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Undetermined sets of point-open games

by

Janusz Pawlikowski (Wrocław)

Abstract. We show that a set of reals is undetermined in Galvin's point-open game iff it is uncountable and has property C'', which answers a question of Gruenhage.

Let X be a topological space. The *point-open game* $G(X)$ of Galvin [G] is played as follows. **B**lack chooses a point $x_0 \in X$, then **W**hite chooses an open set $U_0 \ni x_0$, then **B** chooses a point $x_1 \in X$, then **W** chooses an open set $U_1 \ni x_1$, etc. **B** wins the play $(x_0, U_0, x_1, U_1, \ldots)$ iff $X = \bigcup_n U_n$.

Galvin [G] showed that the Continuum Hypothesis yields a Lusin set X which is undetermined (i.e. for which the game $G(X)$ is undetermined). (A *Lusin set* is an uncountable set of reals which has countable intersection with every meager set.)

Recently Recław [R] showed that every Lusin set is undetermined. Motivated by Recław's result we prove the following.

Theorem. *Let X be a topological space in which every point is $\mathbf{G}_\delta$. Then $G(X)$ is undetermined iff X is uncountable and has property C''.*

Property C'' was introduced by Rothberger (see [M]). A topological space X has *property* C'' if for every sequence $\mathcal{U}_n$ $(n \in \omega)$ of open covers of X there exist $U_n \in \mathcal{U}_n$ such that $X = \bigcup_n U_n$. It is known (see [M] or [FM]) that every Lusin set has property C''.

Clearly, a space with property C'' must be Lindelöf. Martin's Axiom implies that every Lindelöf space of size less than $2^{\aleph_0}$ has property C'' and that there are sets of reals of size $2^{\aleph_0}$ with property C'' (see [M]). Thus, Martin's Axiom yields undetermined sets of reals of size $2^{\aleph_0}$ (Theorem 4 of [G]).

On the other hand, in Laver's [L] model for Borel's conjecture all metric spaces with property C'' are countable (see Note 1). Thus, consistently, all metric spaces are determined.

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The connection between property C'' and point-open games is made transparent by the following dual game $G^*(X)$, due also to Galvin [G]. Now **W** chooses an open cover $\mathcal{U}_0$ of X, then **B** chooses a set $U_0 \in \mathcal{U}_0$, then **W** chooses an open cover $\mathcal{U}_1$ of X, then **B** chooses a set $U_1 \in \mathcal{U}_1$, etc. As before, **B** wins if $X = \bigcup_n U_n$.

Galvin [G] showed that the games $G(X)$ and $G^*(X)$ are equivalent in the sense that $\mathbf{W} \uparrow G(X)$ (has a winning strategy) iff $\mathbf{W} \uparrow G^*(X)$; similarly for **B**. In particular, $G(X)$ is determined iff $G^*(X)$ is.

Let $G^\sigma(X)$ and $G^{*\sigma}(X)$ be games that are played as $G(X)$ and $G^*(X)$ are, but in which **B** wins if $X = \bigcap_n \bigcup_{m>n} U_m$. These games are again equivalent (see [G], Theorem 1). Clearly, $|X| \le \aleph_0 \Rightarrow \mathbf{B} \uparrow G^{*\sigma}(X) \Rightarrow \mathbf{B} \uparrow G^*(X)$, and it is not hard to see that if each point of X is $\mathbf{G}_\delta$, then $\mathbf{B} \uparrow G^*(X) \Rightarrow |X| \le \aleph_0$ (see [G], Theorem 2). Also, $X \notin C'' \Rightarrow \mathbf{W} \uparrow G^*(X) \Rightarrow \mathbf{W} \uparrow G^{*\sigma}(X)$ (for the first implication **W** plays covers that witness $X \notin C''$). We shall prove that $\mathbf{W} \uparrow G^{*\sigma}(X) \Rightarrow X \notin C''$.

First let us play one more game. The game $M^*(X)$ is defined as $G^*(X)$ is but **B** chooses finite subsets $\mathcal{V}_n \subseteq \mathcal{U}_n$. He wins if $\bigcup_n \bigcup \mathcal{V}_n = X$. The σ is introduced as before.

The game is motivated by property M of Menger (see [FM]). A topological space has *property* M if for every sequence $\mathcal{U}_n$ $(n \in \omega)$ of open covers of X there exist finite $\mathcal{V}_n \subseteq \mathcal{U}_n$ such that $\bigcup_n \bigcup \mathcal{V}_n = X$. Clearly, property C'' implies property M.

Lemma 1. *Suppose that X has property M. Then* **W** *has no winning strategy in $M^{*\sigma}(X)$.*

Proof. X is clearly Lindelöf. Without loss of generality, we can assume that **W** plays increasing sequences from which **B** chooses single sets. Then a strategy for **W** can be identified with a family $\{U_\sigma : \sigma \in {}^{<\omega}\omega\}$ such that for every σ, $\{U_{\sigma^\frown i} : i < \omega\}$ is an increasing open cover of X. We seek $s \in {}^\omega\omega$ such that $\forall x \in X\ \exists^\infty n\ x \in U_{s|n}$.

For integers $j > 0$ and $k, m \ge 0$ let

$$V_k(m,j) = \bigcap_{\tau \in {}^m j} U_{\tau^\frown k}\,.$$

Note that $\{V_k(m,j) : k \in \omega\}$ is an increasing open cover of X: given $x \in X$, find k_τ $(\tau \in {}^m j)$ with $x \in U_{\tau^\frown k_\tau}$ and let $k = \max_\tau k_\tau$; then $x \in V_k(m,j)$.

For integers $j > 0$, $k \ge j$ and $m, n \ge 0$ let

$$W_k^n(m,j) = \bigcup_{j=k_0 \le k_1 \le \ldots \le k_{n+1}=k} \ \bigcap_{i=0}^{n} V_{k_{i+1}}(m+i, k_i)\,.$$

Again $\{W_k^n(m,j) : k \ge j\}$ is an increasing open cover of X: given $x \in X$, find $k_1 \ge k_0$ with $x \in V_{k_1}(m,k_0)$, next $k_2 \ge k_1$ with $x \in V_{k_2}(m+1,k_1)$, etc. So, $\{W_k^n(m,j) : k \ge j\}$ $(n \in \omega)$ is a sequence of open covers, and, since $X \in M$, there is $t_{m,j} \in {}^\omega\omega$ such that

$$\forall x \in X\ \exists^\infty n\ x \in W^n_{t_{m,j}(n)}(m,j)\,.$$

Let $s \in {}^\omega\omega$ be strictly increasing such that $\forall m, j\ \forall^\infty n\ s(m+n) \ge t_{m,j}(n)$. Then

$$\forall x \in X\ \forall m\,, j\ \exists n\ x \in W^n_{s(m+n)}(m,j)\,.$$

CLAIM. $\forall x \in X\ \exists^\infty n\ x \in U_{s|n}$.

Proof. Suppose not. Fix x and m with $\forall n\ x \notin U_{s|(m+n)}$. By the choice of s there is n with $x \in W^n_{s(m+n)}(m, s(m))$. So, there are integers

$$s(m) = k_0 \le k_1 \le \ldots \le k_{n+1} = s(m+n)$$

such that

$$x \in \bigcap_{i=0}^{n} V_{k_{i+1}}(m+i, k_i).$$

Now, $s|m \in {}^m k_0$, $x \in V_{k_1}(m, k_0)$ and $x \notin U_{s|m^\frown s(m)}$ yield $k_1 > s(m)$. Next, $s|(m+1) \in {}^{(m+1)}k_1$, $x \in V_{k_2}(m+1, k_1)$ and $x \notin U_{s|(m+1)^\frown s(m+1)}$ yield $k_2 > s(m+1)$. Proceeding in this way we get $k_{n+1} > s(m+n)$, which is a contradiction. ■

It follows that if **W** plays according to $\{U_\sigma : \sigma \in {}^{<\omega}\omega\}$ and **B** according to s, then **B** wins. ■

Now we prove that if $X \in C''$ then **B** can spoil each strategy of **W** in $G^{*\sigma}(X)$. The idea of diagonalization used in the proof is taken from [FM], Lemma 5.1.

LEMMA 2. *Suppose that X has property C''. Then* **W** *has no winning strategy in $G^{*\sigma}(X)$.*

Proof. Again X is Lindelöf and we can identify a strategy for **W** with a family $\{U_\sigma : \sigma \in {}^{<\omega}\omega\}$ of open sets such that $\forall \sigma\ X = \bigcup_i U_{\sigma^\frown i}$. We seek $s \in {}^\omega\omega$ such that $\forall x \in X\ \exists^\infty n\ x \in U_{s|n}$.

For integers $j > 0$, $m \ge 0$ and for $\sigma : j^m \mapsto \omega$ let

$$U_\sigma(m,j) = \bigcap_{\tau \in {}^m j} \bigcup \{U_{\tau^\frown \sigma|i} : 0 < i \le j^m\}.$$

B is sure to cover this set if from round m on he plays according to σ, provided so far he has played numbers $< j$.

CLAIM 1. $\forall m, j$ $U_\sigma(m,j)$*'s form an open cover of X.*

Proof. Fix m and j. Let $x \in X$ be given. Let $\langle \tau_k : k < j^m \rangle$ be an enumeration of ${}^m j$. Define σ by induction: choose $\sigma(0)$ so that $x \in U_{\tau_0{}^\frown \sigma(0)}$, next choose $\sigma(1)$ so that $x \in U_{\tau_1{}^\frown \sigma(0)^\frown \sigma(1)}$, etc. ■

Claim 2. *There are increasing sequences* $\langle j_n : n < \omega\rangle$, $\langle m_n : n < \omega\rangle$ *of integers such that*

$$\forall x \in X\ \exists^\infty n\ \exists\ \sigma : (m_{n+1} - m_n) \mapsto j_{n+1}\ x \in U_\sigma(m_n, j_n)\,.$$

Proof. Let $j_0 = 1$, $m_0 = 0$. We start a game. At the nth round, j_n and m_n are given and **W** plays an open cover

$$U_\sigma(m_n, j_n) \qquad (\sigma : j_n^{m_n} \mapsto \omega)\,.$$

B responds with an integer $j_{n+1} \ge j_n$, but really thinks about $\bigcup\{U_\sigma(m_n, j_n) : \max_i \sigma(i) < j_{n+1}\}$. Then he declares $m_{n+1} = m_n + j_n^{m_n}$.

We view this as the $M^{*\sigma}(X)$ game played by **W** according to a fixed strategy. Since $C'' \Rightarrow M$, by Lemma 1, **B** can spoil this strategy. ∎

For $k_1 < \ldots < k_n < \omega$ and $\sigma_i : (m_{k_i+1} - m_{k_i}) \mapsto j_{k_i+1}$ define

$$W(k_1, \ldots, k_n; \sigma_1, \ldots, \sigma_n) = \bigcap_{i=1}^{n} U_{\sigma_i}(m_{k_i}, j_{k_i})\,.$$

By Claim 2 we see that for every n, $W(k_1, \ldots, k_n; \sigma_1, \ldots, \sigma_n)$'s form an open cover of X. Since $X \in C''$, there are σ_i^n, k_i^n $(n = 1, 2, \ldots; i = 1, \ldots, n)$ such that

$$\forall x \in X\ \exists^\infty n\ x \in W(k_1^n, \ldots, k_n^n; \sigma_1^n, \ldots, \sigma_n^n)\,.$$

Let $l_n \in \{k_1^n, \ldots, k_n^n\} \setminus \{k_1^{n-1}, \ldots, k_{n-1}^{n-1}\}$ and let τ_n be the σ_i^n corresponding to l_n. Then l_n's are distinct and, by the definition of W's, we get

$$\forall x \in X\ \exists^\infty n\ x \in U_{\tau_n}(m_{l_n}, j_{l_n})\,.$$

Now define $s \in {}^\omega\omega$ by

$$s(m_{l_n} + i) = \tau_n(i)\,,$$

for $n \in \omega$ and $i \in \mathrm{dom}(\tau_n)$, and put 0 elsewhere.

Claim 3. $\forall x \in X\ \exists^\infty n\ x \in U_{s|n}$.

Proof. If $x \in U_{\tau_n}(m_{l_n}, j_{l_n})$ then, since $s|m_{l_n} : m_{l_n} \mapsto j_{l_n}$, we get

$$x \in \bigcup\{U_{s|m_{l_n}{}^\frown\tau_n|i} : 0 < i \le m_{l_n+1} - m_{l_n}\}\,.$$

But $s|m_{l_n}{}^\frown\tau_n|i = s|(m_{l_n} + i)$. ∎

It follows that if **W** plays according to $\{U_\sigma : \sigma \in {}^{<\omega}\omega\}$ and **B** plays according to s, then **B** wins. ∎

Call an open cover $\mathcal{U}$ of X *strong* if for each $U \in \mathcal{U}$, the family $\{V \in \mathcal{U} : U \subseteq V\}$ covers X. Galvin showed that for a regular space X, $\mathbf{W} \uparrow G(X)$ iff no strong open cover contains an increasing subcover $\{U_n : n \in \omega\}$ ([GT], Theorem 4). Combining Galvin's theorem with ours we can give a characterization of regular C'' spaces. By a *covering tree* we mean a family

T of finite sequences $\sigma = \langle U_0, \ldots, U_{n-1}\rangle$ ($n \in \omega$) of open subsets of X such that $\forall \sigma \in T\ \forall k < |\sigma|\ \sigma|k \in T$ and $\forall \sigma \in T\ \{U : \sigma^\frown U \in T\}$ covers X.

PROPOSITION 1. *Let X be a regular topological space. Then the following are equivalent.*

(a) *X has property C''.*

(b) *In every covering tree there exists a branch $\langle U_n : n \in \omega\rangle$ with $\bigcup_n U_n = X$ (equivalently, with $\bigcap_m \bigcup_{n>m} U_n = X$).*

(c) *In every strong open cover there exists an increasing subcover $\{U_n : n \in \omega\}$.*

Proof (cf. [GT], Theorem 4). (a)⇒(b) follows by Lemma 2. (b)⇒(c) is easy: given a strong open cover, use increasing finite sequences of its members as a covering tree.

We shall show (c)⇒(a). Assume (c). First, X is Lindelöf. Otherwise any finitely additive open cover with no countable subcover violates (c). Also, X is zerodimensional (has a clopen base). Indeed, there is no continuous function f from X onto $[0,1]$ (otherwise $\{f^{-1}[V] : V \subseteq [0,1]$ open with $[0,1]\setminus V$ uncountable$\}$ violates (c)). For completely regular spaces this means having a clopen base, and X, being Lindelöf regular, is completely regular.

Now, let $\mathcal{U}_n$ ($n \in \omega$) be a sequence of open covers of X. Since X is Lindelöf and zerodimensional, we can assume that each $\mathcal{U}_n$ is countable and consists of pairwise disjoint clopens (see [K], §26).

Let $\mathcal{U}_n = \{U_i^n : i < \omega\}$ (some U_i^n's may be empty). Define $f : X \mapsto {}^\omega\omega$ by

$$f(x)(n) = i \quad \text{iff} \quad x \in U_i^n .$$

Let $V = \bigcup\{U_i^n : |f[U_i^n]| \le \aleph_0\}$. Then $|f[V]| \le \aleph_0$ and since $\forall x \in X\ x \in \bigcap_n U^n_{f(x)(n)}$, it is not hard to see that there exist $t_{2n+1} \in \omega$ ($n \in \omega$) with $V \subseteq \bigcap_n U^{2n+1}_{t_{2n+1}}$. Suppose that $X \setminus V \ne \emptyset$ (otherwise we are done). We can easily refine the covers $\mathcal{U}_{2n}$ (with the help of $\mathcal{U}_{2m}$'s, $m \ge n$) to covers $\mathcal{W}_n$ so that $\mathcal{W}_{n+1}$ is a refinement of $\mathcal{W}_n$ and each $W \in \mathcal{W}_n$ which meets $X \setminus V$ contains at least two sets from $\mathcal{W}_{n+1}$ which meet $X \setminus V$.

Let $V_\sigma = V \cup \bigcup_{n<|\sigma|} W^n_{\sigma(n)}$ for $\sigma \in {}^{<\omega}\omega$ such that each $W^n_{\sigma(n)}$ meets $X \setminus V$. Then V_σ's constitute a strong open cover of X. Also, if $V_\sigma \subseteq V_\tau$ then $\sigma \subseteq \tau$ (because no W^n_k which meets $X \setminus V$ can be covered by finitely many sets taken from different $\mathcal{W}_m$ ($m > n$)).

It follows that from an increasing sequence of V_σ's covering X, which exists by (c), we get $s \in {}^\omega\omega$ with $X \setminus V \subseteq \bigcup_n W^n_{s(n)}$. Since there are t_{2n} ($n \in \omega$) with $W^n_{s(n)} \subseteq U^{2n}_{t_{2n}}$, we get $X \subseteq \bigcup_n U^n_{t_n}$. ■

Rothberger also considered property C', which is defined as C'' is but the covers $\mathcal{U}_n$ are finite. We can define a game corresponding to C' by introducing to G^* the requirement that the covers played by $\mathbf{W}$ are finite. Then an analogue of Lemma 2 is true (see the proof of (a)⇒(b) below). We also have the following. (A tree is *finitely branching* if the set of immediate successors of any node is finite.)

PROPOSITION 2. *Let X be a regular topological space. Then the following are equivalent.*

(a) *X has property C'.*

(b) *In every finitely branching covering tree there exists a branch $\langle U_n : n \in \omega\rangle$ with $\bigcup_n U_n = X$ (equivalently, with $\bigcap_m \bigcup_{n>m} U_n = X$).*

(c) *In every strong open cover $\mathcal{U}$ such that for each $U \in \mathcal{U}$, a finite subfamily of $\{V \in \mathcal{U} : U \subseteq V\}$ covers X, there exists an increasing subcover $\{U_n : n \in \omega\}$.*

Proof. We sketch (a)⇒(b); (b)⇒(c)⇒(a) are proved as in Proposition 1. Suppose that $X \in C'$. Let T be a finitely branching covering tree. For each $n \in \omega$, let $\mathcal{V}_n$ be a common finite refinement of all covers $\mathcal{U}_\sigma =_{\rm df} \{U : \sigma^\frown U \in T\}$ ($\sigma \in T$ and $|\sigma| = n$). Such a refinement exists because there are only finitely many covers to refine. Since $X \in C'$ there is a sequence $V_n \in \mathcal{V}_n$ such that $X = \bigcap_m \bigcup_{n>m} V_n$. Define a branch $\langle U_n : n \in \omega\rangle$ of T by $U_n =$ any $U \supseteq V_n$ such that $\langle U_0, \ldots, U_{n-1}, U\rangle \in T$. ■

Note. 1. It is folklore that every metric space $X \in C''$ is homeomorphic to a subspace of $\mathbb{R}$ (the reals). Such an X is zerodimensional (it cannot be continuously mapped onto $[0, 1]$ as C'' is preserved by continuous images and $[0, 1] \notin C''$; [K], §40). Being Lindelöf, X is separable, and so homeomorphic to a subset of ${}^\omega\omega$. Since every C'' set of reals has strong measure zero, Borel's conjecture implies that every C'' metric space is countable.

2. In the spirit of [R], $X \subseteq \mathbb{R}$ with property C'' can be characterized by any of the following (see [P]):

(a) for any $\mathbf{F}_\sigma$ (equivalently, closed) $A \subseteq \mathbb{R}^2$ with all vertical sections A_x ($x \in X$) meager, $\bigcup_{x\in X} A_x \neq \mathbb{R}$;

(b) for any closed $A \subseteq \mathbb{R}^2$ with all vertical sections A_x ($x \in X$) null, $\bigcup_{x\in X} A_x$ is null.

Also, $X \subseteq \mathbb{R}$ has strong measure zero iff any of the following holds:

(a) for any $\mathbf{F}_\sigma$ (equivalently, closed) $A \subseteq \mathbb{R}^2$ with all vertical sections A_x ($x \in \mathbb{R}$) meager, $\bigcup_{x\in X} A_x \neq \mathbb{R}$ ([AR]);

(b) for any closed $A \subseteq \mathbb{R}^2$ with all vertical sections A_x ($x \in \mathbb{R}$) null, $\bigcup_{x\in X} A_x$ is null ([P]);

(c) for any closed null $D \subseteq \mathbb{R}$, $X + D$ is null ([P]).

3. There exists (in ZFC) an uncountable C'' space in which every point is G_δ. Todorčević [T] has an example of a zerodimensional first countable Hausdorff space of size $\aleph_1$ whose every continuous image into any second countable space (in particular, into ω^ω) is countable.

QUESTION. In Propositions 1 and 2, can one remove the assumption that X is regular (it is used in (c)$\Rightarrow$(a))?

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF WROCŁAW
PL. GRUNWALDZKI 2/4
50-384 WROCŁAW, POLAND
E-mail: PAWLIKOW@PLWRUW11.BITNET

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