to $X$ is also a retract of $Y$.

Recall that a space $X$ is *extremally disconnected* if the closure of every open subset of $X$ is open. If $Y$ is a retract of an extremally disconnected space $X$, then $Y$ is extremally disconnected as well. On the other hand, the well known Balcar–Franek Theorem [1] implies that if $X$ and $Y$ are compact extremally disconnected spaces and $w(X) \le w(Y)$, then $X$ can be embedded in $Y$. The above remarks motivate the following definition: a compact space $X$ is called an *absolute retract in the class of extremally disconnected compact spaces*, briefly $X \in \mathrm{AR}(\text{e.d.})$, whenever $X \in \mathrm{AR}(Y)$ for any compact extremally disconnected space $Y$ such that $w(X) \le w(Y)$.

Here we will be mainly interested in the space $\omega^*$ (the remainder of the Čech–Stone compactification of a countable discrete space or, equivalently, the Stone space of the Boolean algebra $\mathcal{P}(\omega)/\mathrm{Fin}$). A relevant fact is that every extremally disconnected separable compact space can be embedded as a retract in $\omega^*$ (see Lemma 2.2 for $\kappa = \omega$).

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On the other hand, every absolute retract $X$ of $\omega^*$ is separable and hence even extremally disconnected. Indeed, $X$ is a subset of $\beta\omega$, and the latter is homeomorphic to a subset of $\omega^*$.

The previous considerations naturally suggest the question whether every separable extremally disconnected compact space is an absolute retract of $\omega^*$. Originally this question was formulated by D. Maharam [6] in connection with her investigations of lifting.

The negative answer was first given, under (CH), by M. Talagrand [14] and then by A. Szymański [13] under the assumption of Martin's Axiom. The first examples in ZFC were constructed by P. Simon [12] and next by L. Shapiro [9].

The aim of this paper is to investigate the nature of absolute retracts of $\omega^*$. It turns out that the absolute retracts of $\omega^*$ which are of $\pi$-weight not greater than $\omega_1$ have a nice structure, being essentially the Gleason spaces of some Cantor cube.

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**1. Irreducible and semi-open mappings.** All the mappings considered here are assumed to be continuous and all spaces are compact Hausdorff and 0-dimensional. A mapping $f : X \to Y$ is called *irreducible* provided that $F = X$ for any closed set $F \subset X$ such that $f(F) = f(X)$. It is easy to show that for every $f : X \to Y$ there exists a closed set $F \subset X$ such that $f|F$ is irreducible. By Zorn's lemma, it is enough to consider a minimal closed set in $X$ which is mapped onto the whole $f(X)$.

A surjection $f : X \to Y$ is called *semi-open* if $\operatorname{Int} f(U) \neq \emptyset$ for every nonempty open set $U \subset X$. All irreducible surjections are semi-open. Indeed, if $\operatorname{Int} f(U) = \emptyset$ and $U \subset X$ is open, then $f(X - U) = Y$ (recall that all mappings are closed since they are continuous and all spaces are compact).

A closed set $F \subset X$ is called *regular-closed* whenever $F = \operatorname{cl}\operatorname{Int} F$. Clearly, all regular-closed subsets of an extremally disconnected space are clopen. One can easily prove the following lemmas:

LEMMA 1.1. *A (continuous) surjection $f : X \to Y$ is semi-open iff for every clopen set $U \subset X$, $f(U)$ is regular-closed in $Y$.*

LEMMA 1.2. *Every semi-open surjection onto an extremally disconnected space is open.*

The next lemma follows immediately from the above one and from the fact that irreducible open mappings are necessarily one-to-one.

LEMMA 1.3 (Mioduszewski–Rudolf [7]; Porter–Woods [8]). *Every irreducible surjection onto an extremally disconnected space is a homeomorphism.*

LEMMA 1.4. *Assume $f : X \to Y$ and $g : Y \to Z$ are surjections such that $g \circ f$ is semi-open. Then $g$ is semi-open as well. Moreover, if $g$ is irreducible then also $f$ is semi-open.*

The next lemma presents the most important property of irreducible and semi-open mappings.

LEMMA 1.5 (Mioduszewski–Rudolf [7]). *If $f, g : X \to Y$ are semi-open and $h : Y \to Z$ is irreducible and $h \circ f = h \circ g$, then $f = g$.*

COROLLARY 1.6. *If $h : X \to Y$ is irreducible and $g : X \to X$ is such that $h \circ g = h$, then $g$ is the identity.*

This follows immediately from Lemma 1.5 since, by Lemma 1.4, $g$ is semi-open.

A very important role in our considerations is played by the well known Gleason Theorem.

THEOREM 1.7 (Gleason [4]). *If $Z$ is extremally disconnected, then for every $f : Z \to Y$ and every surjection $g : X \to Y$ there exists $h : Z \to X$ such that $f = g \circ h$.*

The *Gleason space* over $X$ is the Stone space $G(X)$ of the Boolean algebra of all regular-open subsets of $X$; a set $U \subset X$ is *regular-open* if $X - U$ is regular-closed or, equivalently, $U = \operatorname{Int} \operatorname{cl} U$. The space $G(X)$ is compact extremally disconnected and admits a canonical *Gleason projection* $G_X : G(X) \to X$ which is an irreducible surjection; see e.g. Comfort and Negrepontis [2] or Porter and Woods [8]. It is easy to see that by Gleason's Theorem and Lemma 1.3, $G(X)$ is unique up to homeomorphism. It is enough to observe that if a composition $f \circ g$ is irreducible then both $f$ and $g$ are. A similar argument leads to the conclusion that for any $f : X \to Y$ there exists $Gf : G(X) \to G(Y)$ such that

$$G_Y \circ Gf = f \circ G_X .$$

Moreover, if $f$ is a semi-open surjection, then $Gf$ is an open surjection and it is unique. Again, we use Gleason's Theorem for the existence of $Gf$. Next we use Lemma 1.4, Lemma 1.2 and, for the uniqueness, Lemma 1.5.

LEMMA 1.8. *If $U \subset G(X)$ is a clopen set and $V \subset X$ is open and such that $\operatorname{cl} V = G_X(U)$, then $U = \operatorname{cl} G_X^{-1}(V)$.*

Proof. First observe that $G_X^{-1}(V) \subset U$. Indeed, in the other case there exists a clopen non-empty set $H \subset G_X^{-1}(V)$ such that $H \cap U = \emptyset$. Then

$$G_X(H) \subset V \subset G_X(U) \subset G_X(G(X) - H) ,$$

which contradicts the irreducibility of $G_X$. Since $G_X^{-1}(V) \subset U$ and $U$ is clopen, $\mathrm{cl}\, G_X^{-1}(V) \subset U$. Clearly, $\mathrm{cl}\, G_X^{-1}(V)$ is clopen and $G_X(\mathrm{cl}\, G_X^{-1}(V)) = G_X(U)$ because $\mathrm{cl}\, V = G_X(U)$. Thus, using the irreducibility of $G_X$ again, we get the required conclusion.

**2. Absolute retracts of $\beta\kappa$.** Recall that all spaces are assumed to be compact and 0-dimensional.

It is well known that if $X$ is homeomorphic to a retract of the Cantor cube $\{0,1\}^\kappa$, then $X \in \mathrm{AR}(\{0,1\}^\kappa)$; here $\kappa$ stands for an infinite cardinal. We shall show that if we replace the Cantor cube by $\beta\kappa$ (= the Čech–Stone compactification of a discrete space of power $\kappa$), then the situation is quite different. To do this, we will need some lemmas.

LEMMA 2.1. *If a compact space $X$ has a* (*continuous*) *mapping onto an extremally disconnected space $Y$, then $Y$ can be embedded as a retract in $X$.*

Proof. Assume $f : X \to Y$ is a surjection. There exists a closed set $Z \subset X$ such that $f|Z$ is irreducible. By Lemma 1.3, $f|Z$ is a homeomorphism and thus $(f|Z)^{-1} \circ f$ is the required retraction.

Recall that for a cardinal $\kappa \geq \omega$, $U(\kappa)$ denotes the space of all uniform ultrafilters over $\kappa$, i.e. $U(\kappa)$ consists of all ultrafilters $p \in \beta\kappa$ such that $|A| = \kappa$ for every $A \in p$. The topology on $U(\kappa)$ is inherited from $\beta\kappa$. Clearly, $U(\omega) = \omega^*$.

LEMMA 2.2. *If $X$ is compact extremally disconnected and $d(X) = \kappa \geq \omega$, then $X$ can be embedded as a retract in both $\beta\kappa$ and $U(\kappa)$.*

Proof. By the Lemma 2.1, it is enough to show that there exists $f : \beta\kappa \to X$ such that $f(U(\kappa)) = X$. To do this, note that there exists $\varphi : \kappa \to X$ such that $\varphi(\kappa)$ is dense in $X$ and $|\varphi^{-1}(x)| = \kappa$ for every $x \in \varphi(\kappa)$. Then the extension of $\varphi$ to $\beta\kappa$ is the desired mapping.

The lemma above implies, in particular, that every separable extremally disconnected compact space can be embedded as a retract in $\beta\omega$ (and also in $\omega^*$). However, as was mentioned above, there are separable extremally disconnected compact spaces that are not absolute retracts of $\beta\omega$. Also, by Lemma 2.1, every extremally disconnected compact space $X$ can be embedded in the Gleason space over $\{0,1\}^\tau$, where $\tau$ is the weight of $X$. In fact, since $X$ can be embedded in $\{0,1\}^\tau$, some closed subspace of $G(\{0,1\}^\tau)$ has a (continuous) mapping onto $X$.

THEOREM 2.3. *For every extremally disconnected compact space $X$ the following conditions are equivalent*:

(a) $X \in \mathrm{AR}(e.d.)$,
(b) $X \in \mathrm{AR}(G(\{0,1\}^\tau))$, *where* $\tau = w(X)$,

(c) $X \in \mathrm{AR}(U(\kappa))$, *where* $\kappa = d(X)$,
(d) $X \in \mathrm{AR}(\beta\kappa)$, *where* $\kappa = d(X)$.

Proof. (a)⇒(d) is obvious.

(d)⇒(c) follows easily from the fact that $U(\kappa)$ is a subspace of $\beta\kappa$. Thus for every $X \subset U(\kappa)$, the restriction to $U(\kappa)$ of a retraction from $\beta\kappa$ onto $X$ is the desired retraction.

(c)⇒(d). It is enough to prove that $G(\{0,1\}^\tau)$ can be embedded in $U(\kappa)$, where $\tau = w(X)$ and $\kappa = d(X)$. Since the Gleason space over $\{0,1\}^{2^\kappa}$ has a dense subset of size $\kappa$ and $\tau \le 2^\kappa$, there exists a (continuous) mapping from $U(\kappa)$ onto $G(\{0,1\}^\tau)$. Thus we can use Lemma 2.1.

(b)⇒(a). Assume $X$ is an extremally disconnected compact space of weight $\kappa$. Since every extremally disconnected compact space can be embedded in the Gleason space over a Cantor cube, it is enough to show that for every embedding of $X$ in $G(\{0,1\}^\tau)$ for some cardinal $\tau$, $X$ is a retract of $G(\{0,1\}^\tau)$. Without loss of generality we can assume that $X \subset G(\{0,1\}^\tau)$. Since $w(X) = \kappa$, there exists a family $\mathcal{R}$ of clopen subsets of $G(\{0,1\}^\tau)$ such that $\{X \cap U : U \in \mathcal{R}\}$ is a base in $X$. On the other hand, since the canonical Gleason mapping $\widetilde{G} : G(\{0,1\}^\tau) \to \{0,1\}^\tau$ is irreducible, $\widetilde{G}(U)$ is regular-closed for every $U \in \mathcal{R}$ (see Lemma 1.1). Since the Cantor cube has the Suslin property, there exists a countable family of clopen subsets of $\{0,1\}^\tau$ whose union is dense in $\widetilde{G}(U)$. Hence there exists a countable set $A(U) \subset \tau$ and an open set $V \subset \{0,1\}^{A(U)}$ such that $\widetilde{G}(U) = \mathrm{cl}\,\pi^{-1}_{A(U)}(V)$, where $\pi_{A(U)}$ is the canonical projection from $\{0,1\}^\tau$ onto $\{0,1\}^{A(U)}$. Let $\mathcal{A} = \bigcup\{A(U) : U \in \mathcal{R}\}$ and let $\pi : \{0,1\}^\tau \to \{0,1\}^{\mathcal{A}}$ be the projection. Then there exists a family $\mathcal{P}$ of open subsets of $\{0,1\}^{\mathcal{A}}$ such that for every $U \in \mathcal{R}$ there exists $V \in \mathcal{P}$ such that

$$\widetilde{G}(U) = \mathrm{cl}\,\pi^{-1}_{\mathcal{A}}(V),$$

where cl denotes closure in $\{0,1\}^\tau$. Hence, by Lemma 1.8, for every $U \in \mathcal{R}$ there exists $V \in \mathcal{P}$ such that

$$(*) \qquad U = \mathrm{cl}\,\widetilde{G}^{-1}(\pi^{-1}_{\mathcal{A}}(V)).$$

Consider the Gleason space $G(\{0,1\}^{\mathcal{A}})$ and the canonical projection $\overline{G} : G(\{0,1\}^{\mathcal{A}}) \to \{0,1\}^{\mathcal{A}}$. By Gleason's Theorem there exists a mapping $\varphi : G(\{0,1\}^\tau) \to G(\{0,1\}^{\mathcal{A}})$ such that

$$\pi_{\mathcal{A}} \circ \widetilde{G} = \overline{G} \circ \varphi.$$

By $(*)$, for every $U \in \mathcal{R}$ there exists $V \in \mathcal{P}$ such that

$$(**) \qquad U = \mathrm{cl}\,\varphi^{-1}(\overline{G}^{-1}(V)).$$

We claim that $\varphi|X$ is one-to-one. In fact, if $x, y \in X$ and $x \ne y$, then there exist $U, U' \in \mathcal{R}$ such that $x \in U$, $y \in U'$ and $U \cap U' = \emptyset$.

By $(**)$, we can find some $V, V' \in \mathcal{P}$ such that $U = \operatorname{cl}\varphi^{-1}(\overline{G}^{-1}(V))$ and $U' = \operatorname{cl}\varphi^{-1}(\overline{G}^{-1}(V'))$. Clearly $V \cap V' = \emptyset$, $\varphi(x) \in \operatorname{cl}\overline{G}^{-1}(V)$ and $\varphi(y) \in \operatorname{cl}\overline{G}^{-1}(V')$. Thus $\varphi(x) \neq \varphi(y)$.

Since $|\mathcal{A}| = \kappa$, by (b), there exists a retraction $r : \{0,1\}^{\mathcal{A}} \to \varphi(X)$. Hence $(\varphi|X)^{-1} \circ r \circ \varphi$ is the required retraction from $G(\{0,1\}^\tau)$ onto $X$.

A special case of the next theorem was obtained by Shapiro [9].

THEOREM 2.4. *If $X$ is a dyadic space, then $G(X) \in \mathrm{AR}(e.d.)$.*

Proof. Assume $Y$ is extremally disconnected compact and $G(X) \subset Y$. Since $X$ is dyadic, there exists a surjection $f : \{0,1\}^\tau \to X$, where $\tau = w(X)$. Let $\widetilde{G} : G(X) \to X$ be the canonical Gleason mapping. By Gleason's Theorem there exists a surjection $g : G(X) \to \{0,1\}^\tau$ such that

$$f \circ g = \widetilde{G}\,.$$

Using Gleason's Theorem again we obtain $\varphi : Y \to G(X)$ such that

$$\widetilde{G} \circ \varphi = f \circ h\,.$$

Then

$$\widetilde{G} \circ (\varphi|G(X)) = f \circ (h|G(X)) = f \circ g = \widetilde{G}\,,$$

and by Lemmas 1.4 and 1.5 we deduce that $\varphi|G(X)$ is the identity; the proof is complete.

**3. Inverse limits versus absolute retracts.** In the sequel we shall use inverse limits of 0-dimensional compact spaces over well ordered sets of indices. An *inverse system* is a system $S = \{X_\alpha, p^\alpha_\beta;\ \beta < \alpha < \tau\}$ such that $p^\beta_\alpha \circ p^\gamma_\beta = p^\gamma_\alpha$ whenever $\alpha < \beta < \gamma < \tau$, $\tau$ is an ordinal, $X_\alpha$ are 0-dimensional compact spaces and $p^\beta_\alpha$ are continuous surjections for all $\alpha, \beta \in \tau$. The *limit* of the inverse system, denoted by $\varprojlim S$ or $\varprojlim\{X_\alpha, p^\alpha_\beta; \beta < \alpha < \tau\}$ consists of all points $(x_\alpha)_{\alpha<\tau}$ of the product of $X_\alpha$'s such that $x_\beta = p^\alpha_\beta(x_\alpha)$ for any $\beta < \alpha < \tau$. For every $\alpha < \tau$, $p_\alpha : \varprojlim S \to X_\alpha$ is the canonical projection, i.e. $p_\alpha((x_\alpha)_{\alpha<\tau}) = x_\alpha$. All the projections are continuous surjections. If a set $\Sigma \subset \tau$ is unbounded in $\tau$ (i.e. for every $\alpha < \tau$ there exists $\beta \in \Sigma$ such that $\alpha \leq \beta$), then the family $\{p_\alpha^{-1}(U) : \alpha \in \Sigma$ and $U$ is open in $X_\alpha\}$ is a base in $\varprojlim S$. Clearly, $\varprojlim S$ is a 0-dimensional compact space. The mappings $p^\alpha_\beta$ are usually called *bonding mappings* or *connecting mappings*. All the inverse systems considered here are assumed to be *continuous*, i.e. on every limit step the space in the system is the inverse limit of the preceding spaces and bonding mappings. An outline of the theory of inverse systems can be found in the book of Engelking [3].

LEMMA 3.1. *Assume $S = \{X_\alpha, p^\alpha_\beta; \beta < \alpha < \tau\}$ and $T = \{Y_\alpha, q^\alpha_\beta; \beta < \alpha < \tau\}$ are inverse systems and $\Sigma \subset \tau$ is an unbounded set for which there* *is an order preserving function* $\varphi : \Sigma \to \tau$ *such that* $\varphi(\Sigma)$ *is unbounded in* $\tau$. *If for every* $\alpha \in \Sigma$ *there exists a surjection* $h_\alpha : X_\alpha \to Y_{\varphi(\alpha)}$ *such that*

$$\alpha < \beta < \tau \quad \textit{implies} \quad h_\alpha \circ p^\beta_\alpha = q^{\varphi(\beta)}_{\varphi(\alpha)} \circ h_\beta ,$$

*then there exists a unique* (*continuous*) *surjection* $h : \varprojlim S \to \varprojlim T$ *such that for every* $\alpha < \tau$,

$$(*) \qquad h_\alpha \circ p_\alpha = q_{\varphi(\alpha)} \circ h ,$$

*where* $p_\alpha$ *and* $q_\alpha$ *are the canonical projections from* $\varprojlim S$ *and* $\varprojlim T$, *respectively. Moreover*, *if all the mappings* $h_\alpha$, *for* $\alpha \in \Sigma$, *are irreducible*, *then so is* $h$.

P r o o f. The first part of the lemma is well known. The mapping $h$ is uniquely determined by $(*)$: for the $\beta$th coordinate of $h((x_\alpha)_{\alpha<\tau})$ we take $q^{\varphi(\alpha)}_\beta(h_\alpha(x_\alpha))$, where $\alpha$ is so large that $\beta < \varphi(\alpha)$; see e.g. [3]. For the proof of the second part of the lemma choose a closed set $F \subset \varprojlim S$ such that $F \neq \varprojlim S$. Then there exists $\alpha \in \Sigma$ and a non-empty open set $U \subset X_\alpha$ such that $F \subset p^{-1}_\alpha(X_\alpha - U)$. Suppose $h(F) = \varprojlim T$. Then

$$X_{\varphi(\alpha)} = q_{\varphi(\alpha)}(h(F)) = h_\alpha(p_\alpha(F)) \subset h_\alpha(X_\alpha - U) ,$$

and we get a contradiction since $h_\alpha$ is irreducible.

The next lemma, due to Shchepin [11], is in fact a converse of the previous one.

LEMMA 3.2 (Shchepin). *Assume* $X = \varprojlim\{X_\alpha, p^\alpha_\beta;\ \beta < \alpha < \tau\}$ *and* $Y = \varprojlim\{Y_\alpha, q^\alpha_\beta; \beta < \alpha < \tau\}$, *where* $\tau$ *is an uncountable regular cardinal and* $w(Y_\alpha) < \tau$ *for every* $\alpha < \tau$. *Then for every mapping* $f : X \to Y$ *there exists a closed unbounded set* $\Sigma \subset \tau$ *such that for every* $\alpha \in \Sigma$ *there is a mapping* $f_\alpha : X_\alpha \to Y_\alpha$ *with*

$$f_\alpha \circ p_\alpha = q_\alpha \circ f .$$

LEMMA 3.3. *Assume* $S = \{X_\alpha, p^\alpha_\beta; \beta < \alpha < \tau\}$ *is an inverse system such that the cofinality of* $\tau$ *is greater than the Suslin number of* $\varprojlim S$. *If there exists an unbounded set* $\Sigma \subset \tau$ *such that* $X_\alpha$ *is extremally disconnected for all* $\alpha \in \Sigma$, *then* $\varprojlim S$ *is extremally disconnected.*

P r o o f. Let $U$ and $V$ be disjoint open subsets of $\varprojlim S$. Since the family $B = \{p^{-1}_\alpha(W) : \alpha \in \Sigma$ and $W$ is open in $X_\alpha\}$ is a base in $\varprojlim S$, there exist families $P, R \subset B$ consisting of disjoint sets such that $\bigcup P$ is dense in $U$ and $\bigcup R$ is dense in $V$. Then there exists some $\delta \in \Sigma$ such that $P, R \subset \{p^{-1}_\delta(W) : W$ is open in $X_\delta\}$, because the power of both $P$ and

$R$ does not exceed the cofinality of $\tau$. Since $X_\delta$ is extremally disconnected, $\mathrm{cl}(\bigcup R) \cap \mathrm{cl}(\bigcup P) = \emptyset$. Hence $\mathrm{cl}\, U \cap \mathrm{cl}\, V = \emptyset$; the proof is complete.

Now we are ready to prove the main theorem:

THEOREM 3.4. *Assume $X \in \mathrm{AR}(e.d.)$ and the $\pi$-weight of $X$ is an uncountable regular cardinal. Then $X$ is homeomorphic to the Gleason space over $\varprojlim\{X_\alpha, p^\alpha_\beta; \beta < \alpha < \kappa\}$, where $\kappa = \pi w(X)$, all the connecting mappings are semi-open and each $X_\alpha$ is compact 0-dimensional of weight less than $\kappa$.*

Proof. First note that $\{0,1\}^\kappa$ can be represented as the limit of the inverse system $S = \{\{0,1\}^\alpha, \pi^\beta_\alpha;\ \alpha \le \beta < \kappa\}$, where $\{0,1\}^\alpha$ is endowed with the product topology and the connecting mappings $\pi^\beta_\alpha$ are just the projections, i.e. $\pi^\beta_\alpha(x) = x|\alpha$ for all $x \in \{0,1\}^\beta$ and all $\alpha < \beta$. Clearly, the system $S$ is continuous and all the connecting mappings are open. By induction we will construct a continuous *Gleason system* over $S$, i.e. a continuous inverse system $G(S) = \{Z_\alpha, p^\beta_\alpha; \alpha \le \beta < \kappa\}$ together with a family $\{g_\alpha : \alpha < \kappa\}$ of irreducible mappings such that:

(1) $Z_0$ consists of a single point and $g_0 : Z_0 \to \{0,1\}^0 = \{\emptyset\}$ is constant,

(2) if $\alpha < \kappa$ is a limit ordinal then $Z_\alpha = \varprojlim\{Z_\beta, p^\delta_\gamma; \gamma < \delta < \alpha\}$, $p^\alpha_\beta$ are the projections from the inverse limit $Z_\alpha$ onto $Z_\beta$, for all $\beta < \alpha$, and $g_\alpha$ is the unique function from $Z_\alpha$ onto $\{0,1\}^\alpha$ induced by $\{g_\beta : \beta < \alpha\}$, i.e. $g_\beta \circ p^\alpha_\beta = \pi^\alpha_\beta \circ g_\alpha$ for all $\beta < \alpha$,

(3) if $\alpha = \beta + 1$, where $\beta$ is a limit ordinal, then $Z_\alpha = G(\{0,1\}^\beta)$, $g_\alpha$ is the Gleason projection from $G(\{0,1\}^\beta)$ onto $\{0,1\}^\beta$ and $p^\alpha_\beta$ is the unique mapping such that $g_\beta \circ p^\alpha_\beta = g_\alpha$,

(4) if $\alpha = \beta + 1$ and $\beta$ is a successor ordinal, then we set $Z_\alpha = Z_\beta \times \{0,1\}$ and $p^\alpha_\beta(x,i) = x$ for all $(x,i) \in Z_\alpha$, and we define $g_\alpha : Z_\alpha \to \{0,1\}^\beta = \{0,1\}^\gamma \times \{0,1\}$, where $\gamma + 1 = \beta$, by $g_\alpha(x,i) = (g_\beta(x), i)$.

Note that $p^\alpha_\beta$ in (3) exists by Gleason's Theorem and is irreducible since $g_\alpha$ is. Thus, if $\beta$ is a limit ordinal, then $Z_{\beta+1}$ is in fact the Gleason space over $Z_\beta$. However, the mapping $g_\alpha$ in (2) is irreducible by Lemma 3.1. Concerning (4), observe that $g_{\alpha+1}$ is irreducible since $g_\alpha$ is irreducible and $p^{\alpha+1}_\alpha$ is a surjection.

Now consider the space $Z = \varprojlim G(S)$. By (1)–(4) and Lemma 3.1, there exists an irreducible mapping $g : Z \to \{0,1\}^\kappa$ such that for any $\alpha < \kappa$ we have

$$\pi_\alpha \circ g = g_{\alpha+1} \circ p_{\alpha+1}\,, \tag{5}$$

where $p_{\alpha+1}$ is the canonical projection from $Z$ onto $Z_{\alpha+1}$; see the diagram below, where $\alpha$ is a limit ordinal less than $\kappa$.

$$\begin{tikzcd}
 & G(\{0,1\}^\alpha) \arrow[d, leftrightarrow] & & \\
Z_\alpha \arrow[d, "g_\alpha"] & Z_{\alpha+1} \arrow[l] \arrow[dl, "g_{\alpha+1}"] & Z_{\alpha+2} = Z_{\alpha+1} \times \{0,1\} \arrow[l] \arrow[d, "g_{\alpha+2}"] & Z \arrow[l, "p_{\alpha+2}"'] \arrow[d, "g"] \\
\{0,1\}^\alpha & & \{0,1\}^{\alpha+1} = \{0,1\}^\alpha \times \{0,1\} \arrow[ll] & \{0,1\}^\kappa \arrow[l, "\pi_{\alpha+1}"']
\end{tikzcd}$$

Since $g$ is irreducible and $\{0,1\}^\kappa$ is ccc, $Z$ is also ccc. Hence, by Lemma 3.3, $Z$ is extremally disconnected (since $\mathrm{cf}(\kappa) > \omega$). Thus $Z$ is homeomorphic to $G(\{0,1\}^\kappa)$.

Assume $\widetilde{X} \in \mathrm{AR}(\text{e.d.})$ and $\pi w(\widetilde{X}) = \kappa = \mathrm{cf}(\kappa) > \omega$. Then there exists a 0-dimensional compact space $X$ such that $w(X) = \kappa$ and $\widetilde{X}$ is homeomorphic to the Gleason space of $X$. Indeed, one can set $X$ to be the Stone space of the Boolean algebra generated by a family of clopen subsets of $\widetilde{X}$ which is a $\pi$-base of power $\kappa$. Since $w(X) \le \kappa$, we can assume that $X \subset \{0,1\}^\kappa$. For every $\alpha < \kappa$ we set $X_\alpha = \pi_\alpha(X)$ and for every $\alpha \le \beta < \kappa^+$ we define $s_\alpha^\beta = \pi_\alpha^\beta | X_\beta$. Then $X_\alpha \subset \{0,1\}^\alpha$ for every $\alpha < \kappa$ and

$$X = \varprojlim \{X_\alpha, s_\alpha^\beta; \alpha \le \beta < \kappa\}\,.$$

By transfinite induction we define a sequence $\{Y_\alpha : \alpha < \kappa\}$ such that for any $\alpha < \kappa$ the following conditions hold:

(6) $Y_\alpha$ is a closed subset of $Z_\alpha$,

(7) for every limit ordinal $\alpha < \kappa$, $g_\alpha(Y_\alpha) = X_\alpha$ and $h_\alpha = g_\alpha | Y_\alpha$ is irreducible,

(8) for every successor ordinal $\alpha = \beta + 1$, $g_\alpha(Y_\alpha) = X_\beta$ and $h_\alpha = g_\alpha | Y_\alpha$ is irreducible,

(9) for every $\beta < \alpha$, $p_\beta^\alpha(Y_\alpha) = Y_\beta$.

Assume that $Y_\beta$ is defined for every $\beta < \alpha$.

If $\alpha$ is a limit ordinal we set $Y_\alpha = \varprojlim \{Y_\beta, p_\beta^\gamma | X_\gamma;\ \beta < \gamma < \alpha\}$. Since $Z_\alpha$ is the inverse limit of $Z_\beta$'s for $\beta < \alpha$, $Y_\alpha \subset Z_\alpha$ and $g_\alpha(Y_\alpha) = X_\alpha$, it follows by Lemma 3.1 that $h_\alpha = g_\alpha | Y_\alpha$ is irreducible.

If $\alpha = \beta + 1$ and $\beta$ is a limit ordinal we choose a closed set $Y_\alpha \subset (p_\beta^{\beta+1})^{-1}(Y_\beta)$ such that $p_\beta^{\beta+1}(Y_\alpha) = Y_\beta$ and $p_\beta^{\beta+1} | Y_\alpha$ is irreducible. Hence, by (3), $g_\alpha | Y_\alpha$ is irreducible as a composition of irreducible mappings.

If $\alpha = \beta + 1$ and $\beta$ is a successor ordinal then, by (4), $g_\alpha$ is a map from $Z_\alpha$ onto $\{0,1\}^\gamma \times \{0,1\}$, where $\gamma + 1 = \beta$. Since $X_\beta$ is closed in $\{0,1\}^\gamma \times \{0,1\}$, the sets $F_i = \pi_\alpha^\beta((\{0,1\}^\gamma \times \{i\}) \cap X_\beta)$ are closed for any $i \in \{0,1\}$ and $F_0 \cup F_1 = X_\gamma$. We choose closed sets $K_0, K_1 \subset Y_\beta$ such that $g_\beta(K_i)=F_i$ and $g_\beta|K_i$ is irreducible for any $i\in\{0,1\}$. Then we set $Y_\alpha=(K_0\times\{0\})\cup(K_1\times\{1\})$. By (4), $g_\alpha|Y_\alpha$ is irreducible and $g_\alpha(Y_\alpha)=X_\alpha$. Clearly, $K_0\cup K_1=Y_\beta$ because $g_\beta|Y_\beta$ is irreducible. Thus $p^\alpha_\beta(Y_\alpha)=Y_\beta$, which completes the construction of $Y_\alpha$'s.

Hence we get an inverse system $\{Y_\alpha,t^\beta_\alpha;\alpha<\beta<\kappa\}$ such that $Y_\alpha$ is a closed subset of $Z_\alpha$ and $t^\beta_\alpha=p^\beta_\alpha|Y_\beta$ for every $\alpha<\beta$. Consider the space

$$Y=\varprojlim\{Y_\alpha,t^\beta_\alpha;\ \alpha<\beta<\kappa\}\,.$$

Clearly, $Y\subset Z$ and $g|Y$ is an irreducible mapping of $Y$ onto $X$. For every limit ordinal $\alpha<\kappa$, $Y_{\alpha+1}$ is a closed subspace of an extremally disconnected space $Z_{\alpha+1}$. Since $h_{\alpha+1}$ is irreducible and $X_\alpha$ is ccc (because $X$ is ccc), $Y_{\alpha+1}$ is ccc as well. Thus $Y_{\alpha+1}$ is extremally disconnected. By Lemma 3.3, $Y$ is also extremally disconnected and so it is homeomorphic to $\widetilde{X}$. Hence there exists a retraction $r:Z\to Y$. By Lemma 3.2 there exists a closed unbounded set $\varSigma\subset\kappa$ such that for every $\alpha\in\varSigma$ there exists $\varphi_\alpha:Z_\alpha\to X_\alpha$ satisfying

$$\varphi_\alpha\circ p_\alpha=s_\alpha\circ(g|Y)\circ r\,; \tag{10}$$

see the diagram below, where $\alpha\in\varSigma$.

$$\begin{tikzcd}
Z_\alpha \arrow[dd, bend right=60, "\varphi_\alpha"'] & Z_{\alpha+1} \arrow[l, "p^{\alpha+1}_\alpha"'] & Z \arrow[l, "p_{\alpha+1}"'] \arrow[d, "\cup\ r"] \\
Y_\alpha \arrow[u, phantom, "\bigcup"] \arrow[d, "h_\alpha"'] & Y_{\alpha+1} \arrow[u, phantom, "\bigcup"] \arrow[l, "t^{\alpha+1}_\alpha"'] \arrow[dl, "h_{\alpha+1}"] & Y \arrow[l, "t_{\alpha+1}"'] \arrow[d, "g|Y"] \\
X_a & X_{\alpha+1} \arrow[l, "s^{\alpha+1}_\alpha"'] & X \arrow[l, "s_{\alpha+1}"']
\end{tikzcd}$$

Without loss of generality we may assume that $\varSigma$ consists of limit ordinals. By Gleason's Theorem and the fact that $Z_{\alpha+1}$ is extremally disconnected, there exists $r_{\alpha+1}:Z_{\alpha+1}\to Y_{\alpha+1}$ such that

$$\varphi_\alpha\circ p^{\alpha+1}_\alpha=h_{\alpha+1}\circ r_{\alpha+1}\,. \tag{11}$$

We assert that $r_{\alpha+1}$ is a retraction. To see this, choose $y\in Y_{\alpha+1}$. There exists $x\in Y$ such that $t_{\alpha+1}(x)=p_{\alpha+1}(x)=y$. By (10) and (11) we get

$$\begin{aligned}h_{\alpha+1}(r_{\alpha+1}(y))&=h_{\alpha+1}(r_{\alpha+1}(p_{\alpha+1}(x)))\\&=\varphi(p^{\alpha+1}_\alpha(p_{\alpha+1}(x)))=\varphi_\alpha(p_\alpha(x))=s_\alpha(g(r(x)))\,.\end{aligned}$$

Thus, since $r$ is a retraction, we have

$$h_{\alpha+1}(r_{\alpha+1}(y))=s_\alpha(g(x))=h_{\alpha+1}(t_{\alpha+1}(x))=h_{\alpha+1}(y)\,.$$

Therefore, $h_{\alpha+1}\circ(r_{\alpha+1}|Y_{\alpha+1})=h_{\alpha+1}$, which implies $r_{\alpha+1}|Y_\alpha=\mathrm{id}$, because $h_{\alpha+1}$ is irreducible (see Corollary 1.6).

We claim that $p_{\alpha+1}$ is an open mapping for every $\alpha \in \Sigma$. Since $Z_{\alpha+1}$ is extremally disconnected, it is enough to show that $p_{\alpha+1}$ is semi-open; see Lemma 1.2. Since $\pi_\alpha$ is open and $g$ is irreducible, it follows by (5) that $g_{\alpha+1} \circ p_{\alpha+1}$ is semi-open. Hence, by Lemma 1.4, $p_{\alpha+1}$ is semi-open.

Now we shall prove that $p_{\alpha+1}|T$, where $T = (p^{\alpha+1})^{-1}(Y_{\alpha+1})$, is an open mapping from $T$ onto $Y_{\alpha+1}$. Indeed, if $U \subset Z$ is open, then

$$p_{\alpha+1}(U \cap T) = p_{\alpha+1}(U \cap (p^{\alpha+1})^{-1}(Y_{\alpha+1})) = p_{\alpha+1}(U) \cap Y_{\alpha+1}$$

is an open subset of $Y_{\alpha+1}$, because $p_{\alpha+1}$ is open. Hence $h_{\alpha+1} \circ (p_{\alpha+1}|T)$ is a semi-open map from $T$ onto $X_\alpha$. On the other hand, by (10), (5), (11), (8) and (3) we have

$$s_\alpha \circ g \circ (r|T) = h_{\alpha+1} \circ (p_{\alpha+1}|T)\,.$$

Hence, by Lemma 1.4, $s_\alpha$ is semi-open for every $\alpha \in \Sigma$. It follows that for any $\alpha, \beta \in \Sigma$ such that $\alpha < \beta$, $s^\beta_\alpha$ is semi-open. Since $\Sigma$ is unbounded in $\kappa$, $X = \varprojlim\{X_\alpha, s^\beta_\alpha; \alpha < \beta \text{ and } \alpha, \beta \in \Sigma\}$, where all the connecting mappings $s^\beta_\alpha$ are semi-open and for every $\alpha < \kappa$, $X_\alpha$ is a 0-dimensional compact space with $w(X_\alpha) < \kappa$. The proof is complete.

THEOREM 3.5. *If $X$ is dense in itself and the $\pi$-weight of $X$ is not greater than $\omega_1$, then $X \in \mathrm{AR}(e.d.)$ iff it is homeomorphic either to $G(\{0,1\}^\omega)$ or to $G(\{0,1\}^{\omega_1})$ or to their disjoint union.*

Proof. Assume $X \in \mathrm{AR}(\mathrm{e.d})$. Then, by Theorem 3.4, $X$ is homeomorphic to the Gleason space over $\overline{X} = \varprojlim\{X_\alpha, p^\alpha_\beta; \beta < \alpha < \omega_1\}$, where all $X_\alpha$'s are compact metrizable and 0-dimensional and $p^\alpha_\beta$ are semi-open for all $\beta < \alpha < \omega_1$. Now we shall use the following lemma due to Shapiro [10]: if $f : X' \to Y'$ is a semi-open surjection and $X'$ and $Y'$ are 0-dimensional metrizable compact spaces, then there exists a 0-dimensional metrizable compact space $Z$ and mappings $g : X' \to Z$ and $h : Z \to Y'$ such that $h$ is open, $g$ is irreducible and $f = h \circ g$. Then, by induction, we construct an inverse system $S = \{Y_\alpha, q^\alpha_\beta; \beta < \alpha < \omega_1\}$ such that all the $Y_\alpha$ are 0-dimensional metrizable compact spaces, the $q^\alpha_\beta$ are open surjections and for every $\alpha < \omega_1$ there exists an irreducible surjection $h_\alpha : X_\alpha \to Y_\alpha$ such that $q^\alpha_\beta \circ h_\alpha = h_\alpha \circ p^\alpha_\beta$ whenever $\beta < \alpha < \omega_1$ (see the diagram).

$$\begin{tikzcd}
X_0 \arrow[d] & X_1 \arrow[l] \arrow[d] & \cdots \arrow[l] & X_\alpha \arrow[l] \arrow[d, "h_\alpha"] & X_{\alpha+1} \arrow[l, "p^{\alpha+1}_\alpha"'] \arrow[d, "h_{\alpha+1}"] & \cdots \arrow[l] & \overline{X} \arrow[l] \arrow[d, "h"] \\
Y_0 & Y_1 \arrow[l] & \cdots \arrow[l] & Y_\alpha \arrow[l] & Y_{\alpha+1} \arrow[l, "q^{\alpha+1}_\alpha"'] & \cdots \arrow[l] & Y \arrow[l]
\end{tikzcd}$$

By Haydon's Theorem [5], $Y$ is homeomorphic to a retract of a Cantor cube $\{0,1\}^\alpha$, $\alpha \le \omega_1$. In particular, $Y$ is dyadic. On the other hand $X$ is homeomorphic to $G(Y)$; see §1. Thus, by another theorem of Shapiro [10], $X$ is homeomorphic either to $G(\{0,1\}^\omega)$ or to $G(\{0,1\}^{\omega_1})$ or to their disjoint union.

The converse implication follows directly from Theorem 2.4.

One can easily show that every extremally disconnected compact space is the disjoint union of a dense-in-itself extremally disconnected compact space and the Čech–Stone compactification of a discrete space. Clearly, if $X \in \mathrm{AR}(\mathrm{e.d.})$, then the set of isolated points of $X$ is countable since $X$ is a retract of the Gleason space over a Cantor cube. On the other hand, $\beta\omega \in \mathrm{AR}(\mathrm{e.d.})$ because it is homeomorphic to the Gleason space over the convergent sequence, and hence to the Gleason space over a dyadic space. Thus we can restrict our consideration to the spaces that are dense in itself.

COROLLARY 3.6. *If $X \subset \omega^*$ is dense in itself and the $\pi$-weight of $X$ is not greater than $\omega_1$, then $X \in \mathrm{AR}(\omega^*)$ iff $X$ is homeomorphic either to $G(\{0,1\}^\omega)$ or to $G(\{0,1\}^{\omega_1})$ or to their disjoint union.*

Remark 3.7. Recall that every separable extremally disconnected compact space can be embedded in $\omega^*$. On the other hand, if $X \in \mathrm{AR}(\omega^*)$, then $X \in \mathrm{AR}(\beta\omega)$ and thus $X$ is necessarily separable. However, one can show that not all separable compact subspaces of $\omega^*$ are absolute retracts of $\omega^*$. It suffices to construct a separable compact space $X$ of $\pi$-weight $\omega_1$ such that $G(X)$ is not homeomorphic to the Gleason space of a cube. The examples constructed by Shapiro [9] and Szymański [13] are just of this kind.

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