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On strong liftings for projective limits

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Abstract. We discuss the permanence of strong liftings under the formation of projective limits. The results are based on an appropriate consistency condition of the liftings with the projective system called "self-consistency", which is fulfilled in many situations. In addition, we study the relationship of self-consistency and completion regularity as well as projective limits of lifting topologies.

Introduction. Only recently in [19] the general theory of inductive limits of (topological) measure spaces was developed by N. D. Macheras and at the same time the permanence of strong lifting was established for inductive limits. For much longer time the projective limit of measure spaces is in common use (see e.g. Bochner [4], Choksi [5], Musiał [25], Rao [29]) but there seems to be no discussion of the permanence of strong liftings for general projective limits. Only for finite or countable products there exist permanence results in the forthcoming paper [22].

As in [22] our main concern in this paper is with conditions of compatibility of the liftings in the factors with the lifting on the limit or product. The notion of the "consistent lifting" of M. Talagrand [30] seems to be the first example on this line. Talagrand's paper is only for finite products in which all factors must be equal. [22] gives consistency conditions for finite and countable products with different factors, and our basic sufficient condition for the existence of strong liftings on projective limits in this paper, the so- called "self-consistency" (see Section 1 for definitions), may be read as a strengthening of Talagrand's consistent lifting, i.e. to be precise in his special instance it is a condition in terms of the generators of the product σ-algebra while ours is a condition on the whole σ-algebra. Remark 2.2(iii) gives a list of projective systems which allow self- consistent liftings. Among them are always countable systems provided all factors have the universally strong lifting property (USLP for short). The basic existence result for

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strong liftings on projective limits is Theorem 2.3. It can be seen that self-consistency is not a necessary condition for the existence of strong liftings (see Remark 3.3).

For products there is a well-known but somewhat elusive relationship between completion regularity and the existence of strong liftings (see e.g. [6]). By Theorem 3.1, self-consistency is sufficient (but again not necessary by Remark 3.3) for the permanence of completion regularity under projective limits of compact spaces and more generally for Hausdorff completely regular spaces in the presence of sequential maximality. Corollary 3.2 gives a permanence result for strong Baire liftings in projective limits.

In Section 4 we study the projective limit for lifting topologies. In terms of lifting topologies equivalent conditions can be given for the existence of a strong lifting on the projective limit (see Theorem 4.1 and its corollaries). Section 5 contains the specialization to products, thus extending the results in [22] from countable to uncountable products (see Theorem 5.3).

One consequence of the basic result (Theorem 5.3) is the existence of a strong lifting if all factors have the USLP (see Theorem 5.6). This theorem comprises the classical result of [14] and [16].

There exist projective limit Radon measures, e.g. the Wiener measure on $\mathbb{I}^{\mathfrak{c}}$ where $\mathbb{I} := [-\infty, +\infty]$ in which every lifting is almost strong, i.e. they have the USLP but there exists no strong lifting which can be represented as a projective limit of strong liftings.

1. Preliminaries. We assume throughout that every topological space X is Hausdorff completely regular. The σ-field of Borel (resp. Baire) sets over X, $\mathcal{B}(X)$ (resp. $\mathcal{B}_0(X)$), is the one generated by all open subsets of X (resp. by all bounded continuous functions on X). By a Borel (resp. Baire) measure on X we mean a finite, nonnegative countably additive set function defined on $\mathcal{B}(X)$ (resp. $\mathcal{B}_0(X)$).

Let (Ω, Σ, μ) be a finite measure space, i.e. Ω is a set, Σ a σ-field of subsets of Ω, and μ a nonnegative real-valued countably additive measure on Σ. Throughout we assume that $0 < \mu(\Omega) < \infty$. We write $\Sigma^\wedge$ for the Carathéodory completion of Σ with respect to μ. The canonical extension of μ to $\Sigma^\wedge$ will again be denoted by μ.

Let (T, Λ) be a *measurable space*, i.e. T is a set and Λ a σ-field of subsets of T. A mapping f from Ω into T is called Σ-Λ- *measurable* iff $f^{-1}(B) \in \Sigma$ for all $B \in \Lambda$. $\mathcal{L}^\infty(\Omega, \Sigma, \mu)$ or just $\mathcal{L}^\infty(\Omega, \mu)$ is the space of all bounded Σ- $\mathcal{B}(\mathbb{R})$-measurable functions on Ω, where $\mathbb{R}$ denotes the set of all real numbers.

For a complete finite measure space (Ω, Σ, μ) a *lifting* on $\mathcal{L}^\infty(\Omega, \mu)$ is a linear mapping ϱ^* from $\mathcal{L}^\infty(\Omega, \mu)$ into $\mathcal{L}^\infty(\Omega, \mu)$ with the following properties:

(I) $\varrho^*(f) = f$ a.e. (μ),
(II) $f = g$ a.e. (μ) implies $\varrho^*(f) = \varrho^*(g)$,
(III) $\varrho^*(1) = 1$ where 1 is the function identically equal to 1 on Ω,
(IV) $f \geq 0$ a.e. (μ) implies $\varrho^*(f) \geq 0$,
(V) $\varrho^*(fg) = \varrho^*(f)\varrho^*(g)$

(cf. [14, p. 34]). A *lifting* on Σ is a mapping ϱ from Σ into Σ with the following properties:

(I′) $\varrho(A) = A$ a.e. (μ),
(II′) $A = B$ a.e. (μ) implies $\varrho(A) = \varrho(B)$,
(III′) $\varrho(\Omega) = \Omega, \varrho(\emptyset) = \emptyset$,
(IV′) $\varrho(A \cap B) = \varrho(A) \cap \varrho(B)$,
(V′) $\varrho(A \cup B) = \varrho(A) \cup \varrho(B)$

(cf. [14, p. 35]). A mapping φ from Σ into Σ is called a *lower density* (or just a *density*) for (Ω, Σ, μ) if it satisfies (I′)–(IV′) (cf. [14, p. 36]).

We note that for any lifting ϱ on Σ there exists exactly one lifting ϱ^* on $\mathcal{L}^\infty(\Omega, \mu)$ such that $\varrho^*(1_A) = 1_{\varrho(A)}$ for $A \in \Sigma$ and vice versa (cf. [14, pp. 35, 36]). For simplicity we write $\varrho^* = \varrho$ throughout.

A *Radon measure* μ on X is a nonnegative real-valued Borel measure on $\mathcal{B}(X)$ such that for each Borel set E in $\mathcal{B}(X)$,

$$\mu(E) = \sup\{\mu(K) : K \subseteq E,\ K \text{ compact}\}.$$

A Borel measure μ on X is called:

(i) a *category measure* iff the Borel null sets and the Borel sets of first category are the same. Then $(X, \mathcal{B}^\wedge(X), \mu)$ is called a *category measure space* (cf. Oxtoby [28], p. 86);

(ii) *regular* iff it satisfies one of the following equivalent conditions:

(I) $\mu(B) = \sup\{\mu(F) : F \subseteq B,\ F \text{ closed}\}$,
(II) $\mu(B) = \inf\{\mu(U) : B \subseteq U,\ U \text{ open}\}$,

for all $B \in \mathcal{B}(X)$.

A regular Borel measure μ (or Baire measure μ_0) on a compact space X is called *completion regular* iff the completion of the Baire restriction μ_0 of μ coincides with the completion of μ (or the completion of μ_0 coincides with the completion of its regular Borel extension μ). The terminology is due to Halmos [13], p. 230.

We shall use the fact that every τ-additive Baire measure μ_0 on a completely regular space X has a unique extension to a τ- additive Borel measure μ (cf. [17] for the proof of this result and for definition of τ-additivity).

A lifting ϱ for $(X, \mathcal{B}_0^\wedge(X), \mu)$ is called a *strong Baire lifting* iff $\varrho(h) = h$ for each $h \in \mathcal{C}_{\mathrm{b}}(X)$, where $\mathcal{C}_{\mathrm{b}}(X)$ is the set of all bounded continuous functions on X.

1.1. Definitions. A *topological measure space* is a quadruple $(X, \mathcal{T}, \Sigma, \mu)$ where (X, Σ, μ) is a measure space and $\mathcal{T}$ is a topology on X with $T \subseteq \Sigma$.

A lifting ϱ for a complete topological probability space $(X, \mathcal{T}, \Sigma, \mu)$ is called *almost strong* iff there exists $N \in \Sigma$ with $\mu(N) = 0$ and $\varrho(f)(x) = f(x)$ for all $f \in \mathcal{C}_{\mathrm{b}}(X)$ and all $x \in X \setminus N$. The space (X, T, Σ, μ) has the *universal strong lifting property* (USLP for short) iff each lifting ϱ for μ is almost strong.

1.2. Definition. A family of sets $(X_\alpha)_{\alpha\in I}$ is said to be a *projective system relative to mappings* $f_{\alpha\beta}$, $\alpha, \beta \in I$, iff

(i) I is a directed set with respect to the ordering relation $\leq$; if $\alpha \leq \beta$ and $\alpha \neq \beta$, then we write $\alpha < \beta$;

(ii) the mappings $f_{\alpha\beta} : X_\beta \to X_\alpha$ are defined for each $\alpha, \beta \in I$ such that $\alpha \leq \beta$;

(iii) $f_{\alpha\gamma} = f_{\alpha\beta} \circ f_{\beta\gamma}$, whenever $\alpha \leq \beta \leq \gamma$, and $f_{\alpha\alpha}$ is the identity mapping.

We use the notation $(X_\alpha, f_{\alpha\beta}, I)$ for such a system. The set

$$X := \Big\{(x_\alpha)_{\alpha\in I} \in \prod_{\alpha\in I} X_\alpha : f_{\alpha\beta}(x_\beta) = x_\alpha,\ \alpha < \beta\Big\}$$

is the *projective limit* of $(X_\alpha, f_{\alpha\beta}, I)$. In symbols $X = \lim\mathrm{proj}_{\alpha\in I} X_\alpha$.

1.3. Definition. A family $(X_\alpha, \Sigma_\alpha)_{\alpha\in I}$ of measurable spaces (resp. $(X_\alpha, \mathcal{T}_\alpha)_{\alpha\in I}$ of topological spaces) is said to be a *projective system relative to mappings* $f_{\alpha\beta}$, $\alpha, \beta \in I$, iff

(i) $(X_\alpha, f_{\alpha\beta}, I)$ is a projective system,

(ii) $f_{\alpha\beta}$ is Σ_β-Σ_α- measurable (resp. $\mathcal{T}_\beta$-$\mathcal{T}_\alpha$-continuous) for $\alpha\leq\beta$, $\alpha, \beta\in I$.

We use the notation $(X_\alpha, \Sigma_\alpha, f_{\alpha\beta}, I)$ (resp. $(X_\alpha, \mathcal{T}_\alpha, f_{\alpha\beta}, I)$) for such a system. If Σ (resp. $\mathcal{T}$) is the smallest σ-field (resp. topology) in X relative to which the canonical projections f_α from X into X_α, defined by $f_\alpha((x_\beta)_{\beta\in I}) = x_\alpha$, are Σ-Σ_α-measurable (resp. $\mathcal{T}$-$\mathcal{T}_\alpha$-continuous), then Σ (resp. $\mathcal{T}$) is called the *projective limit σ-field* (resp. *topology*) of $(\Sigma_\alpha)_{\alpha\in I}$ (resp. $(\mathcal{T}_\alpha)_{\alpha\in I}$). In symbols $\Sigma = \lim\mathrm{proj}_{\alpha\in I} \Sigma_\alpha$ (resp. $\mathcal{T} = \lim\mathrm{proj}_{\alpha\in I} \mathcal{T}_\alpha$).

1.4. Definition. A family $(X_\alpha, \Sigma_\alpha, \mu_\alpha)_{\alpha\in I}$ (resp. $(X_\alpha, \mathcal{T}_\alpha, \Sigma_\alpha, \mu_\alpha)_{\alpha\in I}$) of measure spaces (resp. topological measure spaces) is said to be a *projective system relative to mappings* $f_{\alpha\beta}$, $\alpha, \beta \in I$ iff

(i) $(X_\alpha, \Sigma_\alpha, f_{\alpha\beta}, I)$ (resp. $(X_\alpha, \Sigma_\alpha, f_{\alpha\beta}, I)$ and $(X_\alpha, \mathcal{T}_\alpha, f_{\alpha\beta}, I)$) are projective systems,

(ii) $f_{\alpha\beta}$ is measure preserving, i.e. $\mu_\beta(f_{\alpha\beta}^{-1}(A)) = \mu_\alpha(A)$ for each $\alpha \leq \beta$ and $A \in \Sigma_\alpha$.

We use the notation $(X_\alpha, \Sigma_\alpha, \mu_\alpha, f_{\alpha\beta}, I)$ (resp. $(X_\alpha, \mathcal{T}_\alpha, \Sigma_\alpha, \mu_\alpha, f_{\alpha\beta}, I)$) for such a system. The projective system $(X_\alpha, \Sigma_\alpha, \mu_\alpha, f_{\alpha\beta}, I)$ (resp. $(X_\alpha, \mathcal{T}_\alpha, \Sigma_\alpha, \mu_\alpha, f_{\alpha\beta}, I)$) is *convergent* iff there exists a measure μ on Σ such that

$$\mu(f_\alpha^{-1}(A)) = \mu_\alpha(A) \quad \text{for each } \alpha \in I \text{ and } A \in \Sigma_\alpha$$

(resp. $(X_\alpha, \Sigma_\alpha, \mu_\alpha, f_{\alpha\beta}, I)$ is convergent and the projective limit topology $\mathcal{T}$ of $(\mathcal{T}_\alpha)_{\alpha\in I}$ is contained in Σ). Then $(X, \Sigma, \mu, (f_\alpha)_{\alpha\in I})$ (resp. $(X, \mathcal{T}, \Sigma, \mu, (f_\alpha)_{\alpha\in I})$) is called the *projective limit* of $(X_\alpha, \Sigma_\alpha, \mu_\alpha, f_{\alpha\beta}, I)$ (resp. $(X_\alpha, \mathcal{T}_\alpha, \Sigma_\alpha, \mu_\alpha, f_{\alpha\beta}, I)$). The measure space (X,Σ,μ) (resp. the topological measure space $(X, \mathcal{T}, \Sigma, \mu)$) is called the *projective limit measure space* (resp. *topological measure space*) of $(X_\alpha, \Sigma_\alpha, \mu_\alpha)_{\alpha\in I}$ (resp. $(X_\alpha, \mathcal{T}_\alpha, \Sigma_\alpha, \mu_\alpha)_{\alpha\in I}$).

The projective system $(X_\alpha, \Sigma_\alpha, \mu_\alpha, f_{\alpha\beta}, I)$ (resp. $(X_\alpha, \mathcal{T}_\alpha, \Sigma_\alpha, \mu_\alpha, f_{\alpha\beta}, I)$) is said to be *sequentially convergent* iff for each sequence $J = (\alpha_n)$, $\alpha_1 < \alpha_2 < \ldots$, $\alpha_n \in I$, the projective system $(X_\alpha, \Sigma_\alpha, \mu_\alpha, f_{\alpha\beta}, J)$ (resp. $(X_\alpha, \mathcal{T}_\alpha, \Sigma_\alpha, \mu_\alpha, f_{\alpha\beta}, J)$) is convergent.

All the systems that are used in this paper are projective, so that, in order to simplify the notation, we may use the word "system" instead of "projective system". We also suppose throughout this paper that all canonical projections f_α are surjective.

Finally, let $(X_\alpha, \Sigma_\alpha, \mu_\alpha)$ be a complete measure space and ϱ_α a lifting for μ_α, $\alpha \in I$. The family $(\varrho_\alpha)_{\alpha\in I}$ is called *self-consistent* iff $\varrho_\beta(f_{\alpha\beta}^{-1}(A)) = f_{\alpha\beta}^{-1}(\varrho_\alpha(A))$ for all $\alpha \le \beta$ and $A \in \Sigma_\alpha$.

All unexplained (topological) measure theoretic notions will be those of Halmos [13] and Knowles [17]. Those concerning lifting theory and topology can be found in A. and C. Ionescu Tulcea [14] and Dugundji [8] respectively.

2. The existence theorem

2.1. Proposition. *Let $(X_n, \mathcal{T}_n, \Sigma_n, \mu_n, f_{nm}, \mathbb{N})$ for $\mathbb{N} = \{1, 2, \ldots\}$ be a system of complete topological probability spaces and let $(\varrho_n)_{n\in\mathbb{N}}$ be a self-consistent sequence of strong liftings ϱ_n for μ_n. Suppose that the system $(X_n, \Sigma_n, \mu_n, f_{nm}, \mathbb{N})$ is convergent with projective limit $(X, \Sigma, \mu, (f_n)_{n\in\mathbb{N}})$, and that $\mathcal{T}$ is the projective limit of the topologies $(\mathcal{T}_n)_{n\in\mathbb{N}}$. Then $\mathcal{T} \subseteq \Sigma^\wedge$ and there exists a strong lifting ϱ for $(X, \mathcal{T}, \Sigma^\wedge, \mu)$ such that*

$$\varrho(f_n^{-1}(A)) = f_n^{-1}(\varrho_n(A))$$

for each $n \in \mathbb{N}$ and $A \in \Sigma_n$. In particular, if $\Sigma_n = \mathcal{B}^\wedge(X_n)$ then $\Sigma^\wedge = \mathcal{B}^\wedge(X)$.

Proof. For any $n \in \mathbb{N}$ the class of sets $\Sigma_n^* := f_n^{-1}(\Sigma_n)$ is a field of subsets of X. For $A_n^* \in \Sigma_n^*$, $A_n^* = f_n^{-1}(A_n)$, let $\mu_n^*(A_n^*) := \mu_n(A_n)$. Then (X, Σ_n^*, μ_n^*) is a measure space. Moreover, for $n \le m$, $\Sigma_n^* \subseteq \Sigma_m^*$ and $\mu_n^*(A) = \mu_m^*(A)$ for any $A \in \Sigma_n^*$. Let $(\Sigma_n^*)_\mu$ be the σ-subfield of $\Sigma^\wedge$ generated by $\Sigma_n^* \cup u$ for $u := \{A \in \Sigma^\wedge : \mu(A) = 0\}$, $(\mu_n)^\wedge := \mu|(\Sigma_n^*)_\mu$ and

$\Sigma^* := \bigcup_{n\in\mathbb{N}}(\Sigma_n^*)_\mu$. Then it can be easily seen that $(\Sigma_n^*)_\mu = \{(A\cap N^{\mathrm{c}})\cup (A^{\mathrm{c}}\cap N) : A\in\Sigma_n^*,\ N\in u\}$, Σ^* is a field of subsets of X, and the σ-field $\Sigma^\wedge$ is equal to $\sigma(\Sigma^*)$, the σ-field generated by Σ^*.

Claim 1. *There exists a density φ for $(X,\Sigma^\wedge,\mu)$ such that, for every $n\in\mathbb{N}$ and $A\in\Sigma_n$,*

$$\varphi(f_n^{-1}(A)) = f_n^{-1}(\varrho_n(A)).$$

Define, for every ϱ_n, a lifting ϱ_n^* for $(X,(\Sigma_n^*)_\mu,(\mu_n)^\wedge)$ by $\varrho_n^*(A^*) = f_n^{-1}(\varrho_n(A))$, where $A^*\in(\Sigma_n^*)_\mu$ and $A\in\Sigma_n$ with $A^* = f_n^{-1}(A)$ a.e. $(\mu_n)^\wedge$.

For all $n,m\in\mathbb{N}$, $n\le m$, we have

$$\varrho_m^*|(\Sigma_n^*)_\mu = \varrho_n^*.$$

Indeed, for $A_n^*\in(\Sigma_n^*)_\mu$ there exists a set $A_n\in\Sigma_n$ such that $A_n^* = f_n^{-1}(A_n)$ a.e. $(\mu_n)^\wedge$, $n\in\mathbb{N}$. So, we get

$$\begin{aligned}\varrho_m^*(A_n^*) &= \varrho_m^*(f_n^{-1}(A_n)) = \varrho_m^*(f_m^{-1}(f_{nm}^{-1}(A_n))) = f_m^{-1}(f_{nm}^{-1}(\varrho_n(A_n)))\\ &= f_n^{-1}(\varrho_n(A_n)) = \varrho_n^*(f_n^{-1}(A_n)) = \varrho_n^*(A_n^*).\end{aligned}$$

Thus there exists a density φ for $(X,\Sigma^\wedge,\mu)$ such that $\varphi(f_n^{-1}(A)) = f_n^{-1}(\varrho_n(A))$ for each $n\in\mathbb{N}$ and $A\in\Sigma_n$ (cf. [12], Proposition 2 for example).

Claim 2. *There exists a strong lifting ϱ for $(X,\mathcal{T},\Sigma^\wedge,\mu)$ such that*

$$\varrho(f_n^{-1}(A)) = f_n^{-1}(\varrho_n(A))$$

for each $n\in\mathbb{N}$ and $A\in\Sigma_n$.

By the theorem of von Neumann [26] (see also Traynor [31], Theorem 3, p. 268) there exists a lifting ϱ for $(X,\Sigma^\wedge,\mu)$ such that $\varphi(A)\subseteq\varrho(A)$ for each $A\in\Sigma^\wedge$, and $\varrho(f_n^{-1}(B)) = f_n^{-1}(\varrho_n(B))$ for each $n\in\mathbb{N}$ and $B\in\Sigma_n$.

Now let $\mathfrak{T}_{\varrho_n} := \{A_n\in\Sigma_n : A_n\subseteq\varrho_n(A)\}$, $n\in\mathbb{N}$, be the "lifting topology" on X_n. Since for each set $A_n\in\mathfrak{T}_{\varrho_n}$ we have

$$f_{nm}^{-1}(A_n)\subseteq f_{nm}^{-1}(\varrho_n(A_n)) = \varrho_m(f_{nm}^{-1}(A_n)),$$

i.e. $f_{nm}^{-1}(A_n)\in\mathfrak{T}_{\varrho_m}$, each mapping f_{nm} is $\mathfrak{T}_{\varrho_m}$-$\mathfrak{T}_{\varrho_n}$-continuous. In the same way it can be proved that each f_n is $\mathfrak{T}_\varrho$-$\mathfrak{T}_{\varrho_n}$-continuous. Thus we get

$$(*)\qquad \mathcal{T}\subseteq\lim_{n\in\mathbb{N}}\operatorname{proj}\mathfrak{T}_{\varrho_n}\subseteq\mathfrak{T}_\varrho\subseteq\mathcal{B}(\mathfrak{T}_\varrho) = \Sigma^\wedge$$

(for the equality cf. e.g. [14]). Consequently, ϱ is a strong lifting for $(X,\mathcal{T},\Sigma^\wedge,\mu)$.

In particular, if $\Sigma_n = \mathcal{B}^\wedge(X_n)$, $n\in\mathbb{N}$, then relations $(*)$ and the obvious one, $\Sigma^\wedge\subseteq\mathcal{B}^\wedge(X)$, imply that $\Sigma^\wedge = \mathcal{B}^\wedge(X)$.

2.2. Remarks. (i) By using the same arguments as in the proof of the above proposition it can be proved that (a) the density φ in the proof is strong, and (b) the lifting ϱ and the density φ are strong even if every ϱ_n is only a strong density.

(ii) Clearly the trivial system $(X_\alpha, \Sigma_\alpha, \mu_\alpha, f_{\alpha\beta}, I)$ where $X_\alpha = X$, $\Sigma_\alpha = \Sigma$, $\mu_\alpha = \mu$ for each $\alpha \in I$, (X, Σ, μ) is a probability space and each $f_{\alpha\beta}$ is the identical mapping on X is convergent. The first nontrivial example of a convergent system was given by Kolmogorov who obtained a projective limit probability measure on the infinite cartesian product of unit intervals (cf. e.g. [5, p. 321]). Bochner [4, pp. 118–119] extended the result of Kolmogorov by proving the existence of the projective limit space for an arbitrary system of topological spaces with measures approximated by compact sets. Generalizations of the result of Bochner can be found e.g. in [5], [25], and [27].

(iii) There are projective systems $(X_\alpha, \mathcal{T}_\alpha, \mathcal{B}^\wedge(X_\alpha), \mu_\alpha, f_{\alpha\beta}, I)$ of topological probability spaces for which there exist self-consistent systems of (almost) strong liftings as the following examples show:

(a) Each X_α is only a completely regular space, μ_α has the USLP and $I = \mathbb{N}$. We construct inductively a self-consistent sequence $(\varrho_n)_{n\in\mathbb{N}}$ of almost strong liftings in the following way: Let ϱ_1 be an almost strong lifting for μ_1. Define ω_2 by

$$\omega_2(g \circ f_{12}) := \varrho_1(g) \circ f_{12}\,,$$

where $g \in \mathcal{L}^\infty(X_1, \mu_1)$. Then ω_2 is an unambiguously defined almost strong lifting on $\{g \circ f_{12} : g \in \mathcal{L}^\infty(X_1, \mu_1)\}$ which can be extended to an almost strong lifting ϱ_2 for μ_2 (cf. [1], proof of Theorem 2.3, or [21], Lemma 2.1). In the same way we construct for a given almost strong lifting ϱ_n for μ_n an almost strong lifting ϱ_{n+1} for μ_{n+1} such that

$$\varrho_{n+1}(g \circ f_{n(n+1)}) := \varrho_n(g) \circ f_{n(n+1)}$$

where $g \in \mathcal{L}^\infty(X_n, \mu_n)$. Obviously the system (ϱ_n) is self-consistent.

(b) Each X_α is an extremally disconnected compact space, each $(X_\alpha, \mathcal{B}^\wedge(X_\alpha), \mu_\alpha)$ is a category probability space (such spaces X_α are exactly the hyperstonian spaces of Dixmier [7] (cf. [11], Satz 9.6)) with $\mathrm{supp}(\mu_\alpha) = X_\alpha$, $\alpha \in I$, and each ϱ_α is the natural strong lifting for μ_α. We mention that the projective limit of hyperstonian spaces is not in general a hyperstonian space (cf. [20], §4).

(c) Each $(X_\alpha, \mathcal{T}_\alpha, \mathcal{B}^\wedge(X_\alpha), \mu_\alpha)$ is a topological probability space such that each X_α is an extremally disconnected Baire space where each set of the first category is closed, and each $(X_\alpha, \mathcal{B}^\wedge(X_\alpha), \mu_\alpha)$ is a category measure space. Then, for each $\alpha \in I$, there exists a unique strong lifting ϱ_α for μ_α such that $\mathcal{T}_\alpha = \mathfrak{T}_{\varrho_\alpha}$ (cf. [11], Satz 8.33).

If $\alpha \le \beta$ put

$$H_{\beta,0} := \{\widetilde{f} \circ f_{\alpha\beta} : \widetilde{f} \in \mathcal{L}^\infty(X_\alpha, \mu_\alpha)\}\,.$$

Let $H_\beta := \{g \in \mathcal{L}^\infty(X_\beta, \mu_\beta) : g = g'$ a.e. (μ_β) for some $g' \in H_{\beta,0}\}$. Define a lifting ϱ'_β on H_β by means of the equation

$$(*) \qquad \varrho'_\beta(\widetilde{f} \circ f_{\alpha\beta}) = \varrho_\alpha(\widetilde{f}) \circ f_{\alpha\beta}$$

for $\widetilde{f} \in \mathcal{L}^\infty(X_\alpha, \mu_\alpha)$. Then ϱ'_β is $\mathcal{H}_\beta$-strong, i.e. $\varrho'_\beta(g) = g$ for each $g \in \mathcal{H}_\beta := \{h \circ f_{\alpha\beta} : h \in \mathcal{C}_{\mathrm{b}}(X_\alpha)\}$. In both cases (b) and (c) the restriction of the strong lifting ϱ_β to H_β is equal to ϱ'_β. Therefore in $(*)$ we may replace ϱ'_β by ϱ_β, which is the self-consistency we have to prove.

Next we deal with the general projective system $(X_\alpha, \mathcal{T}_\alpha, \Sigma_\alpha, \mu_\alpha, f_{\alpha\beta}, I)$. We put

$$D(I) := \{(\alpha_n)_{n\in\mathbb{N}} : \alpha_n \in I \text{ for } n \in \mathbb{N} \text{ and } \alpha_1 < \alpha_2 < \ldots\}.$$

For each $M = (\alpha_n)_{n\in\mathbb{N}}$ in $D(I)$ we denote by f_M the canonical projection from X to $X_M = \lim\mathrm{proj}_{n\in\mathbb{N}} X_{\alpha_n}$. We say that the system $(X_\alpha, f_{\alpha\beta}, I)$ is *sequentially maximal* iff f_M is a surjection for each M in $D(I)$ (cf. e.g. [25]).

Using Proposition 2.1 we obtain the following result.

2.3. Theorem. *Let $(X_\alpha, \mathcal{T}_\alpha, \Sigma_\alpha, \mu_\alpha, f_{\alpha\beta}, I)$ be a system of complete topological probability spaces. Suppose $(X_\alpha, \Sigma_\alpha, \mu_\alpha, f_{\alpha\beta}, I)$ is convergent with projective limit $(X, \Sigma, \mu, (f_\alpha)_{\alpha\in I})$, $\mathcal{T}$ is the projective limit of $(\mathcal{T}_\alpha)_{\alpha\in I}$, and $(\varrho_\alpha)_{\alpha\in I}$ is a self-consistent family of strong liftings ϱ_α for $(X_\alpha, \mathcal{T}_\alpha, \Sigma_\alpha, \mu_\alpha)$. Then $\mathcal{T} \subseteq \Sigma^\wedge$ and there exists a strong lifting ϱ for $(X, \mathcal{T}, \Sigma^\wedge, \mu)$ such that*

$$(\mathrm{L}) \qquad \varrho(f_\alpha^{-1}(A)) = f_\alpha^{-1}(\varrho_\alpha(A))$$

for all $\alpha \in I$ and $A \in \Sigma_\alpha$. In particular, if $\Sigma_\alpha = \mathcal{B}^\wedge(X_\alpha)$ then $\Sigma^\wedge = \mathcal{B}^\wedge(X)$.

Proof. For each $M = (\alpha_n)_{n\in\mathbb{N}}$ in $D(I)$, $(X_{\alpha_n}, \mathcal{T}_{\alpha_n}, \Sigma_{\alpha_n}, \mu_{\alpha_n}, f_{\alpha_n\alpha_m}, \mathbb{N})$ is a system of complete topological probability spaces such that $(X_{\alpha_n}, \Sigma_{\alpha_n}, \mu_{\alpha_n}, f_{\alpha_n\alpha_m}, \mathbb{N})$ is convergent with projective limit $(X_M, \Sigma_M, \mu_M, (f_{\alpha_n M})_{n\in\mathbb{N}})$ and $\mu \circ f_M^{-1} = \mu_M$ (cf. [25], Proposition 2.3). Let $\mathcal{T}_M$ be the projective limit topology of $(\mathcal{T}_{\alpha_n})_{n\in\mathbb{N}}$.

By Remark 2.2(i) there exists a strong density φ_M for $(X_M, \mathcal{T}_M, \Sigma_M^\wedge, \mu_M)$ such that

$$\varphi_M(f_{\alpha_n M}^{-1}(A)) = f_{\alpha_n M}^{-1}(\varrho_{\alpha_n}(A))$$

for any $\alpha_n \in M$ and $A \in \Sigma_{\alpha_n}$. In particular, $\Sigma_M^\wedge = \mathcal{B}^\wedge(X_M)$ if $\Sigma_{\alpha_n} = \mathcal{B}^\wedge(X_{\alpha_n})$.

We now introduce an order relation in $D(I)$ as follows: for $M = (\alpha_n)$, $N = (\beta_n)$ in $D(I)$,

$$M \le N \quad \text{iff} \quad \alpha_n \le \beta_n \text{ for each } n \in \mathbb{N}.$$

For (x_{β_n}) in X_N we set

$$f_{MN}((x_{\beta_n})) = (f_{\alpha_n\beta_n}(x_{\beta_n})).$$

It follows that $f_M = f_{MN} \circ f_N$ and $\mu_N \circ f_{MN}^{-1} = \mu_M$ since $\mu_M = \mu \circ f_M^{-1} = \mu \circ f_N^{-1} \circ f_{MN}^{-1} = \mu_N \circ f_{MN}^{-1}$.

Now we claim that for $M = (\alpha_n)$, $N = (\beta_n) \in D(I)$,

$$M \le N \quad \text{implies} \quad \varphi_N \circ f_{MN}^{-1} = f_{MN}^{-1} \circ \varphi_M\,.$$

Indeed, let $A \in \Sigma_M^\wedge$ and denote by $E_{\alpha_n}(1_A)$ the conditional expectation of 1_A with respect to the σ-field $(\Sigma_{\alpha_n}^*)_{\mu_M}$, $n \in \mathbb{N}$, defined in the proof of Proposition 2.1. For all $n, k \in \mathbb{N}$ we have $(A_{k,\alpha_n})^* := \{E_{\alpha_n}(1_A) > 1-1/k\} \in (\Sigma_{\alpha_n}^*)_{\mu_M}$. It is easily seen that

$$(*) \qquad f_{MN}^{-1}((A_{k,\alpha_n})^*) = ((f_{MN}^{-1}(A))_{k,\beta_n})^* \quad \text{a.e. } (\mu_{\beta_n})^\wedge\,.$$

On the other hand, it is well known that

$$\varphi_M(A) = \bigcap_{k\in\mathbb{N}} \bigcup_{n\in\mathbb{N}} \bigcap_{m\ge n} \varrho_{\alpha_m}^*((A_{k,\alpha_m})^*)$$

where $\varrho_{\alpha_m}^*$ are defined in the proof of Proposition 2.1 (cf. for example [11], Lemma 4.6, p. 64). Hence,

$$(1) \qquad \begin{aligned} f_{MN}^{-1}(\varphi_M(A)) &= \bigcap_{k\in\mathbb{N}} \bigcup_{n\in\mathbb{N}} \bigcap_{m\ge n} f_{MN}^{-1}(\varrho_{\alpha_m}^*((A_{k,\alpha_m})^*)) \\ &= \bigcap_{k\in\mathbb{N}} \bigcup_{n\in\mathbb{N}} \bigcap_{m\ge n} f_{MN}^{-1}(\varrho_{\alpha_m}^*(f_{\alpha_m M}^{-1}(A_{k,\alpha_m})) \\ &= \bigcap_{k\in\mathbb{N}} \bigcup_{n\in\mathbb{N}} \bigcap_{m\ge n} f_{MN}^{-1}(f_{\alpha_m M}^{-1}(\varrho_{\alpha_m}(A_{k,\alpha_m}))) \\ &= \bigcap_{k\in\mathbb{N}} \bigcup_{n\in\mathbb{N}} \bigcap_{m\ge n} f_{\beta_m N}^{-1}(f_{\alpha_m\beta_m}^{-1}(\varrho_{\alpha_m}(A_{k,\alpha_m}))). \end{aligned}$$

On the other hand,

$$(2) \qquad \begin{aligned} \varphi_N(f_{MN}^{-1}(A)) &\overset{(*)}{=} \bigcap_{k\in\mathbb{N}} \bigcup_{n\in\mathbb{N}} \bigcap_{m\ge n} \varrho_{\beta_m}^*(f_{MN}^{-1}((A_{k,\alpha_m})^*)) \\ &= \bigcap_{k\in\mathbb{N}} \bigcup_{n\in\mathbb{N}} \bigcap_{m\ge n} \varrho_{\beta_m}^*(f_{MN}^{-1}(f_{\alpha_m M}^{-1}(A_{k,\alpha_m}))) \\ &= \bigcap_{k\in\mathbb{N}} \bigcup_{n\in\mathbb{N}} \bigcap_{m\ge n} \varrho_{\beta_m}^*(f_{\beta_m N}^{-1}(f_{\alpha_m\beta_m}^{-1}(A_{k,\alpha_m}))) \\ &= \bigcap_{k\in\mathbb{N}} \bigcup_{n\in\mathbb{N}} \bigcap_{m\ge n} f_{\beta_m N}^{-1}(\varrho_{\beta_m}(f_{\alpha_m\beta_m}^{-1}(A_{k,\alpha_m}))) \\ &= \bigcap_{k\in\mathbb{N}} \bigcup_{n\in\mathbb{N}} \bigcap_{m\ge n} f_{\beta_m N}^{-1}(f_{\alpha_m\beta_m}^{-1}(\varrho_{\alpha_m}(A_{k,\alpha_m})))\,. \end{aligned}$$

From (1) and (2) we deduce our claim.

Furthermore, we have $\Sigma = \bigcup_{M\in D(I)} f_M^{-1}(\Sigma_M)$ and for any $M,N\in D(I)$, $M\le N$,

$$f_M^{-1}(\Sigma_M)\subseteq f_N^{-1}(\Sigma_N).$$

Now, for $M\in D(I)$, define a density φ_M^* for $(X,(f_M^{-1}(\Sigma_M^\wedge))_\mu,(\mu_M)^\wedge)$, where $(f_M^{-1}(\Sigma_M^\wedge))_\mu$ is the σ-subfield of $\Sigma^\wedge$ generated by $f_M^{-1}(\Sigma_M^\wedge)\cup u$ for $u:=\{A\in\Sigma^\wedge:\mu(A)=0\}$, and $(\mu_M)^\wedge:=\mu|(f_M^{-1}(\Sigma_M^\wedge))_\mu$, by

$$\varphi_M^*(A^*):=f_M^{-1}(\varphi_M(A))$$

for any $A^*\in(f_M^{-1}(\Sigma_M^\wedge))_\mu$ and $A\in\Sigma_M^\wedge$ with $A^*=f_M^{-1}(A)$ a.e. $(\mu_M)^\wedge$ (see the proof of Proposition 2.1). Since $\varphi_N^*|(f_M^{-1}(\Sigma_M^\wedge))_\mu=\varphi_M^*$ for $M\le N$, there exists a density φ for $(X,\Sigma^\wedge,\mu)$ such that

$$\varphi(f_M^{-1}(A))=f_M^{-1}(\varphi_M(A))\quad \text{for } A\in\Sigma_M^\wedge.$$

The result now follows as in the proof of Proposition 2.1.

If for a self-consistent family $(\varrho_\alpha)_{\alpha\in I}$ of liftings ϱ_α for $(X_\alpha,\Sigma_\alpha,\mu_\alpha)$, $\alpha\in I$, and for a lifting ϱ for $(X,\Sigma^\wedge,\mu)$ the equality (L) of the above theorem is true we write $\varrho=\lim\mathrm{proj}_{\alpha\in I}\varrho_\alpha$ and call ϱ a *projective limit* of the family $(\varrho_\alpha)_{\alpha\in I}$.

2.4. Remarks. (i) There are systems of topological probability spaces without any self-consistent family of strong liftings. Fremlin's simplification of Losert's [18] celebrated counter-example to the strong lifting conjecture gives such a system. Indeed, let μ be Fremlin's Radon probability measure on $X:=\{0,1\}^{\aleph_2}$ which has no strong lifting and is supported by X (cf. [10]).

The set I of all finite subsets of $\aleph_2$ forms a directed set under inclusion; $(X,(f_\alpha)_{\alpha\in I})$ is the projective limit of $(X_\alpha,f_{\alpha\beta},I)$ where $X_\alpha=\prod X_i$, $X_i=\{0,1\}$, and $f_{\alpha\beta}$ (resp. f_α) is the canonical projection from X_β onto X_α (resp. from X onto X_α) for $\alpha\le\beta$, $\alpha,\beta\in I$ (resp. $\alpha\in I$). If μ_α is the image measure $\mu\circ f_\alpha^{-1}$ on $\mathcal{B}^\wedge(X_\alpha)$ then μ is the projective limit of the system $(\mu_\alpha)_{\alpha\in I}$.

Now assume that there exists a self-consistent family $(\varrho_\alpha)_{\alpha\in I}$ of strong liftings ϱ_α for μ_α. Then by Theorem 2.3 there exists a strong lifting ϱ for μ; this yields a contradiction and hence there cannot exist any self-of strong liftings for the system $(X_\alpha,\mathcal{T}_\alpha,\mathcal{B}^\wedge(X_\alpha),\mu_\alpha,f_{\alpha\beta},I)$.

(ii) It is well known that the projective limit μ of a system (μ_α) of τ-additive measures μ_α is, in general, not such a measure, even if each μ_α is Radon (cf. [27], p. 331 and [24], Theorem 4.6).

Combining Theorem 2.3 and [2], Proposition 3, we conclude that the existence of a self-consistent family $(\varrho_\alpha)_{\alpha\in I}$ of strong liftings ϱ_α for μ_α is sufficient in order to preserve the τ-additivity of measures under the formation of projective limits.

(iii) The above example of Moran from (ii) together with [3], Theorem 4.2 and Corollary 6.1, and [1] shows that projective limits of (strongly) measure compact spaces (resp. (strongly) lifting compact spaces) are, in general, not even measure compact (resp. lifting compact). (For definitions of the above notions see [1], [3] and [24].)

(iv) The relation $\mathcal{B}^\wedge(X) = \Sigma^\wedge$ is not true in general. For example, the Wiener measure μ (or the measure of the Brownian motion process) defined on $\mathbb{I}^{[0,1]} = \mathbb{I}^{\mathfrak{c}}$ where $\mathbb{I} := [-\infty, +\infty]$ is a projective limit of Borel measures μ_α defined on $\mathbb{I}^\alpha$ for $\alpha \in I$ and I is the family of all finite nonvoid subsets of $[0,1]$. Denote by $(\mathbb{I}^{\mathfrak{c}}, \Sigma^\wedge, \mu)$ the completed projective limit space of the measure spaces $(\mathbb{I}^\alpha, \mathcal{B}^\wedge(\mathbb{I}^\alpha), \mu_\alpha)$ where $\mu_\alpha = \mu \circ f_\alpha^{-1}$ and $f_\alpha : \mathbb{I}^{\mathfrak{c}} \to \mathbb{I}^\alpha$ are the canonical projections for all $\alpha \in I$. Since μ is not completion regular (cf. [6]) it follows that $\mathcal{B}_0^\wedge(X) \subset \mathcal{B}^\wedge(X)$ properly. On the other hand, $\mathcal{B}_0(X) = \Sigma$ (cf. e.g. [15]). Thus $\mathcal{B}^\wedge(X) \subsetneq \Sigma^\wedge$. This means that the open sets are not measurable for the infinite product while they are for the finite products.

(v) The converse of Theorem 2.3 is true in the following sense: Let $(X_\alpha, \mathcal{T}_\alpha, \Sigma_\alpha, \mu_\alpha, f_{\alpha\beta}, I)$ be a system of complete topological probability spaces. Suppose that $(X_\alpha, \Sigma_\alpha, \mu_\alpha, f_{\alpha\beta}, I)$ is convergent with projective limit $(X, \Sigma, \mu, (f_\alpha)_{\alpha\in I})$ and $\mathcal{T}$ is the projective limit of $(\mathcal{T}_\alpha)_{\alpha\in I}$. If $\mathcal{T} \subseteq \Sigma^\wedge$, $(\varrho_\alpha)_{\alpha\in I}$ is a family of liftings ϱ_α for μ_α ($\alpha \in I$), and ϱ is a strong lifting for μ such that condition (L) from Theorem 2.3 holds true then all ϱ_α ($\alpha \in I$) are necessarily strong. Indeed, for given $U \in \mathcal{T}_\alpha$ for some $\alpha \in I$ we have $f_\alpha^{-1}(U) \subseteq \varrho(f_\alpha^{-1}(U)) = f_\alpha^{-1}(\varrho_\alpha(U))$. This implies $U \subseteq \varrho_\alpha(U)$.

3. Permanence of completion regularity.

Theorem 2.3 provides a basis for discussing the permanence of completion regularity for projective limits. The Wiener measure shows that a projective limit of completion regular measures is not in general such a measure and at the same time it shows that a projective limit of measures with the strong Baire lifting property need not be such a measure (see Remark 2.4 (iv) in combination with [2], Prop. 3).

For preparation we need the following lemma.

Lemma. *Let $(X, \Sigma, \mu, (f_\alpha)_{\alpha\in I})$ and $(X, \Sigma_1, \mu_1, (f_\alpha)_{\alpha\in I})$ be the projective limits of the systems $(X_\alpha, \Sigma_\alpha, \mu_\alpha, f_{\alpha\beta}, I)$ and $(X_\alpha, \Sigma_\alpha^\wedge, \mu_\alpha, f_{\alpha\beta}, I)$ respectively. Suppose that the system $(X_\alpha, f_{\alpha\beta}, I)$ is sequentially maximal. Then $\Sigma^\wedge = \Sigma_1^\wedge$.*

Proof. Clearly, given a system $(X_\alpha, \Sigma_\alpha, \mu_\alpha, f_{\alpha\beta}, I)$ the family $(X_\alpha, \Sigma_\alpha^\wedge, \mu_\alpha, f_{\alpha\beta}, I)$ is also a system, $\Sigma \subseteq \Sigma_1$, and the restriction of μ_1 to Σ coincides with μ. We only have to show that $\Sigma_1 \subseteq \Sigma^\wedge$. Let $A \in \bigcup_{\alpha\in I} f_\alpha^{-1}(\Sigma_\alpha^\wedge)$. There exist $\alpha \in I$ and $A_\alpha \in \Sigma_\alpha^\wedge$ with $A = f_\alpha^{-1}(A_\alpha)$. So there exist $E_\alpha, F_\alpha \in \Sigma_\alpha$ such that $E_\alpha \subseteq A_\alpha \subseteq F_\alpha$ and $\mu_\alpha(F_\alpha \setminus E_\alpha) = 0$. Consequently, $f_\alpha^{-1}(E_\alpha), f_\alpha^{-1}(F_\alpha) \in \Sigma$, $f_\alpha^{-1}(E_\alpha) \subseteq A \subseteq f_\alpha^{-1}(F_\alpha)$ and $\mu(f_\alpha^{-1}(F_\alpha) \setminus f_\alpha^{-1}(E_\alpha)) = 0$, i.e. $A \in \Sigma^\wedge$.

3.1. THEOREM. *Let* $(X_\alpha, \mathcal{T}_\alpha, f_{\alpha\beta}, I)$ (*resp.* $(X_\alpha, \mathcal{B}_0(X_\alpha), \mu_{\alpha,0}, f_{\alpha\beta}, I)$) *be a system of compact spaces* (*resp. Baire probability spaces*) *with projective limit* $(X, \mathcal{T}, (f_\alpha)_{\alpha\in I})$ (*resp.* $(X, \Sigma_0, \mu_0, (f_\alpha)_{\alpha\in I})$). *Suppose that the regular Borel extension* μ_α *of* $\mu_{\alpha,0}$, $\alpha \in I$, *is completion regular*, *and that* $(\varrho_\alpha)_{\alpha\in I}$ *is a self-*$(X_\alpha, \mathcal{T}_\alpha, \mathcal{B}^\wedge(X_\alpha), \mu_\alpha)$. *Then* $\Sigma_0^\wedge = \mathcal{B}^\wedge(X)$, *there exists a strong lifting* ϱ *for* $(X, \mathcal{T}, \mathcal{B}^\wedge(X), \mu_0)$ *with*

$$(*) \qquad \varrho(f_\alpha^{-1}(A)) = f_\alpha^{-1}(\varrho_\alpha(A))$$

for all $\alpha \in I$ *and* $A \in \mathcal{B}^\wedge(X_\alpha)$, *and* μ_0 *is completion regular.*

P r o o f. By [5], p. 325, the system $(X_\alpha, \mathcal{B}_0(X_\alpha), \mu_{\alpha,0}, f_{\alpha\beta}, I)$ is convergent with

$$(1) \qquad \Sigma_0 = \mathcal{B}_0(X)\,.$$

Again by [5], Theorem 2.2, the system $(X_\alpha, \mathcal{B}(X_\alpha), \mu_\alpha, f_{\alpha\beta}, I)$ (resp. $(X_\alpha, \mathcal{B}^\wedge(X_\alpha), \mu_\alpha, f_{\alpha\beta}, I)$) is convergent; denote by $(X, \Sigma, \mu, (f_\alpha)_{\alpha\in I})$ (resp. $(X, \Sigma_1, \mu_1, (f_\alpha)_{\alpha\in I})$) its projective limit. We may apply Theorem 2.3 to deduce that

$$(2) \qquad \Sigma_1^\wedge = \mathcal{B}^\wedge(X)$$

and that there exists a strong lifting ϱ for $(X, \mathcal{T}, \mathcal{B}^\wedge(X), \mu_1)$ with property $(*)$.

Since each μ_α is completion regular it follows that

$$\Sigma_1 = \lim_{\alpha\in I} \operatorname{proj} \mathcal{B}_0^\wedge(X_\alpha)\,.$$

Hence applying the above lemma we get

$$(3) \qquad \Sigma_1^\wedge = \Sigma_0^\wedge\,,$$

and $\mu_1 = \mu_0$. Thus ϱ is strong for $(X, \mathcal{T}, \mathcal{B}^\wedge(X), \mu_0)$.

Finally, from (1)–(3) it follows that $\mathcal{B}_0^\wedge(X) = \mathcal{B}^\wedge(X)$, i.e. the completion regularity of μ_0.

R e m a r k. The above theorem remains true if the spaces X_α are only Hausdorff completely regular, $(X_\alpha, f_{\alpha\beta}, I)$ is sequentially maximal, $\mathcal{B}_0(X) = \Sigma_0$, and the measures μ_α are Radon for all $\alpha \in I$. The proof is the same with the exception of the convergence of $(X_\alpha, \mathcal{T}_\alpha, \mathcal{B}_0(X_\alpha), \mu_\alpha, f_{\alpha\beta}, I)$, which follows e.g. from [5], Theorem 2.1.

3.2. COROLLARY. *Let* $(X_\alpha, \mathcal{T}_\alpha, f_{\alpha\beta}, I)$ (*resp.* $(X_\alpha, \mathcal{B}_0(X_\alpha), \mu_{\alpha,0}, f_{\alpha\beta}, I)$) *be a system of compact spaces* (*resp. Baire probability spaces*) *with projective limit* $(X, \mathcal{T}, (f_\alpha)_{\alpha\in I})$ (*resp.* $(X, \Sigma, \mu, (f_\alpha)_{\alpha\in I})$). *Suppose that* $(\varrho_\alpha)_{\alpha\in I}$ *is a* *self-consistent family of strong Baire liftings ϱ_α for $\mu_{\alpha,0}$. Then there exists a strong Baire lifting ϱ for μ such that*

$$(*)\qquad \varrho(f_\alpha^{-1}(A)) = f_\alpha^{-1}(\varrho_\alpha(A))$$

for all $\alpha \in I$ and $A \in \mathcal{B}_0^\wedge(X_\alpha)$, and $\mathcal{B}_0^\wedge(X) = \Sigma^\wedge$.

Proof. According to [2], Proposition 3, each $\mu_{\alpha,0}$ is completion regular and therefore each ϱ_α is a strong lifting for $(X_\alpha, \mathcal{T}_\alpha, \mathcal{B}^\wedge(X_\alpha), \mu_\alpha)$, where μ_α is the regular Borel extension of $\mu_{\alpha,0}$. Thus by Theorem 3.1, $\mathcal{B}^\wedge(X) = \Sigma^\wedge$, μ is completion regular and there exists a strong lifting for $(X, \mathcal{T}, \mathcal{B}^\wedge(X), \mu)$ with property $(*)$. Consequently, ϱ is a Baire strong lifting.

3.3. Remark. The self-consistency of the strong liftings ϱ_α in Theorem 2.3 as well as in Theorem 3.1 is not necessary for the existence of a strong lifting for the Borel extension of the projective limit measure μ or for the completion regularity of μ as the following examples show.

The Wiener measure μ in Remark 2.4(iv) has a strong lifting ϱ but it is not completion regular (cf. [6]). If ϱ were a projective limit of a self-consistent family of strong liftings ϱ_α for μ_α then by Theorem 3.1, μ should be completion regular, which contradicts what we mentioned at the beginning of this section.

On the other hand, Fremlin's simplification [10] of Losert's counterexample to the strong lifting conjecture gives a Radon probability measure on $[0,1]^{\aleph_2}$ which is completion regular but it has no strong lifting. The above measure is a projective limit of completion regular measures without any family of self-consistent strong liftings (compare Remark 2.4(i)).

4. Lifting topologies. In the following we give conditions equivalent to the existence of a strong lifting which is a projective limit in terms of lifting topologies. For a complete probability space (Ω, Σ, μ), one can associate with every lifting ϱ for μ two so-called lifting topologies $\mathfrak{T}_\varrho := \{A \in \Sigma : A \subseteq \varrho(A)\}$ and $\mathcal{T}_\varrho := \{\bigcup_i \varrho(A_i) : A_i \in \Sigma \text{ for } i \in I\}$. We used $\mathfrak{T}_\varrho$ in the proof of Proposition 2.1. The topologies $\mathfrak{T}_\varrho$ and $\mathcal{T}_\varrho$ are extremally disconnected, $\mathcal{T}_\varrho \subseteq \mathfrak{T}_\varrho$ and $\mathcal{C}_{\mathrm{b}}(\Omega, \mathcal{T}_\varrho) = \mathcal{C}_{\mathrm{b}}(\Omega, \mathfrak{T}_\varrho) = \{f \in \mathcal{L}^\infty(\mu) : f = \varrho(f)\}$ (see [14]).

4.1. Theorem. *Let $(X_\alpha, \Sigma_\alpha, \mu_\alpha, f_{\alpha\beta}, I)$ be a convergent system of complete probability spaces with projective limit $(X, \Sigma, \mu, (f_\alpha)_{\alpha\in I})$. Let $(\varrho_\alpha)_{\alpha\in I}$ be a self- consistent family of liftings ϱ_α for μ_α, and ϱ a lifting for μ. Then the following conditions are equivalent*:

(i) *The projective limit topology $\mathcal{T}$ of $(\mathcal{T}_{\varrho_\alpha})_{\alpha\in I}$ is contained in $\Sigma^\wedge$ and ϱ is strong with respect to $\mathcal{T}$.*

(ii) *The projective limit topology $\mathfrak{T}$ of $(\mathfrak{T}_{\varrho_\alpha})_{\alpha\in I}$ is contained in $\Sigma^\wedge$ and ϱ is strong with respect to $\mathfrak{T}$.*

(iii) *ϱ is the projective limit of $(\varrho_\alpha)_{\alpha\in I}$.*

(iv) $\mathcal{T} \subseteq \mathcal{T}_\varrho$.
(v) $\mathfrak{T} \subseteq \mathfrak{T}_\varrho$.

Proof. Using the self-consistency of $(\varrho_\alpha)_{\alpha\in I}$ and the same arguments as in the proof of Proposition 2.1 we conclude that $(X_\alpha, \mathcal{T}_{\varrho_\alpha}, \Sigma_\alpha, \mu_\alpha, f_{\alpha\beta}, I)$ and $(X_\alpha, \mathfrak{T}_{\varrho_\alpha}, \Sigma_\alpha, \mu_\alpha, f_{\alpha\beta}, I)$ are convergent systems of complete topological probability spaces with projective limits $(X, \mathcal{T}, \Sigma, \mu, (f_\alpha)_{\alpha\in I})$ and $(X, \mathfrak{T}, \Sigma, \mu, (f_\alpha)_{\alpha\in I})$ respectively.

((i) or (ii))$\Rightarrow$(iii). Let $\mathcal{T} \subseteq \Sigma^\wedge$ and ϱ be strong with respect to $\mathcal{T}$. For $\alpha \in I$ and $h_\alpha \in \mathcal{C}_{\rm b}(X_\alpha, \mathcal{T}_{\varrho_\alpha}) = \mathcal{C}_{\rm b}(X_\alpha, \mathfrak{T}_{\varrho_\alpha})$ we have $h_\alpha \circ f_\alpha \in \mathcal{C}_{\rm b}(X, \mathcal{T}) \subseteq \mathcal{C}_{\rm b}(X, \mathfrak{T})$ and hence

$$(*) \qquad \varrho(h_\alpha \circ f_\alpha) = h_\alpha \circ f_\alpha = \varrho_\alpha(h_\alpha) \circ f_\alpha\,.$$

For $A_\alpha \in \Sigma_\alpha$ ($\alpha \in I$) the sets $f_\alpha^{-1}(A_\alpha)$ and $f_\alpha^{-1}(\varrho_\alpha(A_\alpha))$ differ only by a set of μ- measure zero. Therefore $\varrho(f_\alpha^{-1}(A_\alpha)) = \varrho(f_\alpha^{-1}(\varrho_\alpha(A_\alpha))) = f_\alpha^{-1}(\varrho_\alpha(A_\alpha))$ where the latter equality follows from $(*)$ for $h_\alpha = 1_{\varrho_\alpha(A_\alpha)}$ in $\mathcal{C}_{\rm b}(X_\alpha, \mathcal{T}_{\varrho_\alpha}) = \mathcal{C}_{\rm b}(X_\alpha, \mathfrak{T}_{\varrho_\alpha})$, $\alpha \in I$.

(iii)$\Rightarrow$((i) or (ii)) follows from Theorem 2.3.

The equivalences (i)$\Leftrightarrow$(iv) and (ii)$\Leftrightarrow$(v) hold true by [14], Theorem 3, p. 64.

4.2. Corollary. *Let* $(X_\alpha, \mathcal{T}_\alpha, \mathcal{B}^\wedge(X_\alpha), \mu_\alpha, f_{\alpha\beta}, I)$ *be a system of topological probability spaces as in Remark* 2.2(iii)(c) *with all probability measures* μ_α ($\alpha \in I$) *of full support. Suppose that* $(X, \mathcal{T}, (f_\alpha)_{\alpha\in I})$ *and* $(X, \Sigma, \mu, (f_\alpha)_{\alpha\in I})$ *are the projective limits of the systems* $(X_\alpha, \mathcal{T}_\alpha, f_{\alpha\beta}, I)$ *and* $(X_\alpha, \mathcal{B}^\wedge(X_\alpha), \mu_\alpha, f_{\alpha\beta}, I)$ *respectively, and* ϱ *is a lifting for* μ. *Then the following conditions are equivalent.*

(i) $\mathcal{T} \subseteq \Sigma^\wedge$ *and* ϱ *is strong with respect to* $\mathcal{T}$.
(ii) ϱ *is a projective limit of* $(\varrho_\alpha)_{\alpha\in I}$ *where each* ϱ_α *is the unique lifting for* μ_α *such that* $\mathcal{T}_\alpha = \mathfrak{T}_{\varrho_\alpha}$.
(iii) $\mathcal{T} \subseteq \mathfrak{T}_\varrho$.

Proof. By Remark 2.2(iii)(c) the family $(\varrho_\alpha)_{\alpha\in I}$ is self-consistent. So we may apply Theorem 4.1 to deduce the equivalence of (i)–(iii).

4.3. Corollary. *Let* $(X_\alpha, \mathcal{T}_\alpha, \mathcal{B}^\wedge(X_\alpha), \mu_\alpha, f_{\alpha\beta}, I)$ *be a system of topological probability spaces where* X_α *is compact extremally disconnected for each* $\alpha \in I$, μ_α *is a diffuse measure with full support, and* $(X_\alpha, \mathcal{B}^\wedge(X_\alpha), \mu_\alpha)$ *is a category probability space. Denote by* $(X, \mathcal{T}, (f_\alpha)_{\alpha\in I})$ *the projective limit of* $(X_\alpha, \mathcal{T}_\alpha, f_{\alpha\beta}, I)$. *Then the system* $(X_\alpha, \mathcal{B}^\wedge(X_\alpha), \mu_\alpha, f_{\alpha\beta}, I)$ *is convergent with projective limit* $(X, \Sigma, \mu, (f_\alpha)_{\alpha\in I})$ *and for a lifting* ϱ *for* μ *the following conditions are equivalent.*

(i) $\mathcal{T} \subseteq \Sigma^\wedge$ *and* ϱ *is strong with respect to* $\mathcal{T}$.

(ii) *The projective limit topology* $\mathfrak{T}$ *of* $(\mathfrak{T}_{\varrho_\alpha})_{\alpha\in I}$, *where each* ϱ_α *is the unique strong lifting for* $(X_\alpha, \mathcal{T}_\alpha, \mathcal{B}^\wedge(X_\alpha), \mu_\alpha)$, *is contained in* $\Sigma^\wedge$ *and* ϱ *is strong with respect to* $\mathfrak{T}$.

(iii) ϱ *is a projective limit of* $(\varrho_\alpha)_{\alpha\in I}$.

(iv) $\mathcal{T} \subseteq \mathcal{T}_\varrho$.

(v) $\mathfrak{T} \subseteq \mathfrak{T}_\varrho$.

Moreover, *if one of the above conditions is valid then* μ *is completion regular.*

P r o o f. According to [28], Theorem 22.3, each μ_α is Radon and completion regular. Thus by [5], Theorem 2.2, the system $(X_\alpha, \mathcal{T}_\alpha, \mathcal{B}^\wedge(X_\alpha), \mu_\alpha, f_{\alpha\beta}, I)$ is convergent. On the other hand, $\mathcal{T}_{\varrho_\alpha} = \mathcal{T}_\alpha \subseteq \mathfrak{T}_{\varrho_\alpha}$ for all $\alpha \in I$ (cf. [22], Remark 2) and by Remark 2.2(iii)(b), $(\varrho_\alpha)_{\alpha\in I}$ is self-consistent. So we may apply Theorem 4.1 to deduce the equivalence of (i)–(v).

Moreover, if (iii) is valid the completion regularity of μ follows from Theorem 3.1.

R e m a r k. In general we have $\mathcal{T} \subset \mathcal{T}_\varrho$ properly by Remark 2.2(iii)(b). Spaces as those of the system in Corollary 4.2 are given by the hyperstonian space derived from a diffuse probability space, e.g. the hyperstonian space of the Lebesgue measure space on $[0,1]$ will do (see [9]).

5. Products. Our next aim is to apply our results to products of topological probability spaces. Let $I \neq \emptyset$ be an arbitrary index set. For each $i \in I$ let $\mathcal{T}_i$ (resp. Σ_i) be a topology (resp. σ-field) in $X_i \neq \emptyset$, and let μ_i be a measure on Σ_i. For each nonempty subset J of I let $X_J := \prod_{i\in J} X_i$ be the product of $(X_i)_{i\in J}$, $\Sigma_J := \prod_{i\in J} \Sigma_i$ be the product σ-field in X_J, $\mathcal{T}_J := \prod_{i\in J} \mathcal{T}_i$ be the product topology in X_J, and $\mu_J := \prod_{i\in J} \mu_i$ be the product measure on Σ_J. For $\emptyset \neq J \subseteq K \subseteq I$ let f_{JK} be the canonical projection from X_K onto X_J given by $f_{JK}((x_j)_{j\in K}) = (x_j)_{j\in J}$. Put $X := X_I$, $\Sigma := \Sigma_I$, $\mathcal{T} := \mathcal{T}_I$, $\mu := \mu_I$, and $f_J := f_{JI}$ $(\emptyset \neq J \subseteq I)$. For $f_{\{i\}J}$ (resp. $f_{\{i\}}$) we write f_{iJ} (resp. f_i), for simplicity.

In particular, for topological measure spaces $(X_i, \mathcal{T}_i, \Sigma_i, \mu_i)$, $i \in [n] := 1, \ldots, n$, $n \in \mathbb{N}$, we write

$$X_{[n]} := \prod_{i=1}^n X_i, \quad \Sigma_{[n]} := \prod_{i=1}^n \Sigma_i, \quad \mathcal{T}_{[n]} := \prod_{i=1}^n \mathcal{T}_i, \quad \mu_{[n]} := \prod_{i=1}^n \mu_i\,.$$

Finally, put $\mathcal{F}(I) := \{\alpha : \alpha \subseteq I,\ \alpha \text{ finite}\}$.

The following lemma, proved in [22], Section 3, Theorem 1, will turn out to be useful for the proofs of the next results.

5.1. Lemma. *Let* $(X_i, \mathcal{T}_i, \mathcal{B}(X_i), \mu_i)$, $i \in [n]$, *be topological probability spaces and* $(X_{[n]}, \Sigma_{[n]}, \mu_{[n]})$ (*resp.* $\mathcal{T}_{[n]}$) *be the product of the probability spaces* $(X_i, \mathcal{B}(X_i), \mu_i)$ (*resp. of the topologies* $\mathcal{T}_i$), $i \in [n]$. *Suppose that* *there exists a strong lifting* ϱ_i *for* $(X_i, \mathcal{T}_i, \mathcal{B}^\wedge(X_i), \mu_i)$, $i \in [n]$, *and a lifting* $\varrho_{[n]}$ *for* $\mu_{[n]}$ *such that*

$$\varrho_{[n]}(f_i^{-1}(A)) = f_i^{-1}(\varrho_i(A))$$

for all $i \in [n]$ *and* $A \in \mathcal{B}^\wedge(X_i)$. *Then* $\varrho_{[n]}$ *is strong for* $(X_{[n]}, \mathcal{T}_{[n]}, \Sigma_{[n]}^\wedge, \mu_{[n]})$ *and* $\Sigma_{[n]}^\wedge = \mathcal{B}^\wedge(X_{[n]})$. *Moreover, if* X_i *is compact and* μ_i *is completion regular for each* $i \in [n]$ *then* μ *is completion regular.*

The assertion that $\varrho_{[n]}$ is strong is also true for products of spaces $(X_i, \mathcal{T}_i, \Sigma_i, \mu_i)$ with $\mathcal{T}_i \subseteq \Sigma_i$ and Σ_i complete ($i \in [n]$). The proof is the same. Using now Lemma 5.1 and the same arguments as in the proof of Corollary 3.2 one can easily deduce the following result.

5.2. COROLLARY. *Let* $(X_i, \mathcal{T}_i)$, $i \in [n]$, *be compact topological spaces*, μ_i *Baire probability measures on* X_i, *and* $(X_{[n]}, \Sigma_{[n]}, \mu_{[n]})$ (*resp.* $\mathcal{T}_{[n]}$) *the product of the probability measure spaces* $(X_i, \mathcal{B}_0(X_i), \mu_i)$ (*resp. of the topologies* $\mathcal{T}_i$), $i \in [n]$. *Suppose that there exists a strong Baire lifting* ϱ_i *for* μ_i, $i \in [n]$, *and a lifting* $\varrho_{[n]}$ *for* $\mu_{[n]}$ *such that*

$$\varrho_{[n]}(f_i^{-1}(A)) = f_i^{-1}(\varrho_i(A))$$

for all $i \in [n]$ *and* $A \in \mathcal{B}_0(X_i)$. *Then* $\varrho_{[n]}$ *is a strong Baire lifting for* $\mu_{[n]}$.

The next result extends Theorem 1, Section 3 of [22] to uncountable products.

5.3. THEOREM. *Let* $(X_i, \mathcal{T}_i, \Sigma_i, \mu_i)_{i\in I}$ *be a family of complete topological probability spaces and* (X, Σ, μ) (*resp.* $\mathcal{T}$) *the product of* (X_i, Σ_i, μ_i) (*resp.* $\mathcal{T}_i$), $i \in I$. *Suppose that* $(\varrho_\alpha)_{\alpha\in\mathcal{F}(I)}$ *is a family of liftings* ϱ_α *for* μ_α *such that*

$$\varrho_\beta(f_{\alpha\beta}^{-1}(A)) = f_{\alpha\beta}^{-1}(\varrho_\alpha(A))$$

for all $\alpha, \beta \in \mathcal{F}(I)$, $\alpha \subseteq \beta$, $A \in \Sigma_\alpha^\wedge$, *and such that each* ϱ_i *is strong for* μ_i. *Then there exists a strong lifting* ϱ *for* $(X, \mathcal{T}, \Sigma^\wedge, \mu)$ *with*

$$\varrho(f_\alpha^{-1}(A)) = f_\alpha^{-1}(\varrho_\alpha(A))$$

for all $\alpha \in \mathcal{F}(I)$ *and* $A \in \Sigma_\alpha^\wedge$. *In particular, if* $\Sigma_i = \mathcal{B}^\wedge(X_i)$ *then* $\Sigma^\wedge = \mathcal{B}^\wedge(X)$ *and if* X_i *is compact and* μ_i *is completion regular for each* $i \in I$ *then* μ *is completion regular.*

Proof. The set $\mathcal{F}(I)$ forms a directed set under inclusion; the family $(X_\alpha, \mathcal{T}_\alpha, \Sigma_\alpha, \mu_\alpha, f_{\alpha\beta}, I)$ forms a system of topological probability spaces such that (X, Σ, μ) (resp. $\mathcal{T}$) can be identified with the projective limit of $(X_\alpha, \Sigma_\alpha, \mu_\alpha)$ (resp. $\mathcal{T}_\alpha$), $\alpha \in \mathcal{F}(I)$ (cf. [15], VI, Proposition 5.4).

According to Lemma 5.1, ϱ_α is strong for $(X_\alpha, \mathcal{T}_\alpha, \Sigma_\alpha^\wedge, \mu_\alpha)$ for each $\alpha \in \mathcal{F}(I)$. Hence by Theorem 2.3 there exists a strong lifting ϱ for $(X, \mathcal{T}, \Sigma^\wedge, \mu)$

such that

$$\varrho(f_\alpha^{-1}(A)) = f_\alpha^{-1}(\varrho_\alpha(A))$$

for all $\alpha \in \mathcal{F}(I)$ and $A \in \Sigma_\alpha^\wedge$.

In particular, if $\Sigma_i = \mathcal{B}^\wedge(X_i)$ then by Lemma 5.1, $\Sigma_\alpha = \mathcal{B}^\wedge(X_\alpha)$ and by Theorem 2.3, $\Sigma^\wedge = \mathcal{B}^\wedge(X)$.

Moreover, if each X_i is compact and each μ_i is completion regular then by Lemma 5.1 each μ_α is completion regular and therefore by Theorem 3.1, μ is completion regular.

5.4. Corollary. *Let $(X_i, \mathcal{T}_i)$, $i \in I$, be compact topological spaces, μ_i Baire probability measures on X_i, and (X, Σ, μ) (resp. $\mathcal{T}$) the product of $(X_i, \mathcal{B}_0(X_i), \mu_i)$ (resp. $\mathcal{T}_i$), $i \in I$. Suppose that $(\varrho_\alpha)_{\alpha\in\mathcal{F}(I)}$ is a family of liftings ϱ_α for μ_α with*

$$\varrho_\beta(f_{\alpha\beta}^{-1}(A)) = f_{\alpha\beta}^{-1}(\varrho_\alpha(A))$$

for all $\alpha, \beta \in \mathcal{F}(I)$, $\alpha \subseteq \beta$, $A \in \mathcal{B}_0^\wedge(X_\alpha)$, and such that each ϱ_i is a strong Baire lifting for μ_i. Then there exists a strong Baire lifting ϱ for μ with

$$\varrho(f_\alpha^{-1}(A)) = f_\alpha^{-1}(\varrho_\alpha(A))$$

for all $\alpha \in \mathcal{F}(I)$ and $\alpha \in \mathcal{B}_0^\wedge(X_\alpha)$.

Proof. As shown in the proof of Theorem 5.3, $(X, \Sigma, \mu, (f_\alpha)_{\alpha\in\mathcal{F}(I)})$ (resp. $\mathcal{T}$) is the projective limit of $(X_\alpha, \mathcal{B}_0(X_\alpha), \mu_\alpha, f_{\alpha\beta}, \mathcal{F}(I))$ (resp. of $(\mathcal{T}_\alpha)_{\alpha\in\mathcal{F}(I)}$). By [2], Proposition 3, each μ_i is completion regular and therefore each ϱ_i is strong for $(X_i, \mathcal{T}_i, \mathcal{B}^\wedge(X_i), \mu_i)$. Thus Theorem 5.3 yields the desired result.

5.5. Theorem. *Let $(X_i, \mathcal{T}_i, \Sigma_i, \mu_i)$ $(i = 1, 2)$ be complete topological probability spaces, (X, Σ, μ) the completed product of (X_i, Σ_i, μ_i), and $\mathcal{T}$ the product topology $\mathcal{T}_1 \times \mathcal{T}_2$. Suppose that ϱ_1 is a strong lifting for μ_1, ϱ_2 is an almost strong lifting for μ_2 with exceptional μ_2- null set N_2, μ_2 has full support, and π is a lifting for μ with $\pi = \varrho_1 \otimes \varrho_2$, i.e. $\pi(f_1 \otimes f_2) = \varrho_1(f_1)\varrho_2(f_2)$, where $f_i \in \mathcal{L}^\infty(X_i, \mu_i)$ and $f_1 \otimes f_2 := (f_1 \circ p_1) \cdot (f_2 \circ p_2)$, with p_i the canonical projections from $X_1 \times X_2$ onto X_i $(i = 1, 2)$. Then there exist strong liftings $\varrho_2^\wedge$ for μ_2 and $\pi^\wedge$ for μ such that $\pi^\wedge = \varrho_1 \otimes \varrho_2^\wedge$, for any $f_2 \in \mathcal{L}^\infty(X_2, \mu_2)$, $\varrho_2^\wedge(f_2)|N_2^c = \varrho_2(f_2)|N_2^c$, and $\mathcal{T} \subseteq \Sigma$. If $N := X_1 \times N_2$ then $N \in \Sigma$, $\mu(N) = 0$ and $\pi^\wedge(f)|N^c = \pi(f)|N^c$ for any $f \in \mathcal{L}^\infty(X, \mu)$.*

Moreover, if $\Sigma_i = \mathcal{B}_0^\wedge(X_i)$, μ_i $(i = 1, 2)$ is completion regular, and $\mathcal{B}_0(X) = \mathcal{B}_0(X_1) \otimes \mathcal{B}_0(X_2)$ then μ is completion regular.

Proof. For $x_2 \in N_2$ let χ_{x_2} be a character on $L^\infty(X_2, \mu_2)$ such that $\chi_{x_2}([f_2]) := f_2(x_2)$ for $f_2 \in \mathcal{C}_\mathrm{b}(X_2)$ and put for any $f_2 \in \mathcal{L}^\infty(X_2, \mu_2)$,

$$\varrho_2^\wedge(f_2)(x_2) := \begin{cases} \varrho_2(f_2)(x_2) & \text{for } x_2 \in X_2' := X_2 \setminus N_2, \\ \chi_{x_2}([f_2]) & \text{for } x_2 \in N_2. \end{cases}$$

Then $\varrho_2^\wedge$ is strong for μ_2 (cf. [14], p. 127). Let

$$L^\infty(X_1,\mu_1)\otimes L^\infty(X_2,\mu_2) := \Big\{\sum_{i=1}^n [f_i\otimes g_i] : f_i\in \mathcal{L}^\infty(X_1,\mu_1),\ g_i\in\mathcal{L}^\infty(X_2,\mu_2),\ i=1,\ldots,n\Big\}$$

and let $\mathcal{A}$ be the closure of $L^\infty(X_1,\mu_1)\otimes L^\infty(X_2,\mu_2)$ in $L^\infty(X,\mu)$. We define a linear, multiplicative functional $\chi^0_{(x_1,x_2)}$ on $L^\infty(X_1,\mu_1)\otimes L^\infty(X_2,\mu_2)$ by means of

$$\chi^0_{(x_1,x_2)}\Big(\sum_{i=1}^n [f_i\otimes g_i]\Big) := \sum_{i=1}^n \varrho_1(f_i)(x_1)\chi_{x_2}([g_i])$$

for any $f_i\in\mathcal{L}^\infty(X_1,\mu_1)$, $g_i\in\mathcal{L}^\infty(X_2,\mu_2)$, $x_1\in X_1$, $x_2\in N_2$.

Denote by $\chi^\wedge_{(x_1,x_2)}$ the continuous extension of $\chi^0_{(x_1,x_2)}$ on $\mathcal{A}$, which is a character on the closed subalgebra $\mathcal{A}$ of $L^\infty(X,\mu)$. By [14], Chapter VIII, Prop. 1, there exists a character $\chi_{(x_1,x_2)}$ on $L^\infty(X,\mu)$ such that $\chi_{(x_1,x_2)}|\mathcal{A} = \chi^\wedge_{(x_1,x_2)}$, in particular

$$(*)\qquad \chi_{(x_1,x_2)}([f_1\otimes f_2]) = \varrho_1(f_1)(x_1)\chi_{x_2}([f_2])$$

for $f_i\in\mathcal{L}^\infty(X_i,\mu_i)$ $(i=1,2)$, $x_1\in X_1$, $x_2\in N_2$.

Next define for any $f\in\mathcal{L}^\infty(X,\mu)$,

$$\pi^\wedge(f)(x_1,x_2) := \begin{cases}\pi(f)(x_1,x_2) & \text{for } (x_1,x_2)\in X_1\times X_2',\\ \chi_{(x_1,x_2)}([f]) & \text{for } (x_1,x_2)\in N.\end{cases}$$

Then $\pi^\wedge$ is a lifting for μ and for $f_i\in\mathcal{L}^\infty(X_i,\mu_i)$ $(i=1,2)$, if $(x_1,x_2)\in X_1\times X_2'$, and therefore $x_2\in X_2'$, we have

$$\begin{aligned}\pi^\wedge(f_1\otimes f_2)(x_1,x_2) &= \pi(f_1\otimes f_2)(x_1,x_2) = \varrho_1(f_1)(x_1)\varrho_2(f_2)(x_2)\\ &= \varrho_1(f_1)(x_1)\varrho_2^\wedge(f_2)(x_2) = (\varrho_1\otimes\varrho_2^\wedge)(f_1\otimes f_2)(x_1,x_2),\end{aligned}$$

and if $(x_1,x_2)\in N$, i.e. $x_2\in N_2$, then

$$\begin{aligned}\pi^\wedge(f_1\otimes f_2)(x_1,x_2) &= \chi_{(x_1,x_2)}([f_1\otimes f_2]) \overset{(*)}{=} \varrho_1(f_1)(x_1)\chi_{x_2}([f_2])\\ &= \varrho_1(f_1)(x_1)\varrho_2^\wedge(f_2)(x_2)\\ &= (\varrho_1\otimes\varrho_2^\wedge)(f_1\otimes f_2)(x_1,x_2),\end{aligned}$$

i.e. $\pi^\wedge = \varrho_1\otimes\varrho_2^\wedge$. Applying now [22], Section 3, Th. 1, we conclude that $\mathcal{T}\subseteq\Sigma$ and $\pi^\wedge$ is strong for μ. The relations $\pi^\wedge(f)|N^{\mathrm c} = \pi(f)|N^{\mathrm c}$ for any $f\in\mathcal{L}^\infty(X,\mu)$ and $\varrho_2^\wedge(f_2)|N_2^{\mathrm c} = \varrho_2(f_2)|N_2^{\mathrm c}$ for any f_2 in $\mathcal{L}^\infty(X_2,\mu_2)$ follow immediately from the definitions of $\pi^\wedge$ and $\varrho_2^\wedge$ respectively. The completion regularity of μ follows from [22], Section 3, Th. 1.

5.6. Theorem. *Let $(X_i,\mathcal{T}_i,\mathcal{B}^\wedge(X_i),\mu_i)_{i\in J}$ be a family of topological probability spaces such that $(X_i,\mathcal{T}_i,\mathcal{B}^\wedge(X_i),\mu_i)$ has the USLP and μ_i has full support for each $i\in J$. Then the completed product $(X,\mathcal{T},\Sigma^\wedge,\mu)$ of $(X_i,\mathcal{T}_i,\mathcal{B}^\wedge(X_i),\mu_i)_{i\in J}$ has a strong lifting and $\Sigma^\wedge = \mathcal{B}^\wedge(X)$. Moreover, if μ_i is* *completion regular for each $i \in J$, and $\mathcal{B}_0(X_J)$ is the product of the σ-fields $\mathcal{B}_0(X_i)$ $(i \in J)$ then μ is completion regular.*

Proof. Let $\mathcal{L}$ be the set of all pairs (I, ϱ_I) where $I \subseteq J$ and ϱ_I is a strong lifting for $(X_I, \mathcal{T}_I, \mathcal{B}^\wedge(X_I), \mu_I)$. We order the set $\mathcal{L}$ as follows:

$$(I', \varrho_{I'}) \le (I'', \varrho_{I''}) \quad \text{iff} \quad I' \subseteq I'' \text{ and } \varrho_{I''} \circ f_{I'I''}^{-1} = f_{I'I''}^{-1} \circ \varrho_{I'} .$$

We will show that $\mathcal{L}$ is inductive for the above order relation. Let $\Phi = (I(j), \varrho_{I(j)})_{j \in H}$ be a totally ordered family of elements of $\mathcal{L}$ (we suppose that $j' \le j''$ iff $(I(j'), \varrho_{I(j')}) \le (I(j''), \varrho_{I(j'')})$). Let $I = \bigcup_{j \in H} I(j)$. It is easy to show that $(X_{I(j)}, \mathcal{T}_{I(j)}, \mathcal{B}^\wedge(X_{I(j)}), \mu_{I(j)}, f_{I(j)I(j')}, H)$ is a system of topological probability spaces, $(\varrho_{I(j)})_{j \in H}$ is a self- consistent system of strong liftings $\varrho_{I(j)}$ for $\mu_{I(j)}$, and $(X_I, \mathcal{T}_I, \Sigma_I, \mu_I)$ is the projective limit of the system $(X_{I(j)}, \mathcal{T}_{I(j)}, \mathcal{B}^\wedge(X_{I(j)}), \mu_{I(j)})_{j \in H}$. By Theorem 2.3 there exists a strong lifting ϱ_I for μ_I such that

$$\varrho_I \circ f_{II(j)}^{-1} = f_{II(j)}^{-1} \circ \varrho_{I(j)} \quad \text{for all } j \in H,$$

and $\Sigma_I^\wedge = \mathcal{B}^\wedge(X_I)$. Thus (I, ϱ_I) is a majorant for Φ. We now apply Zorn's lemma and obtain a maximal element (M, ϱ_M) in $\mathcal{L}$. It is sufficient to prove $M = J$. Assume that $M \ne J$ and $i \in J \setminus M$. Applying [22], Theorem 4, Section 2, we find a lifting ϱ_i for μ_i and a lifting $\varrho_{M \cup \{i\}}$ for the product of the probability measures μ_M and μ_i such that $\varrho_{M \cup \{i\}} = \varrho_i \otimes \varrho_M$. Since each $(X_i, \mathcal{T}_i, \mathcal{B}^\wedge(X_i), \mu_i)$, $i \in J$, has the USLP, each ϱ_i is almost strong. So by Theorem 5.5, there exist strong liftings $\varrho_i^\wedge$ for μ_i and $\varrho_{M \cup \{i\}}^\wedge$ for $\mu_{M \cup \{i\}}$ such that $\varrho_{M \cup \{i\}}^\wedge = \varrho_M \otimes \varrho_i^\wedge$ and $\Sigma_{M \cup \{i\}}^\wedge = \mathcal{B}^\wedge(X_{M \cup \{i\}})$, therefore

$$\varrho_{M \cup \{i\}}^\wedge \circ f_{M, M \cup \{i\}}^{-1} = f_{M, M \cup \{i\}}^{-1} \circ \varrho_M .$$

Hence $(M \cup \{i\}, \varrho_{M \cup \{i\}}^\wedge)$ is a strict majorant of (M, ϱ_M), contradicting the maximality of (M, ϱ_M). Thus $J = M$ follows and $\varrho_M = \varrho_J$ is a strong lifting for the product space $(X, \mathcal{T}, \mathcal{B}^\wedge(X), \mu) = (X, \mathcal{T}, \Sigma^\wedge, \mu)$.

Moreover, suppose that μ_i is completion regular for each $i \in J$ and $\mathcal{B}_0(X_J)$ is the product of the σ-fields $\mathcal{B}_0(X_i)$ $(i \in J)$. Let $\mathcal{L}'$ be the set of all pairs (I, ϱ_I) where $I \subseteq J$, ϱ_I is a strong lifting for $(X_I, \mathcal{T}_I, \mathcal{B}^\wedge(X_I), \mu_I)$ and μ_I is completion regular. Using the same argument as above and applying Theorem 3.1 instead of Theorem 2.3 we get the completion regularity of μ.

The following classical result (compare [14], [16] and [23]) is an immediate consequence of Theorem 5.6.

5.7. Corollary (A. and C. Ionescu Tulcea [14], Kakutani [16], and Maharam [23]). *Let $(X_i, \mathcal{T}_i, \mathcal{B}^\wedge(X_i), \mu_i)_{i \in J}$ be a family of topological probability spaces such that each X_i is a compact metric space, and each μ_i has full support. Then the completed product $(X, \mathcal{T}, \Sigma^\wedge, \mu)$ of $(X_i, \mathcal{T}_i, \mathcal{B}^\wedge(X_i), \mu_i)_{i \in J}$ has a strong lifting, $\Sigma^\wedge = \mathcal{B}^\wedge(X)$ and μ is completion regular.*

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