Undetermined sets of point-open games

by

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Abstract. We show that a set of reals is undetermined in Galvin’s point-open game iff it is uncountable and has property $C''$, which answers a question of Gruenhage.

Let $X$ be a topological space. The point-open game $G(X)$ of Galvin [G] is played as follows. Black chooses a point $x_0 \in X$, then White chooses an open set $U_0 \ni x_0$, then Black chooses a point $x_1 \in X$, then White chooses an open set $U_1 \ni x_1$, etc. Black wins the play $(x_0, U_0, x_1, U_1, \ldots)$ iff $X = \bigcup_n U_n$.

Galvin [G] showed that the Continuum Hypothesis yields a Lusin set $X$ which is undetermined (i.e. for which the game $G(X)$ is undetermined). (A Lusin set is an uncountable set of reals which has countable intersection with every meager set.)

Recently Reclaw [R] showed that every Lusin set is undetermined. Motivated by Reclaw’s result we prove the following.

Theorem. Let $X$ be a topological space in which every point is $G_δ$. Then $G(X)$ is undetermined iff $X$ is uncountable and has property $C''$.

Property $C''$ was introduced by Rothberger (see [M]). A topological space $X$ has property $C''$ if for every sequence $U_n$ ($n \in \omega$) of open covers of $X$ there exist $U_n \in U_n$ such that $X = \bigcup_n U_n$. It is known (see [M] or [FM]) that every Lusin set has property $C''$.

Clearly, a space with property $C''$ must be Lindelöf. Martin’s Axiom implies that every Lindelöf space of size less than $2^{\aleph_0}$ has property $C''$ and that there are sets of reals of size $2^{\aleph_0}$ with property $C''$ (see [M]). Thus, Martin’s Axiom yields undetermined sets of reals of size $2^{\aleph_0}$ (Theorem 4 of [G]).

On the other hand, in Laver’s [L] model for Borel’s conjecture all metric spaces with property $C''$ are countable (see Note 1). Thus, consistently, all metric spaces are determined.

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The connection between property $C''$ and point-open games is made transparent by the following dual game $G^*(X)$, due also to Galvin [G]. Now $W$ chooses an open cover $U_0$ of $X$, then $B$ chooses a set $U_0 \in U_0$, then $W$ chooses an open cover $U_1$ of $X$, then $B$ chooses a set $U_1 \in U_1$, etc. As before, $B$ wins if $X = \bigcup_n U_n$.

Galvin [G] showed that the games $G(X)$ and $G^*(X)$ are equivalent in the sense that $W \uparrow G(X)$ (has a winning strategy) iff $W \uparrow G^*(X)$; similarly for $B$. In particular, $G(X)$ is determined iff $G^*(X)$ is.

Let $G^*(X)$ and $G^{*\sigma}(X)$ be games that are played as $G(X)$ and $G^*(X)$ are, but in which $B$ wins if $X = \bigcap_n \bigcup_{m>\tau} U_m$. These games are again equivalent (see [G], Theorem 1). Clearly, $|X| \leq \aleph_0 \Rightarrow B \uparrow G^{*\sigma}(X) \Rightarrow B \uparrow G^*(X)$, and it is not hard to see that if each point of $X$ is $G_\delta$, then $B \uparrow G^*(X) \Rightarrow |X| \leq \aleph_0$ (see [G], Theorem 2). Also, $X \not\in C'' \Rightarrow W \uparrow G^*(X) \Rightarrow W \uparrow G^{*\sigma}(X)$ (for the first implication $W$ plays covers that witness $X \not\in C''$).

We shall prove that $W \uparrow G^{*\sigma}(X) \Rightarrow X \not\in C''$.

First let us play one more game. The game $M^*(X)$ is defined as $G^*(X)$ is but $B$ chooses finite subsets $V_n \subseteq U_n$. He wins if $\bigcup_n \bigcup V_n = X$. The $\sigma$ is introduced as before.

The game is motivated by property $M$ of Menger (see [FM]). A topological space has property $M$ if for every sequence $\mathcal{U}_n$ $(n \in \omega)$ of open covers of $X$ there exist finite $\mathcal{V}_n \subseteq \mathcal{U}_n$ such that $\bigcup_n \bigcup \mathcal{V}_n = X$. Clearly, property $C''$ implies property $M$.

**Lemma 1.** Suppose that $X$ has property $M$. Then $W$ has no winning strategy in $M^{*\sigma}(X)$.

**Proof.** $X$ is clearly Lindelöf. Without loss of generality, we can assume that $W$ plays increasing sequences from which $B$ chooses single sets. Then a strategy for $W$ can be identified with a family $\{U_\sigma : \sigma \in <\omega \}$ such that for every $\sigma$, $\{U_\sigma^{-i} : i < \omega\}$ is an increasing open cover of $X$. We seek $s \in <\omega$ such that $\forall x \in X \exists^\infty n x \in U_{s[n]}$.

For integers $j > 0$ and $k, m \geq 0$ let

$$V_k(m, j) = \bigcap_{\tau \in ^m j} U_{\tau \sim k}.$$  

Note that $\{V_k(m, j) : k \in \omega\}$ is an increasing open cover of $X$: given $x \in X$, find $k_\tau (\tau \in ^m j)$ with $x \in U_{\tau \sim k_\tau}$, and let $k = \max \tau k_\tau$; then $x \in V_k(m, j)$.

For integers $j > 0$, $k \geq j$ and $m, n \geq 0$ let

$$W_k^n(m, j) = \bigcup_{j = k_0 \leq k_1 \leq \ldots \leq k_{n+1} = k} \bigcap_{i = 0}^n V_{k_{i+1}}(m + i, k_i).$$

Again $\{W_k^n(m, j) : k \geq j\}$ is an increasing open cover of $X$: given $x \in X$, find $k_1 \geq k_0$ with $x \in V_{k_1}(m, k_0)$, next $k_2 \geq k_1$ with $x \in V_{k_2}(m + 1, k_1)$,
Lemma 5.1. \( G \) to \( \sigma \) next choose \( \{ \text{a family} \} \). So, \( G \) strategy in \( \{ \text{etc.} \} \). Let \( s \in ^\omega \omega \) be strictly increasing such that \( \forall m, j \forall \omega x \geq t_{m, j}(n) \). Then
\[ \forall x \in X \forall m, j \exists n x \in W^n_{s(m + n)}(m, j). \]

**Claim.** \( \forall x \in X \exists \omega x \in U_{s[n]} \).

**Proof.** Suppose not. Fix \( x \) and \( m \) with \( \forall n x \notin U_{s[m + n]} \). By the choice of \( s \) there is \( n \) with \( x \in W^n_{s(m + n)}(m, s(m)) \). So, there are integers
\[ s(m) = k_0 \leq k_1 \leq \ldots \leq k_{n + 1} = s(m + n) \]
such that
\[ x \in \bigcap_{i=0}^{n} V_{k_{i+1}}(m + i, k_i). \]
Now, \( s[m] \in m k_0, x \in V_{k_0}(m, k_0) \) and \( x \notin U_{s[m] - s(m)} \) yield \( k_1 > s(m) \). Next, \( s[m + 1] \in (m + 1) k_1, x \in V_{k_2}(m + 1, k_1) \) and \( x \notin U_{s[m + 1] - s(m + 1)} \) yield \( k_2 > s(m + 1) \). Proceeding in this way we get \( k_{n + 1} > s(m + n) \), which is a contradiction. \( \blacksquare \)

It follows that if \( W \) plays according to \( \{ U_{\sigma} : \sigma \in \omega \} \) and \( B \) according to \( s \), then \( B \) wins. \( \blacksquare \)

Now we prove that if \( X \in C'' \) then \( B \) can spoil each strategy of \( W \) in \( G^\sigma(X) \). The idea of diagonalization used in the proof is taken from [FM], Lemma 5.1.

**Lemma 2.** Suppose that \( X \) has property \( C'' \). Then \( W \) has no winning strategy in \( G^\sigma(X) \).

**Proof.** Again \( X \) is Lindelöf and we can identify a strategy for \( W \) with a family \( \{ U_{\sigma} : \sigma \in \omega \} \) of open sets such that \( \forall \sigma X = \bigcup_i U_{\sigma \sim i} \). We seek \( s \in ^\omega \omega \) such that \( \forall x \in X \exists \omega n x \in U_{s[n]} \).

For integers \( j > 0, m \geq 0 \) and for \( \sigma : j^m \to \omega \) let
\[ U_{\sigma}(m, j) = \bigcup_{\tau \in j^m} \bigcup_{0 < i \leq j^m} U_{\tau_{\sim \sigma} \sim i}. \]

**Claim 1.** \( \forall m, j U_{\sigma}(m, j) \)'s form an open cover of \( X \).

**Proof.** Fix \( m \) and \( j \). Let \( x \in X \) be given. Let \( \langle \tau_k : k < j^m \rangle \) be an enumeration of \( m^j \). Define \( \sigma \) by induction: choose \( \sigma(0) \) so that \( x \in U_{\tau_0 \sim \sigma(0)} \), next choose \( \sigma(1) \) so that \( x \in U_{\tau_1 \sim \sigma(0) \sim \sigma(1)} \), etc.
Claim 2. There are increasing sequences \( \langle j_n : n < \omega \rangle \), \( \langle m_n : n < \omega \rangle \) of integers such that
\[
\forall x \in X \exists n \exists \sigma : (m_{n+1} - m_n) \mapsto j_{n+1} \ x \in U_\sigma(m_n,j_n).
\]

Proof. Let \( j_0 = 1, m_0 = 0 \). We start a game. At the \( n \)th round, \( j_n \) and \( m_n \) are given and \( W \) plays an open cover
\[
U_\sigma(m_n,j_n) \ (\sigma : j_n^{m_n} \mapsto \omega).
\]

\( B \) responds with an integer \( j_{n+1} \geq j_n \), but really thinks about \( \bigcup \{U_\sigma(m_n,j_n) : \max \sigma(i) < j_{n+1}\} \). Then he declares \( m_{n+1} = m_n + j_{n+1}^{m_n} \).

We view this as the \( M^\sigma(X) \) game played by \( W \) according to a fixed strategy. Since \( C'' \Rightarrow M \), by Lemma 1, \( B \) can spoil this strategy. \( \blacksquare \)

For \( k_1 < \ldots < k_n < \omega \) and \( \sigma_i : (m_{k_i+1} - m_{k_i}) \mapsto j_{k_i+1} \) define
\[
W(k_1,\ldots,k_n;\sigma_1,\ldots,\sigma_n) = \bigcap_{i=1}^n U_{\sigma_i}(m_{k_i},j_{k_i}).
\]

By Claim 2 we see that for every \( n \), \( W(k_1,\ldots,k_n;\sigma_1,\ldots,\sigma_n) \)'s form an open cover of \( X \). Since \( X \in C'' \), there are \( \sigma^n_1, k^n_1 (n = 1,2,\ldots ; i = 1,\ldots,n) \) such that
\[
\forall x \in X \exists n \ x \in W(k^n_1,\ldots,k^n_n;\sigma^n_1,\ldots,\sigma^n_n).
\]

Let \( l_n \in \{k^n_1,\ldots,k^n_n\} \setminus \{k^n_1-1,\ldots,k^n_n-1\} \) and let \( \tau_n \) be the \( \sigma^n_i \) corresponding to \( l_n \). Then \( \tau_n \)'s are distinct and, by the definition of \( W \)'s, we get
\[
\forall x \in X \exists n \ x \in U_{\tau_n}(m_{l_n},j_{l_n}).
\]

Now define \( s \in \omega^\omega \) by
\[
s(m_{l_n} + i) = \tau_n(i),
\]
for \( n \in \omega \) and \( i \in \text{dom}(\tau_n) \), and put 0 elsewhere.

Claim 3. \( \forall x \in X \exists n \ x \in U_{s|n} \).

Proof. If \( x \in U_{\tau_n}(m_{l_n},j_{l_n}) \) then, since \( s|m_{l_n} : m_{l_n} \mapsto j_{l_n} \), we get
\[
x \in \bigcup \{U_{s|m_{l_n} \mapsto \tau_n} : 0 < i \leq m_{l_n+1} - m_{l_n}\}.
\]

But \( s|m_{l_n} \mapsto \tau_n|j \mapsto s(m_{l_n} + i) \). \( \blacksquare \)

It follows that if \( W \) plays according to \( \{U_\sigma : \sigma \in \omega^\omega\} \) and \( B \) plays according to \( s \), then \( B \) wins. \( \blacksquare \)

Call an open cover \( \mathcal{U} \) of \( X \) strong if for each \( U \in \mathcal{U} \), the family \( \{V \in \mathcal{U} : U \subseteq V\} \) covers \( X \). Galvin showed that for a regular space \( X \), \( W \upharpoonright G(X) \) iff no strong open cover contains an increasing subcover \( \{U_n : n \in \omega\} \) ([GT], Theorem 4). Combining Galvin's theorem with ours we can give a characterization of regular \( C'' \) spaces. By a covering tree we mean a family
are equivalent.

\[ \bigcup_{n} U_n = X \] contains at least two sets from \( W \) so that \( U \) easily refine the covers \( \bigcap_{n} \bigcup_{n>m} U_n = X \).

\[ \forall T \] \( T \) finitely additive open cover with no countable subcover violates \( (c) \). Also, \( \tau \) \( \Xi \) has a clopen base, and \( X \) being Lindelöf regular, is completely regular.

Now, let \( U_n \) \( n \in \omega \) be a sequence of open covers of \( X \). Since \( X \) is Lindelöf and zerodimensional, we can assume that each \( U_n \) is countable and consists of pairwise disjoint clopens (see [K], §26).

Let \( U_n = \{ U^n_i : i < \omega \} \) (some \( U^n_i \)'s may be empty). Define \( f : X \mapsto \omega^\omega \) by

\[ f(x)(n) = i \iff x \in U^n_i \] where \( f \) is zero-dimensional (has a clopen base). Indeed, there is no continuous \( f \) from \( X \) onto \([0,1]\) (otherwise \( \{ f^{-1}[V] : V \subseteq [0,1] \text{ open with } |V| \text{ uncountable} \} \) violates \( (c) \)).

We shall show \((c)\Rightarrow(a)\). Assume \((c)\). First, \( X \) is Lindelöf. Otherwise any finitely additive open cover with no countable subcover violates \((c)\). Also, \( X \) is zerodimensional (has a clopen base).

Indeed, there is no continuous function \( f \) from \( X \) onto \([0,1]\) (otherwise \( \{ f^{-1}[V] : V \subseteq [0,1] \text{ open} \} \) violates \((c)\)). For completely regular spaces this means having a clopen base, and \( X \), being Lindelöf regular, is completely regular.

Let \( V = \bigcup \{ U^n_i : |f[U^n_i]| \leq \aleph_0 \} \). Then \( |f[V]| \leq \aleph_0 \) and since \( \forall x \in X \) \( x \in \bigcap_n U^n_{f(x)(n)} \), it is not hard to see that there exist \( t_{2n+1} \in \omega \) \( n \in \omega \) with \( V \subseteq \bigcap_n U^n_{t_{2n+1}}. \) Suppose that \( X \setminus V \neq \emptyset \) (otherwise we are done). We can easily refine the covers \( U_{2n} \) (with the help of \( U'_{2m} \)'s, \( m \geq n \)) to covers \( W_n \) so that \( W_{n+1} \) is a refinement of \( W_n \) and each \( W \) in \( W_n \) which meets \( X \setminus V \) contains at least two sets from \( W_{n+1} \) which meet \( X \setminus V \).

Let \( V_{\sigma} = V \cup \bigcup_{n < |\sigma|} W^n_{\sigma(n)} \) for \( \sigma \in <\omega \) such that each \( W^n_{\sigma(n)} \) meets \( X \setminus V \). Then \( V_{\sigma} \) constitute a strong open cover of \( X \). Also, if \( V_{\sigma} \subseteq V_{\tau} \) then \( \sigma \subseteq \tau \) (because no \( W^n_k \) which meets \( X \setminus V \) can be covered by finitely many sets taken from different \( W_m \) \( m > n \)).

It follows that from an increasing sequence of \( V_{\sigma} \)'s covering \( X \), which exists by \((c)\), we get \( s \in \omega \) with \( X \setminus V \subseteq \bigcup_n W^n_{s(n)} \). Since there are \( t_{2n} \) \( n \in \omega \) with \( W^n_{s(n)} \subseteq U^n_{t_{2n}} \), we get \( X \subseteq \bigcup_n U^n_{t_{2n}} \).

Rothberger also considered property \( C' \), which is defined as \( C'' \) is but the covers \( U_n \) are finite. We can define a game corresponding to \( C' \) by
introducing to $G^*$ the requirement that the covers played by $W$ are finite. Then an analogue of Lemma 2 is true (see the proof of (a)$\Rightarrow$(b) below). We also have the following. (A tree is **finitely branching** if the set of immediate successors of any node is finite.)

**Proposition 2.** Let $X$ be a regular topological space. Then the following are equivalent.

(a) $X$ has property $C'$.

(b) In every finitely branching covering tree there exists a branch $\langle U_n : n \in \omega \rangle$ with $\bigcup_n U_n = X$ (equivalently, with $\bigcap_m \bigcup_{n>m} U_n = X$).

(c) In every strong open cover $U$ such that for each $U \in U$, a finite subfamily of $\{V \in U : U \subseteq V\}$ covers $X$, there exists an increasing subcover $\{U_n : n \in \omega\}$.

**Proof.** We sketch (a)$\Rightarrow$(b); (b)$\Rightarrow$(c)$\Rightarrow$(a) are proved as in Proposition 1.

Suppose that $X \in C'$. Let $T$ be a finitely branching covering tree. For each $n \in \omega$, let $V_n$ be a common finite refinement of all covers $U_\sigma \overset{def}{=} \{U : \sigma \upharpoonright U \in T\}$ ($\sigma \in T$ and $|\sigma| = n$). Such a refinement exists because there are only finitely many covers to refine. Since $X \in C'$ there is a sequence $V_n \in V_n$ such that $X = \bigcap_m \bigcup_{n>m} V_n$. Define a branch $\langle U_n : n \in \omega \rangle$ of $T$ by $U_n =$ any $U \supseteq V_n$ such that $\langle U_0, \ldots, U_{n-1}, U \rangle \in T$.

**Note. 1.** It is folklore that every metric space $X \in C''$ is homeomorphic to a subspace of $\mathbb{R}$ (the reals). Such an $X$ is zero-dimensional (it cannot be continuously mapped onto $[0, 1]$ as $C''$ is preserved by continuous images and $[0, 1] \not\in C''$; [K], §40). Being Lindelöf, $X$ is separable, and so homeomorphic to a subset of $^\omega \omega$. Since every $C''$ set of reals has strong measure zero, Borel's conjecture implies that every $C''$ metric space is countable.

2. In the spirit of [R], $X \subseteq \mathbb{R}$ with property $C''$ can be characterized by any of the following (see [P]):

(a) for any $F_\sigma$ (equivalently, closed) $A \subseteq \mathbb{R}^2$ with all vertical sections $A_x (x \in X)$ meager, $\bigcup_{x \in X} A_x \not\in \mathbb{R}$;

(b) for any closed $A \subseteq \mathbb{R}^2$ with all vertical sections $A_x (x \in X)$ null, $\bigcup_{x \in X} A_x \not\in \mathbb{R}$.

Also, $X \subseteq \mathbb{R}$ has strong measure zero iff any of the following holds:

(a) for any $F_\sigma$ (equivalently, closed) $A \subseteq \mathbb{R}^2$ with all vertical sections $A_x (x \in \mathbb{R})$ meager, $\bigcup_{x \in X} A_x \not\in \mathbb{R}$ ([AR]);

(b) for any closed $A \subseteq \mathbb{R}^2$ with all vertical sections $A_x (x \in \mathbb{R})$ null, $\bigcup_{x \in X} A_x \not\in \mathbb{R}$ ([P]);

(c) for any closed null $D \subseteq \mathbb{R}$, $X + D$ is null ([P]).
3. There exists (in ZFC) an uncountable $C''$ space in which every point is $G_{δ}$. Todorčević [T] has an example of a zerodimensional first countable Hausdorff space of size $ℵ_1$ whose every continuous image into any second countable space (in particular, into $ω^ω$) is countable.

**Question.** In Propositions 1 and 2, can one remove the assumption that $X$ is regular (it is used in (c)$⇒$(a))? 

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**References**


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