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Homology lens spaces and Dehn surgery on homology spheres

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Abstract. A homology lens space is a closed 3-manifold with $\mathbb{Z}$-homology groups isomorphic to those of a lens space. A useful theorem found in [Fu] states that a homology lens space M^3 may be obtained by an $(n/1)$-Dehn surgery on a homology 3-sphere if and only if the linking form of M^3 is equivalent to $(1/n)$. In this note we generalize this result to cover all homology lens spaces, and in the process offer an alternative proof based on classical 3-manifold techniques.

1. Introduction. Throughout this paper, all homology is with $\mathbb{Z}$-coefficients. When α is an oriented closed curve in a manifold, we let $[\alpha]$ represent the corresponding homology element. When no confusion can arise, we leave it to the reader to choose an orientation. The following simple fact will make recognition of homology lens spaces and homology 3-spheres especially easy.

Observation 1.1. *Let M^3 be a closed, connected 3-manifold with $H_1(M^3) \cong \mathbb{Z}_n$ $(n \geq 1)$. Then $H_3(M^3) \cong \mathbb{Z}$ (so M^3 is orientable) and $H_2(M^3) = 0$. In particular, if $n \geq 2$ then M^3 is a homology lens space, and if $n = 1$ then M^3 is a homology sphere.*

Proof. Duality implies that every orientable closed 3-manifold has trivial Euler characteristic. Using the orientable double cover, the same must be true for non-orientable closed 3-manifolds. By our hypothesis $\beta_1(M^3) = 0$, so by arithmetic, $\beta_3(M^3) \neq 0$, and so M^3 is orientable. Duality and universal coefficients imply the triviality of $H_2(M^3)$. ∎

If M^3 is a homology lens space and $|H_1(M^3)| = n$, we call M^3 a *homology n-lens space*. We say that M^3 is *$\mathbb{Z}$-homology equivalent to $L(n,m)$* if there is a map $f : M^3 \to L(n,m)$ which induces $\mathbb{Z}$-homology isomorphisms in all dimensions. Since we cannot expect such maps to have "homology inverses",

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we define two homology n-lens spaces to be *$\mathbb{Z}$-homology equivalent* if there is a common lens space to which both are $\mathbb{Z}$-homology equivalent.

OBSERVATION 1.2. *If M^3 is a homology n-lens space, then a map $f : M^3 \to L(n,m)$ induces $\mathbb{Z}$-homology isomorphisms in all dimensions iff* $|\deg(f)| = 1$.

Proof. Suppose $\deg(f) = \pm 1$. Then clearly f_* is an isomorphism in dimensions 0 and 3. Since degree ± 1 maps induce surjections on first homology (see [Ol]) and since the first homology groups are finite, f_* is an isomorphism in dimension 1. All other homology groups are trivial. The reverse implication is obvious. ∎

The main result from [L-S], which is key to the present paper, may now be restated as follows:

THEOREM 1.3 (Luft–Sjerve). *Each homology lens space is $\mathbb{Z}$-homology equivalent to a lens space $L(n,m)$, which is uniquely determined up to homotopy equivalence.*

A classical theorem on lens spaces (see e.g. [Co], p. 96) states that $L(n,m)$ and $L(n,m')$ are homotopy equivalent iff $mm' = \pm b^2 \pmod{n}$ for some integer b. This gives a neat partition of actual lens spaces into homotopy equivalence classes. Theorem 1.3 says that these same classes form the foundation for a partition of homology lens spaces into $\mathbb{Z}$-homology equivalence classes with each class containing at least one actual lens space.

A more traditional way to distinguish between homology lens spaces is via the *linking form* $\lambda : \mathrm{Torsion}(H_1(M^3)) \times \mathrm{Torsion}(H_1(M^3)) \to \mathbb{Q}/\mathbb{Z}$. In our special case, λ may be represented by a 1×1 matrix (m/n) where n is the order of $H_1(M^3)$ and m is the intersection number between a generator $[\alpha]$ of $H_1(M^3)$ and a surface in M^3 with boundary consisting of n (oriented) copies of α. See [Fu] for details. Later it will be clear that the two approaches are equivalent.

2. Dehn surgery on homology spheres. Let T be a solid torus in S^3, and let α be a non-separating simple closed curve in ∂T. If T' is a solid torus disjoint from S^3, and $g : \partial T' \to \partial T$ is a homeomorphism taking a meridian of T' onto α, we say that the adjunction space $M = (S^3 - \mathrm{int}(T)) \cup_g T'$ is *the result of performing a Dehn surgery on $T \subseteq S^3$*. It is well known that the homeomorphism type of M is completely determined by α, and moreover, if μ is an oriented meridian of T and λ is the unique (up to isotopy) longitude of T which bounds a Seifert surface in $S^3 - \mathrm{int}(T)$ with some orientation chosen, then α uniquely determines relatively prime integers p, q such that $[\alpha] = [p\mu + q\lambda]$ in $H_1(\partial T)$. We call the corresponding surgery a (p/q)-*Dehn surgery on T*. Since $p/q = (-p)/(-q)$, the orientation chosen for α is insignificant. The following lemma shows, among other things, that we may extend this notation to Dehn surgery in a homology 3-sphere.

LEMMA 2.1. *Let M^3 be a homology 3-sphere or a homology lens space, $T \subseteq M^3$ be a solid torus whose core generates $H_1(M^3)$, and let X denote $M^3 - \mathrm{int}(T)$. Then*:

(a) *$H_k(X) \cong \mathbb{Z}$ if $k = 0, 1$ and $H_k(X) \cong 0$ otherwise*,

(b) *$H_1(X)$ is generated by a simple closed curve α in ∂T, where, if M^3 is a homology sphere, α may be chosen to be a meridian of T*,

(c) *there is an orientable surface F, properly embedded in X, such that ∂F is connected and intersects α transversally in a single point. Moreover, ∂F is uniquely determined up to isotopy in ∂T, and*

(d) *any simple closed curve γ in ∂T which meets ∂F transversally in a single point generates $H_1(X)$.*

Proof. Since core(T) generates $H_1(M^3)$, $H_1(M, T) = 0$, so we have

$$(*)\qquad \begin{array}{ccccccccccc} \ldots \to & H_2(M) & \to & H_2(M,T) & \to & H_1(T) & \to & H_1(M) & \to & H_1(M,T) & \to \ldots \\ & \parallel & & & & \parallel & & \parallel & & \parallel & \\ & 0 & & & & \mathbb{Z} & & \mathbb{Z}_n & & 0 & \end{array}$$

This forces $H_2(M, T) \cong \mathbb{Z}$. By excision, $H_2(X, \partial X) \cong \mathbb{Z}$, so by duality $H^1(X) \cong \mathbb{Z}$. By universal coefficients $\mathrm{Free}(H_1(X)) \cong \mathbb{Z}$. As $H_1(X, \partial X) \cong 0$, $H^2(X) \cong 0$, so $\mathrm{Torsion}(H_1(X)) \cong 0$. Thus $H_1(X) \cong \mathbb{Z}$. Similarly $H_2(X) \cong H^1(X, \partial X) \cong \mathrm{Free}(H_1(X, \partial X)) \cong 0$. That $H_k(X) = 0$ for $k > 2$ is clear.

To verify (b), note that since $H_1(X, \partial X) = 0$, inclusion induces a surjection: $H_1(\partial X) \to H_1(X)$. Therefore $H_1(X)$ is generated by a closed loop α in ∂X. Since $[\alpha]$ is not divisible in $H_1(X)$, $[\alpha]$ is not divisible as an element of $H_1(\partial X)$, and thus may be represented by an embedded loop.

If M^3 is a homology sphere, consider the Mayer–Vietoris sequence

$$\begin{array}{ccccccccc} \ldots \to & H_2(M) & \to & H_1(\partial T) & \xrightarrow{(i_*, j_*)} & H_1(T) \oplus H_1(X) & \to & H_1(M) & \to \ldots \\ & \parallel & & \parallel & & \parallel & & \parallel & \\ & 0 & & \mathbb{Z} \oplus \mathbb{Z} & & \mathbb{Z} \oplus \mathbb{Z} & & 0 & \end{array}$$

Let ω be a meridian of T and τ an arbitrary longitude. Then $i_*([\tau])$ is a generator of $H_1(T)$, and $i_*([\omega]) = 0$. Since (i_*, j_*) is surjective, $j_*([\omega])$ must generate $H_1(X)$, so we may let $\alpha = \omega$.

To find F, first construct a map $f : X \to S^1$ inducing an isomorphism on first homology and sending α homeomorphically onto S^1. Adjust f so that $f|_\alpha$ is a homeomorphism and there is a point $x_0 \in S^1$ at which f is transversal. Let F be he component of $f^{-1}(x_0)$ which has non-empty boundary. By construction ∂F is connected and intersects α transversally in a single point. Any other loop β in ∂T can be expressed as $[k \cdot \alpha + l \cdot \partial F]$

in $H_1(\partial T)$ for integers k and l. If β bounds an orientable surface embedded in X, then $[\beta] = 0 = [\partial F]$ and $[\alpha] \neq 0$ in $H_1(X)$, so k must be zero. Since β is embedded, $l = 1$, hence, β is isotopic to ∂F in ∂T.

Now let γ be a simple closed curve in ∂T which meets ∂F transversally in a single point. Then $[\alpha] = [k \cdot \gamma + l \cdot \partial F]$ for some integers k, l. But ∂F is null-homologous in X, so $[\alpha] = [k\gamma]$ in $H_1(X)$. Since $[\alpha]$ generates, so does $[\gamma]$. ∎

Note. In case M^3 is a homology sphere, T can be an arbitrary solid torus in M^3, and by (b) a meridian μ generates $H_1(X)$. Thus ∂F is a longitude of T which we call the *preferred longitude* and denote by λ. We may now use the pair μ, λ to specify Dehn surgeries on T with fractions as promised earlier.

3. Main results

THEOREM 3.1. *A 3-manifold M^3 is $\mathbb{Z}$-homology equivalent to $L(n,m)$ iff there exists a homology sphere Σ^3 and a solid torus $T \subseteq \Sigma^3$ on which an (n/m)-Dehn surgery yields M^3.*

Proof. Let $f : M^3 \to L(n,m)$ be a $\mathbb{Z}$-homology equivalence, and $L(n,m) = V_1 \cup V_2$ be a genus 1 Heegard splitting, where a meridian ω_2 of V_2 is identified with a simple closed curve σ in ∂V_1 homologous to $m\omega_1 + n\tau_1$ where ω_1, τ_1 is a meridian-longitude pair for V_1. Notice that $\mathrm{core}(V_i)$ generates $H_1(L(n,m))$ for each i. By [Wa] we may adjust f so that $f^{-1}(V_2)$ is a solid torus T in M^3 and $f|_T$ is a homeomorphism onto V_2. Then $\mathrm{core}(T)$ generates $H_1(M^3)$. Adjusting f further, if necessary, we may assume that for a meridional disk D in V_1 with $\omega_1 = \partial D$, $f^{-1}(D)$ is an orientable surface in $M^3 - \mathrm{int}(T)$. Let F be the component of this surface with $\partial F = f^{-1}(\omega_1)$. Lemma 2.1 applies to $X = M^3 - \mathrm{int}(T)$, so $H_1(X)$ is generated by $\alpha = f^{-1}(\tau_1)$. Attaching a solid torus T' to X with a meridian being sent to α produces a homology sphere Σ^3. Removing T' from Σ^3 and replacing it with T gives us M^3 again. Now α is a meridian of T', ∂F is the preferred longitude of T' in Σ^3, and $f^{-1}(\sigma)$ $(= f^{-1}(\omega_2))$ is a meridian of T; therefore, replacing T' with T is an (n/m)-Dehn surgery on Σ^3.

For the reverse implication, let $T \subseteq \Sigma^3$ be a solid torus in a homology sphere, μ a meridian of T, F a Seifert surface for T, $\lambda = \partial F$ the preferred longitude of T, and $X = \Sigma^3 - \mathrm{int}(T)$. Let M^3 be the manifold obtained by performing an (n/m)-Dehn surgery which replaces T with T'. Lemma 2.1 makes it easy to see that M^3 is a homology n-lens space. We now construct a degree ± 1 map $f : M^3 \to L(n,m)$. Let $L(n,m) = V_1 \cup V_2$ as above. Let $f|_{T'}$ take T' homeomorphically onto V_2 so that a meridian μ' of T' is taken to ω_2 $(= \sigma)$ and λ is taken to ω_1 on ∂V_1 $(= \partial V_2)$. This can be done since $|\lambda \cap \mu'| = m = |\omega_1 \cap \omega_2|$. Send F onto a meridional disk D in V_1 bounded by ω_1 by crushing out all but a collar on ∂F. Extend this to take a product neighborhood, $N(F)$, of F in X to a product neighborhood, $N(D)$, of D in V_1. Since $V_1 - \mathrm{int}(N(D))$ is a 3-ball we may extend this map to take the remainder of X into $V_1 - \mathrm{int}(N(D))$. Since f is a homeomorphism over V_2, $|\deg(f)| = 1$. ∎

We now translate this result into one involving linking forms.

COROLLARY 3.2 (Generalization of [Fu, Theorem 1]). *A homology lens space M^3 has linking form equivalent to (m/n) for relatively prime integers m and n iff M^3 may be obtained by a single (n/m)-Dehn surgery on a homology sphere.*

Proof. It is clear that a homology lens space created by an (n/m)-Dehn surgery on a homology sphere has linking form equivalent to (m/n). Now suppose M^3 has linking form (m/n). Then M^3 is a homology n-lens space which by Theorems 1.3 and 3.1 is $\mathbb{Z}$-homology equivalent to $L(n, m')$ and may be obtained by an (n/m')-Dehn surgery on a homology sphere for some m'. Then (m'/n) also represents the linking form of M^3, so (m/n) and (m'/n) are equivalent. We may conclude that $mm' = \pm b^2 \pmod n$ for some integer b. It follows that $L(n, m)$ and $L(n, m')$ are homotopy equivalent, so M^3 is $\mathbb{Z}$-homology equivalent to $L(n, m)$. By Theorem 3.1, M^3 may be obtained by an (n/m)-Dehn surgery on a homology sphere. ∎

Remark. It should now be obvious that our partition of homology lens spaces into Z-homology equivalence classes is the same as that obtained via linking forms.

4. Homology $S^1 \times S^2$'s. Since nearly all of machinery is in place, we include a version of Theorem 3.1 for 3-manifolds with the same $\mathbb{Z}$-homology as $S^1 \times S^2$.

LEMMA 4.1. *Let M^3 be a homology $S^1 \times S^2$, $T \subseteq M^3$ a solid torus whose core generates $H_1(M^3)$, λ a longitude of T, and $X = M^3 - \mathrm{int}(T)$. Then*:

(a) *$H_k(X) \cong \mathbb{Z}$ for $k = 0, 1$ and $H_k(X) = 0$ otherwise*,

(b) *$[\lambda]$ generates $H_1(X)$, and*

(c) *there is a surface F, properly embedded in X, such that ∂F is a meridian of T.*

Proof. Verification of (a) is similar to that of Lemma 2.1(a). To see that $[\lambda]$ generates $H_1(X)$ consider the inclusion induced homomorphism $H_1(X) \to H_1(M^3)$. Since $[\lambda]$ generates $H_1(M^3)$ this map is surjective, hence it is an isomorphism. Therefore, $[\lambda]$ generates $H_1(X)$.

Existence of an embedded surface F is verified as in Lemma 2.1(c). Since ∂F is trivial in $H_1(X)$, part (b) implies that ∂F can only be a meridian. ∎

THEOREM 4.2. *Every homology $S^1 \times S^2$ may be obtained by a single* (0/1)*-Dehn surgery on some homology sphere Σ^3.*

Proof. Let M^3 be a homology $S^1 \times S^2$ and $T \subseteq M^3$ a solid torus whose core generates $H_1(M^3)$. Using the result and notation of Lemma 4.1, attach a solid torus T' to X with a meridian being identified to λ. The result is a homology sphere, Σ^3. Since the preferred longitude of T' in Σ^3 is a meridian of T, reversing this surgery amounts to a (0/1)-Dehn surgery on Σ^3.

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