FUNDAMENTA
MATHEMATICAE
144 (1994)

Composants of the horseshoe

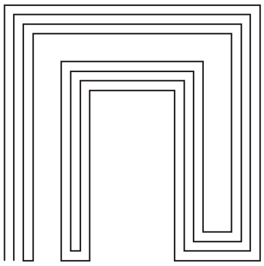
by

Christoph Bandt (Greifswald)

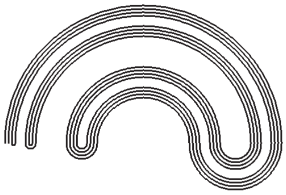
Abstract. The horseshoe or bucket handle continuum, defined as the inverse limit of the tent map, is one of the standard examples in continua theory as well as in dynamical systems. It is not arcwise connected. Its arcwise components coincide with composants, and with unstable manifolds in the dynamical setting. Knaster asked whether these composants are all homeomorphic, with the obvious exception of the zero composant. Partial results were obtained by Bellamy (1979), Dębski and Tymchatyn (1987), and Aarts and Fokkink (1991). We answer Knaster's question in the affirmative. The main tool is a very simple type of symbolic dynamics for the horseshoe and related continua.

1. Introduction. The "bucket handle" $K = K_2$ was constructed in 1922 as a union of half-circles with endpoints in Cantor's middle third set C. Kuratowski ([10], cf. [11]) attributed this idea to Knaster (while Knaster in the same volume 3 of Fundamenta Mathematicae gives credit to Kuratowski for the corresponding construction of K_3). A topologically equivalent definition of K had already been given in 1911 by Z. Janiszewski in his Paris thesis (see [9]). In connection with dynamical systems, the space K and related spaces have become known in the sixties as the "horseshoe"—the attractor $K = \bigcap f^n(Q)$ of a suitably chosen nonlinear map $f : Q \to Q$ from the square to itself (Fig. 1). The action of f seems obvious when you imagine how a very strong man would form a horseshoe from a rectangular iron plate (cf. [6], 13.3). However, it is not an easy matter to define f on Q. The original paper by Smale [14] considers only the middle part of Q where K has the product structure of a Cantor set by an interval, and the term "horseshoe" has often been used in a wider sense. Only in 1985, did Misiurewicz [12] give detailed constructions of diffeomorphisms f in three-dimensional manifolds and homeomorphisms in the plane which yield K as an attractor. Szczechla [15] constructed a diffeomorphism f in the plane with K as attractor which is C^∞ at all but a finite number of points. In the present paper, we shall take "horseshoe", rather than "bucket handle", as a name for K.

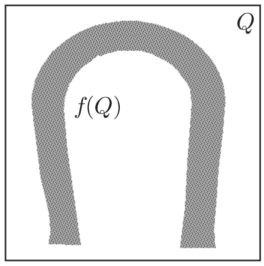
1991 *Mathematics Subject Classification*: Primary 54H20; Secondary 58F12, 54F15.



Janiszewski 1911



Knaster 1922



Smale 1967

Fig. 1

Technically, it is most convenient to define K_2 as the inverse limit of the tent map $g : [0,1] \to [0,1]$, $g(x) = 2x$ for $x \le 1/2$ and $g(x) = 2(1-x)$ for $x \ge 1/2$ [8, 13, 16]. For a continuous map $h : X \to X$ the inverse limit $(\widehat{X}, \widehat{h})$ is defined as the subset of the product space $X^{\mathbb{N}}$ given by

$$\widehat{X} = \{(\ldots x_3 x_2 x_1) \mid h(x_i) = x_{i-1} \text{ for } i \ge 2\}$$

with the map

$$\widehat{h}(\ldots x_3 x_2 x_1) = (\ldots x_2 x_1 h(x_1))\,.$$

The bucket handle is not arcwise connected since from one point of C we can only reach countably many points by finite unions of half-circles. Thus there are uncountably many arcwise components of K which coincide with composants (see [11]), and also with unstable manifolds in the dynamical setting. We shall use the term "composant". There is one special composant which contains the point zero of C which is an endpoint of the composant. All the other composants contain only cutpoints, their topology is coarser than that of the real line.

In the fifties (or even earlier), Knaster asked in his seminar whether all non-zero composants of K are homeomorphic. This question first appeared in print in 1979 when Bellamy [4] analyzed the way the composants are permuted by the shift map $\widehat{h}$. (This will become clear at the end of Sec. 2.) The subsequent papers of Dębski and Tymchatyn, Aarts and Fokkink [5, 7, 1] stated, among other things, that homeomorphisms of K will not permute more composants than iterates of the shift, and that there are continuous bijections between the composants.

The purpose of this note is to give an affirmative answer to Knaster's question. We shall confine ourselves to K_2 although most arguments apply to more general spaces.

Theorem. *All non-zero composants of K_2 are homeomorphic.*

The corresponding assertion for the solenoid, a quite related space, is trivial since we know it is a group. The problem for K_2 is the lack of a good analytical description. We shall start by introducing coordinates for K_2 in a rather simple way. Before going into details, let me thank the referee for a number of useful comments.

2. Coordinates for the horseshoe. It is well known that $[0,1]$ is the quotient space of the Cantor set $C=\{0,1\}^{\mathbb{N}}$ by the equivalence relation which identifies the sequences $w0\overline{1}$ and $w1\overline{0}$ for each 0-1-word $w=w_1\dots w_n$, and no others. These are the binary numbers. $\overline{0}$ denotes a periodic sequence, and $\{0,1\}^*$ will denote the set of all 0-1-words including the empty word. We need another representation of $[0,1]$ as a quotient of C which comes from kneading theory.

PROPOSITION 1. $[0,1]=C/\sim$, *where* $b,c\in C$ *are equivalent if either* $b=c$, *or* $b=w01\overline{0}$ *and* $c=w11\overline{0}$ *for some* 0-1-*word* w, *or* $b=w11\overline{0}$ *and* $c=w01\overline{0}$.

This is the representation of points $x\in[0,1]$ by their itineraries with respect to the tent map, $x\rightsquigarrow i_0i_1i_2\dots$, with $i_k=0$ if $g^k(x)\in[0,1/2]$ and $i_k=1$ if $g^k(x)\in[1/2,1]$. Thus the point 0 as a fixed point of g has itinerary $\overline{0}$, and the point 1 with $g(1)=0$ has itinerary $1\overline{0}$. Each point in $[0,1]$ has one itinerary, except for $1/2$ which has the two itineraries $01\overline{0}$ and $11\overline{0}$, and the preimages of $1/2$ under the maps g^n, which have two itineraries $w01\overline{0}$ and $w11\overline{0}$. It is easy to see that each 0-1-sequence appears as itinerary of a unique point $x\in[0,1]$, and that the corresponding map from C onto $[0,1]$ is continuous, which proves the proposition (cf. [2]).

An equivalence relation $\sim$ on $C=\{0,1\}^{\mathbb{N}}$ was called (strongly) shift-invariant in [2] if $i_1i_2i_3\ldots\sim j_1j_2j_3\ldots$ implies $i_2i_3\ldots\sim j_2j_3\ldots$ as well as $i_0i_1i_2\ldots\sim i_0j_1j_2\ldots$ for $i_0\in\{0,1\}$. In the terminology of dynamical systems, this means that the projection $\pi:C\to X=C/\sim$ is a semiconjugacy from the shift map $\sigma(i_1i_2i_3\ldots)=i_2i_3\ldots$ on C to a map $h:X\to X$. In other words, $\pi\sigma=h\pi$. We write $(X,h)=(C,\sigma)/\sim$.

In our example, $(X,h)=([0,1],g)$. We could say that the tent map is obtained from the shift σ on C by identification according to the formula

$$01\overline{0}\sim 11\overline{0}\,.$$

A single structural formula for a dynamical system! It will turn out that this formula describes even the structure of K completely. Let $D=\{0,1\}^{\mathbb{Z}}$ denote the space of two-sided 0-1-sequences with the shift map $\sigma(\ldots x_2x_1\underline{x_0}x_{-1}\ldots)=\ldots x_1x_0\underline{x_{-1}}x_{-2}\ldots$, the zero coordinate being underlined.

PROPOSITION 2. *If* $\sim$ *is a shift-invariant equivalence relation with* $(X,h)=(C,\sigma)/\sim$, *then the inverse limit is a corresponding quotient of the two-*

sided shift:

$$(\widehat{X},\widehat{h})=(D,\sigma)/\approx$$

where $(s_n)_{n\in\mathbb{Z}}\approx(t_n)_{n\in\mathbb{Z}}$ *if there is* m *with* $s_n=t_n$ *for* $n\ge m$ *and* $s_{m-1}s_{m-2}\ldots\sim t_{m-1}t_{m-2}\ldots$

For the proof, note that we can always choose $m\ge 0$. For each $m\ge 0$, the space

$$X_m=\{s_{m-1}s_{m-2}\ldots\mid s_k\in\{0,1\}\text{ for }k\in\mathbb{Z},\ k<m\}/\sim$$

with the shift map $\sigma(s_{m-1}s_{m-2}\ldots)=s_{m-2}s_{m-3}\ldots$ is conjugate to (X,h). Thus the inverse sequences

$$\ldots\to X_m\xrightarrow{\sigma}X_{m-1}\xrightarrow{\sigma}\ldots\to X_1\xrightarrow{\sigma}X_0\quad\text{and}\quad\ldots\to X\xrightarrow{h}X\xrightarrow{h}\ldots\to X$$

are identical in the category of dynamical systems. So their limits must coincide. The elements of $\lim X_m=\{(\ldots x_m\ldots x_0)\mid\sigma(x_m)=x_{m-1}\}$ can be written as $(s_n)_{n\in\mathbb{Z}}$. A sequence $(t_n)_{n\in\mathbb{Z}}$ represents the same point as $(s_n)_{n\in\mathbb{Z}}$ iff for some $m\ge 0$, both $s_{m-1}s_{m-2}\ldots$ and $t_{m-1}t_{m-2}\ldots$ represent the same point in X_m.

This proposition implies Theorem 3.1 of Holte [8] for the important case when the f_n and g_n are equal. The one-dimensional maps considered there are those which can be described by a finite number m of "structural formulas" with eventually periodic sequences. The space of symbol sequences is then a Markov subshift on m symbols instead of $\{0,1\}^\infty$.

Now we know that K is the quotient of the two-sided shift with respect to the structural formula $01\overline{0}\sim 11\overline{0}$, and we look for the composants. It is convenient to think of K as a union of intervals, and to concentrate on the "integer" identification points $\ldots s_{m+1}s_m01\overline{0}\sim\ldots s_{m+1}s_m11\overline{0}$ with $m>0$, where intervals are linked together. The "fractional" identification points inside the intervals will play no role hereafter. If two points $(x_k)_{k\in\mathbb{Z}}$ and $(y_k)_{k\in\mathbb{Z}}$ can be connected by an arc, there are only finitely many integer identification points in-between, so x and y have a common left tail: there is m with $x_k=y_k$ for all $k>m$. On the other hand, by the remark on X_m, any two points with common left tail are contained in an arc in K. Consequently, we have

PROPOSITION 3. *Each left-infinite sequence* $s=\ldots s_3s_2s_1$ *describes one composant in* K: *the set of two-sided sequences which have a left tail common with* s. *Two sequences* s,t *describe the same composant iff they have a common tail.*

Now it is easy to formulate Bellamy's results in [4] on the action of the shift on composants: The zero composant given by $\ldots 000=\overline{0}$ and the composant given by $\overline{1}$ are fixed, the composants given by (eventually) periodic s are periodic under the shift, and all other composants will have an infinite orbit.

3. On the structure of a composant. Now let us fix a sequence $s = \ldots s_3 s_2 s_1$ which does not start with $\overline{0}$, and denote the corresponding composant of K by I. It will be convenient to consider I as a copy of the real line, consisting of unit length intervals I_v^0, $v = v_m \ldots v_1 \in \{0,1\}^*$, defined by

$$I_v^0 = \{(x_k)_{k\in\mathbb{Z}} \mid x_k = s_k \text{ for } k > m,\ x_k = v_k \text{ for } k = 1, \ldots, m\}\,.$$

To avoid ambiguity, we can require $v_m \neq s_m$, but sometimes it is more intuitive to describe neighbouring intervals by words v of the same length. Figure 2 shows both versions, where the short-hand notation refers to $s = \ldots 0100$.

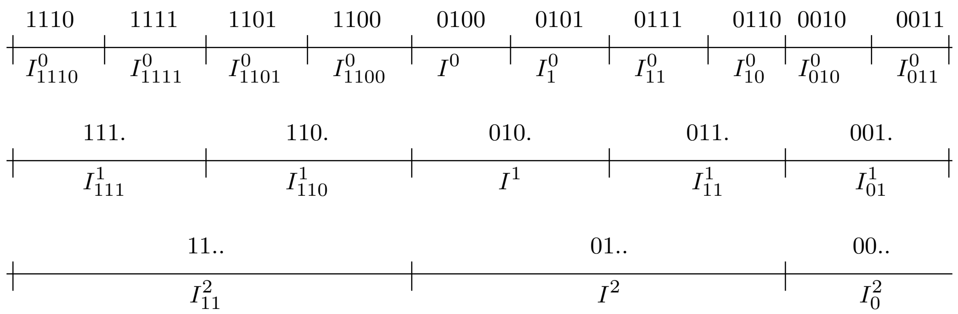


Fig. 2

Similarly, we can assemble the line I from those larger intervals I_v^n of length 2^n which are obtained by neglecting the digits $x_n, \ldots, x_1$:

$$I_v^n = \{(x_k)_{k\in\mathbb{Z}} \mid x_k = s_k \text{ for } k > m+n,\ x_{n+k} = v_k \text{ for } k = 1, \ldots, m\}\,.$$

The incidence of neighbouring words follows from the identification rule for the endpoints $\ldots v\overline{0}$ and $\ldots v1\overline{0}$ of I_v^n. Words with neighbouring intervals have the form $v_m \ldots v_2 0$ and $v_m \ldots v_2 1$, or $v_m \ldots v_3 01$ and $v_m \ldots v_3 11$, or $v_m \ldots v_{k+2} 010^k$ and $v_m \ldots v_{k+2} 110^k$ for some $k \geq 1$. Thus, the "up and down" of the horseshoe is now formalized by saying that the digit v_1 changes periodically as

$$0\text{-}1\text{-}1\text{-}0,$$

$v_2 v_1$ follows the pattern

$$00\text{-}01\text{-}11\text{-}10\text{–}10\text{-}11\text{-}01\text{-}00,$$

and $v_3 v_2 v_1$ repeat as follows:

$$000\text{-}001\text{-}011\text{-}010\text{-}110\text{-}111\text{-}101\text{-}100\text{–}100\text{-}101\text{-}111\text{-}110\text{-}010\text{-}011\text{-}001\text{-}000.$$

It is not hard to prove the following more general

FACT 1. *For every word* $u = u_m \ldots u_1$ *and every* n, *the intervals of type* I^n_{wu}, $w \in \{0,1\}^*$, *appear with period* 2^{m+1} *in the sequence of all* I^n_v. *Among* 2^{m+1} *consecutive intervals* I^n_v *there are exactly two of type* I^n_{wu}, *and they have different orientation.*

Let us say that I^n_v has *positive orientation* if the point $\ldots s_{m+n+1} v_m \ldots \ldots v_1 000 \ldots$ (which corresponds to 0 in the unit interval) is the left endpoint of I^n_v.

FACT 2. *For fixed* $s = \ldots s_3 s_2 s_1$, *the intervals* $I^0, I^1, I^2, \ldots$ *form a nested sequence.* I^n *is the left half of* I^{n+1} *if* I^n *has positive orientation*, *right if negative. Moreover*, (*orientation of* I^{n+1}) $= (-1)^{s_{n+1}}$(*orientation of* I^n). *Thus*, *if* $s_{n+2} = 1$, *then* I_n *is one of the middle quarters of* I_{n+2}. *Consequently*, *if* $\overline{0}$ *is not a left tail of* s, *the union of all* I^n *is the line* I.

Our proof makes use of these partitions of the "line" I, and of the "translations" φ^n_v which map I^n onto I^n_v (the φ^n_v may also be "reflections" $x \mapsto -x + c$, in some cases). We shall apply the Euclidean metric—which is determined by the division of I into the I^0_v and into smaller binary subintervals—to define linear mappings between intervals of I. However, we keep in mind that the topology of I coincides with the usual topology of the line only on compact intervals. There are "unbounded" sequences on I which converge. In the following, we describe some kind of neighbourhood systems of points in I.

Those intervals I^0_v which come near to our origin interval I^0, in terms of the topology of I, will be called "return intervals". To be more precise, let us fix a sequence of integers $0 = n_1 < n_2 < n_3 < \ldots$ With respect to this sequence, $I^{n_k}_v$ is called a *return interval* if the last $d_k = n_{k+1} - n_k$ symbols of v agree with the corresponding letters of s:

$$v_{d_k} \ldots v_1 = s_{n_{k+1}} \ldots s_{n_k+1} \,.$$

$I^{n_k}_v$ is a *close return interval* if even the last $d_k + d_{k+1} = n_{k+2} - n_k$ symbols of v coincide with the corresponding symbols in s.

Since the convergence on I is coordinatewise convergence of sequences, up to identification of some special points, we can now describe which sequences $x^{(p)}$ in I will converge to a given x. There is an n with $x \in \operatorname{int} I^n$. For each $n_k > n$, the $x^{(p)}$ must eventually belong to return intervals $I^{n_k}_{v^{(p)}}$, and the position of $x^{(p)}$ in these intervals must stabilize:

$$\lim_{p\to\infty} (\varphi^{n_k}_{v^{(p)}})^{-1}(x^{(p)}) = x \quad \text{ in } I^{n_k} \,.$$

This condition is necessary and sufficient. It suffices to require the stabilizing condition only for one n_k. It is also possible to replace "return intervals" by "close return intervals".

4. How to construct a homeomorphism. We shall construct a homeomorphism f from an arbitrary composant I characterized by s to the particular composant J with characteristic sequence $\overline{1} = \ldots 111$. All definitions and statements above apply to $s = \overline{1}$, and we shall write J^n_w, ψ^n_w for $s = \overline{1}$ and I^n_v, φ^n_v for the arbitrarily chosen s.

Since s does not start with $\overline{0}$, it is easy to choose a sequence $0 = n_1 < n_2 < \ldots$ with

(i) $d_k = n_{k+1} - n_k \geq 10 + k$ for $k \geq 1$,
(ii) $s_{n_k} s_{n_k - 1}$ is either 01 or 11 for $k \geq 2$.

The one-to-one map $f : I \to J$ which we will construct has the following properties with respect to the chosen sequence, for all k:

(a) f maps I^{n_k} continuously onto J^{n_k}.

(b) f maps each close return interval $I^{n_k}_v$ onto a return interval $J^{n_k}_w$ in the same way as I^{n_k} is mapped onto J^{n_k}—that is, $f\varphi^{n_k}_v(z) = \psi^{n_k}_w f(z)$ for $z \in I^{n_k}$.

(c) f^{-1} maps each close return interval $J^{n_k}_w$ onto a return interval $I^{n_k}_v$ in the same way as it maps J^{n_k} onto I^{n_k}.

Let us first show that (a), (b) imply that $f : I \to J$ is continuous—then f^{-1} must also be continuous by (a) and (c). Suppose $x^{(p)}$ converges to x in I, where x is an interior point of I^{n_l}. For $k \geq l$, there is p_k such that all $x^{(p)}$ with $p > p_k$ belong to close return intervals $I^{n_k}_{v^{(p)}}$, and $z^{(p)} = (\varphi^{n_k}_{v^{(p)}})^{-1}(x^{(p)})$ converges to x. By (b), all $f(x^{(p)})$ belong to return intervals $J^{n_k}_{w^{(p)}}$. Since f is continuous on I^{n_k}, the sequence $f(z^{(p)})$ converges to $f(x)$ in J^{n_k}. From the equation in (b) it follows that $f(z^{(p)}) = (\psi^{n_k}_{w^{(p)}})^{-1} f(x^{(p)})$. We have shown that $f(x^{(p)})$ converges in J to $f(x)$, so f is continuous.

Now we shall inductively construct f on the intervals I^{n_k} so that (a)–(c) hold true. We fix one of the two linear mappings from I^{n_1} onto J^{n_1}. Next, we extend f to a homeomorphism from I^{n_2} onto J^{n_2}, in such a way that the two intervals of $I^{n_2} \setminus I^{n_1}$ are mapped linearly onto the two respective intervals of $J^{n_2} \setminus J^{n_1}$. We have to show that this is possible.

For each n, J^n divides into four intervals J^{n-2}_w, and J^{n-2} is one of the middle quarters. Thus $J^{n_2} \setminus J^{n_1}$ does really consist of two intervals, the length of which is at least one quarter and at most three quarters of J^{n_2}. By (ii), the same holds for I^{n_2}. Thus $f : I^{n_2} \to J^{n_2}$ is defined as a piecewise linear bijection, and f and f^{-1} are Lipschitz maps with Lipschitz constant 3.

Let us proceed by induction. Intervals of the form $I^{n_k}_v$, $J^{n_k}_w$ will be said to have *order* k. Suppose $f : I^{n_k} \to J^{n_k}$ is already defined and satisfies (a)–(c). We have to define f on the remaining two intervals of $I^{n_{k+1}}$ so that (a)–(c) are valid.

5. Counting return intervals. We have to place the close return intervals, so let us count them. By definition, each interval of order $m+1$ contains 2^{d_m} intervals of order m, exactly one return interval of order m and exactly one close return interval of order $m-1$. Thus for one close return interval of order $m-1$ we have 2^{d_m} return intervals of order $m-1$. The density of close return intervals is so much lower than that of return intervals, that it should not be difficult to satisfy (b) and (c), even if the image interval is three times shorter than the domain.

The difficulty is that $I^{n_{k+1}}$ contains close return intervals of all orders from 1 up to $k-2$ (for order $k-1$ we only have $I^{n_{k-1}}$). When we first choose the images of the large close return intervals of order $k-2$, we are left with almost 2^{d_k} intervals in $I^{n_{k+1}}$ and uniquely corresponding intervals in $J^{n_{k+1}}$ on which f has still to be defined. The length ratio of domain and image interval could be considerably larger than 3, and after some further steps it could happen that one of the remaining image intervals is so short that it does not contain return intervals. We show how to avoid this situation.

LEMMA. *Let $\widetilde{I}$ be a union of c consecutive intervals of order m in I, and $\widetilde{J}$ a union of d consecutive intervals of order m in J, and let the length ratio of these two line segments be $q = \max\{c/d, d/c\} < 4$. Assume that between a return interval of order m and an endpoint in $\widetilde{I}$ and $\widetilde{J}$ there are at least 2^{d_m-2} intervals of order m. Finally, let an orientation on $\widetilde{I}$ and $\widetilde{J}$ be given which says which is the "left" endpoint, and let an orientation for each interval of order $m-1$ in $\widetilde{I}$ and $\widetilde{J}$ be specified, in an alternating way so that two neighbouring intervals always have different orientation.*

Then there are partitions of $\widetilde{I}$ and $\widetilde{J}$ into finitely many subintervals and a correspondence between the first, second, ..., k-th elements (counted from the left) of these partitions such that

(1) *Each close return interval of order $m-1$ in $\widetilde{I}$ or $\widetilde{J}$ is a partition element, and corresponds to a return interval of order $m-1$ in the other partition which has the same orientation.*

(2) *The other partition elements are unions of consecutive intervals of order $m-1$. Between an endpoint of the partition interval and the next return interval of order $m-1$ inside that partition interval, there are at least $2^{d_{m-1}-2}$ other intervals. The length ratio of two corresponding partition elements is at most $q+2^{-m}$.*

This lemma allows us to construct f on $I^{n_{k+1}}$. We first apply it with $m = k-1$ and $q = 3$, taking as $\widetilde{I}$ each of the two parts $I^{n_{k+1}} \setminus I^{n_k}$ which are unions of intervals of order $k = m+1$. Since a return interval of order m is in one of the middle quarters of the larger interval of order $m+1$ (in the same way as I^{n_m} is contained in $I^{n_{m+1}}$), there are at least $2^{d_m}/4$

intervals of order m between such a return interval and an endpoint of $\widetilde{I}$. The points $\widetilde{I}\cap I^{n_k}$ and $\widetilde{J}\cap J^{n_k}$ have to be on the same side (both left or both right), and intervals of order $m-1=k-2$ on I and J are oriented in an alternating way so that the already constructed $f: I^{n_{k-2}}\to J^{n_{k-2}}$ is orientation-preserving.

The lemma gives a correspondence of the close return intervals of order $k-2$ to return intervals on the other side. Since the orientation is preserved, we can simply transfer the definition of $f: I^{n_{k-2}}\to J^{n_{k-2}}$ to each pair of such intervals. So far, (a)–(c) are satisfied.

By (2), we can again apply the lemma to each pair of the remaining intervals, with $m=k-3$ and $q=3+2^{-(k-2)}$. As a result, we get rid of return intervals of order $k-3$, and, by induction, of all return intervals of orders down to 1. By geometric series, q stays below 4. f is defined on $I^{n_{k+1}}$, and by induction on all I, in such a way that (a)–(c) hold. Thus $f: I\to J$ will be a homeomorphism, and f, f^{-1} will be Lipschitz with respect to the Euclidean metric, with constant 4.

It remains to show the lemma. Taking left endpoints as zero and intervals of order m as units of measurement, we define a linear scale on $\widetilde{I}$ and $\widetilde{J}$. Let $a_1,\ldots,a_n$ and $b_1,\ldots,b_p$ be the coordinates of left endpoints of the close return intervals of order $m-1$ in $\widetilde{I}$ and $\widetilde{J}$, respectively. Assume $c>d$, and let $b_i'=b_iq$ and $a_i'=a_i/q$ be the points corresponding to b_i and a_i under the unique linear orientation-preserving correspondence between $\widetilde{I}$ and $\widetilde{J}$. The idea of the proof is to shift the b_i' and a_i' to nearby left endpoints of return intervals.

Note that $a_{i+1}-a_i\ge 2^{d_m-2}$ since two close return intervals of order $m-1$ are contained in different return intervals of order m, which are in the middle quarters of two different intervals of order $m+1$. The assumption of the lemma says this remains true if the endpoints of $\widetilde{I}$ are denoted by a_0, a_{n+1}. Similarly, $b_{i+1}'-b_i'\ge b_{i+1}-b_i\ge 2^{d_m-2}$. Consider the a_i, the b_i' and the endpoints of $\widetilde{I}$ as vertices of a partition $\mathcal{P}$ of $\widetilde{I}$. Then at least one of any two neighbouring intervals of $\mathcal{P}$ is larger than 2^{d_m-3}. For the partition $\mathcal{Q}$ of $\widetilde{J}$ induced by the b_i and a_i', at least one of two neighbouring intervals is larger than $2^{d_m-3}/q$.

Now we shift each b_i' in $\widetilde{I}$ to the left or to the right to the next left endpoint of a return interval of order $m-1$ which has the same orientation as the close return interval given by b_i. Since every interval of order m contains one return interval of order $m-1$, our shift does not exceed two units to either side. In choosing left and right we take care of shifting the border points of short intervals (that is, intervals of length $\le 2^{d_m-3}$) to the inside. In $\widetilde{J}$, we shift the a_i' at most two units to the left or right, to a left endpoint of some return interval of order $m-1$ and of the same orientation as the close return interval next to a_i. Here we require that both border points of intervals of length smaller than $2^{d_m-3}/q$ are shifted to the outside.

This procedure results in partitions $\mathcal{P}', \mathcal{Q}'$ of $\widetilde{I}$ and $\widetilde{J}$ with the same number of elements, where the first, second, ..., kth intervals from the left correspond to each other in an obvious manner. All pairs of corresponding intervals in these partitions, except the first, have at their left corner a close return interval in $\widetilde{I}$ and a return interval in $\widetilde{J}$, or conversely. Considering these left parts as separate partition elements, we obtain the partitions described in the lemma.

We conclude with proving the last two assertions of (2). Take a return interval of order $m-1$ in one of the partition sets of type (2). It is contained in one of the middle quarters of the corresponding interval of order m, which contains $2^{d_{m-1}}$ intervals of order $m-1$. We only have to prove that the larger interval is also in the partition set. For the leftmost and rightmost partition sets this follows from the first assumption, and for the other partition sets from the fact that two return intervals of order $m-1$ are in different intervals of order m.

For pairs of intervals which have been termed "short", the length ratio has become smaller than q. For the other pairs of intervals, the maximum relative increase of the I-part by the shift procedure is $4/2^{d_m-3}$, and the strongest possible relative decrease of the J-part is $4q/2^{d_m-3}$. With $d_m \geq 10+m$ we get, for each $m \geq 1$,

$$q' \leq q \cdot \frac{1+2^{5-d_m}}{1-2^{7-d_m}} \leq q \cdot \frac{1+2^{-5-m}}{1-2^{-3-m}} = q + \frac{5q}{4(2^{3+m}-1)} \leq q + 2^{-m} .$$

References

[1] J. M. Aarts and R. J. Fokkink, *On composants of the bucket handle*, Fund. Math. 139 (1991), 193–208.

[2] C. Bandt and K. Keller, *A simple approach to the topological structure of fractals*, Math. Nachr. 154 (1991), 27–39.

[3] C. Bandt and T. Retta, *Topological spaces admitting a unique fractal structure*, Fund. Math. 141 (1992), 257–268.

[4] D. P. Bellamy, *Homeomorphisms of composants*, Houston J. Math. 5 (1979), 313–318.

[5] W. Dębski and E. D. Tymchatyn, *Homeomorphisms of composants in Knaster continua*, Topology Proc. 12 (1987), 239–256.

[6] K. Falconer, *Fractal Geometry*, Wiley, New York, 1990.

[7] R. J. Fokkink, *The structure of trajectories*, Dissertation, Delft, 1991.

[8] S. E. Holte, *Generalized horseshoe maps and inverse limits*, Pacific J. Math. 156 (1992), 297–306.

[9] Z. Janiszewski, *Oeuvres Choisies*, PWN, Warszawa, 1962.

[10] K. Kuratowski, *Théorie des continues irréductibles entre deux points I*, Fund. Math. 3 (1922), 200–231.

[11] —, *Topology*, Vol. II, PWN, Warszawa, and Academic Press, New York, 1968.

[12] M. Misiurewicz, *Embedding inverse limits of interval maps as attractors*, Fund. Math. 125 (1985), 23–40.

[13] S. B. Nadler, *Continuum Theory*, Marcel Dekker, New York, 1992.

[14] S. Smale, *Differentiable dynamical systems*, Bull. Amer. Math. Soc. 73 (1967), 747–817.

[15] W. Szczechla, *Inverse limits of certain interval mappings as attractors in two dimensions*, Fund. Math. 133 (1989), 1–23.

[16] W. T. Watkins, *Homeomorphic classification of certain inverse limit spaces with open bonding maps*, Pacific J. Math. 103 (1982), 589–601.

FR MATHEMATIK/INFORMATIK
ARNDT-UNIVERSITÄT
17487 GREIFSWALD, GERMANY

Received 2 February 1993;
in revised form 25 August and 30 November 1993