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Accessibility of typical points for invariant measures of positive Lyapunov exponents for iterations of holomorphic maps

by

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Abstract. We prove that if A is the basin of immediate attraction to a periodic attracting or parabolic point for a rational map f on the Riemann sphere, if A is completely invariant (i.e. $f^{-1}(A) = A$), and if μ is an arbitrary f-invariant measure with positive Lyapunov exponents on ∂A, then μ-almost every point $q \in \partial A$ is accessible along a curve from A. In fact, we prove the accessibility of every "good" q, i.e. one for which "small neighbourhoods arrive at large scale" under iteration of f.

This generalizes the Douady–Eremenko–Levin–Petersen theorem on the accessibility of periodic sources.

We prove a general "tree" version of this theorem. This allows us to deduce that on the limit set of a geometric coding tree (in particular, on the whole Julia set), if the diameters of the edges converge to 0 uniformly as the generation number tends to ∞, then every f-invariant probability ergodic measure with positive Lyapunov exponent is the image, via coding with the help of the tree, of an invariant measure on the full one-sided shift space.

The assumption that f is holomorphic on A, or on the domain U of the tree, can be relaxed and one need not assume that f extends beyond A or U.

Finally, we prove that if f is polynomial-like on a neighbourhood of $\overline{\mathbb{C}} \setminus A$, then every "good" $q \in \partial A$ is accessible along an external ray.

Introduction. Let $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be a rational map of the Riemann sphere $\overline{\mathbb{C}}$. Let $J(f)$ denote its Julia set. We say a periodic point p of period m is *attracting* (a *sink*) if $|(f^m)'(p)| < 1$, *repelling* (a *source*) if $|(f^m)'(p)| > 1$ and *parabolic* if $(f^m)'(p)$ is a root of unity. We say that $A = A_p$ is an *immediate basin of attraction* to a sink or a parabolic point p if A is a component of

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$\overline{\mathbb{C}} \setminus J(f)$ such that $f^{nm}|_A \to p$ as $n \to \infty$ and $p \in A_p$ for p attracting, and $p \in \partial A$ for p parabolic.

We call $q \in \partial A$ *good* if there exist real numbers $r > 0$, $\kappa > 0$, δ with $0 < \delta < r$ and an integer $\Delta > 0$ such that for every n large enough,

$$\sharp\{\text{good times}\}/n \geq \kappa\,. \tag{0.0}$$

We call here $\overline{n}$ ($0 \leq \overline{n} \leq n$) a *good time* if for each $0 \leq l \leq \overline{n} - \Delta$ the component $B_{\bar{n},l}$ of $f^{-(\bar{n}-l)}(B(f^{\bar{n}}(q), r))$ containing $f^l(q)$ satisfies

$$B_{\bar{n},l} \subset B(f^l(q), r - \delta)\,. \tag{0.1}$$

In the definition of good q we also assume that

$$\lim_{\bar{n}\to\infty} \operatorname{diam} B_{\bar{n},0} \to 0 \tag{0.2}$$

with lim taken over good $\overline{n}$'s.

Finally, in the definition of good q we assume that for each good $\overline{n}$,

$$f^{-\bar{n}}(A) \cap B_{\bar{n},0} \subset A\,. \tag{0.3}$$

We shall prove the following

Theorem A. *Every good $q \in \partial A$ is accessible from A, i.e. there exists a continuous curve $\gamma : [0,1] \to \overline{\mathbb{C}}$ such that $\gamma([0,1)) \subset A$ and $\gamma(1) = q$.*

Theorem A generalizes the Douady–Eremenko–Levin–Petersen theorem on the accessibility of periodic sources. Note that in the case of periodic sources one obtains curves of finite lengths along which a periodic q is accessible (see Section 1). Condition (0.1) holds for all $\overline{n}$'s in the case where q is a periodic source. Condition (0.3) is true if A is the basin of attraction to ∞ for f a polynomial, and more generally if A is completely invariant, i.e. $f^{-1}(A) = A$.

Condition (0.3) in the case of a source is equivalent to Petersen's condition [Pe].

Under the assumption of the complete invariance of A, μ-almost every point (for μ an invariant probability measure with positive Lyapunov exponents) is good, hence accessible (cf. Corollary 0.2).

In fact, we shall introduce in Section 2 a weaker definition of good q and prove Theorem A with that weaker definition. In that weaker definition parabolic periodic points in ∂A are good. The traces of telescopes built there can sit in an arbitrary interpetal, so one obtains the accessibility in each interpetal. One obtains in particular Theorem 18.9 of [Mi1].

Note that the above conditions of being good are already quite weak. In particular, we do not exclude critical points in $B_{\bar{n},l}$.

For example, every point in ∂A is good if A is the basin of attraction to ∞ for a polynomial $z \mapsto z^2 + c$ which is non-renormalizable, with c outside the "cardioid". This is Yoccoz–Branner–Hubbard theory (see [Mi2]). (In this case, however, Theorem A is worthless because one proves directly the local connectedness of ∂A, in particular one proves the existence of an infinite telescope.)

Note that complete invariance of A, the basin of attraction to a sink, does not imply that f is polynomial-like on a neighbourhood of $\overline{\mathbb{C}} \setminus A$. (Polynomial-like maps were first defined and studied in [DH].) In [P4] an example of degree 3 is described, of the form $z \to z^2 + c + \frac{b}{z-a}$, with a completely invariant basin of attraction to ∞, not simply connected, with only 2 critical points in the basin.

We prove in the present paper a theorem more general than Theorem A, namely a theorem on the accessibility along branches of a *geometric coding tree*. We now recall basic definitions from [P1, P2, PUZ, PS].

Let U be an open connected subset of the Riemann sphere $\overline{\mathbb{C}}$. Consider any holomorphic mapping $f : U \to \overline{\mathbb{C}}$ such that $f(U) \supset U$ and $f : U \to f(U)$ is a proper map. Write $\operatorname{Crit} f = \{z : f'(z) = 0\}$. This is the set of *critical points* for f. Suppose that $\operatorname{Crit} f$ is finite. Consider any $z \in f(U)$. Let $z^1, \ldots, z^d$ be all the f-preimages of z in U where $d = \deg f \geq 2$. (We emphasize that we consider here, in contrast to other papers, only the full tree, i.e. not just some preimages but all preimages of z in U.)

Consider smooth curves $\gamma^j : [0,1] \to f(U)$, $j = 1, \ldots, d$, joining z and z^j (i.e. $\gamma^j(0) = z$, $\gamma^j(1) = z^j$), such that there are no critical values for iterations of f in $\bigcup_{j=1}^d \gamma^j$, i.e. $\gamma^j \cap f^n(\operatorname{Crit} f) = \emptyset$ for every j and $n > 0$. We allow self-intersections of each γ^j.

Let $\Sigma^d := \{1, \ldots, d\}^{\mathbb{Z}^+}$ denote the one-sided shift space and σ the shift to the left, i.e. $\sigma((\alpha_n)) = (\alpha_{n+1})$. We consider the standard metric on Σ^d,

$$\varrho((\alpha_n), (\beta_n)) = \exp(-k((\alpha_n), (\beta_n))),$$

where $k((\alpha_n), (\beta_n))$ is the least integer for which $\alpha_k \neq \beta_k$.

For every sequence $\alpha = (\alpha_n)_{n=0}^\infty \in \Sigma^d$ we define $\gamma_0(\alpha) := \gamma^{\alpha_0}$. Suppose that for some $n \geq 0$, for every $0 \leq m \leq n$, and all $\alpha \in \Sigma^d$, the curves $\gamma_m(\alpha)$ are already defined. Suppose that for $1 \leq m \leq n$ we have $f \circ \gamma_m(\alpha) = \gamma_{m-1}(\sigma(\alpha))$, and $\gamma_m(\alpha)(0) = \gamma_{m-1}(\alpha)(1)$.

Define the curves $\gamma_{n+1}(\alpha)$ so that the previous equalities hold by taking suitable components of the f-preimages of the curves γ_n. For every $\alpha \in \Sigma^d$ and $n \geq 0$ define $z_n(\alpha) := \gamma_n(\alpha)(1)$.

For every $n \geq 0$ denote by $\Sigma_n = \Sigma_n^d$ the space of all sequences of elements of $\{1, \ldots, d\}$ of length $n+1$. Let π_n denote the projection $\pi_n : \Sigma^d \to \Sigma_n$ defined by $\pi_n(\alpha) = (\alpha_0, \ldots, \alpha_n)$. As $z_n(\alpha)$ and $\gamma_n(\alpha)$ depend only on $(\alpha_0, \ldots, \alpha_n)$, we can consider z_n and γ_n as functions on Σ_n.

The graph $\mathcal{T} = \mathcal{T}(z, \gamma^1, \ldots, \gamma^d)$ with vertices z and $z_n(\alpha)$ and edges $\gamma_n(\alpha)$ for all $n \geq 0$ is called a *geometric coding tree* with root at z. For every $\alpha \in \Sigma^d$ the subgraph composed of $z, z_n(\alpha)$ and $\gamma_n(\alpha)$ for all $n \geq 0$

is called a *geometric branch* and denoted by $b(\alpha)$. The branch $b(\alpha)$ is called *convergent* if the sequence $\gamma_n(\alpha)$ converges to a point in $\mathrm{cl}\, U$. We define the *coding map* $z_\infty : \mathcal{D}(z_\infty) \to \mathrm{cl}\, U$ by $z_\infty(\alpha) := \lim_{n\to\infty} z_n(\alpha)$ on the domain $\mathcal{D} = \mathcal{D}(z_\infty)$ of all α's for which $b(\alpha)$ is convergent.

In Sections 1–3, for any curve (maybe with self-intersections) $\gamma : I \to \overline{\mathbb{C}}$ where I is a closed interval in $\mathbb{R}$, we call γ restricted to any subinterval of I (maybe degenerate) a *part* of γ. Consider now γ on $J_1 \subset [0,1]$ and γ' on $J_2 \subset [0,1]$ with either both γ and γ' being parts of one $\gamma_n(\alpha)$, $J_1 \cap J_2 = \emptyset$, J_1 between 0 and J_2 , or γ a part of $\gamma_{n_1}(\alpha)$ and γ' a part of $\gamma_{n_2}(\alpha)$ where $n_1 < n_2$. Let $\Gamma : [0, n_2 - n_1 + 1] \to \overline{\mathbb{C}}$ be the concatenation of $\gamma_{n_1}, \gamma_{n_1+1}, \ldots, \gamma_{n_2}$. We call the restriction of Γ to the convex hull of $J_1 \subset [0,1]$ and $J_2 \subset [n_2 - n_1, n_2 - n_1 + 1]$ (we identified here $[0,1]$ with $[n_2 - n_1, n_2 - n_1 + 1]$) the *part of* $b(\alpha)$ *between* γ *and* γ' .

For every continuous map $F : X \to X$ of a compact space X denote by $M(F)$ the set of all probability F-invariant measures on X. In the case where X is a compact subset of the Riemann sphere $\overline{\mathbb{C}}$ and the map F extends holomorphically to a neighbourhood of X and $\mu \in M(F)$ we can consider, for μ-a.e. x, the *Lyapunov characteristic exponent*

$$\chi(F, x) = \lim_{n\to\infty} \frac{1}{n} \log |(F^n)'(x)|$$

(derivative in the standard spherical metric on $\overline{\mathbb{C}}$).

If μ is ergodic then for μ-a.e. x,

$$\chi(F, x) = \chi_\mu(F) = \int \log |F'| \, d\mu \, .$$

Since in this paper we discuss properties of μ-a.e. point, it is enough to consider only ergodic measures, because by the Rokhlin Decomposition Theorem every $\mu \in M(F)$ can be decomposed into ergodic measures.

Define

$$M_{\mathrm{e}}^{\chi+}(F) = \{\mu \in M(F) : \mu \text{ ergodic}, \ \chi_\mu(F) > 0\} \, ,$$
$$M_{\mathrm{e}}^{\mathrm{h}+}(F) = \{\mu \in M(F) : \mu \text{ ergodic}, \ \mathrm{h}_\mu(F) > 0\} \, ,$$

where h_μ denotes measure-theoretic entropy.

From the Ruelle Theorem it follows that $\mathrm{h}_\mu(F) \le 2\chi_\mu(F)$ (see [R]), so $M_{\mathrm{e}}^{\mathrm{h}+}(F) \subset M_{\mathrm{e}}^{\chi+}(F)$.

The basic theorem concerning convergence of geometric coding trees is the following:

Convergence Theorem. 1. *Every branch, except branches in a set of Hausdorff dimension* 0 *in the metric* ϱ *on* Σ^d, *is convergent* (*i.e.* $\mathrm{HD}(\Sigma^d \setminus \mathcal{D}) = 0$). *In particular, for every* $\nu \in M_{\mathrm{e}}^{\mathrm{h}+}(\sigma)$ *we have* $\nu(\Sigma^d \setminus \mathcal{D}) = 0$, *so the measure* $(z_\infty)_*(\nu)$ *makes sense.*

2. *For every* $z \in \mathrm{cl}\, U$, $\mathrm{HD}(z_\infty^{-1}(\{z\})) = 0$. *Hence for every* $\nu \in M(\sigma)$ *we have* $\mathrm{h}_{\nu_\varphi}(\sigma) = \mathrm{h}_{(z_\infty)_*(\nu_\varphi)}(\bar{f}) > 0$ (*provided we assume that there exists a continuous extension* $\bar{f}$ *of* f *to* $\mathrm{cl}\, U$).

The proof of this theorem can be found in [P1] and [P2] under some assumptions on slow convergence of $f^n(\mathrm{Crit}\, f)$ to γ^j as $n \to \infty$, and in [PS] in full generality (even with $f^n(\mathrm{Crit}\, f) \cap \gamma^j \neq \emptyset$ allowed).

Let $\widehat{\Lambda} \subset \mathrm{cl}\, U$ denote the set of all limit points of $f^{-n}(z)$, $n \to \infty$. Analogously to the case $q \in \partial A$ we say that $q \in \widehat{\Lambda}$ is *good* if f extends holomorphically to a neighbourhood of $\{f^n(q) : n = 0, 1, \ldots\}$ (we use the same symbol f to denote the extension) and conditions (0.0′), (0.1′), (0.2′) and (0.3′) hold. These are defined similarly to (0.0)–(0.3), with A replaced by U and ∂A replaced by $\widehat{\Lambda}$.

Again recall that we shall give a precise weaker definition of q good in Section 2 and prove Theorem B with that weaker definition. That definition will not require extending f beyond U.

THEOREM B. *Let* $f : U \to \overline{\mathbb{C}}$ *be a holomorphic mapping and* $\mathcal{T}$ *be a geometric coding tree in* U *as above. Suppose*

$$\operatorname{diam} \gamma_n(\alpha) \to 0 \quad \text{as } n \to \infty \tag{0.4}$$

uniformly in $\alpha \in \Sigma^d$. *Then each good* $q \in \widehat{\Lambda}$ *is the limit point of a branch* $b(\alpha)$.

Using a lemma belonging to Pesin's theory (see Section 2) we prove that μ-a.e. q is good and easily obtain the following

COROLLARY 0.1. *Let* $f : U \to \overline{\mathbb{C}}$ *be a holomorphic mapping and* $\mathcal{T}$ *be a geometric coding tree in* U *such that condition* (0.4) *holds. If* μ *is a probability measure on* $\widehat{\Lambda}$ *and the map* f *extends holomorphically from* U *to a neighbourhood of* $\mathrm{supp}\, \mu$ *so that* $\mu \in M_{\mathrm{e}}^{\chi+}(f)$, *then for* μ*-almost every* $q \in \widehat{\Lambda}$ *satisfying* (0.3′) *there exists* $\alpha \in \Sigma^d$ *such that* $b(\alpha)$ *converges to* q. *In particular*, *every* μ *such that* μ*-a.e.* q *satisfies* (0.3′) *is the* $(z_\infty)_*$*-image of a measure* $m \in M(\sigma)$ *on* Σ^d.

Note that Corollary 0.1 concerns in particular every μ with $\mathrm{h}_\mu(f) > 0$. Assuming that f extends holomorphically to a neighbourhood of $\widehat{\Lambda}$ and referring also to the Convergence Theorem we see that $(z_\infty)_*$ maps $M_{\mathrm{e}}^{\mathrm{h}+}(\sigma)$ onto $M_{\mathrm{e}}^{\mathrm{h}+}(f|_{\widehat{\Lambda}})$ preserving entropy.

The question whether this correspondence is *onto* is stated in [P3]. Thus Corollary 0.1 answers this question in the affirmative under the additional assumptions (0.3′) and (0.4).

We do not know whether this correspondence is finite-to-one except for measures supported by orbits of periodic sources for which the answer is positive (see Proposition 1.2).

Two special cases are of particular interest. The first one corresponds to Theorem A:

Corollary 0.2. *Let $f:\overline{\mathbb{C}}\to\overline{\mathbb{C}}$ be a rational mapping and A be a completely invariant basin of attraction to a sink or a parabolic point. Then for every $\mu\in M_{\mathrm{e}}^{\chi+}(f|_{\partial A})$, μ-a.e. $q\in\partial A$ is accessible from A.*

Corollary 0.3. *Let $f:\overline{\mathbb{C}}\to\overline{\mathbb{C}}$ be a rational mapping*, $\deg f=d$, *and $\mathcal{T}=\mathcal{T}(z,\gamma^1,\ldots,\gamma^d)$ be a geometric coding tree. Assume* (0.4). *Let $\mu\in M_{\mathrm{e}}^{\chi+}(f)$. Then for μ-a.e. q there exists $\alpha\in\Sigma^d$ such that $b(\alpha)$ converges to q.*

In Theorem A and Corollary 0.2, in the case of f being a polynomial (or a polynomial-like map) and A the basin of attraction to ∞, accessibility of a point along a curve often automatically implies accessibility along an external ray. For A simply connected this follows from Lindelöf's Theorem. External rays are defined as the images under the standard Riemann map of rays $t\zeta$, $\zeta\in\partial\mathbb{D}$, $1<t<\infty$.

If A is not simply connected one should first define external rays in the absence of the Riemann map. This is done in [GM] and [LevS] for f a polynomial, and in [LevP] in the polynomial-like situation. We recall these definitions in Section 3.

In Section 3, we prove the following

Theorem C. *Let $W_1\subset W$ be open*, *connected*, *simply connected domains in $\overline{\mathbb{C}}$ such that* $\mathrm{cl}\, W_1\subset W$, *and let $f:W_1\to W$ be a polynomial-like map. Define $K=\bigcap_{n\ge0}f^{-n}(W)$. Then every good $q\in\partial K$ is accessible along an external ray in $W\setminus K$.*

An alternative way to prove accessibility along an external ray is to use Lindelöf's Theorem somehow, as in the simply connected case. This is performed in [LevP]. It is proved there that if q is accessible along a curve in $W\setminus K$ and q belongs to a periodic or preperiodic component $K(q)$ of K then it is accessible along an external ray.

Note also that for every $q\in\partial K$, if $K(q)$ is one point then q is accessible along an external ray. This is easy (see [GM, Appendix] and [LevP]).

Remark 0.4 (Proof of Theorem A from B and Corollary 0.2 from 0.1). We do not know how to get rid of the assumption (0.4) in Theorem B and Corollary 0.1. In Theorem A and Corollary 0.2 this condition is guaranteed automatically. More precisely, to deduce Theorem A from B and Corollary 0.2 from 0.1 we consider an arbitrary tree $\mathcal{T}=\mathcal{T}(z,\gamma^1,\ldots,\gamma^d)$ in A, where $d=\deg f|_A$, so that $\gamma^j\cap\bigcup_{n>0}f^n(\mathrm{Crit}\, f)=\emptyset$ and $p\notin\bigcup_{j=1}^d\gamma^j$. Only critical points in A count here. The forward orbits of these critical points converge to p, hence the following condition holds:

$$\Big(\bigcup_{j=1}^{d}\gamma^j\Big)\cap \operatorname{cl}\bigcup_{n>0} f^n(\operatorname{Crit} f)=\emptyset\,. \tag{0.5}$$

Hence we can take open discs $U^j\supset\gamma^j$ such that

$$\bigcup_{j=1}^{d} U^j\cap \operatorname{cl}\bigcup_{n>0} f^n(\operatorname{Crit} f)=\emptyset$$

and consider univalent branches $F_n(\alpha)$ of f^{-n} mapping a suitable γ^j to $\gamma_n(\alpha)$. $\{F_n(\alpha)\}_{\alpha,n}$ is a normal family of maps. If it had a non-constant limit function G then we would find an open domain $V\subset U$ in the range of G such that $f^{n_t}(V)\subset U^j$ for an integer j, as $n_t\to\infty$. On the other hand, $f^{n_t}(V)\to p$. If we assumed $p\notin U^j$ we would arrive at a contradiction. This proves (0.4). Finally, by the complete invariance of A every $q\in\widehat{\Lambda}$ satisfies (0.3) and we have $\widehat{\Lambda}=\partial A$.

In Corollary 0.3, to find $\mathcal{T}$ such that (0.4) holds it is enough to assume that the forward limit set of $f^n(\operatorname{Crit} f)$ does not dissect $\overline{\mathbb{C}}$, because then we find $\mathcal{T}$ so that (0.5) holds, which easily implies (0.4).

We believe, however, that in the proof of Corollary 3 one can omit (0.4), or maybe often find a tree such that (0.4) holds.

Remark 0.5. Observe that there are examples where (0.4) does not hold. Take for example z in a Siegel disc or z being just a sink. Even if $J(f)=\overline{\mathbb{C}}$ one should be careful: for M. Herman's examples

$$z\mapsto \lambda z\frac{z-a}{1-\bar{a}z}\Big/\frac{z-b}{1-\bar{b}z},\qquad |\lambda|=1,\ a\neq 0\neq b,\ a\approx b$$

(see [H1]), the unit circle is invariant and for a branch in it (0.4) fails. These examples are related to the notion of neutral sets (see [GPS]).

Remark 0.6. The assumption that f is holomorphic on U (or A) can be replaced by the assumption that f is just a continuous map, a branched cover over $f(U)\supset U$.

However, without the holomorphy of f we do not know how the assumption (0.4) could be verified.

Remark 0.7. The fact that in, say, Theorem A we do not need to assume that f extends holomorphically beyond the basin A suggests that maybe the assumption (0.3) is substantial and without it the accessibility in Theorem A is not true. We have in mind here the analogous situation of a Siegel disc with boundary not simply connected, where the map is only smooth beyond it (see [H2]). Accessibility of periodic sources in the boundary of A in the absence of the assumption (0.3) is a famous open problem and we think that if the answer is positive one should substantially use in the proof the holomorphy of f outside A.

The paper is organized as follows: in Section 1 we prove Theorem B for q a periodic source, in Section 2 we deal with the general case. The case of sources was known in the polynomial-like and parabolic situations ([D], [EL], [Pe]). The general case contains the case of sources but it is more tricky (though not more complicated) so we decided to separate the case of sources to make the paper more understandable. Section 3 is devoted to Theorem C.

1. Accessibility of periodic sources

THEOREM D. *Let $f : U \to \overline{\mathbb{C}}$ be a holomorphic map and $\mathcal{T}(z, \gamma^1, \ldots, \gamma^d)$ be a geometric coding tree in U, with $d = \deg f|_U$. Assume* (0.4). *Next assume that f extends holomorphically to a neighbourhood of a family of points $q_0, \ldots, q_{n-1} \in \widehat{\Lambda}$ in such a way that this family is a periodic repelling orbit of period n for this extension* (*the extension is also denoted by f*).

Assume finally that there exists a neighbourhood V of $q = q_0$ on which f^n is linearizable and if F is its inverse on V such that $F(q) = q$ then

$$F(V \cap U) \subset U . \tag{1.1}$$

Then there exists a periodic $\alpha \in \Sigma^d$ such that $b(\alpha)$ converges to q. Moreover, *the convergence is exponential*, *in particular the curve which is the body of $b(\alpha)$ is of finite length.*

Proof. As usual, we can suppose that q is a fixed point by passing to the iterate f^n if $n > 1$.

Assume that $q \neq z$. We shall deal with the case $q = z$ later.

Let h denote a linearizing map, i.e. h conjugates f on a neighbourhood of $\operatorname{cl} V$ to $z \to \lambda z$ with $\lambda = f'(q)$, and maps q to $0 \in \mathbb{C}$.

Replace if necessary the set V by a smaller neighbourhood of q so that $z \notin V$ and $\partial V = h^{-1} \exp(\{\Re\xi = a\})$ for a constant $a \in \mathbb{R}$.

For every set $K \subset (\operatorname{cl} V) \setminus \{q\}$ consider its diameter in the radial direction (with origin at q) in the logarithmic scale, i.e. the diameter of the projection of the set $\log h(K)$ to the real axis. This will be denoted by $\operatorname{diam}_{\Re \log} K$.

For every $m \geq 0$ write

$$R_m := h^{-1} \exp(\{\zeta \in \mathbb{C} : a - (m+1) \log |\lambda| < \Re\zeta < a - m \log |\lambda|\})$$

and

$$V_m := h^{-1} \exp(\{\zeta \in \mathbb{C} : \Re\zeta < a - m \log |\lambda|\}) .$$

Observe the following important property of $\gamma_n(w)$'s, $n \geq 0$, $w \in \Sigma^d$:

For every $\varepsilon > 0$ there exists $N(\varepsilon)$ such that if a component γ of $\gamma_n(w) \cap R_m$ satisfies

$$\operatorname{diam}_{\Re \log} \gamma > \varepsilon \log |\lambda| \quad \text{and} \quad z_n(w) \in V_m \tag{1.2}$$

then

$$0 \le n - m < N(\varepsilon)\,. \tag{1.3}$$

Indeed, by (1.2) for every $t = 0, 1, \ldots, m$ we have $f^t(z_n(w)) \in V_{m-t}$ so $f^t(z_n(w)) \neq z$. Hence $n \ge m$. On the other hand, we have

$$\varepsilon \le \operatorname{diam}_{\Re \log} \gamma = \operatorname{diam}_{\Re \log} f^m(\gamma) \le \operatorname{Const} \operatorname{diam} f^m(\gamma)\,.$$

So from (0.4) and from the estimate $\operatorname{diam} f^m(\gamma_n(w)) = \operatorname{diam} \gamma_{n-m}(\sigma^m(w)) \ge \varepsilon$, we deduce that $n - m$ is bounded by a constant depending only on ε. This proves (1.3).

Fix topological discs $U^1, \ldots, U^d$ which are neighbourhoods of $\gamma^1, \ldots, \gamma^d$ respectively such that $\bigcup_{i=1}^{N(\varepsilon)} f^i(\operatorname{Crit} f) \cap U^j = \emptyset$ for every $j = 1, \ldots, d$.

(There is a minor inaccuracy here because this concerns the case where the curves γ^j are embedded. If they have self-intersections we should cover them by families of small discs and later lift them by branches of f^{-t} one by one along each curve.)

For every γ which is part of $\gamma_n(w)$ satisfying (1.2) we can consider

$$W_1 = F_{n-(m-1)}(\sigma^{m-1}(w))(U^j)\,,$$

which is a neighbourhood of $f^{m-1}(\gamma_n(w))$. We have used here the notation $F_t(v)$ for the branch of f^{-t} mapping γ^j to $\gamma_t(v)$, $v \in \Sigma^d$. Here $j = v_t$.

Next consider the component W_2 of $W_1 \cap V$ containing $f^{m-1}(\gamma)$. Using Koebe's Bounded Distortion Theorem we can find a disc

$$W(\gamma) = B(x, \operatorname{Const} \varepsilon \lambda^{-m}) \tag{1.4}$$

in $F^{m-1}(W_2)$ with $x \in \gamma$ such that f^n maps $W(\gamma)$ univalently into U^j. We take Const such that

$$\operatorname{diam}_{\Re \log} W(\gamma) < \tfrac{1}{2} \log |\lambda|\,. \tag{1.5}$$

(Note that this part is easier if (0.5) is assumed. Then we just consider U^j's disjoint from $\operatorname{cl} \bigcup_{n=1}^{\infty} f^n(\operatorname{Crit} f)$.)

By the definition of $\widehat{\Lambda}$ there exist $n_0 \ge 0$ and $\alpha \in \Sigma^d$ such that $\gamma_{n_0}(\alpha) \cap V \neq \emptyset$. By (1.1) there exist $\beta_1, \beta_2, \ldots$ in $\{1, \ldots, d\}$ such that for each $k \ge 0$ we have

$$F^k(b(\alpha)) = b(\beta_k, \beta_{k-1}, \ldots, \beta_1, \alpha)\,.$$

More precisely, we consider an arbitrary component $\widehat{\gamma}$ of $\gamma_{n_0}(\alpha) \cap V$ and extend F^k from it holomorphically along $b(\alpha)$.

Denote for abbreviation the concatenation $\beta_k \beta_{k-1} \ldots \beta_1 \alpha$ by $k]\alpha$.

Denote also $F^k(\widehat{\gamma})$ by $\widehat{\gamma}_{k]}$ and the part of $\gamma_{n_0+k}(k]\alpha)$ between $\widehat{\gamma}_{k]}$ and $z_{n_0+k-1}(k]\alpha)$ by $\gamma_{k]}$.

For each $k \ge 0$ denote by $\mathcal{N}_k$ the set of all pairs of integers (t, m) such that $0 \le t \le k + n_0$, $0 < m < k$ and either $\gamma_t(k]\alpha)$ satisfies (1.2) for γ a part of $\gamma_t(k]\alpha)$ if $t < k + n_0$, or $\gamma_t(k]\alpha)$ satisfies (1.2) except that we do not assume $z_t(k]\alpha) \in V_m$ for γ a part of $\gamma_{k]}$ if $t = k + n_0$. Additionally we assume

$$\{\text{the part of } b(\alpha) \text{ between } \gamma \text{ and } \widehat{\gamma}_{k]}\} \subset V_m\,. \tag{1.6}$$

We write in this case $W(\gamma) = W_{k,t,m}$ and $\gamma = \gamma_{k,t,m}$. Figure 1 illustrates our definitions.

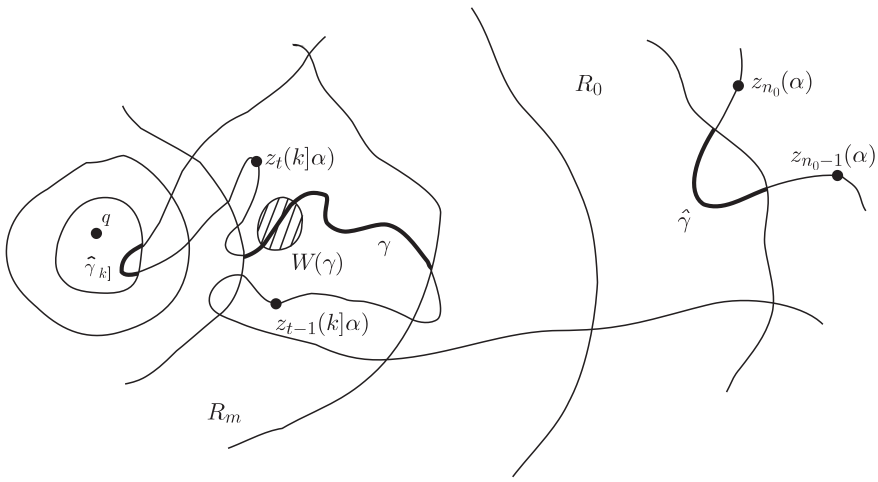

Fig. 1

We now have two possibilities:

1. For every $k_2 > k_1 \ge 0$, $0 < m_1 < k_1$, $0 < m_2 < k_2$ and $0 \le T \le k_1 + n_0$ such that $(T, m_1) \in \mathcal{N}_{k_1}$, $(T, m_2) \in \mathcal{N}_{k_2}$, if there is equality of the Tth entries $(k_1]\alpha)_T = (k_2]\alpha)_T$, then

$$W_{k_1,T,m_1} \cap W_{k_2,T,m_2} = \emptyset\,.$$

(Equality of the Tth entries means that $f^T(W_{k_1,T,m_1})$, $f^T(W_{k_2,T,m_2})$ are in the same U^j.)

2. Case 1 does not hold, which obviously implies the existence of T and the other integers as above such that $\pi_T(k_1]\alpha) = \pi_T(k_2]\alpha)$ (i.e. the blocks of $k_1]\alpha$ and $k_2]\alpha$ from 0 to T are the same).

Later we shall prove that case 1 leads to a contradiction. Now we prove that case 2 allows us to find a periodic branch converging to q, which proves our theorem.

Define $K = k_2 - k_1$. Recall that

$$\pi_T(\sigma^K(k_2]\alpha)) = \pi_T(k_1]\alpha) = \pi_T(k_2]\alpha)\,.$$

Denote $k_2]\alpha$ by ϑ. By the above we get

$$f^K(z_{T+K}(\vartheta)) = z_T(\vartheta)\,.$$

Writing this with the help of F which is the inverse of f on V so that $F(q) = q$ we have $F^K(z_T(\vartheta)) = z_{T+K}(\vartheta)$. We also know that $\gamma := \bigcup_{t=T+1}^{T+K} \gamma_t(\vartheta)$, which is a curve joining $z_T(\vartheta)$ and $z_{T+K}(\vartheta)$, is contained in V by (1.6).

Hence the curve $\Gamma := \bigcup_{n\ge 0} F^{nK}(\gamma)$ is the body of the part starting from the Tth vertex of the periodic branch $(\vartheta_0, \ldots, \vartheta_{K-1}, \vartheta_0, \ldots, \vartheta_{K-1}, \vartheta_0, \ldots)$.

To finish the proof of Theorem D we should now eliminate the disjointness case 1. We just prove there is not enough room for that case to hold.

For every $k \ge 0$ define

$$A_k^+ := \{m : 0 < m < k,\ \text{there exists } t \text{ such that } (t,m) \in \mathcal{N}_k\}\,.$$

Let $A_k^- := \{1, \ldots, k-1\} \setminus A_k^+$.

As $\gamma_{k+n_0}(k]\alpha)$ intersects V_k (at $\widehat{\gamma}_{k]}$), each R_m, $0 < m \le k-1$, is fully intersected by the curve built from the components γ of the intersection of R_m with the curves $\gamma_t(k]\alpha), t = 0, \ldots, k+n_0-1$, and $\gamma_{k]}$ such that (1.6) is satisfied.

For each such γ, $\operatorname{diam}_{\Re\log}\gamma < \varepsilon\log\lambda$ so to cross R_m one needs at least $[\varepsilon^{-1}]+1$ edges $\gamma_t(k]\alpha)$ (where $\gamma_{k+n_0}(k]\alpha)$ means $\gamma_{k]}$). We have only $k+n_0+1$ edges at our disposal so

$$\sharp A_k^- \varepsilon^{-1} \le 2(k+n_0+1)\,.$$

The coefficient 2 accounts for the possibility that one γ intersects R_m and R_{m+1}, where $m, m+1 \in A_k^-$ (it cannot intersect more than two R_m's because $\operatorname{diam}_{\Re\log}\gamma < \varepsilon$).

Hence

$$\sharp A_k^- \le 2(k+n_0+1)\varepsilon\,.$$

So

$$\sharp A_k^+ \ge k - 2(k+n_0+1)\varepsilon - 1 \ge k(1-3\varepsilon) \tag{1.7}$$

for k large enough.

From now on, fix $\varepsilon = 1/4$. Fix an arbitrarily large k_0. Let

$$\mathcal{N}^+ = \bigcup_{0\le k\le k_0} \{(k,(t,m)) : (t,m) \in \mathcal{N}_k\}\,.$$

Observe that each point $\xi \in V$ belongs to at most

$$4dN(1/4) \tag{1.8}$$

sets $W(k,t,m)$ where $(k,(t,m)) \in \mathcal{N}^+$.

Indeed, if $W(k_1,t_1,m_1)\cap W(k_2,t_2,m_2)\neq\emptyset$ then $|m_1-m_2|\leq 1$ by (1.5), and by (1.3) we have

$$|m_i-t_i|<N(1/4), \quad i=1,2,$$

hence

$$|t_1-t_2|<2N(1/4).$$

(If $t_i=k_i+n_0$ for $i=1$ or 2 we cannot in fact refer to (1.3). The trouble is with its $n-m>0$ part, because we do not know whether $z_{k_i+n_0}\in V_{m_i}$. But then directly $m_i<k_i\leq t_i$.)

But we assumed (this is our case 1) that for any fixed t, m and j all the sets $W(k,t,m)$ with the tth entry of $k]\alpha$ equal to j, and k varying, are pairwise disjoint. This finishes the proof of the estimate (1.8).

The conclusion from (1.8) and (1.4) is that because of lack of room, $\sharp\mathcal{N}^+<\mathrm{Const}\,k_0$.

On the other hand, (1.7) gives

$$\sharp\mathcal{N}^+\geq\sum_{k=0}^{k_0}\sharp A_k^+\geq k_0^2(1-3\varepsilon).$$

We have arrived at a contradiction for $\varepsilon=1/4$ and k_0 large enough.

The disjointness case 1 is eliminated. Theorem D in the case $z\neq q$ is proved.

Consider the case $z=q$. Then, unless $\gamma^j\equiv q$ in which case the assertion is trivial, the role of z in the above proof can be played by arbitrary $w^j\in\gamma^j\setminus\{q\}$. Formally on the level 0 we now have d^2 curves joining each w^j to the preimages of w^i in $\gamma_1((j,i))$. ■

Remark 1.1. Under the assumption $z\neq q$ and moreover $q\notin\bigcup_{j=1}^d\gamma^j$ (which is the case when we apply Theorem B to prove Theorem A) observe that there exists a constant M such that for every $n\geq 0$ and $\vartheta\in\Sigma^d$ we have $\mathrm{diam}_{\Re\log}\gamma_n(\vartheta)<M$.

Indeed, let $m=m_1\geq 0$ be the smallest integer such that $\gamma_n(\vartheta)$ intersects R_m and let m_2 be the largest one. Suppose that $m_2-m_1>1$. Then by (1.3), $n<m_1+1+N(1)$ and $m_2\leq n$. (The role of $z_n(\vartheta)$ in the proof of this part of (1.3) is played by $V_{m_2}\cap\gamma_n(\vartheta)$.) Thus $m_2-m_1\leq N(1)$.

This observation allows one to modify (simplify) slightly the proof of Theorem B. One does not need (1.6) then.

PROPOSITION 1.2. *Every branch $b(\alpha)$ converging to a periodic source q is periodic* (*i.e. α is periodic*). *There are only a finite number of α's such that $b(\alpha)$ converges to q.*

Proof. Suppose $z\neq q$ and $b(\alpha)$ converges to q. Take a neighbourhood V of q, arbitrarily small. Then the constant n_0 (see the proof of Theorem D) will depend on it. However, the above proof shows that

$$\pi_T(k_1]\alpha) = \pi_T(k_2]\alpha)$$

for $k_1 - k_2$ bounded by a constant independent of n_0. Observe also that $z \neq q$ implies that $T \to \infty$ as V shrinks to q. So there exists a finite block of symbols β such that $\alpha = \beta\beta\beta\ldots\beta\alpha'$ (α' infinite) with arbitrarily many β's. So α is periodic. This consideration also gives a bound for the period of α, hence it proves finiteness of the set of α's with $b(\alpha)$ converging to q. ∎

Note that with some additional effort we could obtain an estimate for the number of branches converging to q. For q in the boundary of the basin of attraction to a sink this estimate should give the so-called Pommerenke–Levin–Yoccoz inequality (see for example [Pe]).

2. Theorem B and Corollary 0.1. Given a holomorphic map $f : U \to \overline{\mathbb{C}}$ and $\mathcal{T} = \mathcal{T}(z, \gamma^1, \ldots, \gamma^d)$ a geometric coding tree in U as in the introduction we shall give a more general definition of $q \in \widehat{\Lambda}$ being good.

Let us start with some preliminary definitions:

DEFINITION 2.1. $D \subset U$ is called n_0-*significant* if there exist $\alpha \in \Sigma^d$ and $0 \leq n \leq n_0$ such that $\gamma_n(\alpha) \cap D \neq \emptyset$.

DEFINITION 2.2. For every $\delta, \kappa > 0$ and integer $k > 0$ a pair of sequences $(D_t)_{t=0,1,\ldots,k}$ and $(D_{t,t-1})_{t=1,\ldots,k}$ is called a *telescope* or a (δ,κ,k)-*telescope* if each D_t is an open connected subset of U, there exists a strictly increasing sequence of integers $0 = n_0, n_1, \ldots, n_k$ such that each $D_{t,t-1}$ is a nonempty component of $f^{-(n_t - n_{t-1})}(D_t)$ contained in D_{t-1} (of course $f^{n_t - n_{t-1}}$ can have critical points in $D_{t,t-1}$),

$$t/n_t > \kappa \quad \text{for each } t\,, \tag{2.0}$$

and

$$\operatorname{dist}(\partial_U^{\text{ess}} D_{t,t-1}, \partial_U D_{t-1}) > \delta\,. \tag{2.1}$$

Here the subscript U means the boundary in U, and the *essential boundary* $\partial_U^{\text{ess}} D_{t,t-1}$ is defined as $\partial_U D_{t,t-1} \setminus \bigcup_{n=1}^{n_t - n_{t-1}} f^{-n}(\partial U)$.

DEFINITION 2.3. A (δ,κ,k)-telescope is called n_0-*significant* if D_k is n_0-significant.

DEFINITION 2.4. For any (δ,κ,k)-telescope we can choose inductively sets $D_{t,l}$, where $l = t-2, t-3, \ldots, 0$, by taking $D_{t,l-1}$ to be a component of $f^{-(n_l - n_{l-1})}(D_{t,l})$ in $D_{t-1,l-1}$. We call the sequence of the resulting sets

$$D_{k,0} \subset D_{k-1,0} \subset \ldots \subset D_{1,0} \subset D_0$$

the *trace* of the telescope.

Definition 2.5. We call $q \in \widehat{\Lambda}$ *good* if there exist $\delta, \kappa > 0$, an integer $n_0 \geq 0$ and a sequence of n_0-significant (δ,κ,k)-telescopes Tel^k, $k = k_1, k_2, \ldots \to \infty$, with traces $D^k_{k,0} \subset D^k_{k-1,0} \subset \ldots \subset D^k_0$ respectively (we add the superscript k to the notation of each object relative to the telescope Tel^k) such that

$$D^k_{l,0} \to q \quad \text{as } l \to \infty \text{ uniformly in } k\,. \tag{2.2}$$

Remark 2.6. If $q \in \widehat{\Lambda}$ is good in the sense of the introduction (i.e. conditions $(0.0')$–$(0.3')$ are satisfied) then it is of course good in the above sense. Indeed, we choose each Δ's good time and denote these times by $n_0, n_1, \ldots$; of course then κ in (2.0) is κ/Δ for the old κ from (0.0).

For each k we define a telescope Tel^k by taking as D^k_k an arbitrary n_0-significant component of $B(f^{n_k}(q), r) \cap U$. Such a component exists with n_0 depending only on r because the set of all vertices of the tree $\mathcal{T}$ is by definition dense in $\widehat{\Lambda}$. Then inductively for each $0 \leq t < k$ we choose as D^k_t the component of $B(f^{n_t}(q), r) \cap U$ containing a component $D^k_{t+1,t}$ of $f^{-(n_{t+1}-n_t)}(D^k_{t+1})$ (such a component $D^k_{t+1,t}$ exists by $(0.3')$). By $(0.2')$ any choice of traces will be OK for (2.2).

Of course in the case of $U = A$, a basin of immediate attraction to a sink or a parabolic point, one can build telescopes with $D^k_{t,l}$ containing no critical points, but there is no reason for that to be possible in general.

Proof of Theorem B. Let $q \in \widehat{\Lambda}$ be a good point according to the definition above. Fix constants δ, κ and n_0 and a sequence of (δ,κ,k)-telescopes and their traces, $k = 0, 1, \ldots$, as in Definition 2.5.

We can assume that $z \notin D^k_0$ or at least that each γ^j, $j = 1, \ldots, d$, has a point outside D^k_0. If this is not so then either there exists l such that each γ^j has a point outside $D^k_{l,0}$ for every k, in which case in the considerations below we should consider $m \geq l$ rather than $m > 0$, or else there exists j such that $\gamma^j \equiv q$, in which case obviously $b(j,j,j,\ldots)$ converges to q.

Denote $D^k_{m,0} \setminus D^k_{m+1,0}$ by R^k_m for $m = 0, 1, \ldots, k-1$, and $D^k_{k,0}$ by R^k_k. These sets replace the annuli from Section 1.

Choose for each k a curve $\gamma_{n(k)}(\alpha^k)$ for $\alpha^k \in \Sigma^d$ and $n(k) \leq n_0$ intersecting D^k_k. Choose a part $\widehat{\gamma}^k$ of $\gamma_{n(k)}(\alpha^k)$ in this intersection.

As in Section 1 there exists $k]\alpha^k = \beta^k_0 \beta^k_1 \ldots \beta^k_{n^k_k - 1} \alpha^k \in \Sigma^d$ such that $\gamma_{n(k)+n^k_k}(k]\alpha^k)$ intersects $D^k_{k,0}$ and moreover it contains a part $\widehat{\gamma}_{k]}$ which is a lift of $\widehat{\gamma}^k$ by f^{n_k}. Denote the part of $\gamma_{n(k)+n^k_k}(k]\alpha^k)$ between $z_{n(k)+n^k_k-1}(k]\alpha^k)$ and $\widehat{\gamma}_{k]}$ by $\gamma_{k]}$.

Fix an integer $E > 0$ to be specified later.

Define $\mathcal{N}_k$ to be the set of pairs (t,m) such that $0 < m < k, 0 \leq t \leq n^k_k + n(k)$, there exist integers $E_1, E_2 \geq 0$ with $E_1 + E_2 < E$ such that

$\gamma_{t+E_2}(k]\alpha^k) \cap R^k_{m+1} \neq \emptyset$, $\gamma_{t-E_1}(k]\alpha^k) \cap R^k_{m-1} \neq \emptyset$ and there exists a part $\gamma(t,m)$ of $\gamma_t(k]\alpha^k)$ in R^k_m, or of $\gamma_{k]}$ if $t = n^k_k + n(k)$, such that

$$\text{(2.3)} \qquad \{\text{the part of } b(k]\alpha) \text{ between } \gamma(t,m) \text{ and } \widehat{\gamma}_{k]}\} \subset D^k_{m,0}\,,$$

analogously to (1.6) (see Figure 2).

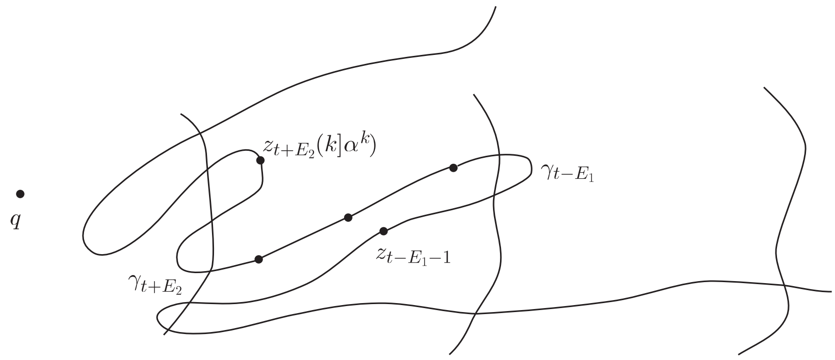

Fig. 2

We claim that analogously to the right hand inequality of (1.3), for $(t,m) \in \mathcal{N}_k$ we have

$$\text{(2.4)} \qquad t \leq n^k_{m+1} + E + N(\delta/E)$$

where $N(\varepsilon) := \sup\{n\text{: there exists } \alpha \in \Sigma^d \text{ such that } \operatorname{diam}\gamma_n(\alpha) \geq \varepsilon\}$. (The number $N(\varepsilon)$ is finite by (0.4).)

Indeed, denote by Γ the part of the concatenation of $\gamma_l(k]\alpha^k), l = t - E_1, \ldots, t+E_2$, in R^k_m joining R^k_{m-1} and R^k_{m+1}. Suppose that $t - E_1 \geq n^k_m$ (otherwise the claim is proved). Then $f^{n^k_m}(\Gamma)$ joins a point $\xi \in \partial_U D^k_{m+1,m}$ lying on the curve

$$f^{n^k_m}(\gamma_{t'}(k]\alpha^k)) = \gamma_{t'-n^k_m}(\sigma^{n^k_m}(k]\alpha^k)), \qquad t - E_1 \leq t' \leq t + E_2\,,$$

to $\partial_U D^k_m$.

If $\xi \notin \partial^{\text{ess}}_U D^k_{m+1,m}$, then

$$t' < n^k_{m+1}\,.$$

Indeed, otherwise there exists $n \leq n^k_{m+1}$ such that $f^n(\xi) \in \partial U$. This is already outside U so the trajectory of ξ hits $\bigcup \gamma^j$ before time n^k_{m+1}.

If $\xi \in \partial^{\text{ess}}_U D^k_{m+1,m}$ then by (2.1) at least one of the curves $f^{n^k_m}(\gamma_l(k]\alpha^k))$, $t - E_1 \leq l \leq t + E_2$, has diameter not less than δ/E. Hence

$$l - n_m \leq N(\delta/E)\,.$$

In both cases (2.4) is proved.

Define

$$A^+_k := \{m : 0 < m < k,\ \text{there exists } t \text{ such that } (t,m) \in \mathcal{N}_k\}$$

and

$$A_k^- := \{1,\ldots,k-1\} \setminus A_k^+ .$$

As each set R_m^k for $m \in A_k^-$ is crossed by a part of $b(k]\alpha^k)$ between γ^j for some j and $\widehat{\gamma}_{k]}$ which consists of at least E edges and one edge cannot serve for more than two R_m^k's we obtain, similarly to the inequality preceding (1.7),

$$E \cdot \sharp A_k^- \le 2(n_k^k + n(k) + 1) .$$

Hence using (2.0) we obtain

$$\sharp A_k^+ \ge k - 1 - \frac{2}{E}(n(k) + n_k^k + 1) \ge k\left(1 - \frac{3}{E\kappa}\right) . \tag{2.5}$$

From now on, fix $E > 3/\kappa$ and set $\eta = 1 - 3/(E\kappa) > 0$.

For every $0 < M \le k$ define

$$A_k^+(M) := \{m \in A_k^+ : m < M\} .$$

We claim that there exists $M_0 > 0$, not depending on k, such that for every $M \ge M_0$, $M \in A_k^+$ we have

$$\sharp A_k^+(M) \ge \eta M . \tag{2.6}$$

This means that the property (2.6), which is true for $M = k$ (see (2.5)), extends miraculously to all $M \in A_k^+$ large enough. The proof of this claim is the same as for A_k^+:

Indeed, $M \in A_k^+$ implies the existence of t such that $(t, M) \in \mathcal{N}_k$. By (2.3), $t \le n_{M+1}^k + E + N(\delta/E)$. Next we estimate $\sharp A_k^+(M)$ just as we estimated $\sharp A_k^+$, with $n_k^k + n(k) + 1$ replaced by $n_{M+1}^k + E + N(\delta/E)$. We succeed for all M large enough.

Now we can conclude our proof of Theorem B. Let $M_n := (\frac{1}{2}\eta)^{-n} M_0$. By (2.6) for every $k \ge 0$ and $n \ge 0$ there exists $m \in A_k^+$ such that $M_n \le m < M_{n+1}$.

For each $n = 0, 1, \ldots$ there are only a finite number of blocks of symbols of the form $\pi_t(k]n_k)$ such that $(t, m) \in \mathcal{N}_k$ and $m < M_{n+1}$. This is so by (2.4).

So there are constants $t_0 \ge 0$ and $\mathcal{D}_0 \in \Sigma_{t_0}$ and an infinite set

$$K_0 = \{k \ge 0 : \text{there exists } m \text{ such that}$$
$$M_0 \le m < M_1,\ (t_0, m) \in \mathcal{N}_k,\ \pi_{t_0}(k]\alpha^k) = \mathcal{D}_0\} .$$

In K_0 we find an infinite K_1 etc. by induction. For every $n > 0$ we obtain infinite $K_n \subset K_{n-1}$ and constants t_n, $\mathcal{D}_n$ such that

$$K_n = \{k \in K_{n-1} : \text{there exists } m \text{ such that}$$
$$M_n \le m < M_{n+1},\ (t_n, m) \in \mathcal{N}_k,\ \pi_{t_n}(k]\alpha^k) = \mathcal{D}_n\} .$$

For $\alpha \in \Sigma^d$ such that $\pi_{t_n}(\alpha) = \pi_{t_n}(k]\alpha^k)$, we conclude that $b(\alpha)$ converges to q.

We assumed here that $t_n \to \infty$ as $n \to \infty$. If $\sup t_n = t_* < \infty$ then also $\mathcal{D}_n$ stabilizes at $\mathcal{D}_*$ and by (2.3), $z_{t_*}(\mathcal{D}_*) = q$. Moreover, there exists a sequence of integers $j_1, j_2, \ldots \in \{1, \ldots, d\}$ such that $\gamma_t(\mathcal{D}_*, j_1, j_2, \ldots) \equiv q$ for all $t \geq t_*$ so $b(\mathcal{D}_*, j_1, j_2, \ldots)$ converges to q.

This is not an imaginary case. Consider a source $f(q) = q \in U$ and a tree $\mathcal{T}(q, \gamma^1, \gamma^2)$ such that $\gamma_1 \equiv q$ and γ_2 joins q and $q' \in f^{-1}(q)$, $q \neq q'$. Then the above proof gives $b(2, 1, 1, \ldots)$, a branch for which $\gamma_n((2, 1, 1, \ldots)) \equiv q'$ for every $n \geq 1$. ∎

R e m a r k 2.7. We needed in the above proof neither the left hand side inequality (1.3): $t \geq m - \mathrm{Const}$ for $(t, m) \in \mathcal{N}_k$, nor the sets $W(k, t, m)$. As already mentioned in the introduction, no distortion estimates were needed, i.e. no holomorphy. The holomorphy of f is useful only to verify (0.4).

P r o o f o f C o r o l l a r y 0.1. This follows immediately from Theorem B and the following fact belonging to Pesin's theory:

Let X be a compact subset of $\overline{\mathbb{C}}$ and F be a holomorphic mapping of a neighbourhood of X such that $F(X) = X$. Let $\mu \in M_{\mathrm{e}}^{\chi+}(F)$. Let $(\widetilde{X}, \widetilde{F}, \widetilde{\mu})$ be a natural extension (*inverse limit*) *of (X, F, μ). Denote $\pi : \widetilde{X} \to X$ the projection to the 0-coordinate, and by π_n the projection to the n-th coordinate.*

Then for $\widetilde{\mu}$-a.e. $\widetilde{x} \in \widetilde{X}$ there exists $r = r(\widetilde{x}) > 0$ such that there exist univalent branches F_n of F^{-n} on $B(\pi(\widetilde{x}), r)$ for $n = 1, 2, \ldots$ with $F_n(\pi(\widetilde{x})) = \pi_{-n}(\widetilde{x})$. Moreover, for every λ with $\exp(-\chi_\mu) < \lambda < 1$ (*not depending on $\widetilde{x}$*) *and a constant $C = C(\widetilde{x}) > 0$,*

$$|F_n'(\pi(\widetilde{x}))| < C\lambda^n \quad \textit{and} \quad \frac{|F_n'(\pi(\widetilde{x}))|}{|F_n'(z)|} < C$$

for every $z \in B(\pi(\widetilde{x}), r)$, $n > 0$ (*distances and derivatives in the Riemann metric on $\overline{\mathbb{C}}$*).

Moreover, r and C are measurable functions of $\widetilde{x}$.

To prove Corollary 0.1, observe that the above fact implies the existence of numbers $r, C > 0$ and a set $\widetilde{Y} \subset \widetilde{X}$ of positive $\widetilde{\mu}$-measure such that the above properties hold for every $\widetilde{x} \in \widetilde{Y}$ and for these r and C. Ergodicity of μ implies ergodicity of $\widetilde{\mu}$. So by Birkhoff's Ergodic Theorem there exists a set $\widetilde{Z} \subset \widetilde{X}$ of full measure $\widetilde{\mu}$ such that for each point $\widetilde{x} \in \widetilde{Z}$ the set of times at which the forward orbit of $\widetilde{x}$ by $\widetilde{F}$ hits $\widetilde{Y}$ has positive density. These are good times and $\pi(\widetilde{x})$ is a good point in the sense of the introduction (provided they satisfy $(0.3')$). ∎

3. External rays. Let $W_1 \subset W$ be open, connected, simply connected bounded domains in the complex plane $\mathbb{C}$ such that $\mathrm{cl}\, W_1 \subset W$. Let $f : W_1 \to W$ be a holomorphic proper surjective map of degree $d \geq 2$. We call such an f a *polynomial-like map*. Define $K = \bigcap_{n \geq 0} f^{-n}(W)$. This set is called a *filled-in Julia set* [DH]. We can assume that ∂W is smooth. Let M be an arbitrary smooth function on a neighbourhood of $\mathrm{cl}\, W \setminus W_1$ without critical points, such that $M|_{\partial W} \equiv 0$, $M|_{\partial W_1} \equiv 1$ and $M \circ f = M - 1$ wherever it makes sense. Extend M to $W \setminus K$ by $M(z) = M(f^n(z)) + n$ where n is such that $f^n(z) \in W \setminus W_1$.

Fix τ with $0 < \tau < \pi$ and consider curves $\gamma : [0, \infty) \to (\mathrm{cl}\, W) \setminus K$ with $\gamma(0) \in \partial W$, intersecting lines of constant M at angle τ (this demands fixing orientations), not containing critical points for M and converging to K as the parameter tends to ∞. One can change the standard euclidean metric on $\mathbb{C}$ so that τ is the right angle and think of gradient lines in the new metric. We call such a line a *smooth τ-ray.* Instead of parametrizing such a curve with the gradient flow time we parametrize it by the values of M. Limits of smooth τ-rays are called *τ-rays.* They can pass through critical points of M. (Such a τ-ray enters a critical point along a stable separatrix and leaves it along an unstable one, the closest clockwise or counter-clockwise. If it hits again a critical point for the first time it leaves it along an unstable separatrix on the same side from which it came to the previous critical point; see Fig. 3, and [GM] and [LevP] for a more detailed description.

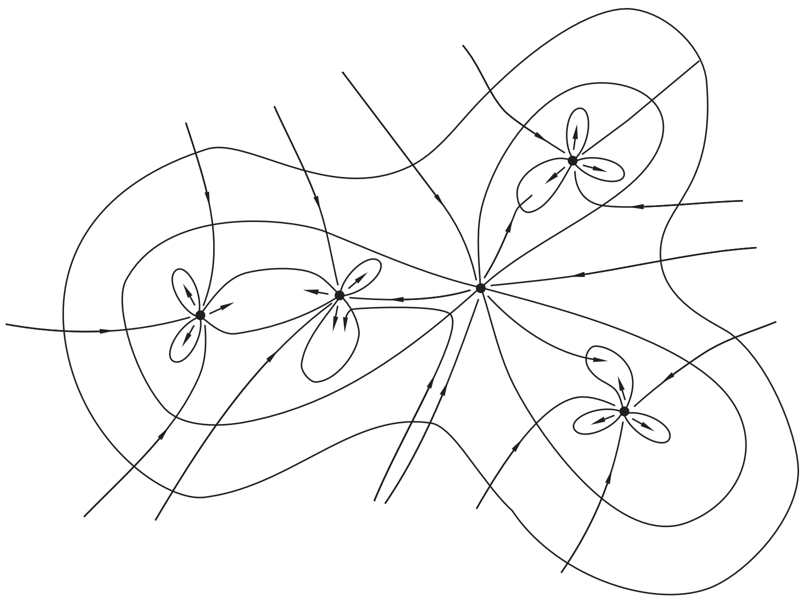

Fig. 3

Proof of Theorem C. Divide each τ-ray γ into pieces γ_n, $n \geq 1$, each joining $f^{-n}(\partial W)$ and $f^{-n-1}(\partial W)$.

One easily proves the fact corresponding to (0.4):

$$\text{length}\,\gamma_n \to 0 \quad \text{as } n \to \infty \tag{3.0}$$

uniformly over τ-rays γ.

The proof is the same as that of the implication (0.5)⇒(0.4) in Remark 0.4. We have univalent branches of f^{-k} for all k on neighbourhoods of γ_n for external rays γ, neighbourhoods not depending on k, for n large enough, because then the critical points of f in $W \setminus K$ do not interfere. There are a finite number of them and their forward trajectories escape out of W.

For our q find significant telescopes Tel^k as in Section 2, where n_0-significant means here that D_k^k intersects $\gamma_{n(k)}^k$ for a τ-ray γ^k and $n(k) \leq n_0$, a constant independent of k. This is possible by (3.0).

Denote by $\gamma^{k]}$ the τ-ray containing a point of $f^{-n_k}(\gamma^k)$ which is in $D_{k,0}^k$.

We consider, similarly to Section 2, (2.3), the set

$$\mathcal{N}_k = \{(t,m) : \text{the same conditions as in Section 2, in particular } \gamma_t^{k]} \cap R_m \neq \emptyset\}.$$

Similarly we define A_k^+ and $A_k^+(M)$, $M \leq k$.

The same miracle that

$$\sharp A_k^+(M) \geq \eta M$$

occurs for $M \geq M_0$, $M \in A_k^+$.

To get it we prove and use the estimate $t \leq m + \mathrm{Const}$ for $(t,m) \in \mathcal{N}_k$.

Because for $M_0 \leq m < M_1$ and $(t,m) \in \mathcal{N}_k$ the integers t are uniformly bounded over all k, by T_0 say, the parts $\gamma'^{k]} = \bigcup_{l=1}^{T_0} \gamma_l^{k]}$ of respective τ-rays $\gamma^{k]}$ have a convergent subsequence and the limit ray (joining levels 0 and T_0) intersects R_m.

Choosing consecutive subsequences we find a limit ray converging to q. ∎

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